Intersection exponents and the multifractal spectrum for measures on Brownian paths

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Abstract. The aim of this paper is to give a survey of recent developments in the multifractal analysis of measures arising in the study of Brownian motion. We first discuss the harmonic measure on a Brownian path, studied by Lawler, and the intersection local time on the intersection of two independent Brownian paths, studied by Klenke and Mörters. In both examples the *intersection exponents* of Brownian motion play a crucial role in the multifractal analysis. Whereas the first example is in line with the multifractal formalism, the second example is not and more subtle effects come into the picture. Then we discuss the occupation measure of Brownian motion, in which case a *refined* multifractal analysis was carried out by Dembo, Peres, Rosen and Zeitouni.

1. Introduction

An interesting fractal geometric characteristic for discrete and continuous models of statistical physics is the *multifractal spectrum*, which originated in the study of turbulence models, see e.g. [33]. Loosely speaking, the multifractal spectrum evaluates the degree of variation in the local intensity of a spatial distribution, or, in other words, it describes quantitatively how frequent a certain type of local irregularity appears in a fractal measure, set or function.

There are many different ways in which this concept can be made precise, see for example the contribution of Lévy-Vehel in this volume. In this paper I would like to focus on the multifractal spectrum as discussed e.g. in [13, 40, 4], where irregularities are described in terms of local dimensions and the size of sets is measured by the Hausdorff dimension. There is a host of literature on this particular type of spectrum spanning almost two decades, the reader is referred to the survey [35] and the extensive list of references given there.

Received by the editors August 30, 2003.
1991 *Mathematics Subject Classification.* Primary 60J65 Secondary 28A80.
Key words and phrases. Brownian motion, occupation measure, harmonic measure, intersection local time, critical exponent, intersection exponent, multifractal spectrum, multifractal formalism, exceptional set, large deviation heuristic.

To give the definition of this type of multifractal spectrum, suppose that μ is a (fractal) measure on $\mathbb{R}^d$. One can define the *local dimension* of the measure μ at the point $x \in \mathbb{R}^d$ as

$$\lim_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r} = a, \tag{1.1}$$

if the limit exists. In words, this is the rate of decay of the measure $\mu(B(x,r))$ as the concentric balls $B(x,r)$ of radius $r \downarrow 0$ zoom into the point x. The value $f(a)$ of the multifractal spectrum is the Hausdorff dimension of the set of points x with local dimension a, i.e.

$$\dim \Big\{ x \in \mathbb{R}^d \, : \, \lim_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r} = a \Big\}. \tag{1.2}$$

In many cases of interest, the limit in (1.1) has to be replaced by liminf or limsup to obtain an interesting nontrivial spectrum. A measure is called *multifractal* if $f(a) > 0$ for more than one value of $a \in \mathbb{R}$, in any of these definitions.

Whereas much of the earlier literature on multifractal theory was focused on the analysis of (mostly deterministic) objects with strong features of self-similarity, and the verification of the multifractal formalism proposed, e.g. in [13], the interest has shifted more recently towards the analysis of stochastic processes by means of multifractal spectra. This trend can be seen in applied fields, like the analysis of time series, see for example [15, 42], as well as in mathematical physics, see [10] for a survey from a physics point of view. In the probability context new patterns have emerged and a lot of interest is going towards measures where the multifractal formalism fails, and towards refinements of the multifractal formalism.

In the present paper we shall review these developments by considering three natural measures on the path of a Brownian motion in dimension $d = 3$.[1] These examples are

- the *harmonic measure* on a Brownian path, studied by Lawler [24, 25, 26],
- the *intersection local times* on the intersection of two independent Brownian path, studied by Klenke and Mörters [18],
- the *occupation measure* on a Brownian path, studied by Dembo, Peres, Rosen and Zeitouni [6, 7], see [5] for an introduction.

The main emphasis will be on the first two examples, as they give us also the opportunity to highlight another important subject, which has emerged in recent years in fractal geometry, namely the relation of Hausdorff dimension and the critical exponents of statistical physics, in this case the *intersection exponents*.

We start the paper with an introduction of intersection exponents in the next section, followed by three sections discussing our three examples. A final section tries to wrap up the developments described in this paper, hinting also at some expectations for future developments.

[1] The focus is on $d = 3$ for better comparison, though some results extend to other dimensions.

2. Intersection exponents

The intersection exponents for random curves measure how likely these curves do not intersect, and maybe should therefore be called *non*-intersection exponents. They contain crucial information about the fractal geometry of the curves, and substantial recent research is devoted to exploiting this information.

