Almost sure Kallianpur-Robbins laws for Brownian motion in the plane

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Abstract. The Kallianpur-Robbins law describes the long term asymptotic behaviour of integrable additive functionals of Brownian motion in the plane. In this paper we prove an almost sure version of this result. It turns out that, differently from many known results, this requires an iterated logarithmic average. A similar result is obtained for the small scales asymptotic by means of an ergodic theorem of Chacon-Ornstein type, which allows an exceptional set of scales.

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1 Introduction and statement of the main theorems

It is a central paradigm in probability theory that statements about the (asymptotic) distribution of stochastic processes can be turned into almost sure or pathwise limit theorems by means of averaging. Beyond the law of large numbers, there are several early examples of such limit theorems, which use logarithmic averages, for example in the work of Lévy [23, p.270] and Erdös and coauthors [13], [17]. Renewed interest in the subject came up, when, independently, Fisher [19], [20], Brosamler [12] and Schatte [30] discovered the almost sure central limit theorem. Interesting recent progress can be found, for example, in the papers of Berkes and Dehling [6], Fahrner and Stadtmüller [18], Ibragimov and Lifshits [22], Berkes, Csaki and Csörgö [5] or the survey of Berkes [3].

The common feature of most of these results is that for a process $\{X_t\}$ we know the asymptotic distribution $P = \lim_{t\to\infty} \mathbb{P}\{X_t \in \bullet\}$ and the corresponding almost sure limit theorem states that, almost surely, for some class of functions f, the logarithmic averages converge,

$$\lim_{t\to\infty} \frac{1}{\log t} \int_1^t f(X_s)\, \frac{ds}{s} = \int f\, dP\,.$$

In this paper we will study the Kallianpur-Robbins law from this point of view. For the formulation of this law we recall the notion and some basic facts about additive functionals, see [29, Ch.X]. Let $\{B_t\}_{t\geq 0}$ be a standard linear or planar Brownian motion realized as the canonical process on d-dimensional Wiener space and denote the left shifts by $\{\theta_t \,:\, t\geq 0\}$. An *additive functional* is a nonnegative, adapted process $\{A(t) \,:\, t \geq 0\}$ on Wiener space, which is almost surely right-continuous and nondecreasing, such that $A(0) = 0$ and, for all $s, t \geq 0$, we have

$$A(s+t) = A(t) + A(s)\circ\theta_t \text{ almost surely.}$$

To each additive functional one can associate a measure ν_A by

$$\nu_A(M) = \int_{\mathbb{R}^d} \mathbb{E}^x \Big[\int_0^1 \mathbf{1}_M(B_s)\, dA(s)\Big]\, d\ell^d(x) \qquad \text{for } M\subset\mathbb{R}^d \text{ Borel.}$$

An additive functional $A(t)$ is called *integrable* if the measure ν_A is finite. The key example of an additive functional is the occupation time of Brownian motion (up to time t) in a Borel set $A \subset \mathbb{R}^d$. Then ν_A is Lebesgue measure restricted to A and integrability means that the Lebesgue measure of A is finite. The *ratio ergodic theorem* states that for each pair A, B of integrable additive functionals with total mass $\|\nu_B\| > 0$,

$$\lim_{t\to\infty} \frac{A(t)}{B(t)} = \frac{\|\nu_A\|}{\|\nu_B\|} \text{ almost surely.} \tag{1}$$

The *Kallianpur-Robbins laws* state that in the case of *linear* Brownian motion the asymptotic distribution of any integrable additive functional $\{A(t)/\sqrt{t}\}$ is the distribution of the modulus of a normally distributed random variable with expectation 0 and variance $\|\nu_A\|^2$. In the case of *planar* Brownian motion the asymptotic distribution of an integrable additive functional $\{A(t)/\log(\sqrt{t})\}$ is exponential with expectation $\|\nu_A\|/\pi$, for a proof see e.g. [29] or [14].

The Kallianpur-Robbins law for *linear* Brownian motion provides a simple example of a limit law that allows an almost sure limit theorem using logarithmic averages. Let $\mathbb{F}$ be the class of all continuous $f : [0,\infty) \to [0,\infty)$, which are bounded by some polynomial. For every integrable additive functional, $\mathbb{W}$-almost surely, for all $f \in \mathbb{F}$,

$$\lim_{t\to\infty} \frac{1}{\log t} \int_1^t f\Big(\frac{A(s)}{\sqrt{s}}\Big)\, \frac{ds}{s} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(a\|\nu_A\|)\, e^{-a^2/2}\, da\,.$$

This can be proved easily in the case that $A(t)$ is the local time at 0, using the scaling property of Brownian motion and Birkhoff's ergodic theorem. By means of (1) it can be extended to arbitrary integrable additive functionals.

It is the principal aim of this paper to prove almost sure Kallianpur Robbins laws for the more interesting case of a planar Brownian motion. Perhaps surprisingly, it turns out that one cannot obtain an almost sure version of the Kallianpur-Robbins law by taking logarithmic averages. However, we shall show that a different averaging technique is successful: Writing $\log^{(n)}$ and $\exp^{(n)}$ for the n-th iterate of the logarithm or exponential, we can define, following a clasical approach of Hardy and Riesz, see e.g. [20], for every $n \geq 1$, the *logarithmic laws of order* n as

$$\frac{1}{\log^{(n-1)}(t)} \int_{\exp^{(n-1)}(0)}^t \delta_{\{s\}}\, \frac{ds}{\prod_{j=0}^{n-2} \log^{(j)}(s)}\ , \text{ for } t > \exp^{(n-1)}(0),$$

where $\delta_{\{s\}}$ is the Dirac measure concentrated in s. Averaging with respect ot these laws includes the *Cesàro average* as logarithmic average of order one, the usual *logarithmic average* as logarithmic average of order two and what we call *iterated logarithmic average* as logarithmic average of order three. The averages are hierarchical: the convergence of averages with respect to the order n logarithmic law implies convergence with respect to all higher order logarithmic laws, but not conversely, see [20]. The log-averages are natural cases of the *Riesz means* $R(\lambda, 1)$. For a discussion of these averaging methods and their relationship, see e.g. [19], [7], [20], [8], [18]. It turns out that the *iterated logarithmic average* is the minimal choice to obtain an almost-sure version of the Kallianpur-Robbins law.

