STRONG CLUMPING OF SUPER-BROWNIAN MOTION IN A STABLE CATALYTIC MEDIUM

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A typical feature of the long time behaviour of continuous super-Brownian motion in a stable catalytic medium is the development of large mass clumps (or clusters) at spatially rare sites. We describe this phenomenon by means of a functional limit theorem under renormalisation. The limiting process is a Poisson point field of mass clumps with no spatial motion component and with infinite variance. The mass of each cluster evolves independently according to a non-Markovian continuous process trapped at mass zero, which we describe explicitly by means of a Brownian snake construction in a random medium. We also determine the survival probability and asymptotic size of the clumps.

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1. Introduction

1.1. **Motivation.** Models of particle movement and branching in random media have been widely studied in the last twenty years. A class which received particular interest are models of measure-valued processes where, heuristically speaking, the individual branching rates of the moving particles depend on the amount of contact between the particle, called the *reactant*, and a singular random medium, called the *catalyst*. In dimension one even very thin catalysts, for example point catalysts, can be considered. A particularly natural choice of a catalytic medium are *stable* random measures Γ on $\mathbb{R}$ of index $0 < \gamma < 1$, which are the prototypes of a singular catalyst with infinite asymptotic density; see formula (3) below, and [7, Subsections 1.3–1.4] for further motivation for this choice of catalytic medium. A rather general one-dimensional model combining super-stable motions of the reactant particles with possibly moving random catalysts, covering the case of the stable medium Γ was developed in [7, 8]. For an up-to-date introduction to catalytic super-Brownian motion, we refer to [10].

Recent research on super-Brownian motion with a stable catalytic medium has lead to several interesting results; we restrict our attention to the case of a Brownian moving reactant, which branches with finite variance in the presence of a non-moving stable catalyst Γ in $\mathbb{R}$. In this case, starting from a finite initial mass, the compact support property was proved in [12], and finite time extinction in [11], see [18] for a quick route. Already in [7], in the case of an infinite initial measure, the long-term clumping behaviour of the reactant was shown in a mass-time-space rescaling limit theorem. It states that at a fixed macroscopic time t the suitably mass-space-rescaled clumps form a random measure with *independent increments*, see [7, Theorem 1.9.4]. But it could not be settled, see [7, p.251], whether or not the clumps are macroscopically spatially isolated, that is whether the limiting measure is carried by a Poisson point field on $\mathbb{R}$ as known in the constant medium case [6].

The main motivation for the present paper was to attack this problem. We show that in fact the clumps *are* isolated, that is, the limiting measure is supported by a homogeneous Poisson point field, see Theorem 1 (ii) below. This is achieved by a refinement of a method of good and bad historical reactant paths, which was developed in [11] and goes back to [13, 17, 26].

Beyond this problem, we describe the mass of the rescaled clumps as a process in macroscopic time. For this purpose we provide a *functional* limit approach, Theorem 1 (i) below, which shows convergence of the rescaled processes on a path space of *continuous* measure-valued processes. The time evolution of these masses is described in terms of exit measures of a Brownian snake in a random medium with a motion process featuring the inverse of the collision local times of the reactant paths with the medium, see Theorem 6 (ii) below. Whereas the clumps of the original process have finite variance given the medium, this property is lost in the limit, a remarkable property conjectured in [7, p.253]. In fact, the clump sizes of the limit have probability tails of index $1 + \gamma < 2$, see Theorem 11 (iii) below. This is in contrast to the constant medium case studied in [6] and due to some form of averaging over the stable catalyst, which has locally infinite expectations. We also determine the (macroscopic) survival probability of clumps, see Theorem 11 (ii).

A main tool for the functional limit theorem is the representation of both the catalytic super-Brownian motion and the limit process in terms of exit measures of a *Brownian snake in the stable medium* Γ. The use of exit measures and subordination for the historical particles to describe general branching mechanisms goes back to [1], though the present paper seems to be the first instance where this approach is used to deal with the case of space-dependent branching in an irregular catalytic medium.

Following the circulation of a preprint version of the present paper, Klenke [22] extended our approach constructing superprocesses with branching rates given by a large class of (strictly) increasing additive functionals, in terms of a Brownian snake construction. In a recent work, Dhersin and Serlet [15] use a quite different approach to construct a modification of the Brownian snake to represent a class of spatially interacting super-Brownian motion including catalytic super-Brownian motion in a medium, which is a measure *equivalent* to Lebesgue measure; the case of a *singular medium* Γ is not covered by their analysis.

Revealing the macroscopically isolated nature of the clumps embeds the present investigation in the realm of the concept of *intermittency.* Roughly speaking, intermittency means in our context that after a long time the catalytic superprocess exhibits a spatially irregular structure consisting of islands of high mass peaks, which are located at great distance from each other. See for instance [19, 20] or [25] for other work in this direction.

1.2. **Statement of the main results.**

1.2.1. *Super-Brownian motion in a stable catalytic medium: preliminaries.* Let $\mathcal{M}(\mathbb{R})$ denote the space of all locally finite (nonnegative) measures on $\mathbb{R}$, equipped with the vague topology generated by the mappings $\varphi \mapsto \langle \mu, \varphi \rangle$, for all $\varphi : \mathbb{R} \to [0, \infty)$ continuous with compact support. Here and throughout the paper we use both notations $\langle \mu, \varphi \rangle$ and $\int_{\mathbb{R}} \varphi \, d\mu$ to denote integrals. There is a sequence $\{\varphi_n : n \geq 1\}$ of such functions such that

$$d(\mu, \nu) := \sum_{n=1}^{\infty} 2^{-n} \Big(\big| \langle \mu, \varphi_n \rangle - \langle \nu, \varphi_n \rangle \big| \wedge 1 \Big), \ \text{ for } \ \mu, \nu \in \mathcal{M}(\mathbb{R}), \tag{1}$$

defines a metric, which makes $\mathcal{M}(\mathbb{R})$ Polish.

Define Φ to be the set of all continuous functions $\varphi : \mathbb{R} \to [0, \infty)$ such that there are constants $a, b > 0$ (depending on φ) with $\varphi(x) \leq a \exp(-bx^2)$ for all $x \in \mathbb{R}$. For all measure-valued processes in this paper we choose the state space to be the space of tempered measures

$$\mathcal{M}_{\text{tem}} := \mathcal{M}_{\text{tem}}(\mathbb{R}) := \Big\{ \mu \in \mathcal{M}(\mathbb{R}) \, : \, \langle \mu, \varphi \rangle < \infty \ \text{ for all } \ \varphi \in \Phi \Big\}. \tag{2}$$

Note that in particular the Lebesgue measure ℓ belongs to $\mathcal{M}_{\text{tem}}$. We let $\mathcal{M}_{\text{tem}} \subseteq \mathcal{M}(\mathbb{R})$ inherit the vague topology of $\mathcal{M}(\mathbb{R})$.

Suppose that Γ is a *stable random measure* on $\mathbb{R}$ of index $0 < \gamma < 1$, i.e. for every measurable $\varphi : \mathbb{R} \to [0, \infty)$ we have

$$\mathsf{E} \big\{ \exp \langle \Gamma, -\varphi \rangle \big\} = \exp \Big(- \int_{\mathbb{R}} \varphi(x)^{\gamma} \, dx \Big). \tag{3}$$

Almost surely, Γ belongs to $\mathcal{M}_{\text{tem}}$. This follows from the fact that the integral on the right hand side of (3) is always finite for $\varphi \in \Phi$. Moreover, Γ is almost surely a purely atomic measure with atoms densely located in $\mathbb{R}$. Note also that Γ is spatially homogeneous and has independent increments.

The measure-valued processes under consideration may be considered as random variables with values in the space $C\big((0, \infty), \mathcal{M}_{\text{tem}}\big)$ of continuous functions $\nu : (0, \infty) \to \mathcal{M}_{\text{tem}}$, where for topological reasons it is sometimes convenient to exclude the time $t = 0$. We endow this space with the topology of *uniform convergence on compact time intervals*, which is induced by the metric

$$\mathbf{d}(\mu, \nu) := \sum_{n=1}^{\infty} 2^{-n} \sup_{1/n \leq t \leq n} d\big(\mu(t), \nu(t)\big), \ \text{ for } \mu, \nu \in C\big((0, \infty), \mathcal{M}_{\text{tem}}\big), \tag{4}$$

and is easily seen to be Polish.

Let $X := X[\Gamma] := \{X_t : t \geq 0\}$ denote the continuous super-Brownian motion in $\mathbb{R}$ in the catalytic random medium Γ. Throughout the paper we refer to probabilities and expectations with respect to the random medium Γ with letters P and E, respectively, and to the probabilities and expectations of the process with given medium Γ by $\mathbb{P}^\Gamma$ and $\mathbb{E}^\Gamma$, sometimes with a subscript indicating the respective starting measure. With this convention, for *given* Γ, the process $X = X[\Gamma]$ is the continuous, time-homogeneous Markov process with Laplace transition functionals

$$\mathbb{E}^\Gamma \left\{ \exp\langle X_t, -\varphi\rangle \mid X_s = \mu \right\} = \exp\langle \mu, -V^\Gamma_{t-s}\varphi\rangle, \quad \text{for } t > s \geq 0, \tag{5}$$

where $\mu \in \mathcal{M}_{\text{tem}}$, $\varphi \in \Phi$, and $V^\Gamma\varphi = \left\{ V^\Gamma_t\varphi(x) : t \geq 0,\, x \in \mathbb{R} \right\}$ is the unique nonnegative solution of the equation

$$V^\Gamma_t\varphi(y) = S_t\varphi(y) - 2\int_0^t ds \int_{\mathbb{R}} p_s(x-y)\left[V^\Gamma_{t-s}\varphi(x)\right]^2 \Gamma(dx), \quad \text{for } t \geq 0,\, y \in \mathbb{R}. \tag{6}$$

Here p denotes the standard heat kernel in $\mathbb{R}$, and $S = \{S_t : t \geq 0\}$ the heat flow semigroup defined by $S_t\varphi(y) = \int_{\mathbb{R}} p_t(x-y)\varphi(x)\,dx$. The nonlinear semigroup $V^\Gamma = \{V^\Gamma_t : t \geq 0\}$ operates in Φ. The interpretation of the process X as a process whose reactant particles branch at site $x \in \mathbb{R}$ with rate $2\,\Gamma(dx)$ corresponds to the fact that, loosely speaking, given Γ, the function $v = V^\Gamma\varphi$ solves the symbolic partial differential equation

$$\frac{\partial}{\partial t}v = \frac{1}{2}\frac{\partial^2}{\partial x^2}v - 2\,\Gamma v^2, \quad \text{with initial condition } v\big|_{t=0} = \varphi. \tag{7}$$

Existence and uniqueness of nonnegative solutions V^Γ of (6) were established in [8], X was constructed as a Markov process in [7, Section 2], and its continuity follows from [9, Corollary 2 (p.257), Proposition 12 (p.230), and Theorem 1(b) (p.235)], even in a stronger topology. Note also that in the case of a finite starting measure the total mass process $\|X\| := \left\{ \|X_t\| : t \geq 0 \right\}$ is a continuous martingale [9, Proposition 3, p.236].

Already from the form of the transition functional (5) it is clear that, given Γ, for $\mu_1, \mu_2 \in \mathcal{M}_{\text{tem}}$, the process $X = X[\Gamma]$ with $X_0 = \mu_1 + \mu_2$ can be constructed as the sum of two (conditionally given Γ) independent copies of $X[\Gamma]$ with $X_0 = \mu_1$ respectively $X_0 = \mu_2$. We frequently refer to this property as the *branching property* of the process X.

We stress the fact that we always use a *quenched* approach in dealing with the model $X = X[\Gamma]$ in the random medium Γ: First the catalyst Γ is sampled, and then the reactant process $X[\Gamma]$ is run, given the catalyst Γ. In particular, the law $\mathbb{P}^\Gamma$ of the reactant is random, and the randomness is inherited from the distribution P of Γ.

1.2.2. *Strong clumping of catalytic super-Brownian motion.* Recall that γ is the index of the stable catalyst Γ. Define the *scaling index*

$$\eta := (\gamma+1)/(2\gamma) \tag{8}$$

and observe that this number is larger than 1. For every $k > 0$ we introduce the renormalised measure-valued process $X^k = X^k[\Gamma] = \{X^k_t : t \geq 0\}$ by

$$X^k_t(B) := k^{-\eta} X_{kt}(k^\eta B), \quad \text{for } B \subseteq \mathbb{R} \text{ Borel}, \ t \geq 0. \tag{9}$$

The next theorem summarises the results on the limiting behaviour of X^k obtained before the present paper.

Theorem 0 (Results of [7, Theorem 1.9.4]).