To define the intersection exponents for Brownian motion in $d = 2, 3$, suppose $n, m \geq 1$ are integers and let $W_1, \ldots, W_{n+m}$ are independent Brownian motions in $\mathbb{R}^d$ started uniformly on $\partial B(0,1)$ and running up to the first exit times $T^i(r)$ from a large ball $B(0,r)$. We divide the motions into two packets and look at the union of the paths in each family,

$$\mathfrak{B}_1(r) = W_1[0, T^1(r)] \cup \ldots \cup W_n[0, T^n(r)] \text{ and}$$
$$\mathfrak{B}_2(r) = W_{n+1}[0, T^{n+1}(r)] \cup \ldots \cup W_{n+m}[0, T^{n+m}(r)].$$

The event that the two packets of Brownian paths fail to intersect has a decreasing probability as $r \uparrow \infty$. Indeed, it is easy, using subadditivity, to show that there exists a constant $0 < \xi_d(n,m) < \infty$ such that

$$\mathbb{P}\big\{\mathfrak{B}_1(r) \cap \mathfrak{B}_2(r) = \emptyset\big\} = r^{-\xi_d(n,m)+o(1)}, \quad \text{as } r \uparrow \infty. \tag{2.1}$$

The numbers $\xi_d(n,m)$ are called the *intersection exponents*. It is not hard to see that this definition is very robust under changes in the way the motions are started, for example if one starts all motions in a packet in the same fixed point (2.1) continues to hold if the starting points of the packets do not agree.

In fact, the approximation of non-intersection probabilities is stronger than (2.1) suggests. Lawler [23] has shown that there exists constants $0 < c < C < \infty$ such that

$$cr^{-\xi_d(n,m)} \leq \mathbb{P}\big\{\mathfrak{B}_1(r) \cap \mathfrak{B}_2(r) = \emptyset\big\} \leq Cr^{-\xi_d(n,m)} \text{ for all } r > 1.$$

The definition of intersection exponents can be extended naturally to one *real* argument. For this purpose define the random variable

$$Z_n(r) = \mathbb{P}\{\mathfrak{B}_1(r) \cap W_{n+1}[0, T^{n+1}(r)] = \emptyset \,\big|\, W_1, \ldots, W_n\big\}.$$

Again one can show that, for every $\lambda > 0$, there exists a constant $\xi_d(n,\lambda)$ such that

$$\mathbb{E}\big\{Z_n(r)^\lambda\big\} = r^{-\xi_d(n,\lambda)+o(1)}, \quad \text{as } r \uparrow \infty. \tag{2.2}$$

Clearly, this is consistent with the definition of $\xi_d(n,\lambda)$ in the case of an integer λ.

Finally, a natural way to extend the definition to the case $\lambda = 0$ is by letting

$$\mathbb{P}\big\{Z_n(r) > 0\big\} = r^{-\xi_d(n,0)+o(1)}, \quad \text{as } r \uparrow \infty.$$

Note that $Z_n(r) = 0$ is the event that the packet of n Brownian motion disconnects the origin from infinity, so that any further Brownian motion cannot reach $\partial B(0,r)$ without intersecting the packet. Therefore the exponents $\xi_d(n,0)$ are also called *disconnection exponents*.

The first researchers interested in the values of the intersection exponents were physicists. They have made the striking conjecture that in the plane many of these exponents are rational numbers, see e.g. Duplantier and Kwon [11] and [10] for a survey. Very recently, Lawler, Schramm and Werner, in a seminal series of papers [28, 29, 30] have been able to verify this rigorously, see [27] for a survey over the series. They have shown that

$$\xi_2(n,\lambda) = \frac{\left(\sqrt{24n+1}+\sqrt{24\lambda+1}-2\right)^2-4}{48}, \quad \textit{for } n \in \mathbb{N}, \lambda \geq 0. \tag{2.3}$$

The proof of (2.3) is based on conformal invariance, and the *stochastic Löwner evolution (SLE).* I shall not go further into the exciting recent developments about SLE, as our examples are in $d = 3$, where no analogue of the stochastic Löwner evolution exists.

Indeed, in $d = 3$ we do not know the values of the intersection exponents explicitly, and it is very likely that they are typically irrational. I briefly review what is currently known about the values of the intersection exponents $\xi_d(n,\lambda)$. The two values explicitly known in dimension $d = 3$ are

- $\xi_3(1,2) = \xi_3(2,1) = 1$. This follows from properties of the harmonic measure, and an intuitive argument for this will be a spin-off of the next section. For a rigorous and accessible argument see [20, Chapter 3].
- $\xi_3(n,0) = 0$ as no disconnection is possible.

