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ON THE MULTIFRACTAL SPECTRUM OF BRANCHING MEASURE ON A GALTON-WATSON TREE

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Abstract

Suppose that μ is the branching measure on the boundary of a supercritical Galton-Watson tree with offspring distribution N such that $E[N \log N] < \infty$ and $P\{N = 1\} > 0$. We determine the multifractal spectrum of μ, using a method different from that proposed in Shieh and Taylor [7], which is flawed.

Keywords: Galton-Watson tree; branching measure; branching set; percolation; Hausdorff dimension; dimension spectrum; multifractal spectrum.

1. Introduction and main results

We look at a Galton–Watson tree T starting from a single progenitor, the root ρ. Let N be its offspring distribution and suppose it satisfies $\mathfrak{m} := EN > 1$ and $E[N \log N] < \infty$. By ∂T we denote the *boundary* of the tree, consisting of all infinite self-avoiding paths in T starting with the root. We equip the boundary ∂T of the tree with the metric d given by $d(u, v) = e^{-n}$, where n is the generation of the last common vertex of $u, v \in \partial T$. For $\xi \in \partial T$, let $B(\xi|n)$ be the open ball with centre ξ and radius e^{-n}.

For every vertex $v \in T$ we let $T(v)$ be the subtree consisting of all successors of v and $T_n(v)$ be the finite tree consisting of all vertices of the first n generations in the subtree $T(v)$. Denoting by $Z_n(v)$ the number of vertices of exactly nth generation in (and relative to) $T(v)$, it is well-known that, almost surely, for every vertex $v \in T$ with $\partial T(v) \neq \emptyset$ the martingale limit

$$W(v) := \lim_{n\to\infty} \frac{Z_n(v)}{\mathfrak{m}^n}$$

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exists in $(0,\infty)$ almost surely on $\partial T \neq \emptyset$. For every vertex v we consider $\partial T(v)$ in the natural way as a subset of ∂T. Then, almost surely on $\partial T \neq \emptyset$, there exists a unique nonzero, finite measure μ on the boundary ∂T of the tree, called the *branching measure*, such that for every vertex v of generation n,

$$\mu\big(\partial T(v)\big) = \frac{W(v)}{\mathfrak{m}^n}. \tag{1}$$

Throughout the paper we abbreviate $W = W(\rho)$, $T_n = T_n(\rho)$ and $Z_n = Z_n(\rho)$. If we let $n \geq 1$ and denote the vertices of nth generation in T by $v(1), \dots, v(Z_n)$ we have

$$\mu\big(\partial T\big) = W = \mathfrak{m}^{-n} \sum_{j=1}^{Z_n} W\big(v(j)\big), \tag{2}$$

and, given T_n, all random variables $W(v(j))$ are iid with the same distribution as W.

Our aim here is to determine the multifractal spectrum of the measure μ under the assumption that $P\{N = 1\} > 0$, which is necessary for the nontriviality of this spectrum, see Mörters and Shieh [6]. We also assume that $P\{N = 1\} < 1$ to avoid trivialities and, for technical simplicity, that $P\{N = 0\} = 0$. The last assumption can be removed easily by pruning the tree.

Theorem 1. (Multifractal spectrum.) *Let $P\{N = 0\} = 0$ and $0 < P\{N = 1\} < 1$. Define $a := \log \mathfrak{m} > 0$ and $\tau := -\log P\{N = 1\}/\log \mathfrak{m}$. Then, for all $a \leq \theta \leq a\big(1+\frac{1}{\tau}\big)$, almost surely,*

$$\dim \Big\{ \xi \in \partial T \,:\, \limsup_{n\to\infty} \frac{\log \mu B(\xi|n)}{-n} = \theta \Big\} = a\Big(\frac{a}{\theta}(1+\tau) - \tau\Big). \tag{3}$$

Remark 1. Our proof also shows that in the critical case $\theta = a\big(1 + \frac{1}{\tau}\big)$, almost surely, the set on the left hand side of (3) is nonempty.