Theorem 1.1 (Time average Kallianpur-Robbins law for planar Brownian motion) *For every integrable additive functional $\{A(t)\}$ of planar Brownian motion, almost surely,*

$$\lim_{t\to\infty} \frac{1}{\log\log t} \int_e^t f\Big(\frac{A(s)}{\log\sqrt{s}}\Big)\, \frac{ds}{s\,\log s} = \int_0^\infty f\Big(\frac{a\|\nu_A\|}{\pi}\Big)\, e^{-a}\, da \qquad \text{for all } f \in \mathbb{F}.$$

One way to interpret the theorem is that we look at a *fixed* typical Brownian path and study the distribution of $A(t)/\log(\sqrt{t})$ with respect to a random choice of the time parameter t picked according to the logarithmic laws of order three. Limit laws for iterated logarithmic averages have appeared in the literature before, for example, in [11], [21] or [24] and more recently in [4].

It is natural to ask whether a pathwise Kallianpur-Robbins law can also be formulated for the random choice of a *scale* instead of time parameter. We restrict our attention to the occupation time of Brownian motion in a shrinking bounded set $B \subset \mathbb{R}^2$, defined by

$$\nu[r] = \ell\{s \in [0,1] \,:\, B_s \in rB\}\,.$$

By the scaling invariance of Brownian motion the distribution of the occupation time of Brownian motion in B up to time t coincides with the distribution of $t\cdot\nu[1/\sqrt{t}]$. This allows a different view of the Kallianpur-Robbins law: Instead of the long term behaviour we can also study the behaviour of the occupation time in a shrinking family of sets and the Kallianpur-Robbins describes the asymptotic distribution of $\{\nu[r]/(r^2\log(1/r))\}$ as $r \downarrow 0$ as exponential with expectation $\ell^2(B)/\pi$.

Although this statement is completely equivalent to the Kallianpur-Robbins law in the classical formulation, it is impossible to derive a scale average Kallianpur Robbins law from the time average Kallianpur-Robbins law: The almost sure Kallianpur-Robbins law is a statement about the occupation time in a set as time increases, which as a *process* has an entirely different distribution from the occupation time in a shrinking set. However, a pathwise Kallianpur-Robbins law for scale averages can be obtained from quite different sources and, again, iterated logarithmic averaging is necessary.

Theorem 1.2 (Scale average Kallianpur-Robbins law for planar Brownian motion) *For every bounded ℓ^2-continuity set $B \subset \mathbb{R}^2$, almost surely,*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} f\Big(\frac{\nu[r]}{r^2\log(1/r)}\Big)\,\frac{dr}{r\,\log(1/r)} = \int_0^\infty f\Big(\frac{a}{\pi}\cdot\ell^2(B)\Big)\,e^{-a}\,da \qquad \textit{for all } f \in \mathbb{F}.$$

Again, holding a typical Brownian path fixed, we can interpret this as a limit result in distribution for the occupation time in a set with a randomly chosen scale factor. The theorem was proved for the special case of the occupation time in a shrinking family of centred balls in [25]. In contrast to the long term situation one cannot easily generalize this to occupation times in other sets, because the natural analogue of the ratio ergodic theorem (1) fails to hold. There is, however, an analogous ergodic theorem of Chacon-Ornstein type if one allows a small set of exceptional scales. The following result closes the gap in the proof of Theorem 1.2, but is also interesting in its own right.

Theorem 1.3 (Ratio ergodic theorem for small scales) *For all bounded ℓ^2-continuity Borel sets $A, B \subseteq \mathbb{R}^2$ with $\ell^2(B) > 0$, almost surely, there is a set $N \subseteq (0,\infty)$ of vanishing iterated logarithmic density, i.e. such that*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \mathbf{1}_N(r)\,\frac{dr}{r\,\log(1/r)} = 0\,,$$

and such that

$$\lim_{\substack{r\to 0\\ r\notin N}} \frac{\nu[r](A)}{\nu[r](B)} = \frac{\ell^2(A)}{\ell^2(B)}\,.$$

The paper is organized as follows: In Section 2 we prove the random time Kallianpur-Robbins law and in Section 3 we indicate the key step in the proof of the ratio ergodic theorem for small scales. Those parts of the proof, which due to limited space cannot be carried out here, are available upon request from the author or on the internet at the address given at reference [26]. We finish the paper by indicating some extensions of our theorems and referring to related work.

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2 The time average Kallianpur-Robbins law

Note that, by (1), Theorem 1.1 needs only to be proved for some fixed integrable additive functional. We choose $A(t)$ to be the occupation time in an arbitrary fixed compact set $A \subset \mathbb{R}^2$ up to time t. The proof is then based on an approximation of the occupation measures by weighted occupation measures. The decay of correlation of these weighted occupation measures for different weight parameters can be calculated and we obtain an almost sure limit theorem for a random choice of the weight parameter. The statement of Theorem 1.1 can be derived from this by approximation. This proof is inspired by the beautiful proof of the weak law in [14].

Define for every weight parameter $\lambda > 0$ a random variable $\tilde{\mu}[\lambda] = \int_0^\infty e^{-\lambda t} \mathbf{1}_{\{B_t \in A\}}\, dt$. It is possible to determine the moments of these random variables explicitly.

Lemma 2.1 *We have*

$$\mathbb{E}\Big\{ \prod_{n=1}^{k} \frac{\tilde{\mu}[\lambda_n](A)}{\log\sqrt{1/\lambda_n}} \Big\} = \Big(\frac{\ell^2(A)}{\pi}\Big)^k \sum_{\gamma\in\Gamma} \prod_{n=1}^{k} \frac{\log(1/\sum_{j=n}^k \gamma_j)}{\log(1/\gamma_n)} + O\Big(\frac{1}{\log(1/\lambda_k)}\Big),$$

as $0 < \lambda_1 < \ldots < \lambda_k \to 0$, where Γ denotes the set of all permutations of $(\lambda_1, \ldots, \lambda_k)$.