Estimates for other values can be derived from this using (2.2), monotonicity and concavity of $\lambda \mapsto \xi_3(n,\lambda)$. A deep result of Lawler [23] shows that this function is even *strictly concave.* As an example we get for $\xi_3(2,2)$ the estimates

$$\xi_3(2,2) > \xi_3(2,1) = 1 \text{ by monotonicity and strict concavity,}$$

and

$$\xi_3(2,2) < 2\xi_3(2,1) = 2 \text{ by strict concavity.}$$

The project of exploiting intersection exponents as a means to describe the fractal geometry of Brownian motion and random walks was pioneered by Lawler [21, 22]. To name two examples, the set of *cut points* of a Brownian path $B[0,1] \subset \mathbb{R}^d$ is the set of points $x = B(t) \in B[0,1]$ such that $B[0,t) \cap B(t,1] = \emptyset$. Lawler [21] showed that the Hausdorff dimension of this set is $2 - \xi_d(1,1)$. Intersection exponents are also crucial in the solution of the famous *Mandelbrot conjecture* about the Brownian frontier, the boundary of the unbounded component of the complement of a planar Brownian curve. Lawler showed in [22] that the dimension of this set is $2 - \xi_2(2,0)$ and together with the value of this exponent obtained 5 years later this shows that the dimension of the Brownian frontier is $4/3$, as predicted by Mandelbrot in 1983. Our first example below can be seen as a multifractal version of this result (however presented here in $d = 3$).

3. The multifractal spectrum of harmonic measure

Suppose $A \subset \mathbb{R}^3$ is a compact, nonpolar set. The harmonic measure on the set A is defined as follows. Suppose $B(y,R)$ is a ball such that A is contained inside this ball. Let $W : [0,\infty) \to \mathbb{R}^d$ be a Brownian motion started in a point chosen uniformly from the sphere $\partial B(y,R)$ and let

$$T = \inf\{t > 0 \,:\, W_t \in A\}$$

be the first hitting time of the set A by a Brownian motion. The law of $W(T)$ can be shown to be independent of the choice of $B(y,R)$. The *harmonic measure* h_A on A is defined by

$$h_A(B) = \mathbb{P}\{W_T \in B,\, T < \infty\} \text{ for all Borel sets } B \subset A.$$

Note that the support of the harmonic measure is contained in the boundary of the unbounded components of A^c, called the *frontier* of A.

Now we look at $A = B[0,1] \subset \mathbb{R}^d$, the path of a Brownian motion and study the harmonic measure $h = h_A$ on this set. The following result about the multifractal spectrum of the measure h_A is due to Lawler [25, 26].

Theorem 3.1 (Lawler 1999)**.** *Define a convex function ξ^* as the Legendre transform of the the strictly concave function $\lambda \mapsto \xi_3(2,\lambda)$, i.e.*

$$\xi^*(a) = \sup_{\lambda>0} \big\{\xi_3(2,\lambda) - a\lambda\big\}.$$

Then, for every a such that $2 - \xi^*(a-1) > 0$, almost surely there exist points $x \in A$ with

$$\lim_{r\downarrow 0} \frac{\log h(B(x,r))}{\log r} = a.$$

Moreover, for these values of a, almost surely,

$$\dim\Big\{x \in A \,:\, \lim_{r\downarrow 0} \frac{\log h(B(x,r))}{\log r} = a\Big\} = 2 - \xi^*(a-1).$$

Remark: The fact that this spectrum is nontrivial is due to the *non-mean field* nature of the intersections, expressed in the *strict* concavity of $\xi_3(2,\cdot)$. ◇

Following [26] I now give a very rough heuristic argument for Theorem 3.1, which is based on the so-called *multifractal formalism* of [13]. Let

$$A(j,n) = B\Big[\frac{j-1}{e^{2n}}, \frac{j}{e^{2n}}\Big], \text{ for } 1 \le j \le e^{2n}.$$

Consider the quantity

$$Q(n,q) = \sum_{j=1}^{e^{2n}} h(A(j,n))^q.$$

This is a random variable, which is intimately related to the multifractal spectrum. Indeed, for large n, we can roughly assume that $A(j,n)$ is the intersection of the Brownian path A with a ball of radius e^{-n} and if the local dimension of the points in this ball is approximately a, then $h(A(j,n)) \approx e^{-an}$. Hence, if we denote by

$$\mathfrak{E}(a) = \Big\{1 \le j \le e^{2n} \,:\, h(A(j,n)) \approx e^{-an}\Big\},$$

we can write

$$Q(n,q) = \sum_a \#\mathfrak{E}(a) e^{-qan},$$

where the sum extends over an appropriate discretisation of the reals. The Laplace method suggests that, for