(i) Convergence. *Starting $X = X[\Gamma]$ in the Lebesgue measure $X_0 = \ell$, for every* fixed t *there is a random variable X_t^∞ defined on some probability space $(\Omega, \mathcal{A}, \mathbf{P}_\ell)$ such that, in P–probability, the following weak convergence of probability measures on $\mathcal{M}_{\text{tem}}$ holds:*

$$\lim_{k\uparrow\infty} \mathbb{P}_\ell^\Gamma\{X_t^k \in \,\cdot\,\} = \mathbf{P}_\ell\{X_t^\infty \in \,\cdot\,\}. \tag{10}$$

(ii) Characterization of the limit. *For every bounded, measurable function $\varphi : \mathbb{R} \to [0, \infty)$ let*

$$U^\Gamma\varphi = \bigl\{U_r^\Gamma\varphi\,(x) : r \geq 0,\ x \in \mathbb{R}\bigr\} \tag{11}$$

be the nonnegative solution of the equation

$$U_r^\Gamma\varphi\,(y) = S_r\varphi\,(y) - 2\int_0^r ds \int_{\mathbb{R}} p_s(x-y)\,\bigl[U_{r-s}^\Gamma\varphi\,(x)\bigr]^2\,\Gamma(dx), \ \ \textit{for } r \geq 0,\, y \in \mathbb{R}, \tag{12}$$

which is constructed in [8, Theorem 2.14]. *Then the Laplace functional of X_t^∞ satisfies*

$$\mathbf{E}_\ell\bigl\{\exp\bigl(-\theta X_t^\infty(A)\bigr)\bigr\} = \exp\bigl(-\ell(A)\,\mathsf{E}U_t^\Gamma\theta\,(0)\bigr), \ \ \textit{for } A \subseteq \mathbb{R} \textit{ Borel},\ \ \theta \geq 0. \tag{13}$$

(iii) Properties of the limit. *$\mathbf{P}_\ell$–almost surely, X_t^∞ is non-degenerate, homogeneous, and has independent increments. Moreover, the scaling procedure is* persistent *in the sense that $\mathbf{E}_\ell X_t^\infty = \ell$.*

The important feature of this result is that the non-degenerate limit is obtained by a different, stronger scaling than in the classical case of a constant medium [6], hence the qualitative nature of the clumping behaviour is different.

Crucial questions about the spatial structure of the limit measures X_t^∞ were left open in [7]. A question of particular interest in this realm was posed in [7, p.251]: The problem is whether or not the X_t^∞ are compound Poisson point fields on $\mathbb{R}$, i.e. whether on the macroscopic level the clumps are spatially separated. Our first main result answers this question in the affirmative. Our second aim in this paper is to give a full description of the spatial and temporal evolution of the field of clumps at a macroscopic level. This requires, as a first step, a functional limit theorem. This question was not investigated in [7] and is particularly interesting, as the limit process turns out to be non-Markovian and continuous. Here is the precise statement:

Theorem 1 (Main result). *Let X be the continuous super-Brownian motion in the stable random medium Γ started with $X_0 = \ell$, and $X^k = X^k[\Gamma] = \{X_t^k : t > 0\}$, for $k > 0$, the renormalised processes defined in* (9).

(i) Functional limit theorem. *In P–probability, the random laws of the renormalised processes $X^k[\Gamma]$ converge weakly on the function space $C\bigl((0,\infty), \mathcal{M}_{\text{tem}}\bigr)$ as $k \uparrow \infty$ to the deterministic law of a limit process $X^\infty = \{X_t^\infty : t > 0\}$ defined on some probability space $(\Omega, \mathcal{A}, \mathbf{P}_\ell)$. The limit process is started in $X_0^\infty = \ell$.*

(ii) Compound Poisson structure. *For each time $t > 0$, the state X_t^∞ of the limit process of part* (i) *is a compound Poisson point field, i.e. a random discrete measure on $\mathbb{R}$ with atoms located in the points of a Poisson point field and with independent identically distributed atomic weights. The temporal development of X^∞ is non-Markovian and as follows: Almost surely, the atoms do not move in space, no new atoms are born, but each atom dies in finite time.*

Remark 2 (The role of $t = 0$)**.** If $Z = \{Z_t : t > 0\}$ is a random variable in $C\big((0,\infty), \mathcal{M}_{\text{tem}}\big)$, and Z_0 a random variable in $\mathcal{M}_{\text{tem}}$, we say that Z is *started in* Z_0, if Z_ε converges to Z_0 *in law,* as $\varepsilon \downarrow 0$. In the functional limit theorem we get convergence of the processes X^k started in $X_0^k = \ell$ (to X^∞ started in $X_0^\infty = \ell$). This requirement can be relaxed slightly but not completely omitted (see Remark 17 below). Nevertheless it is possible to construct the process X^∞ canonically for *any* starting measure $X_0^\infty = \mu \in \mathcal{M}_{\text{tem}}$, see Theorem 6 or Corollary 10 below, leaving *open* the question of sample path continuity of X^∞ at time $t = 0$. Moreover, part (ii) of Theorem 1 holds for the process X^∞ started in any measure $X_0^\infty = \mu \in \mathcal{M}_{\text{tem}}$. ◇

Remark 3 (Loss of the Markov property in the limit)**.** Already from the Laplace transform (13) it is clear that the law of X_t^∞ is *deterministic.* Whereas the conditioned law of X_t, given Γ, is determined by solutions of the log-Laplace equation (7) for the given Γ, the law of the limit X_t^∞ is described only by an *expectation* of certain log-Laplace equation solutions. In this sense, in the limiting model the random medium is "averaged". The finite dimensional distributions of X^∞ (see Corollary 10 below) are determined by an expectation of iterates of solutions to (7). But this expectation operation destroys the *non-linear* semigroup property of solutions to (7) (given Γ). As a consequence, X^∞ is *not* a Markov process although it keeps the branching property, in the sense discussed in the paragraph after (7).

Less formally, whereas the future motion and branching behaviour of an intrinsic reactant particle of X depend only on its present position relative to the catalyst, in the scaling limit the situation changes drastically. By the strong contraction of space, the information about the relative position of a particle to the medium is getting lost in the limit and only some averaged features of Γ remain, which are not sufficient for a Markovian evolution of the limiting reactant. However, it is possible to construct a Markovian process on a larger state space, including both microscopic and macroscopic space information, from which the process X^∞ can be recovered by projection. See Remark 9 below for more details. ◇

Remark 4 (Continuous limit process)**.** Note that convergence in our functional limit theorem holds in the sense of weak convergence of laws on the space $C\big((0,\infty), \mathcal{M}_{\text{tem}}\big)$ of continuous $\mathcal{M}_{\text{tem}}(\mathbb{R})$–valued paths. In particular, the limit process X^∞ is *continuous* on $(0,\infty)$ as well. ◇

1.2.3. *The crossing property.* An interesting path property of the unscaled process X, which enters in the proof of the functional limit theorem and may be of independent interest, is the following *crossing property*, which is reminiscent of the compact support property investigated in [12]. For the precise formulation, denote by $\ell_{(a,b)}$ for $a, b \in \mathbb{R} \cup \{-\infty, \infty\}, a < b,$ the restriction of Lebesgue measure ℓ on $\mathbb{R}$ to the open interval (a, b). We show that for the catalytic super-Brownian motion X started with $\ell_{(0,\infty)}$ the amount of total mass at a time which has travelled across the origin to the nonpositive halfline is bounded in time.

Theorem 5 (Crossing property)**.** *Suppose that* $\{X_t : t \geq 0\}$ *is the catalytic super-Brownian motion in the stable medium* Γ *starting from* $X_0 = \ell_{(0,\infty)}$. *Then, for* P*–almost every* Γ,

$$\sup_{t \geq 0} X_t\big((-\infty, 0]\big) < \infty, \quad \mathbb{P}^\Gamma\text{–}\textit{almost surely.} \tag{14}$$

1.2.4. *Snake representations of* X *and* X^∞. As a further major tool in the proof of Theorem 1 we construct representations of both the original process X and the limiting process X^∞ in terms of exit measures of a Brownian snake in the stable medium Γ. This makes the limit process explicit and permits a comparison of the two processes X and X^∞. As this is also of independent interest, we present the results here.

The idea of using the Brownian snake to represent classical super-Brownian motion is due to Le Gall and has since been generalised to various other types of superprocesses. Bertoin, Le Gall and Le Jan have extended this technique to represent superprocesses with more general, but not space-dependent, branching mechanisms. Roughly speaking, they use individual time-changes for each particle, which allow to pass from one branching mechanism to a different one by subordination on the particle level. References are [23, 24] for the first explicit snake construction and [1] for the extension.

In the present paper we extend this idea to our particular case of a *space-dependent* branching mechanism — recall that in rough terms the branching rate at site x is given by $2\,\Gamma(dx)$. To formulate the result we briefly introduce the basic notation of the Brownian snake $w = w[\Gamma]$ in our random medium case, and its excursion measures $\mathbb{N}^{\Gamma}_{(0,x)}$, both in the quenched situation of a fixed sample Γ of the random medium. More details can be found in Subsection 2.1 below.

To describe our approach, let us first look at a generic reactant particle, which moves along a Brownian path $W = \{W(t) : t \geq 0\}$ in $\mathbb{R}$ until its death, when it is frozen into its current position. Of course, the motion process could as well be described by the two-dimensional Markov process $t \mapsto (t, W(t))$ with phase space $D := [0,\infty) \times \mathbb{R}$. The branching of the reactant particle, however, is governed by its *collision local time* $L_{[\Gamma,W]}$ with the medium Γ, defined by

$$L_{[\Gamma,W]}(r) = \int_{\mathbb{R}} \Gamma(dy) L^{y}(r), \quad \text{for } r \geq 0, \tag{15}$$

where $r \mapsto L^{y}(r)$ is the continuous local time of W at level $y \in \mathbb{R}$. $L_{[\Gamma,W]}$ is a (homogeneous, nondecreasing) continuous additive functional of Brownian motion W. As the positions of the atoms of Γ are dense in $\mathbb{R}$, it is easy to see that $L_{[\Gamma,W]}$ is (strictly) increasing. Moreover, $L_{[\Gamma,W]}(r) \uparrow \infty$ almost surely as $r \uparrow \infty$.

We use the continuous inverse function $L^{-1}_{[\Gamma,W]} : [0,\infty) \to [0,\infty)$ of $L_{[\Gamma,W]}$ to introduce a new time scale for the reactant particle on which its collision local time grows linearly. More precisely, instead of $t \mapsto (t, W(t))$ we define a time-homogeneous continuous Markov process $\xi := \xi[\Gamma] := \{\xi_r : r \geq 0\}$ with values in $D = [0,\infty) \times \mathbb{R}$ by

$$\xi_r := \big(L^{-1}_{[\Gamma,W]}(r),\, W \circ L^{-1}_{[\Gamma,W]}(r)\big), \quad \text{for } r \geq 0, \tag{16}$$

with W started in x (recall that Γ is a fixed sample). The first component of this process can be interpreted as the new individual clock of the Brownian reactant particle, travelling in the fixed medium Γ, and the second component as its position along the new time scale. For all $t > 0$, define the *first exit time*

$$\tau_t := \tau_t(\xi) := \inf\big\{r > 0 : \xi_r \notin [0,t) \times \mathbb{R}\big\} \tag{17}$$

of the path ξ from the domain $D^t := [0,t) \times \mathbb{R}$. At time τ_t the process ξ is in the state $\big(t, W(t)\big)$, and the reactant particle has accumulated the collision local time

$$L_{[\Gamma,W]}(t) = \tau_t \tag{18}$$

and is placed in $W(t)$. Less formally, a single generic reactant particle of X_t is represented by a path ξ stopped at the random time τ_t (instead of W stopped at t).

The Brownian snake can be interpreted as a natural parametrisation of the collection of all reactant particles in the range of X, where each particle is represented by a stopped path. For this purpose, define the *set of stopped paths* by

$$\mathfrak{P} := \Big\{ f \in C\big([0,\infty), D\big) : \text{ there exists } \zeta \geq 0 \text{ with } f(r) = f(\zeta) \text{ for all } r \geq \zeta \Big\}. \tag{19}$$

With every $f \in \mathfrak{P}$ we can associate the *lifetime* $\zeta = \zeta(f)$, which is the minimal $\zeta \geq 0$ such that the path f is constant on $[\zeta, \infty)$. We equip $\mathfrak{P}$ with the metric $\mathfrak{d}$, defined as follows: For $f, \tilde{f} \in \mathfrak{P}$ let $\zeta, \tilde{\zeta}$ be the associated lifetimes and let

$$\mathfrak{d}(f, \tilde{f}) := \big|f(0) - \tilde{f}(0)\big| + |\zeta - \tilde{\zeta}| + \int_0^{\zeta \wedge \tilde{\zeta}} \Big(\sup_{r \in [0,u]} \big|f(r) - \tilde{f}(r)\big| \wedge 1 \Big)\, du. \tag{20}$$

The *Brownian snake* $w = w[\Gamma]$ *rooted in* $(0, x) \in D$ with motion process $\xi = \xi[\Gamma]$ is a certain continuous strong Markov process $w : [0, \infty) \to \mathfrak{P}$ whose state space is the set of all stopped paths $f \in \mathfrak{P}$ with $f(0) = (0, x)$. Brownian snakes with general Markov processes as motion process were constructed in [1].