Proof. For every tuple $x = (x_1, \ldots, x_k)$ we denote $x_1' = x_1$ and $x_k' = x_k - x_{k-1}$ for $k > 1$. Then

$$\begin{aligned}
&\mathbb{E}\Big\{ \prod_{n=1}^{k} \tilde{\mu}[\lambda_n](A) \Big\} \\
&= \int_0^\infty dt_1 \cdots \int_0^\infty dt_k\, e^{-\sum_{n=1}^k \lambda_n t_n} \mathbb{P}\{B_{t_1} \in A, \ldots, B_{t_k} \in A\} \\
&= \sum_{\gamma\in\Gamma} \int_0^\infty dt_1 \int_{t_1}^\infty dt_2 \cdots \int_{t_{k-1}}^\infty dt_k \Big[\int_A \cdots \int_A \prod_{n=1}^k \frac{1}{2\pi t_n'} e^{-\gamma_n t_n} e^{\frac{-\|x_n'\|^2}{2t_n'}}\, dx \Big] \\
&= \frac{1}{(2\pi)^k} \sum_{\gamma\in\Gamma} \int_A \cdots \int_A \Big[\int_0^\infty \frac{dt_1}{t_1'} e^{-\gamma_1 t_1} e^{\frac{-\|x_1'\|^2}{2t_1'}} \int_{t_1}^\infty \cdots \int_{t_{k-1}}^\infty \frac{dt_k}{t_k'} e^{-\gamma_k t_k} e^{\frac{-\|x_k'\|^2}{2t_k'}} \Big]\, dx.
\end{aligned}$$

For the innermost integral we find

$$\int_{t_{k-1}}^\infty \frac{dt_k}{t_k'} e^{-\gamma_k t_k} e^{\frac{-\|x_k'\|^2}{2t_k'}} = e^{-\gamma_k t_{k-1}} \int_0^\infty e^{-\gamma_k t} e^{\frac{-\|x_k'\|^2}{2t}} \frac{dt}{t} = e^{-\gamma_k t_{k-1}} \cdot 2K_0(\sqrt{2\gamma_k}\|x_k'\|),$$

where K_0 is the Bessel function of imaginary argument, which satisfies, see e.g. [31],

$$K_0(s) = \frac{1}{2}\int_0^\infty e^{-t-\frac{s^2}{4t}}\,\frac{dt}{t} = \log(1/s) + O(1) \text{ , for } s \downarrow 0.$$

Iterating this argument we get the following exact expression

$$\mathbb{E}\Big\{ \prod_{n=1}^k \tilde{\mu}[\lambda_n](A) \Big\} = \frac{1}{\pi^k} \sum_{\gamma\in\Gamma} \int_A \cdots \int_A \prod_{n=1}^k K_0\big(\sqrt{2\textstyle\sum_{j=n}^k \gamma_j}\|x_n'\|\big)\, dx_k \ldots dx_1 \,, \tag{2}$$

and asymptotically, as $0 < \lambda_1 < \ldots < \lambda_k \to 0$,

$$\mathbb{E}\Big\{ \prod_{n=1}^k \frac{\tilde{\mu}[\lambda_n](A)}{\log\sqrt{1/\lambda_n}} \Big\} = \Big(\frac{\ell^2(A)}{\pi}\Big)^k \sum_{\gamma\in\Gamma}\prod_{n=1}^k \frac{\log(1/\sum_{j=n}^k \gamma_j)}{\log(1/\gamma_n)} + O\Big(\frac{1}{\log(1/\lambda_k)}\Big)\,.$$

■

Lemma 2.2 *Let $k, l \geq 0$ be integers and $a_1, \ldots, a_N \in \mathbb{R}$. Then, for $0 < \kappa \leq \lambda \to 0$,*

$$\begin{aligned}
&\mathbb{E}\Big\{ \Big(\sum_{n=1}^N \frac{a_n\tilde{\mu}[n\kappa](A)}{\log\sqrt{1/n\kappa}}\Big)^k \Big(\sum_{n=1}^N \frac{a_n\tilde{\mu}[n\lambda](A)}{\log\sqrt{1/n\lambda}}\Big)^l \Big\} \\
&= \Big(\frac{\ell^2(A)}{\pi}\sum_{n=1}^N a_n\Big)^{k+l} \cdot k!l! \cdot \sum_{i=l}^{k+l} \binom{i-1}{l-1} \Big(\frac{\log(1/\lambda)}{\log(1/\kappa)}\Big)^{i-l} + O\Big(\frac{1}{\log(1/\lambda)}\Big)\,.
\end{aligned}$$

Proof. From Lemma 2.1 we get that

$$\begin{aligned}
&\mathbb{E}\Big\{ \Big(\sum_{n=1}^N \frac{a_n\tilde{\mu}[n\kappa](A)}{\log\sqrt{1/n\kappa}}\Big)^k \Big(\sum_{n=1}^N \frac{a_n\tilde{\mu}[n\lambda](A)}{\log\sqrt{1/n\lambda}}\Big)^l \Big\} \\
&= \Big(\frac{\ell^2(A)}{\pi}\cdot\sum_{n=1}^N a_n\Big)^{k+l} \cdot k!l! \cdot \sum_{\gamma\in\Gamma} \frac{\prod_{n=1}^{k+l}\log(1/\sum_{j=n}^{k+l}\gamma_j)}{\log(1/\kappa)^k\log(1/\lambda)^l} + O\Big(\frac{1}{\log(1/\lambda)}\Big)\,,
\end{aligned}$$

as $0 < \kappa < \lambda \to 0$, where Γ denotes the set of all *distinct* permutations of $(\kappa, \overset{k}{\ldots}, \kappa, \lambda, \overset{l}{\ldots}, \lambda)$. Observe now that, as $\kappa \leq \lambda \to 0$,

$$\log(1/\textstyle\sum_{j=n}^{k+l}\gamma_j) = \begin{cases} \log(1/\kappa) + O(1) & \text{if } \gamma_j = \kappa \text{ for all } j \in \{n, \ldots, k+l\}, \\ \log(1/\lambda) + O(1) & \text{if } \gamma_j = \lambda \text{ for some } j \in \{n, \ldots, k+l\}, \end{cases}$$

and hence

$$\sum_{\gamma\in\Gamma}\prod_{n=1}^{k+l}\log(1/\textstyle\sum_{j=n}^{k+l}\gamma_j) = \displaystyle\sum_{i=l}^{k+l}\binom{i-1}{l-1}\frac{\log(1/\lambda)^i}{\log(1/\kappa)^{i-k-l}} + O\Big((\log(1/\kappa))^k(\log(1/\lambda))^{l-1}\Big)\,.$$

Therefore, for $0 \le \kappa \le \lambda \to 0$,

$$\mathbb{E}\Big\{ \Big(\sum_{n=1}^N \frac{a_n \tilde\mu[n\kappa](A)}{\log\sqrt{1/n\kappa}} \Big)^k \Big(\sum_{n=1}^N \frac{a_n \tilde\mu[n\lambda](A)}{\log\sqrt{1/n\lambda}} \Big)^l \Big\}$$
$$= \Big(\frac{\ell^2(A)}{\pi} \sum_{n=1}^N a_n \Big)^{k+l} \cdot k!l! \cdot \sum_{i=l}^{k+l} \binom{i-1}{l-1} \frac{\log(1/\lambda)^i \log(1/\kappa)^{k+l-i}}{\log(1/\lambda)^l \log(1/\kappa)^k} + O\Big(\frac{1}{\log(1/\lambda)}\Big).$$

■

We now use Lemma 2.2 to prove the almost sure convergence of the moments of $\tilde\mu[\lambda](A)/\log\sqrt{1/\lambda}$ with respect to a random choice of λ.