$$f(a) = \lim_{n\to\infty} \frac{1}{n} \log\big(\#\mathfrak{E}(a)\big)$$

we have

$$Q(n,q) \approx \exp\big(n \sup_a \{f(a) - qa\}\big).$$

Suppose now that we can show that, for some function τ,

$$Q(n,q) \approx \exp(n\tau(q)). \tag{3.1}$$

Then we get,

$$\sup_a \big\{f(a) - qa\big\} = \tau(q).$$

As the sets in $\mathfrak{E}(a)$ roughly cover the set of points of local dimension a, the quantity $f(a)$ is at least an upper bound for the dimension spectrum. If everything is sufficiently regular, this bound is sharp. To identify f note that

$$\tau(q) = \sup_a \big\{f(a) - qa\big\} = f^*(q).$$

If f is concave, we may invert this Legendre transform and get

$$f(a) = \tau^*(a) = \inf_{q>0} \big\{\tau(q) + qa\big\}. \tag{3.2}$$

Hence it boils down to finding the τ in (3.1).

Here the intersection exponents are coming into play. We sketch the argument for the upper bound for τ, which is based on the first moment method. First fix the set $A = B[0,1]$ and look at the harmonic measure of a small ball,

$$h_A(B(x,r)) = P\big\{W \textit{ hits } A \textit{ first in } B(x,r)\big\} \textit{ for } x \in A.$$

The probability that the path W hits $B(x,r)$ is $cr^{d-2} = cr$ for some constant $c > 0$. Given that W hits $B(x,r)$, we can look at the time-reversed path starting on the boundary $\partial B(x,r)$ and stopped upon hitting $\partial B(y,R)$. Though this path is strictly speaking no Brownian path, a sufficiently large part of it is equal to a Brownian path conditioned to exit the annulus $B(y,R) \setminus B(x,r)$ at $\partial B(y,R)$. This is good enough for our purpose, and hence the reversed path heuristically makes the *first packet* $\mathcal{B}_1'(r)$ consisting of one Brownian motion. Considering A as a random set, its distribution given $x \in A$ essentially coincides with a two-sided Brownian motion started in x, forming the *second packet* $\mathcal{B}_2'(r)$ of two independent Brownian motions. The Brownian motions in these packets are started on the sphere $\partial B(x,r)$ and stopped when they leave the fixed sphere $\partial B(y,R)$, and by Brownian scaling we easily map this onto the situation in the definition of the intersection exponents, where motions are started on the unit sphere and stopped upon leaving a large sphere of radius R/r. Hence we have

$$\mathbb{E}\big\{ h_A(B(x,r)) \big\} = cr\, P\big\{ \mathcal{B}_1'(r) \cap \mathcal{B}_2'(r) = \emptyset \big\} = r^{1+\xi_3(2,1)+o(1)},$$

and more generally,

$$\mathbb{E}\big\{ h_A(B(x,r))^q \big\} \approx r^{q+\xi_3(2,q)+o(1)}. \tag{3.3}$$

This gives

$$\mathbb{E}Q(n,q) \approx \exp\big(n(2-q-\xi_3(2,q)+o(1))\big).$$

By the Borel-Cantelli lemma we get from this that

$$Q(n,q) \le \exp\big(n(2-q-\xi_3(2,q)+o(1))\big)$$

for all sufficiently large n. Hence

$$\tau(q) \le 2-q-\xi_3(2,q).$$

To obtain the lower bound we have to add a second moment estimate to our first moment estimate (3.3). This is considerably harder, and after substantial work one arrives at

$$\tau(q) = \lim_{n\to\infty} \frac{1}{n} \log Q(n,q) = 2-q-\xi_3(2,q),$$

and hence, from (3.2),

$$f(a) = 2 - \sup_{q>0} \big\{ \xi_3(2,q) - q(a-1) \big\}.$$

Remark: As the total harmonic measure is constant, we have $\tau(1)=0$ and hence our heuristics gives that $\xi_3(2,1)=1$. For a rigorous argument using this idea, see [20].

4. The multifractal spectrum of intersection local time

Let W_1, W_2 be two independent Brownian motions in $\mathbb{R}^3$ with a joint starting point and running for one unit of time. Let $S = W_1[0,1] \cap W_2[0,1]$ be the intersection of the paths, which is a (random) set of Hausdorff dimension 1. This set can be equipped with a finite measure, the intersection local time ℓ, the *(projected) intersection local time*, which can be described symbolically by the formula

$$\ell(A) = \int_A dy \prod_{j=1}^{2} \int_0^1 ds\, \delta_y(W_j(s)), \ \textit{for } A \subset \mathbb{R}^d \ \textit{Borel.} \tag{4.1}$$

Le Gall has shown in [31, 32] that ℓ is the φ-Hausdorff measure on the set S for the gauge function

$$\varphi(x) = cx \log\log(1/x)^2, \ \textit{for } x>0,$$

where c is an unknown deterministic constant. Alternative rigorous constructions of the random measure ℓ are reviewed in [19]. It is well-known that the local dimension of ℓ-almost all points equals 1, which is also the Hausdorff dimension of the intersection set S.