With every $\mathfrak{P}$-valued Markov process we can associate the *lifetime process* $\zeta : [0, \infty) \to [0, \infty)$ defined by $\zeta_s := \zeta(w_s)$. For the Brownian snake w, the lifetime process ζ is by definition a reflected Brownian motion. Moreover, given ζ, two paths w_{s_1} and w_{s_2}, $s_1 < s_2$, agree up to time $m := \min_{s \in [s_1, s_2]} \zeta_s$, and the two continuations $\{w_{s_1}(m + r) : 0 \leq r \leq \zeta_{s_1} - m\}$ and $\{w_{s_2}(m + r) : 0 \leq r \leq \zeta_{s_2} - m\}$ with fixed starting point $w_{s_1}(m) = w_{s_2}(m)$ are independent (see also Figure 1 below). Heuristically, if $m = 0$ the particles represented by w_{s_1} and w_{s_2} belong to different families, whereas if $m > 0$ and $s \in [s_1, s_2]$ satisfies $\zeta_s = m$, the path w_s represents the last common ancestor of w_{s_1} and w_{s_2}.

The constant path $f \in \mathfrak{P}$ given by $f(r) = (0, x)$ for all $r \geq 0$ is a regular recurrent point for the Markov process w. Indeed, this follows immediately from the fact that the lifetime process $\zeta(w)$ is a reflected Brownian motion. Hence we can define $\mathbb{N}^{\Gamma}_{(0,x)}$ to be the suitably normalised *excursion measure* of the Brownian snake w from the constant path $f = (0, x)$; see e.g. [3] for the excursion theory of Markov processes. Every sample of such an *excursion from* $(0, x)$ is a continuous path-valued function $w : [0, \sigma] \to \mathfrak{P}$ for some finite $\sigma = \sigma(w) > 0$, the *length* of the excursion, such that $w_0 = w_\sigma$ is the constant path remaining at $(0, x)$, and w_s is not constant, for each $s \in (0, \sigma)$. Then $\mathbb{N}^{\Gamma}_{(0,x)}$ is a σ-finite measure on the set

$$\mathfrak{W} := \bigcup_{\sigma > 0} C\big([0, \sigma], \mathfrak{P}\big) \tag{21}$$

of path-valued functions. Although it is stretching the usual terminology a bit, we use the words sample and process also in the case of underlying non-probability measures such as $\mathbb{N}^{\Gamma}_{(0,x)}$.

With each excursion $w : [0, \sigma] \to \mathfrak{P}$ we can again associate the *lifetime process* $\zeta : [0, \sigma] \to [0, \infty)$ by letting $\zeta_s := \zeta(w_s)$, which under $\mathbb{N}^{\Gamma}_{(0,x)}$ is a Brownian excursion from 0. Heuristically, an excursion w represents the whole family tree created by a reactant particle, which at time 0 was located at x.

Following [1] or [16], we can define, for every $t > 0$ the *exit local time at level* t (from our original time scale) of an excursion $w \in \mathfrak{W}$ as the process $L^t := L^t[w] := \big\{L^t_s : s \in [0, \sigma]\big\}$ such that

$$L^t_s := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^s 1_{\{\tau_t(w_u) < \zeta_u < \tau_t(w_u) + \varepsilon\}}\, du, \;\text{ for }\; 0 \leq s \leq \sigma, \tag{22}$$

$\mathbb{N}^{\Gamma}_{(0,x)}$–almost surely, where τ_t was defined in (17). The *total exit local time at level* $t > 0$ *of an excursion* w is $L^t_\sigma[w] := L^t_{\sigma[w]}[w]$. Note that the measure associated with the monotone function $s \mapsto L^t_s[w]$ is supported by those s where $\tau_t(w_s) = \zeta_s$ and recall that exactly those paths w_s represent particles of X_t. The *exit measure at level* $t > 0$ is the measure $Z^t := Z^t[w]$

on $\partial D^t := \{t\} \times \mathbb{R}$ defined by

$$\langle Z^t[w], \varphi \rangle := \int_0^\sigma \varphi\big(w_s(\zeta_s)\big)\, L^t_{ds}\,, \ \text{ for } \varphi : \partial D^t \to [0,\infty) \text{ measurable,} \tag{23}$$

where the integral is a Stieltjes integral with respect to the nondecreasing function $s \mapsto L^t_s[w]$. Slightly abusing notation, for fixed $t > 0$, we can identify $\partial D^t = \{t\} \times \mathbb{R}$ with $\mathbb{R}$, that is, we can consider $Z^t[w]$ and φ as a measure respectively a function on $\mathbb{R}$. Such identifications will often be used in the following. The measure $Z^t[w]$ can be interpreted as the spatial distribution of the descendants at time t of a reactant particle, which at time 0 was located as x. The quantity $L^t_\sigma[w]$ describes the mass of the totally produced reactant progeny of this particle at time t.

We now have the means to describe both X and X^∞ in terms of the excursion measures $\mathbb{N}^\Gamma_{(0,x)}$. In the case of a measure $\mu \in \mathcal{M}_{\text{tem}}$ different from Lebesgue measure, (27) below should be understood as the natural definition of X^∞ with $X^\infty_0 = \mu$, defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P}_\mu)$.

Theorem 6 (Snake representations). *Let $\mu \in \mathcal{M}_{\text{tem}}$ be an arbitrary starting measure.*

(i) Representation of X. *Given Γ, let $\Pi = \Pi[\Gamma]$ be a Poisson point field on the space $\mathfrak{W} \times \mathbb{R}$ with intensity measure $\pi = \pi[\Gamma]$ defined by*

$$\pi(dw\, dx) := \int_{\mathbb{R}} \mathbb{N}^\Gamma_{(0,y)}(dw) \otimes \delta_y(dx)\, \mu(dy). \tag{24}$$

Then, for P–almost all Γ, the superprocess $X = X[\Gamma]$ with $X_0 = \mu$ can be represented as

$$\langle X_t\,, \varphi\rangle = \int_{\mathfrak{W}\times\mathbb{R}} \langle Z^t[w], \varphi \rangle\, \Pi(dw\, dx), \tag{25}$$

for all $t > 0$ and $\varphi : \mathbb{R} \to [0,\infty)$ measurable.

(ii) Representation of X^∞. *Let Π^∞ be a Poisson point field on $\mathfrak{W}\times\mathbb{R}$ with intensity measure*

$$\pi^\infty(dw\, dx) := \Big(\int_{\mathcal{M}_{\text{tem}}} \mathbb{N}^\Upsilon_{(0,0)}(dw)\, \mathsf{P}(d\Upsilon) \Big) \otimes \mu(dx). \tag{26}$$

Then the limit process X^∞ with $X^\infty_0 = \mu$ has the representation

$$\langle X^\infty_t, \varphi\rangle = \int_{\mathfrak{W}\times\mathbb{R}} L^t_\sigma[w]\, \varphi(x)\, \Pi^\infty(dw\, dx), \tag{27}$$

for all $t > 0$ and $\varphi : \mathbb{R} \to [0,\infty)$ measurable.

To get a feeling for the representations in Theorem 6 recall from the discussion preceding the theorem that the offspring generated by a single initial particle initially at position $x \in \mathbb{R}$ and catalysed by the *fixed* medium Γ is described by the process $\{Z^t[w] : t \geq 0\}$ under the measure $\mathbb{N}^\Gamma_{(0,x)}(dw)$. Hence (25) is just the representation of $\{X_t : t \geq 0\}$ as the Poissonian sum of the individual families. In fact, it is well-known from the Lévy-Hincin formula that the decomposition of the infinitely divisible random measure X_t under $\mathbb{P}^\Gamma_\mu$ into families yields a Poisson field. Note that only the marginal measures $\Pi(dw \times \mathbb{R})$ enter in the representation (25). We have included the space coordinate into the definition of the Poisson point field Π just in order to simplify a comparison of the intensity measures (24) and (26).

Furthermore, again recalling the previous discussion, the *total mass* of the offspring generated by a single initial particle initially at position $x \in \mathbb{R}$ and catalysed by a *randomly sampled* medium Υ with law P is described by the process $\{L_\sigma^t[w] : t \geq 0\}$ under the annealed measure $\mathsf{E}\mathbb{N}_{(0,x)}^{\Upsilon}(dw)$. Hence the representation (27) decomposes X^∞ into a Poissonian sum of families, each family remains in the position of its ancestor at time $t = 0$ and has a mass evolving according to $\{L_\sigma^t[w] : t \geq 0\}$ under $\mathsf{E}\mathbb{N}_{(0,x)}^{\Upsilon}(dw)$.

The snake representations enable us to make a comparison of X and X^∞, and, moreover, draw a revealing heuristic picture of the limit process X^∞. For both processes, mass is initially spread on $\mathbb{R}$ according to μ. In the case of the original process X, starting from each infinitesimal small mass point $\mu(dx)$ a potential family of reactant particles is evolving, whereas in the case of the limit process X^∞ at each infinitesimal small mass point $\mu(dx)$ a potential macroscopic clump can be created. After an arbitrarily small, positive amount of time locally only finitely many families survive in X, and, similarly, after an arbitrarily small, positive amount of *macroscopic* time locally only finitely many macroscopic clumps survive in X^∞. The further development of the *total mass* of the offspring progeny of any reactant particle in X or of any macroscopic clump in X^∞ is in both cases governed by the laws of $t \mapsto L_\sigma^t[w]$ under the excursion measure.

There are however a number of *significant differences*:

- In X each particle family uses the excursion measures $\mathbb{N}_{(0,x)}^{\Gamma}$ for the same given sample Γ (though around different places x). The clumps of X^∞ however are based on the samples w of the measure $\mathsf{E}\mathbb{N}_{(0,0)}^{\Upsilon}(dw)$ which is independent of the position x of the clump and of the medium sample Γ. For each individual clump the sample w is in fact the result of a two stage experiment: First Υ is sampled with the law P of the stable medium, and then w is sampled according to the law $\mathbb{N}_{(0,0)}^{\Upsilon}(dw)$.
- Whereas the reactant particle families of X have a spatial spread and their motion component is visible, this is not the case with the clumps of X^∞. Macroscopic clumps are mass points, which remain at their original spatial position, only their mass is variable. Indeed, whereas the full measure $Z^t[w]$ enters into the representation (25) of X_t, only $L_\sigma^t[w]$ enters into the representation (27) of X_t^∞, and the spatial structure of $Z^t[w]$ is suppressed. This in particular leads to the *loss of the Markov property* in the limit process X^∞. Heuristically speaking, the clumps have a hidden *micro-life*, governing the branching behaviour, but invisible from the outside, since the excursion measure $\mathbb{N}_{(0,0)}^{\Upsilon}$ in the random medium Υ is used in the *annealed* sense $\mathsf{E}\mathbb{N}_{(0,0)}^{\Upsilon}$ only. This micro-life can be made explicit, see Remark 9 below.

Remark 7 (Continuous versions of $Z[w]$ and $L_\sigma[w]$)**.** As the process X has a continuous version, there is a continuous version of the process $Z[w] := \{Z^t[w] : t > 0\}$ of exit measures as well. We may henceforth assume that $Z[w]$ under $\mathbb{N}_{(0,x)}^{\Gamma}$ is this continuous version. Similarly, from the continuity of the total mass process $\|X\|$ in the case of a finite starting measure, we can see that also the process $L_\sigma[w] := \{L_\sigma^t[w] : t > 0\}$ has a continuous version, which we henceforth use. $\diamond$

Remark 8 (A finiteness property)**.** From the representation (27) it can be seen easily, that X^∞ has the compound Poisson structure stated in Theorem 1 (ii) if and only if

$$\int_{\mathcal{M}_{\mathrm{tem}}} \mathbb{N}_{(0,0)}^{\Upsilon}\{w : L_\sigma^t[w] > 0\} \mathsf{P}(d\Upsilon) < \infty, \quad \text{for } t > 0. \tag{28}$$

This *finiteness property* of π^∞ implies that after an arbitrarily small, positive amount t of macroscopic time, locally only finitely many macroscopic clumps exist. Moreover, together with the continuity of $t \mapsto L_\sigma^t[w]$ it also implies the continuity of X^∞ in the representation (27). $\diamond$

Remark 9 (A Markovian process)**.** As announced in Remark 3 one can obtain the process X^∞ by projection from a Markov process $\tilde{X}^\infty$ with state space $\mathcal{M}(\mathbb{R}^2)$. Here the first component of $(x,y) \in \mathbb{R}^2$ serves as the macroscopic space component and the second as the microscopic one. More precisely, for $\nu \in \mathcal{M}(\mathbb{R}^2)$ let $\tilde{\Pi}^\infty(dw\,dx)$ be a Poisson point field on $\mathfrak{W} \times \mathbb{R}$ with intensity measure

$$\tilde{\pi}^\infty(dw\,dx) := \int_{\mathbb{R}} \int_{\mathcal{M}_{\mathrm{tem}}} \mathbb{N}^{\Upsilon}_{(0,y)}(dw)\, \mathsf{P}(d\Upsilon)\, \nu(dx\,dy). \tag{29}$$

Let $\tilde{X}^\infty_0 := \nu$ and, for $t>0$ and measurable $\varphi : \mathbb{R}^2 \to [0,\infty)$, define

$$\langle \tilde{X}^\infty_t, \varphi\rangle := \int_{\mathfrak{W}\times\mathbb{R}} \int \varphi(x,y)\, Z^t[w](dy)\, \tilde{\Pi}^\infty(dw\,dx). \tag{30}$$

The process $\tilde{X}^\infty = \{\tilde{X}^\infty_t : t \ge 0\}$ is Markovian. This can be checked by using the special Markov property of the Brownian snake with an argument similar to the proof of (b) on page 15. In order to recover the process X^∞ with initial measure $\mu \in \mathcal{M}_{\mathrm{tem}}(\mathbb{R})$, run the process $\tilde{X}^\infty$ with starting point $\mu \otimes \delta_0$ and obtain X^∞_t as the first component marginal of $\tilde{X}^\infty_t$. $\diamond$

From the snake representation (27) of the limit process X^∞, we easily get the Laplace functionals of its finite dimensional marginals — of course, the result is consistent with the representation of the one-dimensional marginals mentioned already in (13).

