Thin and thick points for branching measure on a Galton-Watson tree

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Abstract

Suppose that μ is the branching measure on the boundary of a supercritical Galton-Watson tree with offspring distribution N such that $E[N\log^+ N] < \infty$. We determine the dimension spectrum of *thin points* under the condition $P\{N \le 1\} = 0$ and the dimension spectrum of *thick points* under the condition that either $0 < \sup\{t : E[\exp(tN)] < \infty\} < \infty$ or ess sup $N < \infty$. Our main tool for obtaining lower bounds is to study percolation on the Galton-Watson tree.

Keywords: Galton-Watson tree, branching measure, branching set, exceptional point, thin point, thick point, dimension spectrum, multifractal spectrum.

1 Introduction and main results

1.1 Branching measure on Galton-Watson trees

We consider a Galton–Watson tree T starting from a single progenitor, the root ρ. Let N be its offspring distribution and suppose it satisfies $m := EN > 1$ and $E[N\log^+ N] < \infty$. By ∂T we denote the *boundary* of the tree, or *branching set*, consisting of all infinite self-avoiding paths in T starting with the root.

For every vertex $v \in T$ we let $T(v)$ be the subtree consisting of all successors of v and $T_n(v)$ be the finite tree consisting of all vertices of the first n generations in the subtree $T(v)$. Denoting by $Z_n(v)$ the number of vertices of exactly nth generation in $T(v)$, it is well-known that, almost surely, for every vertex $v \in T$ with $\partial T(v) \neq \emptyset$ the limit

$$W(v) := \lim_{n\to\infty} \frac{Z_n(v)}{m^n}$$

exists in $(0,\infty)$. For every vertex v we consider $\partial T(v)$ in the natural way as a subset of ∂T. Then, almost surely on $\partial T \neq \emptyset$, there exists a natural nonzero, finite measure μ on the boundary ∂T of the tree, called the *branching measure*, such that for every vertex v of generation n,

$$\mu(\partial T(v)) = \frac{W(v)}{m^n}. \tag{1}$$

Throughout the paper we abbreviate $W = W(\rho)$, $T_n = T_n(\rho)$ and $Z_n = Z_n(\rho)$. If we let $n \geq 1$ and denote the vertices of nth generation in T by $v(1), \ldots, v(Z_n)$ we have

$$\mu(\partial T) = W = m^{-n} \sum_{j=1}^{Z_n} W(v(j)), \tag{2}$$

and, given T_n, all random variables $W(v(j))$ are iid with the same distribution as W. This reflects the statistical self-similarity of the branching measure μ.

We equip the boundary ∂T of the tree with the metric d given by $d(u,v) = e^{-n}$, where n is the generation of the last common vertex of $u, v \in \partial T$. By $B(\xi \mid n)$ we denote the closed ball of radius e^{-n} centred in $\xi \in \partial T$. With respect to this metric it is well–known that, almost surely on $\partial T \neq \emptyset$, the Hausdorff dimension of ∂T is $a := \log m$, see Hawkes (1981) and Lyons (1990), the exact dimension gauge was obtained in Liu (1996).

1.2 Tail behaviour

We recall some important parameters derived from the offspring distribution N. Let

$$M_- = \text{essinf } N, \; M_+ = \text{esssup } N \text{ and } \lambda_\pm = 1 - \log m / \log M_\pm.$$

If $M_+ = \infty$ this means that $\lambda_+ = 1$. Excluding the trivial case of deterministic offspring distribution we always have $\lambda_- < 0 < \lambda_+$. The tail behaviour of W in terms of these parameters is well understood.

Theorem 1.0 (Tail behaviour of W) *Suppose the offspring distribution N is not deterministic and satisfies $EN > 1$ and $E[N \log^+ N] < \infty$.*

(i) If and only if $P\{N \leq 1\} = 0$, the value

$$r_- := \liminf_{x \downarrow 0} \frac{-\log P\{W < x\}}{x^{1/\lambda_-}} \tag{3}$$

is positive and finite.