Applying a multifractal formalism as in the previous example suggests that the measure ℓ is *not* multifractal, as the τ-function is linear. Indeed, points where the local dimension is strictly below the typical value 1 do not exist, even in the weaker sense of a liminf in (1.1). In other words there is no nontrivial multifractal spectrum to the left of 1.

In a recent paper, Klenke and Mörters [18] are investigating whether there are points where the local dimension exceeds 1. It turns out that this is indeed the case, if the local dimension is defined in the weaker sense of a limsup in (1.1). These exceptional points arise, loosely speaking, from random fluctuations in the intersection local time which are too subtle to be picked up by the multifractal formalism.

Theorem 4.1 (Klenke and Mörters 2003). *Suppose ℓ is the intersection local time of two Brownian motions in $\mathbb{R}^3$. For any $1 \le a \le \xi/(\xi-1)$, almost surely, there exist points $x \in S$ such that*

$$\limsup_{r\downarrow 0} \frac{\log \ell(B(x,r))}{\log r} = a,$$

for $\xi = \xi_3(2,2)$. Moreover, for these values of a, almost surely,

$$\dim\Big\{x \in S \,:\, \limsup_{r\downarrow 0} \frac{\log \ell(B(x,r))}{\log r} = a\Big\} = \frac{\xi}{a} + 1 - \xi. \tag{4.2}$$

Remark: Here the multifractal spectrum is strictly *convex*, hence it cannot be found by means of the multifractal formalism of the last section, which by its nature always predicts a concave spectrum. $\diamond$

In the following I give a heuristic argument for this result. The crucial ingredient in the proof is an investigation of the *strategy* used by Brownian motion in order to accumulate only a very small amount of mass in a small ball centred in an intersection point $x \in S$.

In order to find the probability of the event $\ell(B(x,r)) \approx r^a$ given $x \in S$ first note that it is intuitively clear that in a typical intersection point $x \in S$ a pair of two-sided Brownian motions are attached. As the motions start in the same point they typically have a lot of intersections right at the start. In order to avoid intersections to the extent that $\ell(B(x,r)) \approx r^a$ it is optimal for the Brownian motions to use the following strategy:

(1) Run normally until you hit $\partial B(x,r^a)$. The typical intersection local time until then is $\approx r^a$, and now one has to act.
(2) Do not intersect before leaving $B(x,r^{1-\varepsilon})$. The probability that this happens is, by scaling and the definition of the intersection exponents,

$$r^{-\xi(1-\varepsilon-a)+o(1)}.$$

(3) Once you reach $\partial B(x,r^{1-\varepsilon})$ never return to $B(x,r)$. The probability that this happens is $\approx 1-r^{\varepsilon}$.

The total probability (or "cost") of this strategy is of order $r^{-\xi(1-a)+o(1)}$, essentially all of which is incurred in the second step. Note that it leads to an empty annulus $B(x,r) \setminus B(x,r^a)$. It can be shown that this strategy is optimal, i.e. no other strategy to get $\ell(B(x,r)) \approx r^a$ has a significantly higher probability.

We now sketch how the upper bound in our spectrum can be deduced from this. As $\dim S = 1$ we can cover the set S by approximately r^{-a} balls of radius r^a. Using the modulus of continuity result for Brownian motion one can see that, in a ball of radius r^{-a}, the probability that the ball contains a point $x \in S$ with $\ell(B(x,r)) \approx r^a$ is approximately the same as the probability that the centre of the ball has this property. Hence, for each ball in our collection, the probability of containing a point $x \in S$ with $\ell(B(x,r)) \approx r^a$ is approximately $r^{-\xi(1-a)}$. Choosing a covering of the set

$$\Big\{x \in S \,:\, \limsup_{r \downarrow 0} \frac{\log \ell(B(x,r))}{\log r} \geq a\Big\}$$

consisting of all these balls for the radii $r = 2^{-n}, n \geq m$ and m fixed, leads to a covering

$$B(x_1, r_1^a), B(x_2, r_2^a), B(x_3, r_3^a), \ldots$$

with expected s-value

$$\mathbb{E}\Big\{\sum_{i=1}^{\infty} (r_i^a)^s\Big\} = \sum_{n=m}^{\infty} 2^{-nas} 2^{na} 2^{n\xi(1-a)},$$

which is finite iff $s > \xi/a + 1 - \xi$. Note that this argument for the upper bound is particularly simple as it uses only a *first moment method*.