Lemma 2.3 *Almost surely, for all $a_1, \dots, a_N \in \mathbb{R}$, we have*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \Big[\sum_{n=1}^N \frac{a_n\, \tilde\mu[n\lambda](A)}{\log\sqrt{1/n\lambda}} \Big]^k \frac{d\lambda}{\lambda \log(1/\lambda)} = k! \Big[\frac{\ell^2(A)}{\pi} \cdot \sum_{n=1}^N a_n \Big]^k. \tag{3}$$

Proof. We first fix $a_1, \dots, a_N$ and look at the expectations. We use the previous lemma to see that

$$\mathbb{E}\Big\{ \sum_{n=1}^N \frac{a_n\, \tilde\mu[n\lambda](A)}{\log\sqrt{1/\lambda}} \Big\}^k = k! \Big[\frac{\ell^2(A)}{\pi} \cdot \sum_{n=1}^N a_n \Big]^k + O\Big(\frac{1}{\log(1/\lambda)}\Big).$$

Because

$$\frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \frac{1}{\log(1/\lambda)} \frac{d\lambda}{\lambda \log(1/\lambda)} \le \frac{1}{\log\log(1/\varepsilon)}, \tag{4}$$

we conclude that, for some $C > 0$,

$$\Big| \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \mathbb{E}\Big\{ \sum_{n=1}^N \frac{a_n \tilde\mu[n\lambda](A)}{\log\sqrt{1/n\lambda}} \Big\}^k \frac{d\lambda}{\lambda \log(1/\lambda)} - k! \Big[\frac{\ell^2(A)}{\pi} \cdot \sum_{n=1}^N a_n \Big]^k \Big| \le \frac{C}{\log\log(1/\varepsilon)}.$$

Using Lemma 2.2, we get as $0 < \kappa \le \lambda \to 0$, for some constant $C > 0$,

$$\mathrm{Var}\Big\{ \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \Big[\sum_{n=1}^N \frac{a_n \tilde\mu[n\lambda](A)}{\log\sqrt{1/n\lambda}} \Big]^k \frac{d\lambda}{\lambda \log(1/\lambda)} \Big\}$$
$$\le 2\Big(\frac{\ell^2(A)}{\pi} \cdot \sum_{n=1}^N a_n \Big)^{2k} \Big(\frac{1}{\log\log(1/\varepsilon)} \Big)^2 \int_\varepsilon^{1/e} \int_\kappa^{1/e} (k!)^2 \Big(\sum_{i=k+1}^{2k} \binom{i-1}{k-1} \Big(\frac{\log(1/\lambda)}{\log(1/\kappa)} \Big)^{i-k}$$
$$+ \frac{C}{\log(1/\lambda)} \Big) \frac{d\lambda}{\lambda \log(1/\lambda)} \frac{d\kappa}{\kappa \log(1/\kappa)}.$$

Using again (4) and

$$\Big(\frac{1}{\log\log(1/\varepsilon)} \Big)^2 \int_\varepsilon^{1/e} \int_\kappa^{1/e} \frac{\log(1/\lambda)}{\log(1/\kappa)} \frac{d\lambda}{\lambda \log(1/\lambda)} \frac{d\kappa}{\kappa \log(1/\kappa)} \le \frac{1}{\log\log(1/\varepsilon)},$$

we conclude that, for a suitable constant $C > 0$,

$$\mathrm{Var}\left\{ \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \left[\sum_{n=1}^N \frac{a_n \tilde{\mu}[n\lambda](A)}{\log \sqrt{1/n\lambda}} \right]^k \frac{d\lambda}{\lambda \log(1/\lambda)} \right\} \le \frac{C}{\log\log(1/\varepsilon)}.$$

By Chebyshev's inequality, for every $\delta > 0$ and sufficiently small ε,

$$\mathbb{P}\left\{ \left| \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \left[\sum_{n=1}^N \frac{a_n \tilde{\mu}[n\lambda](A)}{\log \sqrt{1/n\lambda}} \right]^k \frac{d\lambda}{\lambda \log(1/\lambda)} - k! \left[\frac{\ell^2(A)}{\pi} \cdot \sum_{n=1}^N a_n \right]^k \right| > 2\delta \right\}$$

$$\le \frac{C}{\delta^2 \cdot \log\log(1/\varepsilon)}.$$

Let $\varepsilon_j = \exp(-\exp(j^2))$. Then, by the Borel-Cantelli lemma, almost surely,

$$\lim_{j\to\infty} \frac{1}{\log\log(1/\varepsilon_j)} \int_{\varepsilon_j}^{1/e} \left[\frac{\sum_{n=1}^N a_n \tilde{\mu}[n\lambda](A)}{\log \sqrt{1/\lambda}} \right]^k \frac{d\lambda}{\lambda \log(1/\lambda)} = k! \left[\frac{\ell^2(A)}{\pi} \cdot \sum_{n=1}^N a_n \right]^k.$$

In particular, we get

$$\begin{aligned}
&\limsup_{j\to\infty} \frac{1}{\log\log(1/\varepsilon_j)} \int_{\varepsilon_{j+1}}^{\varepsilon_j} \left| \frac{\sum_{n=1}^N a_n \tilde{\mu}[n\lambda](A)}{\log \sqrt{1/\lambda}} \right|^k \frac{d\lambda}{\lambda \log(1/\lambda)} \\
\le\; & \lim_{j\to\infty} \frac{\log\log(1/\varepsilon_{j+1})}{\log\log(1/\varepsilon_j)} \cdot \lim_{j\to\infty} \frac{1}{\log\log(1/\varepsilon_{j+1})} \int_{\varepsilon_{j+1}}^{1/e} \left[\frac{\sum_{n=1}^N |a_n| \tilde{\mu}[n\lambda](A)}{\log \sqrt{1/\lambda}} \right]^k \frac{d\lambda}{\lambda \log(1/\lambda)} \\
& \quad - \lim_{j\to\infty} \frac{1}{\log\log(1/\varepsilon_j)} \int_{\varepsilon_j}^{1/e} \left[\frac{\sum_{n=1}^N |a_n| \tilde{\mu}[n\lambda](A)}{\log \sqrt{1/\lambda}} \right]^k \frac{d\lambda}{\lambda \log(1/\lambda)} \\
=\; & 0.
\end{aligned}$$