Corollary 10 (Finite dimensional distributions)**.** *The finite dimensional marginals of the limit process X^∞ with initial measure $X^\infty_0 = \mu \in \mathcal{M}_{\mathrm{tem}}$ are determined by*

$$\mathbf{E}_\mu\Big\{\exp\Big(-\sum_{i=1}^{n}\langle X^\infty_{t_i}, \varphi_i\rangle\Big)\Big\} = \exp\Big(-\int_{\mathbb{R}} \mathsf{E}U^{\Gamma}_{t_1,\dots,t_n}\left[\varphi_1(x),\dots,\varphi_n(x)\right](0)\,\mu(dx)\Big), \tag{31}$$

for $0 \le t_1 \le \ldots \le t_n$ and measurable $\varphi_1,\dots,\varphi_n : \mathbb{R} \to [0,\infty)$ for $n \ge 1$. Here $U^\Gamma_{t_1}[a_1] := U^\Gamma_{t_1} a_1$ is taken from (12) *with constant function $\varphi = a_1 \ge 0$, and $U^\Gamma_{t_1,\dots,t_n}[a_1,\dots,a_n]$ is defined inductively: for $n \ge 2$,*

$$U^\Gamma_{t_1,\dots,t_n}[a_1,\dots,a_n] = U^\Gamma_{t_1}\Big[a_1 + U^\Gamma_{t_2-t_1,\dots,t_n-t_1}[a_2,\dots,a_n]\Big], \tag{32}$$

for all $0 \le t_1 \le \ldots \le t_n$ and $a_1,\dots,a_n \ge 0$.

1.2.5. *Further properties of the limit process.* To round out the picture of the limit model X^∞ we describe the major indices related to the survival probability and the tail behaviour of the mass clumps on the macroscopic level. Recall the index $\gamma \in (0,1)$ of our stable medium Γ, and the scaling index η introduced in (8).

Theorem 11 (Properties of the limit process)**.** *Run the limit process X^∞ with initial measure ℓ.*

(i) Self-similarity. *X^∞ satisfies, for every $k>0$,*

$$X^\infty_t(B) = k^{-\eta} X^\infty_{kt}(k^\eta B) \text{ in distribution, for } t \ge 0 \text{ and } B \subseteq \mathbb{R} \text{ Borel.}$$

(ii) Survival probability. *The ratio of the intensities $\lambda(t)$ and $\lambda(s)$ of the Poisson point fields carrying the (non-zero) clumps at various macroscopic times $t > s > 0$, respectively, satisfies*

$$\lambda(s)/\lambda(t) = (t/s)^\eta.$$

Hence, denoting by $\beth_s(t)$ the mass at the macroscopic time t of a clump at time s, the survival probability of $\beth_s(t)$ is given by

$$\mathbf{P}_\ell\big\{\beth_s(t) > 0\big\} = (s/t)^\eta, \text{ for all } t > s > 0.$$

Moreover, we have, for all $t > s > 0$,

$$\daleth_t(t) = (t/s)^{\eta}\, \daleth_s(s) \ \ \textit{in distribution.}$$

(iii) Clump size tails. *The tail behaviour of the clump size $\daleth_t(t)$ is governed by*

$$\mathbf{P}_\ell\big\{\daleth_t(t) > a\big\} \approx t^{\eta(\gamma+1)}\, a^{-\gamma-1} \ \ \textit{as } a \uparrow \infty.$$

Here $\approx$ means that the ratio of the quantities involved is bounded away from zero and infinity as $a \uparrow \infty$ by constants independent of $t > 0$.

Remark 12 (Index of self-similarity)**.** In [21] the process $L_{[W,\Gamma]} = \{L_{[W,\Gamma]}(t) : t \geq 0\}$ is studied under the joint law of the independent pair (W, Γ). The process is shown to be the functional limit of a *random walk in random scenery.* Moreover, $L_{[W,\Gamma]}$ is self-similar with the index η we encounter in Theorem 11 (i). ◇

Remark 13 (Infinite variance)**.** It is quite remarkable, that the clump size is heavy-tailed, in particular it has *infinite variance.* The latter fact was conjectured in [7, Subsection 1.14]. ◇

Remark 14 (Open problem)**.** Note that the intensity $\lambda(t)$ of the carrying Poisson point field at time $t > 0$ occurring in (ii) is positive and finite, but it is an *open* problem to determine its exact value. ◇

1.3. **Outline of the paper.** Here we indicate the further structure of the paper, give a guideline to where various parts of the proofs can be found, and briefly review the main methods of proof.

Section 2 is devoted to those aspects of the paper related to the Brownian snake construction in a random medium. In Subsection 2.1 we establish the Brownian snake representation of super-Brownian motion X in the catalytic medium Γ [Theorem 6 (i)]. Subsection 2.2 contains the proofs of the Laplace functionals in Corollary 10 and the description of X^∞ in Theorem 1 (ii), which both rely on the *definition* of X^∞ in terms of its snake representation [Theorem 6 (ii)]. Both snake representations are used in Subsection 2.3, together with Birkhoff's individual ergodic theorem, to prove the functional limit theorem. The proof also relies on two further steps of independent interest, whose proofs are deferred to Section 3: the finiteness property (28), and the crossing property, Theorem 5.

Section 3 concerns the aspects of proof related to the method of good and bad paths. In Subsections 3.1 and 3.2 we formulate a quantitative extension of this method. The key step is to give an upper bound on the survival probability of the catalytic super-Brownian motion X with a finite starting measure in terms of a quantitative characteristic of the random medium Γ. This is then applied in Subsection 3.3 to prove the crossing property, Theorem 5, and in 3.4 to verify the finiteness statement (28) and thus derive the compound Poisson structure of the limit process X^∞. We would also like to point out that our approach to the method of good and bad paths (other than the approach of [11]) does not rely on the compact support property of catalytic super-Brownian motion established in [12], and conversely seems to be a good starting point for an independent, new probabilistic proof of the compact support property.

Section 4 deals with the more analytical proof techniques. We first investigate the time evolution of the mass of the clumps in our limit model. The calculations of the Poisson intensities and survival probabilities stated in Theorem 11 exploit the natural scaling invariance of the limit process together with the Poisson carrier structure, see Subsection 4.1. The calculation of the tail behaviour in Theorem 11 is based on a Feynman-Kac representation of the solutions of the log-Laplace equation (6), provided in Subsection 4.2 and a simple version of the Tauberian Theorem of Bingham and Doney [2, Theorem 8.16].

2. The Brownian snake approach in the case of a catalytic medium

In this section we prove the snake representations, Theorem 6, and the functional limit theorem, Theorem 1 (i).

2.1. **The Brownian snake representation of catalytic super-Brownian motion.** We now formalize the construction of the Brownian snake and verify the snake representation of X, Theorem 6 (i). As announced, we first take a fixed sample of the catalytic medium Γ. Recall that $L^{-1}_{[\Gamma,W]}$ denotes the inverse function of the collision local time $L_{[\Gamma,W]}$ of a Brownian path W with Γ, which was introduced in (15).

The continuous time-homogeneous Markov process $\xi = \{\xi_r : r \geq 0\}$ on $D = [0,\infty) \times \mathbb{R}$ with starting point $(a,x) \in D$ is defined by

$$\xi_r = \big(a + L^{-1}_{[\Gamma,W]}(r), W \circ L^{-1}_{[\Gamma,W]}(r)\big), \quad \text{for } r \geq 0, \tag{33}$$

where W is a Brownian motion in $\mathbb{R}$ started from $x \in \mathbb{R}$. Let $P_{(a,x)}$ denote the law of ξ started at time $r = 0$ at (a,x) and, for $b \geq 0$, denote by $P^b_{(a,x)}$ the law of the related stopped paths $\{\xi_{r\wedge b} : r \geq 0\}$.

We now define the Brownian snake with motion process ξ, following the construction of the Brownian snake for an arbitrary continuous Markovian motion process in [24].

Consider a stopped path $f \in \mathfrak{P}$ with lifetime $\zeta(f) > 0$ and such that $f(0) = (0,x)$ as introduced around (19). If $0 \leq a \leq \zeta(f)$ and $b \geq a$ we define $\mathbf{Q}_{a,b}(f, d\tilde{f})$ to be the unique probability measure on $\mathfrak{P}$ such that

- $\mathbf{Q}_{a,b}(f, d\tilde{f})$–almost surely $\tilde{f}(r) = f(r)$, for all $r \in [0,a]$,
- the law under $\mathbf{Q}_{a,b}(f, d\tilde{f})$ of $\{\tilde{f}(a+r) : r \geq 0\}$ is the law of $\{\xi_r : r \geq 0\}$ under $P^{b-a}_{f(a)}$.

This transition can be thought of as follows. From its endpoint $\zeta(f)$, the path f is erased backwards in its original time until the absolute time a, and then renewed according to the random motion process ξ, but stopped at the absolute time b. In particular, $\mathbf{Q}_{0,b}(f, d\tilde{f}) = P^b_{(0,x)}(d\tilde{f})$.

The parameters a, b entering into the transition laws $\mathbf{Q}_{a,b}$ are used to control erasing and renewal of the paths. In snake constructions, these parameters are determined continuously by a stochastic process, for the Brownian snake this role is played by a reflected Brownian motion. To be more precise, for $r, s \geq 0$, denote by $\vartheta^r_s(da\,db)$ the joint distribution of the pair $\big(\min_{\tilde{s}\in[0,s]} |B_{\tilde{s}}|, |B_s|\big)$, where $B = \{B_t : t \geq 0\}$ is a Brownian motion on $\mathbb{R}$ with $B_0 = r$. Note that $\vartheta^r_s\big\{(a,b) \in [0,r] \times [0,\infty) : a \leq b\big\} \equiv 1$.

The *Brownian snake with motion process* ξ *and root* $(0,x)$ is defined to be the time-homogeneous continuous strong Markov process $w = \{w_s : s \geq 0\}$ whose transition kernels are given by

$$\mathbf{Q}_s(f, d\tilde{f}) = \int_0^\infty \int_0^\infty \vartheta^{\zeta(f)}_s(da\,db)\, \mathbf{Q}_{a,b}(f, d\tilde{f}), \quad \text{for } s \geq 0 \text{ and } f \in \mathfrak{P} \text{ with } f(0) = (0,x), \tag{34}$$

see [1, Proposition 5]. Recall that the *lifetime process* $\zeta = \{\zeta_s : s \geq 0\}$ is defined by $\zeta_s = \zeta(w_s)$. Under the law of w determined by the transition kernels $\mathbf{Q}_s$, $s \geq 0$, the lifetime process ζ is by construction a reflected Brownian motion with initial state $\zeta_0 = \zeta(w_0)$.