(ii) If and only if either $0 < \sup\{t : E[\exp(tN)] < \infty\} < \infty$ or ess sup $N < \infty$, the value

$$r_+ := \liminf_{x \uparrow \infty} \frac{-\log P\{W > x\}}{x^{1/\lambda_+}} \tag{4}$$

is positive and finite.

Proof: An almost exhaustive discussion of this topic can be found in Bingham (1988). Up to Tauberian arguments provided in Bingham (1988), the *if* case in (i) is Theorem 23 in Dubuc (1971a), the converse is immediate from Theorem 1 in Dubuc (1971b), see also Theorem 3.3 in Harris (1948). By Theorem 3.4 in Harris (1948), ess sup $N < \infty$ implies $0 < r_+ < \infty$. Equivalence of $0 < \sup\{t : E[\exp(tN)] < \infty\} < \infty$ and $0 < \sup\{t : E[\exp(tW)] < \infty\} < \infty$ is Theorem 2.1 in Liu (1996), and this completes the proof of (ii). ■

1.3 Exceptional behaviour

Let us first look at the dimension spectrum of exceptionally *thin points*. Here two fundamentally different cases may occur, depending whether

$$P\{N \le 1\} > 0 \quad \text{or} \quad P\{N \le 1\} = 0\,. \tag{5}$$

In the first case, sometimes called the *Schröder case*, Shieh and Taylor (1999) obtained a nontrivial spectrum by variation of the exponential decay of $\mu B(\xi|n)$. More precisely, they showed that, if $P\{N = 0\} = 0$ and $P\{N = 1\} < 1$ and if all moments of N exist, for $\tau := -\log P\{N = 1\}/a$,

$$\dim\Big\{\xi \in \partial T \,:\, \limsup_{n\to\infty} \frac{\log \mu B(\xi|n)}{-n} = b\Big\} = a\Big(\frac{a}{b}(1+\tau) - \tau\Big) \qquad \text{for } a \le b \le a(1 + \tfrac{1}{\tau}). \tag{6}$$

Here we study the second case of (5), which is sometimes called the *Böttcher case*, where we encounter a radically different situation. Liu (2001) already observed that the spectrum of the type in (6) is trivial in this case, as

$$\limsup_{n\to\infty} \frac{\log \mu B(\xi|n)}{-n} = a, \qquad \text{for } all\ \xi \in \partial T.$$

Our first result shows how a variation on a finer scale exposes nontrivial behaviour.

Theorem 1.1 (Spectrum of thin points) *Under the assumption of Theorem 1.0 (i), we have, almost surely on $\partial T \neq \emptyset$,*

$$\dim\Big\{\xi \in \partial T \,:\, \liminf_{n\to\infty} \frac{\mu B(\xi|n)}{m^{-n} n^{\lambda_-}} = \theta\Big\} = a - r_- \theta^{1/\lambda_-}, \quad \textit{for all } \theta \ge \Big(\frac{a}{r_-}\Big)^{\lambda_-}. \tag{7}$$

Observe that the behaviour of thin points is exceptional as, by Section 8 in Liu (2001),

$$\liminf_{n\to\infty} \frac{\mu B(\xi|n)}{m^{-n}[\log n]^{\lambda_-}} \ge \frac{1}{r_-^{\lambda_-}}, \qquad \text{for } \mu-\text{almost every } \xi \in \partial T. \tag{8}$$

We now turn to the study of exceptionally thick points. Here our aim is to weaken the assumptions for the spectrum of thick points obtained in Shieh-Taylor (1999) substantially. In their paper it was assumed that $P\{W > x\} \approx \exp(-r_+ x^{1/\lambda_+})$, a condition which is unproven in many cases, including the example discussed below, see Section 4 of Bingham (1988). We obtain a result assuming only the (optimal) conditions of Theorem 1.0 (ii).

Theorem 1.2 (Spectrum of thick points) *Under the assumption of Theorem 1.0 (ii) we have, almost surely on $\partial T \neq \emptyset$,*

$$\dim\Big\{\xi \in \partial T \,:\, \limsup_{n\to\infty} \frac{\mu B(\xi|n)}{m^{-n} n^{\lambda_+}} = \theta\Big\} = a - r_+ \theta^{1/\lambda_+}, \quad \textit{for all } 0 \le \theta \le \Big(\frac{a}{r_+}\Big)^{\lambda_+}. \tag{9}$$

Again, the behaviour of thick points is exceptional, see (3.4a) in Liu (1996) for the contrasting behaviour of typical points.