Hence, for $\varepsilon_{j+1} \le \varepsilon \le \varepsilon_j$,

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \left[\frac{\sum_{n=1}^N a_n \tilde{\mu}[n\lambda](A)}{\log \sqrt{1/\lambda}} \right]^k \frac{d\lambda}{\lambda \log(1/\lambda)} \\
\le\; & \lim_{j\to\infty} \frac{1}{\log\log(1/\varepsilon_j)} \int_{\varepsilon_j}^{1/e} \left[\frac{\sum_{n=1}^N a_n \tilde{\mu}[n\lambda](A)}{\log \sqrt{1/\lambda}} \right]^k \frac{d\lambda}{\lambda \log(1/\lambda)} \\
& + \limsup_{j\to\infty} \frac{1}{\log\log(1/\varepsilon_j)} \int_{\varepsilon_{j+1}}^{\varepsilon_j} \left| \frac{\sum_{n=1}^N a_n \tilde{\mu}[n\lambda](A)}{\log \sqrt{1/\lambda}} \right|^k \frac{d\lambda}{\lambda \log(1/\lambda)} \\
=\; & k! \left[\frac{\ell^2(A)}{\pi} \cdot \sum_{n=1}^N a_n \right]^k.
\end{aligned}$$

Similarly, we get

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \left[\frac{\sum_{n=1}^N a_n \tilde{\mu}[n\lambda](A)}{\log \sqrt{1/\lambda}} \right]^k \frac{d\lambda}{\lambda \log(1/\lambda)} \ge k! \left[\frac{\ell^2(A)}{\pi} \cdot \sum_{n=1}^N a_n \right]^k,$$

and hence (3). Finally, we observe that (3) holds almost surely simultaneously for all tuples with rational entries and thus, by approximation, almost surely, for all $a_1, \ldots, a_N \in \mathbb{R}$. ■

We now go back to the study of the additive functionals, using an approximation argument. The case $k = 1$ of the following lemma is Brosamler's Theorem, which was proved in [11] by a different method.

Lemma 2.4

$$\lim_{t\to\infty} \frac{1}{\log\log t} \int_e^t \left[\frac{A(s)}{\log\sqrt{s}}\right]^k \frac{ds}{s\,\log s} = k!\,\left[\frac{\ell^2(A)}{\pi}\right]^k \textit{ almost surely.} \tag{5}$$

Proof. We look at the continuous function $g : [0,1] \to [0,1]$ with $g(x) = 1$ if $x \geq 1/e$ and $g(x) = 0$ if $x < 1/e^2$, which is linear on the interval $(1/e^2, 1/e)$. We can now find, for every $\eta > 0$, polynomials $p = \sum_{n=1}^N p_n x^n$ and $P = \sum_{n=1}^N P_n x^n$ such that, for every $0 \leq x \leq 1$,

$$p(x) \leq g(x) \leq P(x) \leq p(x) + \eta\,.$$

Observe that $g(e^{-\lambda t}) = 1$ if $t \leq 1/\lambda$ and hence

$$\mu[1/\lambda](A) \leq \int_0^\infty g(e^{-\lambda t})\mathbf{1}_{\{B_t\in A\}}\,dt \leq \int_0^\infty P(e^{-\lambda t})\mathbf{1}_{\{B_t\in A\}}\,dt = \sum_{n=1}^N P_n\tilde{\mu}[n\lambda](A).$$

As $\sum P_n \leq 1 + \eta$ we may use Lemma 2.3 to conclude that, for all k,

$$\limsup_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \left[\frac{\mu[1/\lambda](A)}{\log\sqrt{1/\lambda}}\right]^k \frac{d\lambda}{\lambda\,\log(1/\lambda)} \leq k!\,\left[\frac{\ell^2(A)}{\pi}\cdot(1+\eta)\right]^k.$$

Similarly, we have $g(e^{-\lambda t}) = 0$ if $t \geq 2/\lambda$ and hence

$$\mu[2/\lambda](A) \geq \int_0^\infty g(e^{-\lambda t})\mathbf{1}_{\{B_t\in A\}}\,dt \geq \int_0^\infty p(e^{-\lambda t})\mathbf{1}_{\{B_t\in A\}}\,dt = \sum_{n=1}^N p_n\tilde{\mu}[n\lambda](A)\,.$$

Upon observing that $\sum p_n \geq 1 - \eta$, we infer that, for all k,

$$\begin{aligned}
&\liminf_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \left[\frac{\mu[1/\lambda](A)}{\log\sqrt{1/\lambda}}\right]^k \frac{d\lambda}{\lambda\,\log(1/\lambda)}\\
=\ &\liminf_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \left[\frac{\mu[2/\lambda](A)}{\log\sqrt{1/\lambda}}\right]^k \frac{d\lambda}{\lambda\,\log(1/\lambda)}\\
\geq\ &k!\,\left[\frac{\ell^2(A)}{\pi}(1-\eta)\right]^k.
\end{aligned}$$

As these inequalities hold for every $\eta > 0$ we conclude that, almost surely,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \left[\frac{\mu[1/\lambda](A)}{\log\sqrt{1/\lambda}}\right]^k \frac{d\lambda}{\lambda\,\log(1/\lambda)} = k!\,\left[\frac{\ell^2(A)}{\pi}\right]^k.$$

We finally conclude

$$\lim_{t\to\infty} \frac{1}{\log\log t} \int_e^t \left[\frac{\mu[s](A)}{\log\sqrt{s}}\right]^k \frac{ds}{s\,\log s} = k! \left[\frac{\ell^2(A)}{\pi}\right]^k, \tag{6}$$

by means of a change of variable. ∎

To finish the proof of Theorem 1.1 observe that the limits we get in (6) are the moments of an exponential distribution with parameter $\|\nu_A\|/\pi$. Hence we can get the full statement from the method of moments, see e.g. [10]. ∎

3 The scale average Kallianpur-Robbins law

The scale average Kallianpur-Robbins law for planar Brownian motion follows directly from the known case together with Theorem 1.3. We restrict our attention to the key step in the proof of the latter theorem, which is based on a generalization of a method of Ray [28]. He showed that the occupation measure of small balls can be approximated by means of the number of inward crossings of small annuli performed by the Brownian motion before it leaves the unit ball. This argument was the key to the proof of the special case in [25]. We generalize this approach and approximate the occupation measure of small sets *of arbitrary shape* by means of these crossing numbers. This is not easy, because Ray's proof makes substantial use of the symmetry of the balls.