It is more difficult to establish a lower bound, as second moments (or independence in some form) have to be used. A particularly powerful technique to establish lower bounds is the *stochastic co-dimension* method. Based on earlier ideas of Taylor these techniques were developed further, first in [1, 38] and later more systematically by Peres and coauthors, see e.g. [17, 8] or [19]. Nowadays the method represents an elegant and powerful tool in fractal geometry, further evidence of this can be found in the contribution of Xiao in this volume.

The stochastic co-dimension approach suggests that the dimension of a set $A \subset \mathbb{R}^d$ is at least γ if it intersects a certain compact random set $\Gamma[\gamma]$ of dimension $d-\gamma$ with positive probability, and conversely if $\dim A > \gamma$ then A intersects the test set $\Gamma[\gamma]$ with positive probability. Suitable *test sets* are, for example, the *percolation limit sets* with retention parameter $2^{-\gamma}$, see [12] and also [37].

In our proof we assume that the test sets are independent of our Brownian motions and construct, for any $\gamma \in (0,1)$, a compact random set $S^* \subset S \cap \Gamma[\gamma]$, which is nonempty with positive probability, such that

$$\dim U \cap S^* \geq 1 - \gamma \textit{ whenever } U \textit{ open and } U \cap S^* \neq \emptyset.$$

Fix $\delta > 0$ and define open sets

$$U(m,\delta) = \Big\{x \in S \,:\, \ell\big(B(x,r)\big) < r^{a-\delta} \textit{ for some } 0 < r < \tfrac{1}{m}\Big\}.$$

The regularity of S^* is then used to show that for $\gamma = \xi/a + 1 - \xi$ the sets $U(m,\delta) \cap S^*$ are dense in S^*. To get an impression of this, fix $r = 2^{-n}$ and note that given $S^* \neq \emptyset$ by the regularity property, for every open set U which intersects S^* there is a packing of $S^* \cap U$ consisting of approximately $r^{\gamma-1}$ balls of radius r. We now *localise* the problem and look at each ball in the packing separately. For both motions let S be the last entry of the path into the ball before hitting the centre and T the first exit from the ball afterwards. We denote by ℓ^* the intersection local time of the paths in the time intervals $[S,T]$. Then the probability that inside our ball there is some $x \in S^*$ such that

$$\ell^*\big(B(x,r)\big) < r^{a-\delta} \tag{4.3}$$

is of order

$$r^{-\xi(1-a+\delta)} r^{(\gamma+2)(a-\delta-1)} r^{-3(a-\delta-1)}.$$

Here the first factor comes from the probability that a point x, and by continuity every point in a ball around x of radius $\approx r^{a-\delta}$, satisfies (4.3), the second factor comes from the requirement that this ball intersects S^*, and the third factor accounts for the number of such small balls in the fixed ball of radius r. Moreover, thanks to the localisation these events are essentially independent for different balls. Hence the probability that we do *not* find a point $x \in S^*$ satisfying (4.3) for $r = 2^{-n}$ is

$$\big(1 - 2^{n(\xi+\gamma-1)(1-a+\delta)}\big)^{2^{n(1-\gamma)}} \longrightarrow 0, \;\; as\; n \to \infty.$$

To complete the argument one has to show that for most points the localised intersection local time ℓ^* is not different from ℓ. In $d \geq 3$ this can be shown using the transience of Brownian motion

Now, as $U(m,\delta) \cap S^*$, for $m \in \mathbb{N}$, $\delta > 0$, are a nested family of open dense subsets of the complete metric space S^*, by Baire's theorem, the intersection is also dense, and hence nonempty if S^* is nonempty. Hence the intersection set

$$\mathcal{T}(a) := \bigcap_{\substack{m\in\mathbb{N}\\ \delta>0}} U(m,\delta) = \Big\{ x \in S \,:\, \limsup_{r\downarrow 0} \frac{\log \ell(B(x,r))}{\log r} \geq a \Big\}$$

intersects $\Gamma[\gamma]$ with positive probability. On the other hand almost surely $\mathcal{T}(a + \varepsilon) \cap \Gamma[\gamma] = \emptyset$ for all $\varepsilon > 0$ by the upper bound. This shows that (4.2) holds with positive probability, and the full result follows from a zero-one law argument.