Example: We look at the limiting set C of the *fractal percolation process* based on tiling $[0,1)^d$ in k^d congruent cubes and keeping each of them independently with probability $p \in (1/k^d, 1)$, see Chayes (1995) or Peres (1999). We note that C can naturally be regarded as an embedding into $[0,1)^d$ of the boundary of a Galton–Watson process with binomial offspring distribution with parameters k^d and p, and may be equipped with the image measure ν of the branching measure. Denoting by $C_n(x)$ the tile of sidelength k^{-nd} containing x, we get, almost surely on $C \neq \emptyset$,

$$\dim\Big\{x \in C \,:\, \limsup_{n\uparrow\infty} \frac{\nu C_n(x)}{(k^d p)^{-n} n^{\lambda_+}} = \theta\Big\} = a - r_+ \theta^{1/\lambda_+}, \quad \text{for all } 0 \le \theta \le \Big(\frac{a}{r_+}\Big)^{\lambda_+},$$

for $\lambda_+ = -\log p/(d \log k)$ and r_+ as in Theorem 1.0 (ii). The thin point behaviour in this example is covered by the results of Shieh and Taylor (1999). $\square$

As a corollary of our proofs we can improve Theorem 7.1 of Liu (2001), where the same assertions are proved under stronger assumptions, his conditions (6.4) and (5.4).

Corollary 1.3 (Extremal behaviour) *Under the assumption of Theorem 1.0 (i) we have, almost surely on $\partial T \neq \emptyset$,*

$$\min_{\xi\in\partial T} \liminf_{n\to\infty} \frac{\mu B(\xi|n)}{m^{-n} n^{\lambda_-}} = \Big(\frac{a}{r_-}\Big)^{\lambda_-}.$$

Under the assumption of Theorem 1.0 (ii) we have, almost surely on $\partial T \neq \emptyset$,

$$\max_{\xi\in\partial T} \limsup_{n\to\infty} \frac{\mu B(\xi|n)}{m^{-n} n^{\lambda_+}} = \Big(\frac{a}{r_+}\Big)^{\lambda_+}.$$

Remark 1.1: The main problem in the proofs of Theorems 1.1 and 1.2 are the lower bounds. Several methods have been developed to deal with such problems, for example the construction of *Frostman measures* on the exceptional sets (see e.g. Shieh-Taylor (1998) or Dembo et al.(2000a)) and the method of hitting with independent random sets (see e.g. Khoshnevisan et al. (2000), Dembo et al.(2000b)). In the case of Galton-Watson trees T an additional difficulty is coming from the fact that the *environment* T itself is random. Our approach, discussed in detail in Section 3, solves this problem by studying *percolation* on the Galton-Watson tree. This is a particularly elegant variant of the method of hitting with a random set, which is based on the results and ideas of Lyons (1990) and applicable also in the case of *unbounded* offspring distribution.

Remark 1.2: Results similar to Theorem 1.1 or 1.2 have been obtained for other random fractal measures. For example, Shieh and Taylor (1998) study the range of a stable subordinator, Dembo, Peres, Rosen and Zeitouni (2000a, 2000b, 2001) Brownian paths, König and Mörters (2002) intersections of Brownian paths in $\mathbf{R}^3$, and Hambly and Jones (2002) random recursive fractals.

Remark 1.3: The thin and thick points of this paper have exceptionally small or large mass in a sequence of centred balls with radii decreasing to zero. An entirely different problem would be to study points which have exceptional behaviour in *all* small centred balls. This problem remains open.

2 Proof of the upper bounds

In this section we verify the upper bounds in Theorems 1.1 and 1.2, carrying out the details in the case of Theorem 1.1. The argument for Theorem 1.2 is completely analogous and can also be found in Liu and Shieh (1999).