To interpret the dynamics of the snake w, observe that if $s_1 < s_2$ the path w_{s_2} is obtained from w_{s_1} by erasing from its endpoint ζ_{s_1} down to the absolute time $m := \min_{s\in[s_1,s_2]} \zeta_s$ and adding an independent tip of time length $\zeta_{s_2} - m$ at the end. Figure 1 tries to show this. The

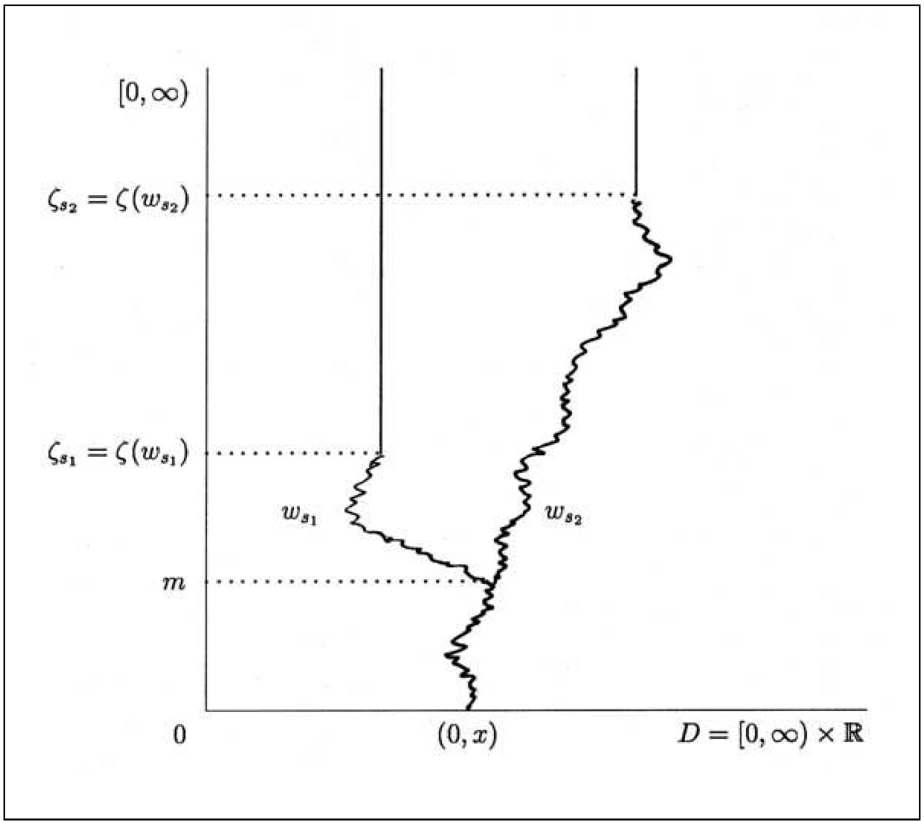

FIGURE 1. Erasing the path w_{s_1} to renew it to w_{s_2}

paths w_{s_1} and w_{s_2}, which are stopped versions of ξ, have to be identical on the time interval $[0, m]$, but to be independent on the intervals $[m, \zeta_{s_1}]$ and $[m, \zeta_{s_2}]$, respectively, except for the common starting point $w_{s_1}(m) = w_{s_2}(m)$. In particular, if $m = 0$, a new path is created, starting again from $(0, x)$. In this case the paths do not have a common part, which means that the reactant particles they represent do not have a common ancestor. This can also interpreted in the sense that the *excursions* from $(0, x)$ of the Markov process w correspond to different families of particles.

To be more precise, note that the constant path $(0, x)$ is a regular recurrent point of the Markov process w. Denote by $\mathbb{N}^{\Gamma}_{(0,x)}$ the *excursion measure* of w from this path, which is a σ-finite measure on the space $\mathfrak{W}$ defined in (21). Under $\mathbb{N}^{\Gamma}_{(0,x)}$ every excursion $w : [0, \sigma] \to \mathfrak{P}$ has a finite length $\sigma(w) = \sigma > 0$. Again we can associate with every excursion $w : [0, \sigma] \to \mathfrak{P}$ a lifetime process $\zeta(w) := \zeta := \{\zeta_s : s \in [0, \sigma]\}$. Observe that under the measure $\mathbb{N}^{\Gamma}_{(0,x)}$ the process ζ is a Brownian excursion of length σ, hence $\zeta_s > 0$ on $(0, \sigma)$. As usual, $\mathbb{N}^{\Gamma}_{(0,x)}$ is normalised such that

$$\mathbb{N}^{\Gamma}_{(0,x)}\Big\{ \sup_{s \in [0,\sigma]} \zeta_s > \varepsilon \Big\} = \frac{1}{2\varepsilon} . \tag{35}$$

At this point it is worth looking back at the definition of the intensity measures $\pi = \pi[\Gamma]$ and π^{∞} in (24) and (26), respectively, and noting that (35) implies that both are in fact σ-finite measures, as needed for the definition of the Poisson point fields.

Proof of Theorem 6 (i). Recall that a sample Γ is fixed. For each measure $\mu \in \mathcal{M}_{\text{tem}}$ we consider a Poisson point field Π with intensity measure π as in (24), defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P}_\mu^\Gamma)$. We have to verify that the process X *defined* on this space by (25) is indeed a catalytic super-Brownian motion in the medium Γ, with initial state $X_0 = \mu$.

Recall the function space Φ introduced before (2). For $\varphi \in \Phi$, $t > 0$, and $x \in \mathbb{R}$, using $\mu = \delta_x$, let

$$U_t\varphi\,(x) := -\log \mathbf{E}_{\delta_x}^\Gamma \{\exp\langle X_t, -\varphi\rangle\}. \tag{36}$$

It suffices to verify the following two points:

(a) $U\varphi := \{U_t\varphi\,(x) :\ t > 0,\ x \in \mathbb{R}\}$ solves equation (6).
(b) For all $0 \leq h < t$ and $\varphi \in \Phi$,

$$\mathbf{E}_\mu^\Gamma \Big\{ \exp\langle X_t\,, -\varphi\rangle \,\Big|\, X_v\,,\, v \leq h \Big\} = \exp\big\langle X_h\,, -U_{t-h}\varphi\big\rangle. \tag{37}$$

Fix $\varphi \in \Phi$ and $t > 0$ for the remaining proof. In order to give the *proof of* (a) we need some more facts concerning the exit measures $Z^t[w]$ of (23) under the excursion measures $\mathbb{N}_{(0,x)}^\Gamma$, see [1, Proposition 6]. For $x \in \mathbb{R}$ and $0 \leq s < t$, define

$$u_t(s,x) := \int_{\mathfrak{W}} \mathbb{N}_{(0,x)}^\Gamma(dw)\Big(1 - \exp\big\langle Z^{t-s}[w], -\varphi\big\rangle\Big), \tag{38}$$

where we identified $\partial D^{t-s} = \{t-s\} \times \mathbb{R}$ with $\mathbb{R}$. Recalling that $P_{(0,x)}$ denotes the law of ξ with $\xi_0 = (0,x)$, this u_t satisfies the equation

$$u_t(0,x) = E_{(0,x)}\big\{\varphi(\xi_{\tau_t}) 1_{\{\tau_t < \infty\}}\big\} - 2E_{(0,x)}\Big\{ \int_0^{\tau_t} \big[u_t(\xi_s)\big]^2\, ds\Big\}, \quad \text{for } x \in \mathbb{R}, \tag{39}$$

with $\tau_t = \tau_t(\xi)$ from (17), using $\xi_{\tau_t} \in \partial D^t = \{t\} \times \mathbb{R}$ and again the identification of ∂D^t and $\mathbb{R}$. On the other hand, by the Laplace functional formula for Poisson point fields, from the definitions (36) and (25) we have

$$U_t\varphi\,(x) = \int_{\mathfrak{W}} \mathbb{N}_{(0,x)}^\Gamma(dw)\Big(1 - \exp\big\langle Z^t[w], -\varphi\big\rangle\Big). \tag{40}$$

Consequently, $u_t(s,x) = U_{t-s}\varphi\,(x)$. Then (39) shows that

$$U_t\varphi\,(x) = E_{(0,x)}\big\{\varphi(\xi_{\tau_t}) 1_{\{\tau_t < \infty\}}\big\} - 2E_{(0,x)}\Big\{ \int_0^{\tau_t} \big[u_t(\xi_s)\big]^2\, ds\Big\}. \tag{41}$$

Recalling that ξ with law $P_{(0,x)}$ can by definition be expressed by a Brownian motion W starting at time 0 from x, whose law we denote by $\mathcal{P}_{0,x}$, and that at time $\tau_t = L_{[\Gamma,W]}(t)$ the process ξ is in the state $\big(t, W(t)\big)$, which is identified with $W(t)$, the identity in (41) can be rewritten as

$$U_t\varphi\,(x) = \mathcal{E}_{0,x}\,\big\{\varphi\big(W(t)\big)\big\} - 2\,\mathcal{E}_{0,x}\bigg\{ \int_0^{L_{[\Gamma,W]}(t)} \Big[U_{t-L^{-1}_{[\Gamma,W]}(s)}\varphi\,\big(W \circ L^{-1}_{[\Gamma,W]}(s)\big)\Big]^2 ds\bigg\}. \tag{42}$$

We now substitute s for $L^{-1}_{[\Gamma,W]}(s)$ in the second term. Thus

$$U_t\varphi\,(x) = \mathcal{E}_{0,x}\,\big\{\varphi\big(W(t)\big)\big\} - 2\,\mathcal{E}_{0,x}\Big\{ \int_0^t \big[U_{t-s}\varphi\,\big(W(s)\big)\big]^2\, dL_{[\Gamma,W]}(s)\Big\}, \tag{43}$$

hence $U\varphi$ solves (6) and we have proved (a).

For the *proof of* (b) we reformulate the statement equivalently in terms of Laplace functionals: For all $0 < t_1 < \cdots < t_{n+1}$ and $\varphi_1, \ldots, \varphi_{n+1} \in \Phi, n \geq 1$,

$$\mathbf{E}_\mu^\Gamma \Big\{ \exp\Big(\sum_{j=1}^{n+1} \langle X_{t_j}, -\varphi_j \rangle \Big) \Big\} = \mathbf{E}_\mu^\Gamma \Big\{ \exp\Big(\sum_{j=1}^{n} \langle X_{t_j}, -\varphi_j \rangle + \langle X_{t_n}, -U_{t_{n+1}-t_n}\varphi_{n+1} \rangle \Big) \Big\}. \tag{44}$$

The main tool for the proof is the *special Markov property* of the exit measures $Z^t[w]$, see for example [1, Proposition 7]. In our particular situation it states that, for all $x \in \mathbb{R}$, $0 < t_1 < \cdots < t_{n+1}$ and $\varphi_1, \ldots, \varphi_{n+1} \in \Phi$, $n \geq 1$, for $\mathbb{N}^\Gamma_{(0,x)}$–almost all w,

$$\begin{aligned} &\mathbb{N}^\Gamma_{(0,x)} \Big\{ \exp\Big[\sum_{j=1}^{n+1} \langle Z^{t_j}, -\varphi_j \rangle \Big] - 1 \,\Big|\, Z^{t_j} = Z^{t_j}[w], 1 \leq j \leq n \Big\} \qquad (45)\\ &= \exp\Big[\sum_{j=1}^{n} \big\langle Z^{t_j}[w], -\varphi_j \big\rangle - \int_\mathbb{R} Z^{t_n}[w](dz) \int_{\mathfrak{W}} \mathbb{N}^\Gamma_{(0,z)}(dv) \Big(1 - \exp\big\langle Z^{t_{n+1}-t_n}[v], -\varphi_{n+1} \big\rangle \Big) \Big] - 1. \end{aligned}$$

Recalling the formula for the Laplace functionals of Poisson point fields we obtain from definition (25) of X, using the special Markov property (45) in the second step, (40) and again definition (25) in the final step,

$$\begin{aligned} &\mathbf{E}_\mu^\Gamma \Big\{ \exp\Big(\sum_{j=1}^{n+1} \langle X_{t_j}, -\varphi_j \rangle \Big) \Big\} \qquad (46)\\ &= \exp\Big[\int_\mathbb{R} \mu(dx) \int_{\mathfrak{W}} \mathbb{N}^\Gamma_{(0,x)}(dw) \Big\{ \exp\Big(\sum_{j=1}^{n+1} \big\langle Z^{t_j}[w], -\varphi_j \big\rangle \Big) - 1 \Big\} \Big] \\ &= \exp\Big[\int_\mathbb{R} \mu(dx) \int_{\mathfrak{W}} \mathbb{N}^\Gamma_{(0,x)}(dw) \Big\{ \exp\Big(\sum_{j=1}^{n} \big\langle Z^{t_j}[w], -\varphi_j \big\rangle \\ &\qquad - \int_\mathbb{R} Z^{t_n}[w](dz) \int_{\mathfrak{W}} \mathbb{N}^\Gamma_{(0,z)}(dv) \Big(1 - \exp\big\langle Z^{t_{n+1}-t_n}[v], -\varphi_{n+1} \big\rangle \Big) \Big) - 1 \Big\} \Big] \\ &= \mathbf{E}_\mu^\Gamma \Big\{ \exp\Big(\sum_{j=1}^{n} \langle X_{t_j}, -\varphi_j \rangle + \big\langle X_{t_n}, -U_{t_{n+1}-t_n}\varphi_{n+1} \big\rangle \Big) \Big\}. \end{aligned}$$

This finishes the proof of (b), and thus completes the proof of Theorem 6 (i).

2.2. **The Brownian snake representation of the limit process.** We now assume that an arbitrary starting measure $\mu \in \mathcal{M}_{\text{tem}}$ is fixed and a Poisson point field Π^∞ with intensity measure π^∞ as in (26) is defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P}_\mu)$. Recall from (35) that π^∞ is σ-finite and hence the Poisson point field is well-defined. We *define* the process X^∞ on this space by (27). In this subsection we show that this process has the properties claimed in Theorem 1 (ii) and Corollary 10. The proof uses (28), which is shown in Subsection 3.4, and (57) which is proved in Subsection 4.1 below. The proof of the convergence $X^k \to X^\infty$ is deferred to Subsection 2.3 below, which then completes the proof of Theorem 6 (ii).