This result was first stated by Shieh and Taylor [7] under the additional asumption that all integer moments of N are bounded, but the proof of the lower bound given there is flawed. Using their notation, the problem lies in the variance assumption (ii) of Proposition 3.1, which is not properly verified. The statement of Remark 3.2 only investigates independence conditional on Γ_n. We do not attempt to fill this gap here, but instead indicate how the method of Mörters and Shieh [6] can be adapted to this problem and give the result in full strength.

2. Proof of the theorem

In this section we discuss a proof of Theorem 1, which uses percolation on trees to study Hausdorff dimension, an idea suggested by Lyons [5]. Our paper is also influenced by the techniques of Khoshnevisan, Peres and Xiao [4] in the Euclidean case.

If T is any tree and $p \in [0,1]$, we attach to each edge e of the tree an independent $\{0,1\}$-valued random variable $X(e)$ with $P\{X(e) = 1\} = p$. We denote by T^* the connected component of the root ρ in the graph consisting of all edges e with $X(e) = 1$. We say that T^* is the result of running *percolation with retention parameter* p on T.

Recall the definition of the *branching number* of a tree, see e.g. Peres [8], and the fact that, for every infinite tree T with branching number b, the Hausdorff dimension of the boundary is $\dim \partial T = \log b$. We use the following lemma of Lyons [5] about the critical parameter for the survival of a tree after percolation.

Lemma 1. *Suppose that T is a tree with branching number b and T^* the random tree obtained from running percolation with retention parameter p on T.*

- *(i) If $p < 1/b$, then $\partial T^* = \emptyset$ almost surely,*
- *(ii) If $p > 1/b$, then $\partial T^* \neq \emptyset$ with positive probability.*

Remark 2. It follows easily from Lemma 1 and Choquet's capacibility theorem, see e.g. Bass [1], that for an analytic set $S \subset \partial T$, if $p < \exp(-\dim S)$, then $S \cap \partial T^* = \emptyset$ almost surely, and if $p > \exp(-\dim S)$, then $\partial T^* \cap S \neq \emptyset$ with positive probability.

Let us now look at a Galton-Watson tree T. We need the following fact from Hawkes [3], see also Lyons [5] for a simpler proof.

Lemma 2. *Let T be a Galton-Watson tree. Then, almost surely on $\partial T \neq \emptyset$, the branching number of T is its mean offspring number $\mathfrak{m}$.*

Remark 3. If we run independent percolation on a Galton-Watson tree T, the unconditional distribution of T^* is again the law of a Galton-Watson tree. If $\mathfrak{m}$ is the mean offspring number of T, then $p\mathfrak{m}$ is the mean offspring number of T^*.

The crucial ingredient towards the multifractal spectrum is the following lemma.

Lemma 3. *For any retention parameter* $1/\mathfrak{m} < p \leq 1$,

$$\lim_{x\downarrow 0} \frac{1}{\log x} \log P\big(\{W < x\} \cap \{\partial T^* \neq \emptyset\}\big) = \tau - \frac{\log p}{\log \mathfrak{m}}.$$

Remark 4. In the case $p = 1$ this is the well-known lower tail asymptotic for W in the so-called Schröder case. All the proofs of this fact, which we found in the literature, rely on substantial work of Dubuc [2]. Our argument is independent of this and, specialized to the case $p = 1$, yields a really simple proof of this fact.

Proof. Throughout the proof we write $p_1 = P\{N = 1\}$ and consider p and p_1 as fixed. First note from (2) that

$$P\{W < \gamma \mathfrak{m}^{-n} \text{ and } \partial T^* \neq \emptyset \,|\, Z_n = 1\} = p^n \, P\{W < \gamma \text{ and } \partial T^* \neq \emptyset\}.$$

For sufficiently large choice of $\gamma > 1$ the latter probability is bounded from below by a constant $c_1 > 0$. Hence, for $n := \lceil \frac{-\log(x/\gamma)}{\log \mathfrak{m}} \rceil$,

$$P\{W < x \text{ and } \partial T^* \neq \emptyset\} \geq c_1 p^n P\{Z_n = 1\} = c_1 (pp_1)^n \geq c_2 x^{\tau - \frac{\log p}{\log \mathfrak{m}}},$$

where $c_2 := c_1 (pp_1)^{1 + \log\gamma/\log\mathfrak{m}}$, and this already settles the lower bound.