To make this precise, let τ be the stopping time given by $\tau = \inf\{t > 0 : |B_t| \geq 1\}$. Denote by μ the occupation measure of the stopped Brownian motion. Fix a number $b > 0$ and define $a_n = e^{-bn}$. For every n we define stopping times $t_0 = 0$, $t_{2k+1} = \inf\{t > t_{2k} : |B_t| > a_{n-1}\}$ and $t_{2k} = \inf\{t > t_{2k-1} : |B_t| < a_n\}$. Define the *crossing numbers* as $N_n = \max\{k \geq 0 : t_{2k} < \tau\}$, the number of inward crossings of the annulus with radii a_{n-1} and a_n.

Theorem 3.1 *Suppose A is a Borel set of the open unit ball $B(0,1)$. Then, almost surely,*

$$\mu\Big(a_n A\Big) = a_n^2 \cdot N_n b \cdot \Big[\frac{\ell^2(A)}{\pi} + o(1)\Big] + o(na_n^2)\ ,\ \textit{as } n \to \infty. \tag{7}$$

From Theorem 3.1 one can see that, unless N_n/n is too small, for large n the ratio of the occupation times in two sets $a_n A$ and $a_n B$ is close to $\ell^2(A)/\ell^2(B)$. Because $\{N_n/n\}$ forms a null-recurrent Markov chain, N_n/n is occasionally small, but by means of the explicitly known transition probabilities of the chain one can show that this happens sufficiently rarely and Theorem 1.3 holds.

We now give the proof of Theorem 3.1. Identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and write $B_t = |B_t| \cdot \exp(i\vartheta_s)$ with $\vartheta_s \in [0, 2\pi)$. Occasionally it is convenient to identify $e^{i\theta} \in \partial B(0,1)$ and $\theta \in [0, 2\pi)$. The random variables M_k^n with values in $[0,2\pi) \times [0, 2\pi)$ are defined by

$$M_k^n := (\Theta_{2k}, \Theta_{2k+1}) := (\vartheta_{t_{2k}}, \vartheta_{t_{2k+1}})\,,$$

where we have omitted the dependence of the stopping times t_k and angles Θ_k on n for notational convenience. For every n the sequence $\{M_k^n\}_{k\geq 1}$ of random variables is a stationary, ergodic

Markov chain. If h_θ is the harmonic measure on $\partial B(0,1)$ at $e^{-b}e^{i\theta}$, then the invariant distribution of $\{M_k^n\}_{k\geq 1}$ is given by

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \delta_{\{(\theta_0,\theta_1)\}}\, dh_{\theta_0}(\theta_1)\, d\theta_0 \,.$$

If g_θ is the harmonic measure on $\partial B(0,1)$ at $e^b e^{i\theta}$, then the transition probabilities of $\{M_k^n\}_{k\geq 1}$ from (θ_0,θ_1) are given by

$$\int_0^{2\pi}\int_0^{2\pi} \delta_{\{(\theta_2,\theta_3)\}}\, dh_{\theta_2}(\theta_3)\, dg_{\theta_1}(\theta_2)\,.$$

We require some basic facts about Brownian motions conditioned to exit the ball $B(0,e^b)$ at a certain point. These facts are provided by well known tools from probabilistic potential theory, see [2] for an excellent introduction.

Lemma 3.2 *For every $(\theta_1,\theta_2)\in[0,2\pi)\times[0,2\pi)$ there is a process $\{X_t(\theta_1,\theta_2)\}_{t\geq 0}$ on a probability space $(\Omega,\mathcal{A},P)$ such that*

(i) if $\{B_t\}_{t\geq 0}$ is a Brownian motion started in $e^{i\theta_1}$, ρ is the first exit time from $B(0,e^b)$ and $\Theta\in[0,2\pi)$ is the exit point defined by $B_\rho = e^b e^{i\Theta}$, then the distribution of the process $\{X_t(\theta_1,\theta_2)\}_{t\geq 0}$ is a conditional distribution of $\{B_t\}_{t\geq 0}$ given $\Theta=\theta_2$,

(ii) if ρ is the first exit time of $\{X_t(\theta_1,\theta_2)\}$ from $B(0,e^b)$, then, for every Borel set $A\subseteq B(0,1)$ the function

$$\begin{array}{rccl} f_A(\,\cdot\,;b): & [0,2\pi)\times[0,2\pi) & \longrightarrow & [0,\infty) \\ & (\theta_1,\theta_2) & \mapsto & \mathbb{E}_P\Big\{\int_0^\rho \mathbf{1}_A(X_t(\theta_1,\theta_2))\,dt\Big\} \end{array}$$

is continuous, where $\mathbb{E}_P$ denotes the expectation with respect to P,

(iii) for every $m\geq 1$ and Borel set $A\subseteq B(0,1)$ the moments $\mathbb{E}_P\big\{\big(\int_0^\rho \mathbf{1}_A(X_t(\theta_1,\theta_2))\,dt\big)^m\big\}$ are bounded in (θ_1,θ_2).

Proof. Let $(\theta_1,\theta_2)\in[0,2\pi)\times[0,2\pi)$. We only have to construct the process $\{X_t\}=\{X_t(\theta_1,\theta_2)\}$ up to the first exit time ρ from $B(0,e^b)$. Define the harmonic function

$$h(x)=\frac{e^{2b}-|x|^2}{|e^b e^{i\theta_2}-x|^2} \text{ for } x\in B(0,e^b).$$

Let $\{X_t\}_{t\geq 0}$ on $(\Omega,\mathcal{A},\tilde P)$ be a Brownian motion started in $X_0=e^{i\theta_1}$ and killed upon exiting the domain $B(0,e^b)$. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration for this Brownian motion with $\mathcal{A}=\bigcup_{t\geq 0}\mathcal{F}_t$ and recall that $\{h(X_{t\wedge\rho})\}_{t\geq 0}$ is an $\{\mathcal{F}_t\}$-martingale. The *h-path transform* of $\{X_t\}$ is defined by

$$P_h(M)=\int_M \frac{h(X_{t\wedge\rho})}{h(X_0)}\,d\tilde P \text{ for } M\in\mathcal{F}_t.$$