More precisely, we show that under the conditions of Theorem 1.0 (i),

$$\dim\Big\{\xi \in \partial T \,:\, \liminf_{n\to\infty} \frac{\mu B(\xi|n)}{m^{-n} n^{\lambda_-}} \le \theta\Big\} \le a - r_- \theta^{1/\lambda_-}, \qquad \text{for all } \theta \ge \Big(\frac{a}{r_-}\Big)^{\lambda_-}. \tag{10}$$

Assume that θ is given as above. It suffices to show that, for any $b > a - r_-\theta^{1/\lambda_-}$, the Hausdorff $b-$dimensional measure of the set

$$\bigcap_{k=1}^{\infty} \bigcup_{n\ge k} \Big\{\xi \in \partial T \,:\, \frac{\mu B(\xi|n)}{m^{-n} n^{\lambda_-}} < (\theta+\varepsilon)\Big\} \tag{11}$$

is zero for sufficiently small $\varepsilon > 0$. For any positive integer k we have a covering of this set by just taking $\partial T(v)$ for all those vertices v of generation $n \ge k$ with the property that,

$$\frac{\mu(\partial T(v))}{m^{-n} n^{\lambda_-}} < (\theta+\varepsilon).$$

Let I_n be the number of sets of diameter e^{-n} in such a covering. Then, using Theorem 1.0 (i) and recalling that $a = \log m$, we see that, for sufficiently large k,

$$\begin{aligned} E\Big[\sum_{n=k}^{\infty} I_n e^{-bn}\Big] &= \sum_{n=k}^{\infty} e^{-bn} m^n P\Big\{\frac{\mu(\partial T(v))}{m^{-n} n^{\lambda_-}} < (\theta+\varepsilon)\Big\} = \sum_{n=k}^{\infty} e^{-bn} e^{an} P\{W < n^{\lambda_-}(\theta+\varepsilon)\} \\ &\le \sum_{n=k}^{\infty} e^{-n(b-a)} e^{-n(r_- - \varepsilon)(\theta+\varepsilon)^{1/\lambda_-}}. \end{aligned} \tag{12}$$

As $b > a - r_-\theta^{1/\lambda_-}$ we can choose a sufficiently small $\varepsilon > 0$ such that the right hand side in (12) is the remainder of a convergent series. Then $\liminf_{k\uparrow\infty} \sum_{n\ge k} I_n e^{-bn} = 0$ almost surely. We infer that the set in (11) has vanishing b-Hausdorff measure, which proves (10).

Finally, to obtain Corollary 1.3, we note that the previous proof also yields that the set on the left hand side of (10) is empty whenever $\theta < (a/r_-)^{\lambda_-}$.

3 Proof of the lower bounds

In this section we discuss the lower bounds in Theorems 1.1 and 1.2 using percolation on trees.

If T is any tree and $0 < p < 1$, we attach to each edge e of the tree an independent $\{0, 1\}$-valued random variable θ_e with $P\{\theta_e = 1\} = p$. We denote by T^* the connected component of the root ρ in the graph consisting of all edges e with $\theta_e = 1$. We say that T^* is the result of running *percolation with retention parameter* p on T.

Recall the definition of the *branching number* of a tree, see Lyons (1990) or Peres (1999), and the fact that, for every infinite tree T with branching number b, the Hausdorff dimension of the boundary is $\dim \partial T = \log b$. We use the following lemma of Lyons (1990) about the critical parameter for the survival of a tree after percolation.

Lemma 3.1 *Suppose that T is a tree with branching number b and T^* the random tree obtained from running percolation with retention parameter p on T.*

(i) If $p < 1/b$, then $\partial T^* = \emptyset$ almost surely,

(ii) If $p > 1/b$, then $\partial T^* \neq \emptyset$ with positive probability.

Let us now look at a Galton-Watson tree T. We need the following fact from Hawkes (1981), see also Lyons (1990) for a simpler proof.

Lemma 3.2 *Let T be a Galton-Watson tree. Then, almost surely on $\partial T \neq \emptyset$, the branching number of T is its mean offspring number m.*

Remark 3.1: If we run independent percolation on a Galton-Watson tree T, the unconditional distribution of T^* is again the law of a Galton-Watson tree. If m was the mean offspring number of T, then pm is the mean offspring number of T^*.