Next, fix $\varepsilon > 0$. We now show that there exists $o(\varepsilon) > 0$ with $\lim_{\varepsilon\downarrow 0} o(\varepsilon) = 0$ such that

$$P\{Z_m \leq \varepsilon m \text{ and } \partial T^* \neq \emptyset\} \leq (p_1 p)^{(1-\varepsilon)m} e^{m o(\varepsilon)} \qquad \text{for all } m \geq 1. \tag{4}$$

Indeed, look at the lexicographically first vertex v_1 of generation m. To each other vertex v_j in generation m associate the last common ancestor of these two vertices. All ancestors of v_1 except those (at most $\lfloor \varepsilon m \rfloor$) common ancestors with its contemporaries have exactly one offspring. Hence

$$P\{Z_m \leq \varepsilon m \text{ and } \partial T^* \neq \emptyset\} \leq (p_1 p)^{m - \lfloor \varepsilon m \rfloor} \binom{m}{m - \lfloor \varepsilon m \rfloor},$$

where the last factor bounds the possible number of configuration of positions of these ancestors. By Stirling's formula this factor is of order $\exp(m o(\varepsilon))$, which shows (4).

Now choose $n := \lceil \frac{-\log x}{\log \mathfrak{m}} \rceil$. We now show that, for any $\varepsilon > 0$, there exists $C > 0$ and an integer k such that

$$P\{W < x, Z_{n+k} > \varepsilon(n+k)\} \leq C (pp_1)^n \qquad \text{for all } x > 0. \tag{5}$$

Indeed, as $W > 0$ almost surely, there exists $\theta > 0$ such that $P\{W < \theta\} \le (pp_1)^{1/\varepsilon}$. We choose $k = \lfloor \log\theta / \log \mathfrak{m} \rfloor$. From (2) we note that

$$\begin{aligned} P\{W < x, Z_{n+k} > \varepsilon(n+k)\} &\le P\Big\{\mathfrak{m}^{-n-k} \sum_{j=1}^{\lceil \varepsilon(n+k) \rceil} W\big(v(j)\big) < \mathfrak{m}^{-n-k}\theta\Big\} \\ &\le P\{W < \theta\}^{\lceil \varepsilon(n+k) \rceil} \le (pp_1)^{n+k} = C(pp_1)^n, \end{aligned}$$

with $C := (pp_1)^k$. To complete the proof note that, by (4) and (5),

$$\begin{aligned} &P\{W < x \text{ and } \partial T^* \neq \emptyset\} \\ &\quad \le P\{W < x, Z_{n+k} > \varepsilon(n+k)\} + P\{Z_{n+k} \le \varepsilon(n+k) \text{ and } \partial T^* \neq \emptyset\} \\ &\quad \le C(pp_1)^n + (p_1 p)^{(1-\varepsilon)(n+k)} e^{(n+k)o(\varepsilon)}, \end{aligned}$$

from which the result follows, as $\varepsilon > 0$ was arbitrary. ∎

For $\theta \ge a$ define a Borel subset of ∂T by

$$A(\theta) = \Big\{\xi \in \partial T \,:\, \limsup_{n\to\infty} \frac{-1}{n} \log \mu B(\xi|n) \ge \theta\Big\}. \tag{6}$$

Proof of the upper bound in (3). By Remark 2 it suffices to show that $A(\theta) \cap \partial T^* = \emptyset$ almost surely, if the retention parameter is chosen satisfying

$$\frac{1}{\mathfrak{m}} < p < \exp(-a((a/\theta)(1+\tau) - \tau)).$$

Note that, for any n and $\varepsilon > 0$, if $A(\theta) \cap \partial T^* \neq \emptyset$ there exists a vertex $v \in T^*$ with $|v| \ge n$ such that $\partial T(v)^* \neq \emptyset$ and $-\log \mu(\partial T(v)) \ge m(\theta - \varepsilon)$. Hence, we obtain an upper bound, using (1) and Lemma 3,