For properties of the h-path transform we refer the reader to [2]. The required process is $\{X_t\}_{0\leq t\leq\rho}$ on the probability space $(\Omega,\mathcal{A},P_h)$. Property (i) is Proposition (2.7) in [2]. To check property (ii) we look at a sequence $b_n \uparrow b$ and denote the first exit time from $B(0, e^{b_n})$ by ρ_n. Then

$$\begin{aligned}
& f_A(\theta_1, \theta_2, ; b) \\
&= \lim_{n\to\infty} \int_0^\infty \int \mathbf{1}_{\{s<\rho_n\}} \mathbf{1}_A(X_s)\, dP_h\, ds = \lim_{n\to\infty} \int_0^\infty \int \mathbf{1}_{\{s<\rho_n\}} \mathbf{1}_A(X_s) \frac{h(X_s)}{h(e^{i\theta_1})}\, d\tilde{P}\, ds \\
&= \lim_{n\to\infty} \int \int_0^{\rho_n} \mathbf{1}_A(X_s) \frac{h(X_s)}{h(e^{i\theta_1})}\, ds\, d\tilde{P} = \lim_{n\to\infty} \int G_{b_n}(e^{i\theta_1}, y) \mathbf{1}_A(y) \frac{h(y)}{h(e^{i\theta_1})}\, dy \\
&= \int_A G_b(e^{i\theta_1}, y) \frac{h(y)}{h(e^{i\theta_1})}\, dy\,,
\end{aligned}$$

where G_b denotes Green's function for the domain $B(0, e^b)$. As the integrand of the last expression is uniformly equicontinuous in (θ_1, θ_2) for all $y \in A$, this implies property (ii). Finally, the boundedness of the moments of ρ follows from the inequality $P_h\{\rho > t\} \le c_1 \exp(-c_2 t)$, for positive constants c_1, c_2 not depending on (θ_1, θ_2), see [1]. ∎

Lemma 3.3

$$\frac{1}{2\pi b} \int_0^{2\pi} \int_0^{2\pi} f_A(\theta_1, \theta_2; b)\, dh_{\theta_1}(\theta_2)\, d\theta_1 = \frac{\ell^2(A)}{\pi}.$$

Proof. Note that $\int_0^{2\pi} f_A(\theta_1, \theta_2; b)\, dh_{\theta_1}(\theta_2) = \int_A G_b(e^{i\theta_1}, y)\, dy$ and $\int_0^{2\pi} G_b(e^{i\theta_1}, y)\, d\theta_1$ is a harmonic function inside $B(0,1)$, which is constant on the boundary. Hence this function is constant and equal to its value at 0, which is $2b$. ∎

We now fix n and let

$$T_k^n = \int_{t_{2k}}^{t_{2k+1}} \mathbf{1}_{\{B_s \in a_n A\}}\, ds\,, \qquad \text{hence} \qquad \mu(a_n A) = \sum_{k=0}^{N_n} T_k^n\,. \tag{8}$$

Conditional on the sequence $M^n = \{M_k^n\}_{k\ge1}$ the T_k^n are independent and T_k^n has the same distribution as $a_n^2 T(\Theta_{2k}, \Theta_{2k+1}; b)$, where

$$T(\theta_1, \theta_2; b) = \int_0^\rho \mathbf{1}_{\{X_t(\theta_1,\theta_2)\in A\}}\, dt\,,$$

and $\{X_t(\theta_1, \theta_2)\}_{t\ge0}$ is as in Lemma 3.2. Hence,

$$\begin{aligned}
& \mathbb{E}\Big\{ \Big(\sum_{k=1}^N (T_k^n - f_A(\Theta_{2k}; \Theta_{2k+1}; b)\cdot a_n^2) \Big)^4 \,\Big|\, M^n \Big\} \\
&= a_n^8 \sum_{k=1}^N \mathbb{E}_P \Big\{ \Big(T(\Theta_{2k}, \Theta_{2k+1}, b) - f_A(\Theta_{2k}, \Theta_{2k+1}; b) \Big)^4 \Big\} \\
&\quad + 6a_n^8 \sum_{k=1}^N \sum_{j=k+1}^N \mathbb{E}_P \Big\{ \Big(T(\Theta_{2k}, \Theta_{2k+1}; b) - f_A(\Theta_{2k}, \Theta_{2k+1}; b) \Big)^2 \Big\} \\
&\quad \times \mathbb{E}_P \Big\{ \Big(T(\Theta_{2j}, \Theta_{2j+1}; b) - f_A(\Theta_{2j}, \Theta_{2j+1}; b) \Big)^2 \Big\}.
\end{aligned}$$

By Lemma 3.2(*iii*), for every positive integer m, $\mathbb{E}_P\{T(\theta_1,\theta_2;b)^m\}$ is bounded in (θ_1,θ_2) and hence the expression above is bounded by a constant multiple of $N^2 a_n^8$. We now apply Markov's inequality and get, for every $\varepsilon > 0$,

$$\mathbb{P}\Big\{\Big|\sum_{k=1}^{N}\Big(T_k^n - f_A(\Theta_{2k},\Theta_{2k+1},b)\cdot a_n^2\Big)\Big| > \varepsilon\cdot n a_n^2 \,\Big|\, M^n\Big\}$$
$$\leq \frac{1}{\varepsilon^4 n^4 a_n^8}\,\mathbb{E}\Big\{\Big[\sum_{k=1}^{N}\Big(T_k^n - f_A(\Theta_{2k},\Theta_{2k+1},b)\cdot a_n^2\Big)\Big]^4\Big| M^n\Big\} \leq C\cdot\frac{N^2}{n^4\varepsilon^4}\,.$$