Proof of Corollary 10. Let μ, $t_1,\dots,t_n$ and $\varphi_1,\dots,\varphi_n$ as in the corollary. By definition (27) of X^∞, recalling the formula for the Laplace functional of a Poisson point field,

$$\mathbf{E}_\mu\Big\{\exp\Big(-\sum_{i=1}^n \langle X^\infty_{t_i},\varphi_i\rangle\Big)\Big\} = \mathbf{E}_\mu\Big\{\exp\Big(-\int_{\mathfrak{W}\times\mathbb{R}} \sum_{i=1}^n L^{t_i}_\sigma[w]\,\varphi_i(x)\,\Pi^\infty(dw\,dx)\Big)\Big\}$$

$$= \exp\Big(\int_{\mathbb{R}}\int_{\mathcal{M}_{\mathrm{tem}}}\int_{\mathfrak{W}} \Big(\exp\big(-\sum_{i=1}^n L^{t_i}_\sigma[w]\,\varphi_i(x)\big)-1\Big)\,\mathbb{N}^{\Upsilon}_{(0,0)}(dw)\,\mathsf{P}(d\Upsilon)\,\mu(dx)\Big). \tag{47}$$

The total mass process of $\{X_t : t\ge 0\}$ started in a finite measure $X_0=\nu$ has, as is easily seen by induction using (5), the Laplace transform

$$\mathbb{E}^\Gamma_\nu\Big\{\exp\big(-\sum_{i=1}^n c_i\|X_{t_i}\|\big)\Big\} = \exp\Big(-\int_{\mathbb{R}} U^\Gamma_{t_1,\dots,t_n}[c_1,\dots,c_n]\,(z)\,\nu(dz)\Big) \tag{48}$$

with $U^\Gamma_{t_1,\dots,t_n}[c_1,\dots,c_n]$ from (32). On the other hand, by the snake representation (25) of X, this Laplace transform can also be written as

$$\mathbb{E}^\Gamma_\nu\Big\{\exp\big(-\sum_{i=1}^n c_i\|X_{t_i}\|\big)\Big\} = \exp\Big(\int_{\mathbb{R}}\int_{\mathfrak{W}}\Big(\exp\big(-\sum_{i=1}^n c_i L^{t_i}_\sigma[w]\big)-1\Big)\,\mathbb{N}^\Gamma_{(0,x)}(dw)\,\nu(dx)\Big). \tag{49}$$

Comparing (48) and (49) as well as taking expectations with respect to the medium Γ,

$$\mathsf{E}\int_{\mathbb{R}}\int_{\mathfrak{W}}\Big(\exp\big(-\sum_{i=1}^n c_i L^{t_i}_\sigma[w]\big)-1\Big)\,\mathbb{N}^\Gamma_{(0,x)}(dw)\,\nu(dx) = -\int_{\mathbb{R}} \mathsf{E}U^\Gamma_{t_1,\dots,t_n}[c_1,\dots,c_n](z)\,\nu(dz). \tag{50}$$

Specializing to $\nu=\delta_0$ gives

$$\int_{\mathcal{M}_{\mathrm{tem}}}\int_{\mathfrak{W}}\Big(\exp\big(-\sum_{i=1}^n c_i L^{t_i}_\sigma[w]\big)-1\Big)\,\mathbb{N}^\Upsilon_{(0,0)}(dw)\,\mathsf{P}(d\Upsilon) = -\mathsf{E}U^\Gamma_{t_1,\dots,t_n}[c_1,\dots,c_n]\,(0). \tag{51}$$

Plugging this into (47) yields the formula stated in Corollary 10.

Proof of Theorem 1 (ii). We still allow an arbitrary starting measure μ, cf. Remark 2. From the definition (27) of X^∞ in terms of the Poisson point field Π^∞ it is clear that, for every $t>0$, the measure X^∞_t is supported by the points of a Poisson point field on $\mathbb{R}$ with intensity measure

$$\Big(\int_{\mathcal{M}_{\mathrm{tem}}} \mathbb{N}^\Upsilon_{(0,0)}\big\{w : L^t_\sigma[w]>0\big\}\,\mathsf{P}(d\Upsilon)\Big)\,\mu(dx). \tag{52}$$

By the finiteness property (28) the factor in front of the measure $\mu(dx)$ is *finite*, say $c>0$. Moreover, the masses of the atoms at these locations are independent with common distribution

$$\frac{1}{c}\int_{\mathcal{M}_{\mathrm{tem}}} \mathbb{N}^\Upsilon_{(0,0)}\big\{L^t_\sigma[w]\in\,\cdot\,;\ L^t_\sigma[w]>0\big\}\,\mathsf{P}(d\Upsilon). \tag{53}$$

This establishes the compound Poisson property.

Suppose that $I\subseteq\mathbb{R}$ is a bounded interval and that $t>0$. Again by (28), $\mathbf{P}_\mu$–almost surely, the point field Π^∞ restricted to the set $\big\{(w,x)\in\mathfrak{W}\times I : L^t_\sigma[w]>0\big\}$ is supported by finitely many points in $\mathfrak{W}\times I$, say

$$(w_1,x_1),\dots,(w_n,x_n),\qquad \text{with } x_1\le\dots\le x_n\,. \tag{54}$$

For every $s\ge t$, the measure X^∞_s is supported by the set $\big\{x_i : 1\le i\le n,\ L^s_\sigma[w_i]>0\big\}$. Hence atoms cannot move in space. To show that no new atoms can be born it would suffice to show that zero is an absorbing state for the process $s \mapsto L_\sigma^s[w_i]$. However, it is easier to argue via the Laplace transform of Corollary 10. Indeed, for all $s, t > 0$,

$$(55) \qquad \mathbf{P}_\mu\big\{X_t^\infty(I) = 0\big\} = \lim_{\theta\uparrow\infty} \exp\big(-\mu(I)\,\mathsf{E}U_t^\Gamma\theta\,(0)\big) = \lim_{\theta\uparrow\infty} \exp\big(-\mu(I)\,\mathsf{E}U_t^\Gamma[\theta + U_s^\Gamma\theta]\,(0)\big)$$
$$= \mathbf{P}_\mu\big\{X_t^\infty(I) = 0 \text{ and } X_{t+s}^\infty(I) = 0\big\}$$

since $0 \leq U_s^\Gamma\theta \leq \theta$ and by monotonicity. This shows that zero is an absorbing state for $t \mapsto X_t^\infty(I)$ and hence also for $t \mapsto L_\sigma^t[w]$.

Finally, for the proof that macroscopic clumps have almost surely finite lifetime, it suffices to show that, for every bounded interval I,

$$(56) \qquad \lim_{t\to\infty} \mathbf{P}_\mu\big\{X_t^\infty(I) = 0\big\} = \lim_{t\to\infty}\lim_{\theta\uparrow\infty} \exp\big(-\mu(I)\,\mathsf{E}U_t^\Gamma\theta\,(0)\big) = 1\,.$$

This does not depend on the starting measure μ, so that we can assume $\mu = \ell$. Now recall the definition of the clump sizes $\beth_s(t)$ introduced in Theorem 11 (ii). In Subsection 4.1 below we show that

$$(57) \qquad \mathbf{P}_\ell\left\{\beth_s(t) > 0\right\} = (s/t)^\eta, \quad \text{for } t > s > 0,$$

which is clearly stronger than (56).

2.3. **The functional limit theorem.** In this section we prove the weak convergence in P–probability of the random distributions of $X^k[\Gamma]$, as $k \uparrow \infty$, which was claimed in Theorem 1 (i). For this purpose we rescale the catalytic medium, but with a *different* spatial scaling, namely

$$(58) \qquad \Gamma^k(\,\cdot\,) := k^{1/(2\gamma)}\Gamma\big(\,\cdot\,/\sqrt{k}\,\big), \text{ for } k > 0.$$

But note that by self-similarity the rescaled stable medium Γ^k has the same distribution as Γ. Our strategy is to look at the distributions of the renormalised process $X^k[\Gamma^k] = \big\{X_t^k[\Gamma^k] : t \geq 0\big\}$ with changing medium Γ^k (instead of Γ), and show, using the representation of Theorem 6 (i), that the random distributions of $X^k[\Gamma^k]$ converge weakly, P–almost surely(!). This clearly implies weak convergence in P–probability of the random distributions of the rescaled processes $X^k[\Gamma]$ in the unscaled medium.

We start by looking at the case of the constant test function $\varphi \equiv 1$, i.e. at the total mass process $t \mapsto \|X_t^k\|$, and start X with the restricted Lebesgue measure, $X_0^k = \ell_{(a,b)}$ for $a < b$ real. The following proposition is the core of our proof of the functional limit theorem. We equip the space $C\big((0,\infty),\mathbb{R}\big)$ with the Polish topology of uniform convergence on compact intervals, which matches the earlier definition of the topology on $C\big((0,\infty),\mathcal{M}_{\rm tem}\big)$.

Proposition 15 (Total mass process). *Fix real numbers $a < b$.*

(i) Convergence. P–*almost surely, the random laws of the renormalised total mass processes $\big\|X^k[\Gamma^k]\big\| = \big\{\|X_t^k[\Gamma^k]\| : t > 0\big\}$ with $X_0^k[\Gamma^k] = \ell_{(a,b)}$ converge weakly on the path space $C\big((0,\infty),\mathbb{R}\big)$ as $k \uparrow \infty$ to the deterministic law of a limit process $X^\infty(a,b) = \{X_t^\infty(a,b) : t > 0\}$.*

(ii) Identification of the limit. *Let $\Pi_{a,b}^\infty$ be a Poisson point field on $\mathfrak{W}$ with intensity measure*

$$\pi_{a,b}^\infty(dw) := (b-a)\int_{\mathcal{M}_{\rm tem}} \mathbb{N}_{(0,0)}^\Upsilon(dw)\,\mathsf{P}(d\Upsilon).$$

Then the limit process satisfies

$$X_t^\infty(a,b) = \int_{\mathfrak{W}} L_\sigma^t[w]\,\Pi_{a,b}^\infty(dw), \quad \textit{for } t>0. \tag{59}$$

Proof. Fix $a<b$. To begin with, we infer from the normalisation condition (35) that $\pi_{a,b}^\infty$ is σ-finite and hence the Poisson point field $\Pi_{a,b}^\infty$ is well-defined. In accordance with (27), we can thus assume that the process $X^\infty(a,b) := \{X_t^\infty(a,b) : t>0\}$ is *defined* by (59), and our aim is to show that P–almost surely the processes $\|X^k[\Gamma^k]\|$ with $X_0^k[\Gamma^k] = \ell_{(a,b)}$ converge in law on $C\big((0,\infty),\mathbb{R}\big)$ to $X^\infty(a,b)$ as $k\uparrow\infty$.

The *first step* is to derive a representation of $\|X^k[\Gamma^k]\|$ as a k–independent functional of a Poisson point field, with k–dependent intensity measure. To do this fix $k>0$ and the medium sample Γ throughout the first step. From the Brownian snake representation of Theorem 6 (i) we infer that

$$\|X_t^k[\Gamma^k]\| = k^{-\eta}\int_{\mathfrak{W}} L_\sigma^{kt}[w]\,\Pi(dw), \quad \text{for } t>0, \tag{60}$$

where $\Pi = \Pi[\Gamma^k]$ is a Poisson point field on $\mathfrak{W}$ with intensity measure $\int_{k^\eta a}^{k^\eta b}\mathbb{N}_{(0,x)}^{\Gamma^k}\,dx$. As the total exit local time $L^{kt}[w]$ of a snake excursion w does not depend on the second component of the motion process ξ, we can equivalently use the intensity measure

$$\int_{k^\eta a}^{k^\eta b}\mathbb{N}_{(0,0)}^{T^x\Gamma^k}\,dx. \tag{61}$$

We *claim* that the distributions of $\{k^{-\eta}L_\sigma^{kt}[w] : t>0\}$ under $\mathbb{N}_{(0,0)}^{T^x\Gamma^k}$ and of $\{L_\sigma^t[w] : t>0\}$ under $k^{-\eta}\mathbb{N}_{(0,0)}^{T^{x/\sqrt{k}}\Gamma}$ coincide.