5. The refined multifractal spectrum of occupation measure

The most natural measure on the Brownian path $\{W(t) \,:\, t \in [0,1]\}$ is of course the occupation measure μ, given by

$$\mu(A) = \int_0^1 1_A(W_s)\, ds.$$

We have neglected this measure so far in this paper, as it is not multifractal. Indeed, one can show that, almost surely,

$$\lim_{r\downarrow 0} \frac{\log \mu(B(x,r))}{\log r} = 2, \text{ for all } x \in W[0,1].$$

This has been known for some time, for example it is implicit in the work of Kaufman [16]. As there is no variation in the local dimension of μ it is natural to look at a finer scale and ask for an almost sure upper and lower hull of $\mu(B(x,r))$ when the point x is varied along the support of μ. More precisely, looking at a the lower hull first, we are asking for a gauge function $\phi : (0,\varepsilon) \to (0,\infty)$ such that

$$0 < \inf_{x\in W[0,1]} \liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{\phi(r)} < \infty.$$

A point $x \in W[0,1]$ is called a *thin point* with thinness $a > 0$ if

$$\liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{\phi(r)} = a.$$

Once ϕ is identified, the *dimension spectrum of thin points* is given by

$$f(a) = \dim \Big\{ x \in W[0,1] : \liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{\phi(r)} = a \Big\}.$$

In [7] the correct gauge function is found, and the spectrum of thin points is identified for the occupation measure of a Brownian path in $d = 3$.

Theorem 5.1 (Dembo, Peres, Rosen and Zeitouni 2000). *Almost surely, we have*

$$1 = \inf_{x\in W[0,1]} \liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{r^2[\log(1/r)]^{-1}} ,$$

and, for every $a \geq 1$, almost surely,

$$\dim \Big\{ x \in W[0,1] : \liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{r^2[\log(1/r)]^{-1}} = a \Big\} = 2 - \frac{2}{a}.$$

Passing to the upper hull, we are looking for a gauge function ψ such that

$$0 < \sup_{x\in W[0,1]} \limsup_{r\downarrow 0} \frac{\mu(B(x,r))}{\psi(r)} < \infty. \tag{5.1}$$

A point $x \in W[0,1]$ is called a *thick point* with thickness $a > 0$ if

$$\limsup_{r\downarrow 0} \frac{\mu(B(x,r))}{\psi(r)} = a,$$

and a *dimension spectrum of thick points* is defined accordingly. In [6] the correct gauge function is found, and the spectrum of thick points is identified for the occupation measure of a Brownian path in $d = 3$.

Theorem 5.2 (Dembo, Peres, Rosen and Zeitouni 2000). *Almost surely,*

$$\sup_{x \in W[0,1]} \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^2[\log(1/r)]} = \frac{16}{\pi^2}, \tag{5.2}$$

and, for all $0 \le a \le \frac{16}{\pi^2}$, *almost surely,*

$$\dim \Big\{ x \in \mathbb{R}^d \, : \, \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^2[\log(1/r)]} = a \Big\} = 2 - \frac{\pi^2}{8}\, a. \tag{5.3}$$

For the proof of these two theorems the authors developed substantial refinements of the tools available for getting lower bounds for the Hausdorff dimension, some of which we have already benefited from in the previous section.

We only indicate briefly the key point in the proof of Theorem 5.2, which, as in Theorem 4.1, is the *strategy* used by the Brownian motion to accumulate exceptionally large mass in a small ball. For any $T > 0$ we look at the occupation measures μ_T for Brownian motion running for T time units,

$$\mu_T(A) = \int_0^T 1_A(W_s)\, ds, \text{ for } A \subset \mathbb{R}^3 \text{ Borel.}$$

Classical results of Ciesielski and Taylor give the upper tail behaviour of the random variable $\mu_\infty(B(0,1))$: For a positive constant $c > 0$,

$$\mathbb{P}\Big\{ \mu_\infty(B(0,1)) \ge x \Big\} \sim c\, e^{-x\pi^2/8}, \text{ as } x \uparrow \infty, \tag{5.4}$$

hence, by scaling, we obtain for small balls $B(0,r)$ and fixed $a > 0$,

$$\mathbb{P}\Big\{ \mu_\infty(B(0,r)) \ge ar^2 \log(1/r) \Big\} \sim c\, r^{a\pi^2/8}, \text{ as } r \downarrow 0.$$

The principal question is, which strategy is used by the Brownian path to accumulate an occupation time of order $r^2 \log(1/r)$ in $B(0,r)$. The answer found in [6] is that this mass is accumulated in a very short time interval after the start of the motion and eventual later returns to this ball do not contribute essentially. Indeed, for $h(r) = r^2 \log(1/r)^6$ they obtain

$$\mathbb{P}\Big\{ \mu_{h(r)}(B(0,r)) \ge ar^2 \log(1/r) \Big\} \sim c\, r^{a\pi^2/8}, \text{ as } r \downarrow 0. \tag{5.5}$$

This *localisation effect* is the crucial effect which leads to Theorem 5.2, we note here that the effect breaks down in the recurrent case of $d = 2$, and in this dimension it is a complete different effect that leads to the dimension spectrum, which in this case was found in [8].