After these preparations we are ready to state the main lemma in our proof.

Lemma 3.3 (Main lemma) *Let T be a Galton-Watson tree with offspring distribution N and $m = EN > 1$ and $E[N \log^+ N] < \infty$. Define*

$$A_-(\theta) = \Big\{ \xi \in \partial T \,:\, \liminf_{n \to \infty} \frac{\mu B(\xi|n)}{m^{-n} n^{\lambda_-}} \leq \theta \Big\} \tag{13}$$

and

$$A_+(\theta) = \Big\{ \xi \in \partial T \,:\, \limsup_{n \to \infty} \frac{\mu B(\xi|n)}{m^{-n} n^{\lambda_+}} \geq \theta \Big\}. \tag{14}$$

(i) Suppose the conditions of Theorem 1.0 (i) are in force. For $\theta \geq (a/r_-)^{\lambda_-}$ define $p = \exp(-\log m + r_- \theta^{1/\lambda_-})$. Then,

$$A_-(\theta) \cap \partial T^* \neq \emptyset, \qquad \textit{almost surely on } \partial T^* \neq \emptyset.$$

(ii) Suppose the conditions of Theorem 1.0 (ii) are in force. Let $0 < \theta \le (a/r_+)^{\lambda_+}$ *and define* $p = \exp(-\log m + r_+ \theta^{1/\lambda_+})$. *Then,*

$$A_+(\theta) \cap \partial T^* \neq \emptyset, \qquad \textit{almost surely on } \partial T^* \neq \emptyset.$$

In the proof of the main lemma we concentrate on the discussion of statement (i), statement (ii) follows from analogous arguments. We first need an extension of Theorem 1.0 (i) studying the probability that $\{W < x\}$ holds jointly with $\{\partial T^* \neq \emptyset\}$.

Lemma 3.4 *Under the assumption of Theorem 1.0 (i) we have*

$$\liminf_{x \downarrow 0} \frac{-\log P(\{W < x\} \cap \{\partial T^* \neq \emptyset\})}{x^{1/\lambda_-}} = r_- . \tag{15}$$

Proof: Inequality $\ge$ is immediate. For the converse fix $\varepsilon > 0$ for the moment. Recall that $p > 1/a$ is fixed and hence, by Lemma 3.1 (*ii*), $P\{\partial T^* \neq \emptyset\} > 0$. Choose c so large that

$$P(\{W < c\} \cap \{\partial T^* \neq \emptyset\}) \ge \tfrac{1}{2} P\{\partial T^* \neq \emptyset\} > 0, \tag{16}$$

and n such that $m^{-n} c < \varepsilon x \le m^{-n+1} c$. Enumerate the vertices of the nth generation of T by $v(1), \ldots, v(Z_n)$. Recall from (2) that $W = m^{-n} \sum_{j=1}^{Z_n} W(v(j))$ and hence

$$\begin{aligned} &\{W < x\} \cap \{\partial T^* \neq \emptyset\} \\ &\supseteq \; \{v(1) \in T^*\} \cap \Big\{ m^{-n} \sum_{j=2}^{Z_n} W(v(j)) < x(1-\varepsilon) \Big\} \cap \Big(\{W(v(1)) < c\} \cap \{\partial T(v(1))^* \neq \emptyset\} \Big), \end{aligned}$$

where $\{\partial T(v)^* \neq \emptyset\}$ means that there exists a path of open edges from v to infinity. The three events on the right are independent, hence we get, using also (16),

$$\begin{aligned} P\Big(\{W < x\} \cap \{\partial T^* \neq \emptyset\}\Big) \;&\ge\; P\{v(1) \in T^*\} P\Big\{ m^{-n} \sum_{j=2}^{Z_n} W(v(j)) < x(1-\varepsilon) \Big\} \\ &\qquad P\Big(\{W(v(1)) < c\} \cap \{\partial T(v(1))^* \neq \emptyset\} \Big) \\ &\ge\; p^n P\Big\{ m^{-n} \sum_{j=1}^{Z_n} W(v(j)) < x(1-\varepsilon) \Big\} \tfrac{1}{2} P\{\partial T^* \neq \emptyset\}. \end{aligned}$$