Indeed, a Brownian scaling of time and space yields for the collision local times the following identity in law:

$$L_{[\Gamma^k,x+W]}(kt) = k^\eta L_{\left[\Gamma,x/\sqrt{k}+W^k\right]}(t), \quad \text{for } t>0, \tag{62}$$

where W^k is defined by $W_t^k = (1/\sqrt{k})W_{tk}$, for $t\ge 0$. We now define a scaling $\mathfrak{W}\to\mathfrak{W}$ mapping w to w^k in such a way that

- the lifetime process ζ^k of w^k is given by $r\mapsto\zeta_r^k = k^{-\eta}\zeta_{k^{2\eta}r}$,
- the motion process of w^k is ξ^k given as

$$r\mapsto\xi_r^k = \Big(L_{\left[\Gamma,x/\sqrt{k}+W^k\right]}^{-1}(r),\; W^k\circ L_{\left[\Gamma,x/\sqrt{k}+W^k\right]}^{-1}(r)\Big). \tag{63}$$

Hence, if w has the distribution $\mathbb{N}_{(0,0)}^{T^x\Gamma^k}$, then w^k has the distribution $k^{-\eta}\mathbb{N}_{(0,0)}^{T^{x/\sqrt{k}}\Gamma}$. Note that $\sigma(w^k) = k^{-2\eta}\sigma(w)$ is the length of the excursion w^k. For the stopping times τ_t we obtain from the formula lines (62), (63) and (18) the relation

$$\tau_{kt}(w_u) = k^\eta\tau_t\big(w_{k^{-2\eta}u}^k\big), \quad \text{for all } u\in[0,\sigma],\ t>0,\ w\in\mathfrak{W}. \tag{64}$$

Looking at the total exit local times $L^{kt}_{\sigma(w)}[w]$ and using (22) and (64) and substitutions $v = k^{-2\eta}u$ and $\delta = k^{-\eta}\varepsilon$ gives, for all $t > 0$,

$$\begin{aligned}
L^{kt}_{\sigma(w)}[w] &= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^{\sigma} 1_{\{\tau_{kt}(w_u)<\,\zeta_u\,<\,\tau_{kt}(w_u)+\varepsilon\}}\,du \\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^{\sigma} 1_{\{k^\eta\tau_t(w^k_{k^{-2\eta}u})<\,\zeta_u\,<\,k^\eta\tau_t(w^k_{k^{-2\eta}u})+\varepsilon\}}\,du \\
&= k^{2\eta}\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^{k^{-2\eta}\sigma} 1_{\{\tau_t(w^k_v)<\,k^{-\eta}\zeta(k^{2\eta}v)\,<\,\tau_t(w^k_v)+k^{-\eta}\varepsilon\}}\,dv \\
&= k^{\eta}\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_0^{\sigma(w^k)} 1_{\{\tau_t(w^k_v)<\,\zeta^k(v)\,<\,\tau_t(w^k_v)+\delta\}}\,dv = k^\eta L^t_{\sigma(w^k)}[w^k].
\end{aligned} \tag{65}$$

Hence, the claim formulated after (61) is proved.

From (60) and this claim we get the representation

$$\big\|X^k_t[\Gamma^k]\big\| = \int_{\mathfrak{W}} L^t_\sigma[w]\,\Pi^k_{a,b}(dw), \ \text{ for } \ t>0, \tag{66}$$

where $\Pi^k_{a,b} = \Pi^k_{a,b}[\Gamma]$ is a Poisson point field on $\mathfrak{W}$ with intensity measure

$$\pi^k_{a,b} = \pi^k_{a,b}[\Gamma] := k^{-\eta}\int_{k^\eta a}^{k^\eta b} \mathbb{N}^{T^{x/\sqrt{k}}\Gamma}_{(0,0)}\,dx = k^{-\rho}\int_{k^\rho a}^{k^\rho b}\mathbb{N}^{T^x\Gamma}_{(0,0)}\,dx, \tag{67}$$

with $\rho := \eta - 1/2 > 0$. This finishes the first step in the proof.

Comparing (59) and (66) we note that $\big\|X^k[\Gamma^k]\big\|$ and the right hand side in (59) are defined by the same functional of a Poisson point field on $\mathfrak{W}$, of course with different intensity measures.

To do the *second step* in the proof and show that P–almost surely the processes $\big\|X^k[\Gamma^k]\big\|$ converge in law on $C\big((0,\infty),\mathbb{R}\big)$ to $X^\infty(a,b)$, one has to verify, by definition of the topology on $C\big((0,\infty),\mathbb{R}\big)$, that for every compact set $I\subset(0,\infty)$, P–almost surely, the processes $\big\{\|X^k_t[\Gamma^k]\| : t\in I\big\}$ converge in law on the space $C(I,\mathbb{R})$ with the uniform topology to $\big\{X^\infty_t(a,b) : t\in I\big\}$. Clearly, it suffices to show this for compact sets of the form $I=[1/n,n]$, so fix an arbitrary positive integer n.

Abbreviate $C_n := C([1/n,n],\mathbb{R})$ and, for $w\in\mathfrak{W}$, let $L_n[w]\in C_n$ denote the function defined by $L_n[w](s) := L^s_\sigma[w]$ for all $s\in[1/n,n]$. By *Birkhoff's Individual Ergodic Theorem* applied to the group of spatial shifts acting ergodically on the stable random measure Γ we obtain, for each measurable $F : C_n\to[0,\infty)$, P–almost surely,

$$\lim_{k\uparrow\infty} k^{-\rho}\int_{k^\rho a}^{k^\rho b}\int_{\mathfrak{W}} F\big(L_n[w]\big)\,\mathbb{N}^{T^x\Gamma}_{(0,0)}(dw)\,dx = (b-a)\int_{\mathcal{M}_{\rm tem}}\mathsf{P}(d\Upsilon)\int_{\mathfrak{W}}F\big(L_n[w]\big)\,\mathbb{N}^{\Upsilon}_{(0,0)}(dw). \tag{68}$$

Define random measures μ_k on C_n by

$$\mu_k(B) := k^{-\rho}\int_{k^\rho a}^{k^\rho b}\int_{\mathfrak{W}} 1_{B\setminus\{0\}}\big(L_n[w]\big)\,\mathbb{N}^{T^x\Gamma}_{(0,0)}(dw)\,dx, \ \text{ for } B\subseteq C_n \text{ Borel.} \tag{69}$$

Define, similarly, a measure μ on C_n by

$$\mu(B) := (b-a)\int_{\mathcal{M}_{\rm tem}}\mathsf{P}(d\Upsilon)\int_{\mathfrak{W}} 1_{B\setminus\{0\}}\big(L_n[w]\big)\,\mathbb{N}^{\Upsilon}_{(0,0)}(dw), \ \text{ for } B\subseteq C_n \text{ Borel.} \tag{70}$$

Note that, by (28), μ_k and μ are finite measures since we did not allow them to have mass at the zero function in C_n.

As the space C_n is Polish, there is a countable family $\{F_m : C_n \to [0,\infty) : m \geq 1\}$ of continuous and bounded functions, which are convergence determining for the weak convergence of finite measures on C_n. This fact together with (68) implies that P–almost surely the measures μ_k converge weakly to the measure μ on C_n.

(66) and (67) together state that $\big\|X^k[\Gamma^k]\big\|_n := \big\{\|X_t^k[\Gamma^k]\| : t \in [1/n, n]\big\}$ is equal in law to the sum of all functions in C_n in the support of a Poisson point field with intensity measure μ_k. By the finiteness of μ and elementary properties of Poisson point fields we infer that P–almost surely this sum converges in distribution to the sum of all functions in the support of a Poisson point field in C_n with intensity measure μ. In other words, P–almost surely, in distribution on the space C_n with the topology of uniform convergence,

$$\lim_{k\uparrow\infty} \big\|X^k[\Gamma_k]\big\|_n = \lim_{k\uparrow\infty} \int_{\mathfrak{W}} L_n[w]\, \Pi_{a,b}^k(dw) = \int_{\mathfrak{W}} L_n[w]\, \Pi_{a,b}^\infty(dw). \tag{71}$$

This finishes the proof of the second step and thus proves the statements (i) and (ii) in the proposition. □

Now we return to the scaled processes $X^k = X^k[\Gamma]$ based on the unscaled medium Γ. In order to be able to deal with the real-valued processes $t \mapsto X_t^k(a,b)$, started in $X_0 = \ell$, we use the crossing property, Theorem 5 (which is proved in Section 3 below) to derive the following corollary.

Corollary 16 (No mass transport on macroscopic scales). *Let (a,b) be a bounded interval and consider the rescaled processes $\{X_t^k : t > 0\}$ with $X_0^k = \ell_{(-\infty,a)}$ or $X_0^k = \ell_{(b,\infty)}$. Then, in P–probability, the processes $\big\{X_t^k(a,b) : t > 0\big\}$ converge in distribution on $C\big((0,\infty),\mathbb{R}\big)$ to the zero function as $k \uparrow \infty$.*

Proof. By translation and (if needed) reflection, we see that it is sufficient to show that, in P–probability, the processes

$$\big\{X_t^k(a-b,0) : t > 0\big\}, \text{ for } X_0 = \ell_{(0,\infty)}, \tag{72}$$

converge in distribution on $C\big((0,\infty),\mathbb{R}\big)$ to the zero function. Now observe that

$$\sup_{t\geq 0} X_t^k(a-b,0) = \sup_{t\geq 0} k^{-\eta} X_{kt}\big(k^\eta(a-b),0\big) \leq k^{-\eta} \sup_{t\geq 0} X_t(-\infty,0] \xrightarrow[k\uparrow\infty]{} 0, \tag{73}$$

$\mathbb{P}^\Gamma$–almost surely, for P–almost all Γ, by Theorem 5. This finishes the proof. □

We now have the means to complete the *proof of Theorem* 1 subject to the proof of the crossing property, Theorem 5, the finite mass property (28), and (57). We start the process X in $X_0 = \ell$ and show the convergence in the sense claimed in Theorem 1 (i) of the rescaled processes X^k to the process X^∞ defined by (27) with starting measure $X_0^\infty = \ell$.

Let again $a < b$. Given Γ, by the branching property, $\{X_t^k(a,b) : t \geq 0\}$ is the sum of independent processes started in $X_0^k = \ell_{(a,b)}$, $\ell_{(b,\infty)}$ and $\ell_{(-\infty,a)}$, respectively. Combining the total mass convergence, Proposition 15 (i), and Corollary 16 we see that, in P–probability, the processes $\big\{X_t^k(a,b) : t \geq 0\big\}$ converge in distribution on $C\big((0,\infty),\mathbb{R}\big)$ to the limit process $\big\{X_t^\infty(a,b) : t \geq 0\big\}$, which is described in Proposition 15 (ii) and coincides, of course, with the limit process X^∞ applied to the interval (a,b).

It remains to lift the result from the indicator functions $1_{(a,b)}$ to any continuous function $\varphi\colon \mathbb{R} \to [0,\infty)$ with bounded support, say contained in (a,b). Let $\delta > 0$. We choose step functions (i.e.

linear combinations of indicator functions on bounded, open intervals) $g, h\colon (a,b) \to [0,\infty)$ with $h \le \varphi \le g$ and $\|g-h\|_\infty < \delta$. Then, for all positive integers n,

$$\sup_{1/n \le t \le n} \langle X_t^k, h\rangle \le \sup_{1/n \le t \le n} \langle X_t^k, \varphi\rangle \le \sup_{1/n \le t \le n} \langle X_t^k, g\rangle. \tag{74}$$

In P–probability, the left and right hand side as well as $\sup_{1/n \le t \le n} \langle X_t^k, g-h\rangle$ converge in distribution as $k \uparrow \infty$ to

$$\sup_{1/n \le t \le n} \langle X_t^\infty, h\rangle \le \sup_{1/n \le t \le n} \langle X_t^\infty, g\rangle \quad \text{and} \quad \sup_{1/n \le t \le n} \langle X_t^\infty, g-h\rangle, \text{ respectively.} \tag{75}$$

Moreover, the latter term is bounded by $\delta \sup_{1/n \le t \le n} X_t^\infty(a,b)$. As, by Proposition 15 (i), the process $X^\infty(a,b)$ has almost surely continuous paths, this can be made arbitrarily small by choice of δ. Recalling the definition (1) of the metric on $\mathcal{M}(\mathbb{R})$, we see that this implies convergence of the processes X^k on $C((0,\infty), \mathcal{M}(\mathbb{R}))$. But the states of the limit process are again in $\mathcal{M}_{\text{tem}}$, because

$$\mathbf{E}_\ell \langle X_t^\infty, \varphi\rangle = \langle \ell, \varphi\rangle < \infty, \text{ for all } \varphi \in \Phi \text{ and } t > 0, \tag{76}$$

recall Theorem 0 (iii), and path continuity. This finishes the proof of Theorem 1 (i).

Remark 17 (Other starting measures)**.** It is possible to start the process X with k–dependent initial measures $X_0 = \mu^{(k)}$ on $\mathbb{R}$ such that $X_0^k \equiv \mu$, where μ is a sufficiently *diffuse* measure in $\mathcal{M}_{\text{tem}}$. For example it is sufficient to require that μ has a continuous density g with the property that, for some constants $a, b \ge 0$,

$$\lim_{x\uparrow\infty} \int_x^\infty \bigl|g(y) - a\bigr|\, dy = \lim_{x\downarrow -\infty} \int_{-\infty}^x \bigl|g(y) - b\bigr|\, dy = 0. \tag{77}$$

To show this, observe that one can extend the proof of convergence easily from the case $X_0^k = \ell$ to $X_0^k = \ell_{(a,b)}$. By uniform approximation from above and below one can then get convergence for all starting measures satisfying (77). However, the functional limit law does *not* hold without any condition for the scaled initial measure $X_0^k = \mu$. Starting, for example, with the counting measure $X_0^k \equiv \sum_{z\in\mathbb{Z}} \delta_z$ does *not* lead to a limit process X^∞ with deterministic law. $\diamond$

3. The method of good and bad paths

In Subsection 3.1 and 3.2 we formulate a quantitative approach to the method of good and bad paths extending a recent result of [11]. The main result of this part, Theorem 22, enables us to prove the crossing property of Theorem 5, in Subsection 3.3, and the compound Poisson property of Theorem 1 (ii), in Subsection 3.4.