It is also worthwhile to look at a *refined* multifractal spectrum, when the measure under investigation has only a one-sided multifractal spectrum. This is the case, for example, in the situation of the intersection local time, studied in the previous section. Recall that in this case there exist points with local dimension exceeding the typical value 1 (these points are exceptionally *thin*!) but no points where the local dimension is smaller than 1. It therefore makes sense to ask for a dimension spectrum of thick points as defined above. This question is answered by König and Mörters in [19].

Theorem 5.3 (König and Mörters 2002)**.** *Almost surely,*

$$\sup_{x\in S} \limsup_{r\downarrow 0} \frac{\ell(B(x,r))}{r[\log(1/r)]^2} = \frac{1}{\theta^2}, \tag{5.6}$$

for a constant θ *given by*

$$\theta = \inf\Big\{ \|\nabla\psi\|_2^2 \,:\, \psi \in C_0^2(\mathbb{R}^d),\ \int_{B(0,1)} \psi^4 = 1 \Big\}.$$

Moreover, for all $0 \le a \le \varrho^2/4$*, almost surely,*

$$\dim\Big\{ x \in S \,:\, \limsup_{r\downarrow 0} \frac{\ell(B(x,r))}{r[\log(1/r)]^2} = a \Big\} = 1 - \sqrt{a}\theta\,.$$

Finally, looking again at the occupation measure μ, recall that our definition of thickness requires large mass only along a sequence of radii $r_n \downarrow 0$. This sequence might be arbitrarily sparse. What happens if we replace the limsup in (5.1) by a liminf? In other words, we are looking for *consistently thick points.* A first task is to identify a gauge function $\varphi : (0,\varepsilon) \to (0,\varepsilon)$ such that

$$0 < \sup_{x\in W[0,1]} \liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{\varphi(r)} < \infty.$$

Dembo, Peres, Rosen and Zeitouni have shown [6, Theorem 1.6] that the right gauge function here is $\varphi(r) = r^2$, which is different from the gauge function $\psi(r) = r^2 \log(1/r)$ in (5.2), but the problem of the dimension spectrum of consistently thick points is still *open.*

6. Summary and Outlook

By looking at three recently studied examples we have illuminated the following three important directions in modern fractal geometry.

- **Multifractal geometry beyond the multifractal formalism.** [9, 14, 39, 18]
  Interesting stochastic examples show that the multifractal formalism fails if the mass in typical small balls is a *heavy-tailed* random variable. In some instances we see a strictly convex spectrum, a behaviour which cannot be picked up by the formalism.
- **Refined multifractal spectrum.** [41, 6, 7, 8, 34, 19, 5]
  In some cases of measures which are not multifractal, or have only a one-sided multifractal spectrum, one can study a refined spectrum and get very detailed information about exceptional sets.
- **Critical exponents and fractal geometry.** [21, 25, 26, 27, 19]
  Critical exponents like the intersection exponents of Brownian motion contain crucial information about the fractal geometry of the models involved.

In some cases this information can be extracted in terms of Hausdorff dimension spectra involving the exponents.

Trying to look into the future of the area, I believe that the following topics deserve attention before the next "Fractal Geometry and Stochastics" conference will reunite many researchers in several years time.

- **Fractal geometry of discrete models.**
  Almost all problems discussed here have a discrete analogue, and in some cases this has already been made explicit, see e.g. [25] for a discrete version of the multifractal spectrum of harmonic measure, and [8] for a discrete version of the refined multifractal spectrum of the occupation measure of a Brownian path, both in $d = 2$. In the latter case this discretisation solved a long-standing conjecture of Erdös and Taylor about the simple random walk in the plane. I believe that in the future fractal geometers will increasingly look into "discrete fractals" and apply their techniques to the fractal phenomena arising, for example, in interacting particle systems and statistical physics.
- **Increasingly complex dimension spectra.**
  Looking at recent results and techniques, for example, of the papers [7, 8] or [17], one cannot help noticing the enormous improvement in the technical standard and the sophistication of the tools presently available. However, I do not believe that an end to this development is reached yet, as important challenges remain open. Among these is the problem of consistently thick points for Brownian motion in $d \geq 3$, mentioned in Section 5, and the problem of the dimension spectrum of local times for Brownian motion in the plane raised in [2].
- **Fractal geometry of nonintersecting curves.**
  The stochastic Löwner evolution introduces a new class of universal nonintersecting random curves in the plane, which arise naturally as scaling limits of important physical models. These objects have a fractal nature which needs to be investigated. This programme has already started and recently Beffara [3] has calculated the Hausdorff dimension of these curves. Still, the fractal geometry of these models will capture some researcher's imaginations for a good while in the future.

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Acknowledgement

I would like to thank the organisers of the conference, Christoph Bandt, Umberto Mosco, and Martina Zähle for their hospitality.

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