From the definition of n we see that p^n decreases at most like a power of x, so that

$$\liminf_{x \downarrow 0} \frac{-\log P(\{W < x\} \cap \{\partial T^* \neq \emptyset\})}{x^{1/\lambda_-}} \le \liminf_{x \downarrow 0} \frac{-\log P\{W < x(1-\varepsilon)\}}{x^{1/\lambda_-}} \le r_- (1-\varepsilon)^{1/\lambda_-},$$

by Theorem 1.0 (i). Letting $\varepsilon \downarrow 0$ yields the result. ∎

Proof of Lemma 3.3: For every positive integer N we let

$$U(N) = \Big\{ \xi \in \partial T \;:\; \text{ there exists } k \ge N \text{ with } \frac{\mu B(\xi|k)}{m^{-k} k^{\lambda_-}} < \theta + \tfrac{1}{N} \Big\}. \tag{17}$$

Our aim is to show that $U(N)\cap\partial T^*$ is dense in ∂T^* almost surely. But we first argue how we can use this to finish the proof of Lemma 3.3 (i). Note that $U(N)\cap\partial T^*$ is relatively open in ∂T^*, which is a compact, hence complete, metric space. By *Baire's Theorem*,

$$\bigcap_{N=1}^{\infty} U(N)\cap\partial T^* \text{ is dense in } \partial T^*.$$

But note that $\bigcap_{N=1}^{\infty} U(N) = A_-(\theta)$, hence $A_-(\theta)\cap\partial T^*$ must be dense in ∂T^* almost surely. This implies that $A_-(\theta)\cap\partial T^*$ is nonempty almost surely conditional on $\partial T^*\neq\emptyset$, which is the required statement.

It remains to show that, for fixed integer N, the set $U(N)\cap\partial T^*$ is dense in ∂T^* almost surely. To prepare the proof recall Lemma 3.4. It is elementary to see that the liminf is also attained through the sequence $x_n = n^{\lambda_-}(\theta+\frac{1}{N})$, hence

$$-r_-(\theta+\tfrac{1}{N})^{1/\lambda_-} = \limsup_{n\to\infty}\frac{1}{n}\log P\Big(\{W<n^{\lambda_-}(\theta+\tfrac{1}{N})\}\cap\{\partial T^*\neq\emptyset\}\Big).$$

Select an increasing sequence (n_j) of integers with

$$P\Big(\{W<n_j^{\lambda_-}(\theta+\tfrac{1}{N})\}\cap\{\partial T^*\neq\emptyset\}\Big)\geq\exp\Big(-n_j(r_-+\tfrac{1}{j})(\theta+\tfrac{1}{N})^{1/\lambda_-}\Big). \tag{18}$$

To prove that $U(N)\cap\partial T^*$ is dense in ∂T^*, we fix some vertex $v\in T$ and need to show that, almost surely conditional on $\partial T(v)^*\neq\emptyset$, we have $U(N)\cap\partial T(v)^*\neq\emptyset$.

Let $\mathcal{G}(n)$ be the σ-field generated by the finite subtree $T_n(v)\subset T(v)$, consisting of the first n generations of $T(v)$, together with the random variables $\{\theta_e : e\in T_n(v)\}$. Let $\mathcal{K}_n$ be the collection of vertices in $T(v)^*$ of generation n and let K_n be the cardinality of $\mathcal{K}_n$. Note that the random variable K_n is $\mathcal{G}(n)$-measurable.