$$\begin{aligned} &P\{A(\theta) \cap \partial T^* \neq \emptyset\} \\ &\quad \le \sum_{m=n}^{\infty} E \sum_{\substack{v \in T^* \\ |v| = m}} P\big\{ -\log \mu(\partial T(v)) \ge m(\theta - \varepsilon) \text{ and } \partial T(v)^* \neq \emptyset \,\big|\, v \in T^* \big\} \\ &\quad = \sum_{m=n}^{\infty} E \sum_{\substack{v \in T^* \\ |v| = m}} P\big\{W \le \exp(m(\log \mathfrak{m} - \theta + \varepsilon)) \text{ and } \partial T^* \neq \emptyset\big\} \\ &\quad \le \sum_{m=n}^{\infty} p^m \mathfrak{m}^m \exp\big(m(\log \mathfrak{m} - \theta + \varepsilon)(\tau - \log p / \log \mathfrak{m} - \varepsilon)\big) \end{aligned}$$

$$= \sum_{m=n}^{\infty} \exp\big(m\big((\log p)\,\tfrac{\theta-\varepsilon}{a} + a + (a-\theta+\varepsilon)(\tau-\varepsilon)\big)\big),$$

recalling that $a = \log \mathfrak{m}$. By our choice of p, we may choose $\varepsilon > 0$ so small that the sum on the right is convergent, and hence the upper bound converges to zero as $n \to \infty$. ■

The main lemma in the proof of the lower bound is the following.

Lemma 4. *For $p \geq \exp(-a((a/\theta)(1+\tau)-\tau))$ we obtain that $A(\theta) \cap \partial T^* \neq \emptyset$ almost surely on $\partial T^* \neq \emptyset$.*

Proof. For every positive integer N we let

$$U(N) = \Big\{\xi \in \partial T \;:\; \text{there exists } n \geq N \text{ with } -\log \mu B(\xi|n) > n(\theta - \tfrac{1}{N})\Big\}. \qquad (7)$$

Our aim is to show that $U(N) \cap \partial T^*$ is dense in ∂T^* almost surely. But we first argue how we can use this to finish the proof of Lemma 4. Note that $U(N) \cap \partial T^*$ is relatively open in ∂T^*, which is a compact, hence complete, metric space. By *Baire's Theorem*,

$$\bigcap_{N=1}^{\infty} U(N) \cap \partial T^* \text{ is dense in } \partial T^*.$$

But note that $\bigcap_{N=1}^{\infty} U(N) = A(\theta)$, hence $A(\theta) \cap \partial T^*$ must be dense in ∂T^* almost surely. This implies that $A(\theta) \cap \partial T^*$ is nonempty almost surely conditional on $\partial T^* \neq \emptyset$, which is the required statement.

It remains to show that, for fixed integer N, the set $U(N) \cap \partial T^*$ is dense in ∂T^* almost surely. For this purpose we fix some vertex $v \in T$ in generation $|v|$ and show that, almost surely conditional on $\partial T(v)^* \neq \emptyset$, we have $U(N) \cap \partial T(v)^* \neq \emptyset$.

Let $\mathcal{G}(n)$ be the σ-field generated by the finite subtree $T_n(v) \subset T(v)$, consisting of the first n generations of $T(v)$, together with the random variables $\{X(e) : e \in T_n(v)\}$. Let $\mathcal{K}_n$ be the collection of vertices in $T(v)^*$ of generation n and let K_n be the cardinality of $\mathcal{K}_n$. Note that the random variable K_n is $\mathcal{G}(n)$-measurable.