Because $\{T_k^n\}_{k\geq 1}$ and N_n are independent conditional on $\{M_k^n\}_{k\geq 1}$ we have

$$\mathbb{P}\Big\{\Big|\sum_{k=1}^{N_n}\Big(T_k^n - f_A(\Theta_{2k},\Theta_{2k+1};b)\cdot a_n^2\Big)\Big| > \varepsilon\cdot n a_n^2 \,\Big|\, M^n\Big\} \leq C\cdot\frac{\mathbb{E}\{N_n^2|M^n\}}{n^4\varepsilon^4}\,,$$

and, together with (8), this gives

$$\mathbb{P}\Big\{\Big|\mu(a_n A) - T_0^n - \sum_{k=1}^{N_n} f_A(\Theta_{2k},\Theta_{2k+1};b)\cdot a_n^2\Big| > \varepsilon\cdot n a_n^2\Big\} \leq C\cdot\frac{\mathbb{E}\{N_n^2\}}{n^4\varepsilon^4} \leq \frac{4C}{n^2\varepsilon^4}\,, \tag{9}$$

using in the last step that $\mathbb{E}N_n^2 \leq 4$. Also T_0^n is bounded by the first time the Brownian motion hits the sphere of radius a_{n-1}. The second moment of this stopping time is bounded by a constant multiple of a_n^4 and thus we get from Markov's inequality, for every $\varepsilon > 0$,

$$\mathbb{P}\{|T_0^n| > \varepsilon\cdot n a_n^2\} < \frac{C}{n^2\varepsilon^2}\,. \tag{10}$$

By the Borel-Cantelli lemma we conclude from (9) and (10) that, almost surely,

$$\mu(a_n A) = \sum_{k=1}^{N_n} f_A(M_k^n;b)\cdot a_n^2 + o(n a_n^2)\ ,\ \text{as } n\to\infty. \tag{11}$$

In the next step we show that, for every $\varepsilon > 0$, if $N_n \geq n\varepsilon$ and n is sufficiently large, then

$$\Big|\sum_{k=1}^{N_n} f_A(M_k^n;b) - N_n b\cdot\frac{\ell^2(A)}{\pi}\Big| \leq \varepsilon\cdot N_n\,. \tag{12}$$

Recall Lemma 3.3 to see that the subtracted term is N_n times the expectation of the summands. Due to the dependence of the chain $\{M_k^n\}_{k\geq 0}$ on n, however, the proof of (12) requires a more subtle argument than the ergodic theorem alone. We will make use of a large deviations principle of Stroock for Markov chains satisfying a uniformity condition, which can be found in [16, Ch.6.3]. The transition densities $\pi((\theta_1,\theta_2),(\theta_3,\theta_4))$ of our Markov chain, which are given in terms of the harmonic measures, satisfy

$$\pi((\theta_1,\theta_2),(\theta_3,\theta_4)) \leq \left[\frac{(1+e^b)(1+e^{-b})}{(1-e^b)(1-e^{-b})}\right]^2\cdot\pi((\theta_1',\theta_2'),(\theta_3,\theta_4)) \ \text{ for all } \theta_i,\,\theta_i'\in[0,2\pi),$$

which means that the uniformity assumption is fulfilled. Also, by Lemma 3.2(ii), the set of all probability measures ν on $[0,2\pi)\times[0,2\pi)$ satisfying $|\int f_A d\nu - \ell^2(A)b/\pi| \geq \varepsilon$ is compact. Hence we infer from the uniform large deviation principle for the empirical measure of $\{M_k^n\}_{k\geq 1}$, see Theorem 6.3.8 and the following remark in [16], that, for every $\varepsilon > 0$ there is some $\delta > 0$ depending, of course, on $\varepsilon > 0$, such that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}\Big\{ \Big| \frac{1}{N} \sum_{k=1}^{N} f_A(M_k^n; b) - \frac{\ell^2(A)b}{\pi} \Big| \geq \varepsilon \Big\} < -\delta \, .$$

Note that the probabilities above are independent of the choice of n, as the distributions of the Markov chains $\{M_k^n\}_{k\geq 1}$ do *not* depend on n. We conclude that, for sufficiently large integers n, denoting by $[n\varepsilon]$ the integer part of $n\varepsilon$,

$$\begin{aligned}
& \mathbb{P}\Big\{ \Big| \frac{1}{N_n} \sum_{k=1}^{N_n} f_A(M_k^n; b) - \ell^2(A)b/\pi \Big| \geq \varepsilon \text{ and } N_n \geq n\varepsilon \Big\} \\
\leq & \sum_{N=[n\varepsilon]}^{\infty} \mathbb{P}\Big\{ \Big| \frac{1}{N} \sum_{k=1}^{N} f_A(M_k^n; b) - \ell^2(A)b/\pi \Big| \geq \varepsilon \Big\} \\
\leq & \sum_{N=[n\varepsilon]}^{\infty} e^{-N\delta} = e^{-\delta[n\varepsilon]} \cdot \frac{1}{1-e^{-\delta}} \, .
\end{aligned}$$

Hence, by the Borel-Cantelli lemma, almost surely, for all but finitely many n,

$$\Big| \sum_{k=1}^{N_n} f_A(\Theta_{2k}, \Theta_{2k+1}; b) - N_n b \cdot \frac{\ell^2(A)}{\pi} \Big| \leq \begin{cases} (\varepsilon\cdot n)\cdot \sup_{\theta_1,\theta_2} f_A(\theta_1,\theta_2;b) & \text{if } N_n < \varepsilon n, \\ (\varepsilon \cdot N_n) & \text{if } N_n \geq \varepsilon n. \end{cases}$$

As this holds for every $\varepsilon > 0$ it implies, together with (11), the required approximation (7). ■

4 Further remarks, extensions and open problems

- The techniques presented here also allow the proof of the divergence of the logarithmic averages in Theorems 1.1 and 1.2, see [25] resp. [26] for more details.
- One can extend our main results in such a way that certain families of additive functionals are covered simultaneously, i.e. with a single zeroset of exceptional paths. An interesting extension, carried out in [26], shows that one can get a limit law simultaneously for all weakly continuous bounded functionals of the occupation *measures*.
- A different kind of pathwise Kallianpur-Robbins law, namely for the random choice of a spatial shift parameter, is described in Section 10.3. of [15].
- An alternative approach to the time average Kallianpur-Robbins law is via the skew-product approximation of planar Brownian motion. This was used in [27] to prove an almost sure version of Spitzer's theorem on the windings of a Brownian motion. This result also requires iterated logarithmic averaging.

- A celebrated result of *Darling and Kac* (see e.g. [9] or [14]) describes the possible limit laws of a class of (not necesarily integrable) additive functionals as the class of Mittag-Leffler laws. The corresponding limit laws may have an almost-sure version either for ordinary logarithmic averages (for example for the occupation time in a stripe of $\mathbb{R}^2$, by the result on linear Brownian motion) or only for iterated logarithmic averages (for example for the occupation time of bounded sets). It would be interesting to see where exactly the boundary between the two regimes is.

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