We define a sequence of $\mathcal{G}(n)$-stopping times $N_1,N_2,\dots$ by $N_0=1$ and

$$N_k=\min\Big\{n_j>N_{k-1} : K_{n_j}\geq\exp\Big(n_j(r_-\theta^{1/\lambda_-}-\tfrac{1}{k})\Big)\Big\}.$$

By Lemma 3.2 and Remark 3.1, almost surely on $\partial T(v)^*\neq\emptyset$, we have

$$\dim\partial T(v)^*=\log pm=r_-\theta^{1/\lambda_-}. \tag{19}$$

This implies that the sequence $\{N_j : j\geq1\}$ of stopping times is indeed infinite almost surely on $\partial T(v)^*\neq\emptyset$. For every vertex $w\in T(v)$ of generation N_j define the event

$$E(w):=\Big\{W(w)<N_j^{\lambda_-}(\theta+\tfrac{1}{N})\text{ and }\partial T(w)^*\neq\emptyset\Big\}=\Big\{\frac{\mu\partial T(w)}{m^{-N_j}N_j^{\lambda_-}}<(\theta+\tfrac{1}{N})\text{ and }\partial T(w)^*\neq\emptyset\Big\}.$$

Given $\mathcal{G}(N_j)$ the events $E(w)$, $w\in\mathcal{K}_{N_j}$ are independent and, by (18), they have probability exceeding

$$\exp\Big(-N_j\Big(r_-+\tfrac{1}{j}\Big)\Big(\theta+\tfrac{1}{N}\Big)^{1/\lambda_-}\Big).$$

Hence the probability that among the vertices $w\in\mathcal{K}_{N_j}$ none satisfies $E(w)$ is less than

$$\Big(1-\exp\Big(-N_j\Big(r_-+\tfrac{1}{j}\Big)\Big(\theta+\tfrac{1}{N}\Big)^{1/\lambda_-}\Big)\Big)^{K_{N_j}}\stackrel{j\to\infty}{\longrightarrow}0,$$

as $K_{N_j} \geq \exp\left(N_j(r_-\theta^{1/\lambda_-} - \frac{1}{j})\right)$. We infer that, almost surely on $\partial T(v)^*$, there exist infinitely many vertices $w \in T(v)^*$ such that $E(w)$ holds. Hence there exists a ray $\xi \in \partial T(v)^* \cap U(N)$ and we are done. ∎

Proof of the lower bounds: We concentrate on the exceptionally thin points, the case of thick points being analogous. For given θ we study the Borel set

$$S(\theta) = \left\{\xi \in \partial T \,:\, \liminf_{n\to\infty} \frac{\mu B(\xi|n)}{m^{-n} n^{\lambda_-}} = \theta\right\}.$$

We run percolation with retention parameter $p = \exp(-\log m + r_-\theta^{1/\lambda_-})$ on T. We have

$$\partial S(\theta) \cap \partial T^* = (A_-(\theta)^* \cap \partial T^*) \setminus \bigcap_{N=1}^{\infty} A_-(\theta - \tfrac{1}{N})\,. \tag{20}$$

Now $A_-(\theta) \cap \partial T^* \neq \emptyset$ almost surely on $\partial T^* \neq \emptyset$, by our main lemma. On the other hand, by (10) we have,

$$\dim A(\theta - \tfrac{1}{N}) \leq a - r_-(\theta - \tfrac{1}{N})^{1/\lambda_-} < a - r_-\theta^{1/\lambda_-} = \log(1/p),$$

so that $A - (\theta - \frac{1}{N}) \cap \partial T^* = \emptyset$ by the first part of Lemma 3.1. We infer that, with positive probability, $S(\theta) \cap \partial T^* \neq \emptyset$. Now Lemma 3.1 (i) and a Choquet capacity argument imply that the Hausdorff dimension of $S(\theta)$ is bounded below by $a - r_-\theta^{1/\lambda_-}$ with positive probability. It remains to observe that $\dim S(\theta) < a - r_-\theta^{1/\lambda_-}$ is a property of a Galton-Watson tree T, which is inherited by all its subtrees $T(v)$ and also holds for all finite trees. Hence, conditional on survival it must have probability zero or one, see e.g. Proposition 2.2 in Peres (1999). This finishes the proof of the lower bound for the dimension spectrum.

Finally, Lemma 3.3 shows that with positive probability there exists a ray $\xi \in \partial T$ with $\liminf_{n\to\infty} \mu B(\xi|n)/m^{-n}n^{\lambda_-} \leq \theta$. Again the complementary event is inherited by all subtrees $T(v)$ and the same zero-one-law as above completes the proof of Corollary 1.3. ∎

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