We define a sequence of $\mathcal{G}(n)$-stopping times $N_1, N_2, \ldots$ by $N_0 = 1$ and, for $k \geq 1$,

$$N_k = \min\Big\{n > N_{k-1} \;:\; K_n \geq \exp\Big(n\big(a - a((a/\theta)(1+\tau)-\tau) - \tfrac{1}{k}\big)\Big)\Big\}.$$

By Lemma 2 and Remark 3, almost surely on $\partial T(v)^* \neq \emptyset$, we have

$$\dim \partial T(v)^* = \log(p\,\mathfrak{m}) \geq a - a((a/\theta)(1+\tau)-\tau). \qquad (8)$$

This implies that $\{N_j : j \geq 1\}$ is a sequence of finite stopping times, almost surely on $\partial T(v)^* \neq \emptyset$. For every vertex $w \in T(v)$ of generation N_j define the event

$$\begin{aligned} E(w) &:= \Big\{ -\log \mu(\partial T(w)) > (N_j + |v|)(\theta - \tfrac{1}{N}) \text{ and } \partial T(w)^* \neq \emptyset \Big\} \\ &= \Big\{ W(w) > \exp\big((N_j + |v|)\big(\log \mathfrak{m} - \theta + \tfrac{1}{N}\big)\big) \text{ and } \partial T(w)^* \neq \emptyset \Big\}. \end{aligned}$$

Given $\mathcal{G}(N_j)$ the events $E(w)$, $w \in \mathcal{K}_{N_j}$ are independent and, by the lower bound in Lemma 3, they have probability exceeding a constant multiple of

$$\begin{aligned} &\exp\Big((N_j + |v|)\big(\log \mathfrak{m} - \theta + \tfrac{1}{N}\big)\big)\big(\tau - \tfrac{\log p}{\log \mathfrak{m}}\big)\Big) \\ &\quad \geq \exp\Big((N_j + |v|)\big(a - \theta + \tfrac{1}{N}\big)\big(\tau + \tfrac{a}{\theta}(1+\tau) - \tau\Big) \\ &\quad = \exp\Big((N_j + |v|)\big(-a + a\big(\tfrac{a}{\theta}(1+\tau) - \tau\big) + \tfrac{a}{N\theta}(1+\tau)\big)\Big). \end{aligned}$$

Hence the probability that among the vertices $w \in \mathcal{K}_{N_j}$ none satisfies $E(w)$ is less than

$$\Big(1 - C \exp\Big(-(N_j + |v|)\big(a - a\big(\tfrac{a}{\theta}(1+\tau) - \tau\big) - \tfrac{a}{N\theta}(1+\tau)\big)\Big)\Big)^{K_{N_j}} \stackrel{j\to\infty}{\longrightarrow} 0\,,$$

as $K_{N_j} \geq \exp\big(N_j\big(a - a((a/\theta)(1+\tau) - \tau) - \tfrac{1}{j}\big)\big)$. We infer that, almost surely on $\partial T(v)^*$, there exist infinitely many vertices $w \in T(v)^*$ such that $E(w)$ holds. Hence there exists a ray $\xi \in \partial T(v)^* \cap U(N)$ and we are done. ■

Proof of the lower bound in (3). For given $\theta \geq a$ we study the Borel set

$$S(\theta) = \Big\{ \xi \in \partial T : \limsup_{n\to\infty} \frac{\log \mu B(\xi|n)}{-n} = \theta \Big\}.$$

We run percolation with retention parameter $p = \exp(-a((a/\theta)(1+\tau) - \tau))$ and note that

$$S(\theta) \cap \partial T^* = (A(\theta) \cap \partial T^*) \setminus \bigcap_{N=1}^{\infty} A(\theta - \tfrac{1}{N})\,. \tag{9}$$

Now $A(\theta) \cap \partial T^* \neq \emptyset$ almost surely on $\partial T^* \neq \emptyset$, by Lemma 4. On the other hand, in order to prove the lower bound we have shown that, $A(\theta - \frac{1}{N}) \cap \partial T^* = \emptyset$ almost surely. We infer that, with positive probability, $S(\theta) \cap \partial T^* \neq \emptyset$. Now, by Remark 2, the Hausdorff dimension of $S(\theta)$ is bounded below by $-\log p = a((a/\theta)(1+\tau) - \tau))$ with positive probability. It remains to observe that $\dim S(\theta) < a((a/\theta)(1+\tau) - \tau))$ is a property of a Galton-Watson tree T, which is inherited by all its subtrees $T(v)$

and also holds for all finite trees. Hence, it must have probability zero or one, see e.g. Proposition 2.2 in Peres [8]. Therefore our lower bound holds with probability one, and this finishes the proof of the lower bound in (3). ■

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