3.1. Regularity of the catalytic medium.

In this subsection we introduce a characteristic quantity $N(\Gamma)$, which measures the regularity of any sample medium Γ of the catalyst. This is used, in Subsection 3.2 below, to formulate an upper bound on the survival probability of the superprocess $X = X[\Gamma]$ at a Brownian stopping time.

For every $n \ge 1$ and purely atomic measure μ we denote by $\pi_n[\mu] \subseteq \mathbb{R}$ the set of spatial positions of the atoms of μ whose weights are in $[2^{-n}, 2^{-n+1})$. For our stable random measure Γ of index $0 < \gamma < 1$ the set $\pi_n[\Gamma]$ is the support of a homogeneous Poisson point field on $\mathbb{R}$ with intensity

$$a_n := c_\gamma\, 2^{\gamma n}, \text{ where } c_\gamma := \frac{1 - 2^{-\gamma}}{\gamma \int_0^\infty \frac{1 - \mathrm{e}^{-r}}{r^{1+\gamma}}\, dr}, \tag{78}$$

see e.g. [8].

Definition 18 (p–perfect medium)**.** Fix a value $\beta \in (0, \gamma \log 2)$ once and for all.

(i) For any positive integer n and positive real k, we denote by $A(n,k)$ the event that all connected components of $[-k,k] \setminus \pi_n[\Gamma]$ are shorter than $\Delta_n := \mathrm{e}^{-\beta n}$.

(ii) For an integer $p \geq 1$, a purely atomic measure $\Theta \in \mathcal{M}_{\mathrm{tem}}(\mathbb{R})$ is called a *p–perfect medium* if all connected components of $\mathbb{R} \setminus \pi_p[\Theta]$ are shorter than Δ_p. ◇

Note that stable measures Γ are almost surely not p–perfect, for any integer $p \geq 1$. But we see in Remark 20 below that, for any fixed bounded window, almost surely there exists a p such that, for all $n \geq p$, the measure Γ agrees with an n–perfect medium on this window. This fact allows us sometimes to study particles in a fixed perfect medium instead of a stable medium, making use of the high degree of regularity a perfect medium has.

We now study the event $A(n,k)$ where the window size k depends on n. More precisely, assume that $\{k(n) : n \geq 1\}$ is an increasing sequence of positive reals with $\log\log k(n) = o(n)$ as $n \uparrow \infty$. Then distances larger then Δ_n occur only with an exponentially small probability.

Lemma 19 (Large gaps in $\pi_n[\Gamma]$)**.** *Let $\varrho = \gamma \log 2 - \beta > 0$. Then there are constants $c, d > 0$ such that, for every $N \geq 1$,*

$$\mathsf{P}\Big\{ \textit{there is an } n \geq N \textit{ such that } A\big(n, k(n)\big) \textit{ fails} \Big\} \leq c \exp\big(-d\, \mathrm{e}^{\varrho N}\big). \tag{79}$$

Remark 20 (p–perfect medium)**.** By the Borel-Cantelli lemma one can see that for a stable measure Γ there is a (random) integer N such that $A(n, k(n))$ holds for every $n \geq N$. Suppose that a bounded interval I is given, then there exists $p \geq N$ with $I \subset [-k(n), k(n)]$ for all $n \geq p$, hence Γ agrees with an n-perfect medium Θ on the set I. In particular, for a Brownian particle W which does not leave I, we have $L_{[\Gamma, W]} = L_{[\Theta, W]}$. ◇

Proof. We adapt results of [11, p.634]. Write $J(n)$ for the number of points in $(-1/2, 1/2) \cap \pi_n[\Gamma]$. Denote by $x_0 := -1/2$ and by $x_1 < x_2 < \ldots$ the points in $(-1/2, \infty) \cap \pi_n[\Gamma]$. Define the distances $y_k := x_k - x_{k-1}$ for $k \geq 1$.

From [11, Subsection 5.2] we quote that there are constants $c_1, c_2 > 0$ such that

$$\mathsf{P}\Big\{ \max_{1 \leq i \leq J(n)+1} y_i > \mathrm{e}^{-\beta n} \Big\} \leq c_1\, \mathrm{e}^{n \gamma \log 2} \exp(-c_2\, \mathrm{e}^{\varrho n}). \tag{80}$$

Estimate (80) is proved in the following way. Recall that $a_n = c_\gamma 2^{\gamma n}$ and note that $y_1, y_2, \ldots$ are an iid sequence of exponential random variables with parameter a_n. A tail estimate for the maximum of exponential distributions taken from extreme value theory gives, see [11, (164)–(167)], that, for suitable constants $c, C > 0$,

$$\mathsf{P}\Big\{ \max_{1 \leq i \leq 2a_n} y_i > \mathrm{e}^{-\beta n} \Big\} \leq C\, \mathrm{e}^{n \gamma \log 2} \exp(-c\, \mathrm{e}^{\varrho n}). \tag{81}$$

As $J(n)$ has a Poisson distribution with parameter a_n, a standard large-deviation estimate for $J(n)$, see [11, (162)], yields, for a constant $k > 0$,

$$\mathsf{P}\Big\{ J(n) + 1 > 2a_n \Big\} \leq \exp\Big(- k e^{n \gamma \log 2} \Big). \tag{82}$$

Now

$$\mathsf{P}\Big\{ \max_{1 \leq i \leq J(n)+1} y_i > \mathrm{e}^{-\beta n} \Big\} \leq \mathsf{P}\Big\{ \max_{1 \leq i \leq 2a_n} y_i > \mathrm{e}^{-\beta n} \Big\} + \mathsf{P}\Big\{ J(n) + 1 > 2a_n \Big\}. \tag{83}$$

As $\varrho < \gamma \log 2$, the term (81) dominates in the sum and one obtains (80).

From this we infer that, for some $c_3, c_4 > 0$,

$$\mathsf{P}\Big\{A(n,1/2)\ \text{fails}\Big\} \le c_3 \exp(-c_4\, \mathrm{e}^{\varrho n}). \tag{84}$$

This probability is invariant under shifts of the interval $(-1/2, 1/2)$ and thus, by taking the union of the events, we obtain for some c_5 , $c_6 > 0$, and all $n \ge 1$,

$$\mathsf{P}\Big\{A\big(n,k(n)\big)\ \text{fails}\Big\} \le c_3 k(n) \exp\big(-c_4\, \mathrm{e}^{\varrho n}\big) \le c_5 \exp\big(-c_6\, \mathrm{e}^{\varrho n}\big). \tag{85}$$

Finally, we take the union of the complements of the events $A(n, k(n))$ over all $n \ge N$ and find a suitable c and $d := c_6$, such that (79) holds. □

From now on we specialize to

$$k(n) := \exp\exp\sqrt{n}, \ \ \text{for}\ \ n \ge 1. \tag{86}$$

Definition 21 (Characteristic $N(\Gamma)$)**.** Define the *characteristic* $N(\Gamma)$ of a medium sample Γ by

$$N(\Gamma) := \min\big\{N \ge 1 :\ A\big(n, k(n)\big)\ \text{holds for all}\ n \ge N\big\}. \tag{87}$$

Using the Borel-Cantelli Lemma and Lemma 19, one can see that $N(\Gamma)$ is a well-defined integer, P–almost surely. ◇

Consequently, from the level $N(\Gamma) = n$ on, in our rapidly increasing window $\big(-k(n), k(n)\big)$, neighbouring Γ–atoms of weight about 2^{-n} have at most a distance Δ_n . We consider samples Γ with small $N(\Gamma)$ as smooth samples. Later in the paper they allow sharper estimates on the survival probabilities of the super-Brownian motions they catalyse.

3.2. **An upper bound on the survival probability.** In this subsection we determine an upper bound on the survival probability of super-Brownian motion in a fixed catalytic medium sample Γ with characteristic $N(\Gamma)$ at certain stopping times. The approach taken here is similar to the key technique of [11], in particular we also work in a historical setting and use a decomposition into good and bad reactant paths, but we have to make a more quantitative approach. Moreover, our argument does not rely on the compact support property of $X[\Gamma]$ from [12]. The most crucial point is that we work out explicitly how the upper bound of the survival probability depends on $N(\Gamma)$.

The terminology "good" and "bad" is motivated by the fact, that we want to have enough killing of the reactant to verify that not too much reactant mass crosses to the left [Theorem 5] or that the macroscopic reactant clumps get isolated [Theorem 1 (ii)]. Loosely speaking, a reactant path is good if it has sufficiently much contact with the catalyst to collect enough branching rate. All this has now to be made precise.

For the further development, we presuppose the reader is familiar with basic ideas and the formalism of historical catalytic super-Brownian motion. Here we closely follow the presentation of [11]. Denote by $Y = Y[\Gamma] = \{Y_t : t \ge 0\}$ the *historical super-Brownian motion* in the catalytic medium Γ with starting measure $\mu \in \mathcal{M}_{\mathrm{tem}}$ defined on a probability space $(\Omega, \mathcal{G}, \tilde{\mathbb{P}}^{\Gamma}_{\mu})$. At the same time we use, for any Brownian stopping time T, Dynkin's stopped measure Y_T and the pre-T σ-field $\mathcal{G}(T)$ for the historical superprocess as introduced in [16] and reviewed in [11, Subsection 3.2]. Denote by $\mathcal{P}_{r,x}$ the distribution of a Brownian path $W = \{W_s : s \ge r\}$ started at time r at $W_r = x$. Note that the collision local time $L_{[\Gamma,W]}(ds)$ is well-defined also for Brownian paths W distributed according to $\mathcal{P}_{r,x}$.

It might be useful to indicate here briefly the formalism of the historical setting, leading to Y, and the meaning of a stopped measure Y_T , for a detailed study we recommend [13] and [16].

Recall that $X_t(dx)$ is a measure on $\mathbb{R}$ which measures the population mass at time t at site x. Thinking of an approximation of X by branching particle systems, to each infinitesimal particle of $X_t(dx)$, there is an ancestor at time 0, and the mass transport along subsequent branchings followed a continuous path W from 0 to t. The idea of the historical process is to represent the infinitesimal particle at time t not only by its position, but by this path. In order to define all the states Y_t, $t \geq 0$, of the historical process on a common space, W is considered as a continuous path defined on $[0,\infty)$ but stopped at time t. Thus, Y_t is a measure on the set of paths which is supported by the paths stopped at time t. The medium Γ does not play a role at this stage since it influences the branching only but not the migration of surviving mass. Of course, the different paths W in the support of Y_t are coupled, expressing family relations of the population at time t.

If $T = T[W]$ is a stopping time on the (filtered) space of continuous paths $W : [0,\infty) \to \mathbb{R}$. Then Y_T is a measure on the paths W stopped at time $T < \infty$. Note that in a branching particle system approximation of X and Y, it makes immediately sense to stop all the particles' evolution, if along their ancestry path lines the stopping time T is reached. We stress the fact that T is a stopping time for the underlying motion process, Brownian motion in our case, and not for the historical superprocess Y.

Let I be an interval, T a stopping time as above and denote

$$\mathcal{W}(I,T) := \Big\{ W : [0,\infty) \to \mathbb{R} \text{ continuous} : \ W(0) \in I \text{ and } T[W] < \infty \Big\}. \tag{88}$$

We introduce two formal Hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$ on an increasing sequence $T_n \uparrow T$ of stopping times of W with $T_0 = 0$. Suppose a positive integer p, a sequence of positive thresholds l_n and small reals $\varepsilon_1, \varepsilon_2 > 0$ are given. For every nonnegative integer n and purely atomic measure $\Theta \in \mathcal{M}_{\mathrm{tem}}$ define the set $B(n) = B(n)[\Theta]$ of *bad paths* for the medium Θ on the time interval $[T_n, T_{n+1}]$ as the set of paths $W \in \mathcal{W}(I, T_{n+1})$ satisfying

$$\int_{T_n}^{T_{n+1}} L_{[\Theta,W]}(ds) < l_n . \tag{89}$$

Recall the definition of the sequence $k(n)$ from (86).

Hypothesis $\mathbf{H_1}$: The sequence of stopping times satisfies the Hypothesis $\mathbf{H_1}(p,\varepsilon_1)$ if

$$\sum_{n=0}^{\infty} 2^{n+1} \mathcal{P}_{0,x} \Big\{ \sup_{T_n \leq s \leq T_{n+1} < \infty} |W_s| > k(n+p) \Big\} \leq \varepsilon_1 , \text{ for all } x \in I. \tag{90}$$

Hypothesis $\mathbf{H_2}$: The sequence of stopping times satisfies the Hypothesis $\mathbf{H_2}(p, l_{.}, \varepsilon_2)$ if there exists a sequence of positive reals b_n such that

$$\sum_{n=0}^{\infty} b_n \leq \varepsilon_2 , \tag{91}$$

and, for every $n \geq 0$ and $(n+p)$-perfect medium Θ, we have

$$\mathcal{P}_{T_n, W(T_n)} \Big\{ \tilde{W} \in B(n)[\Theta] \Big\} \leq b_n, \text{ for } W \in \mathcal{W}(I, T_n). \tag{92}$$

Roughly speaking, $\mathbf{H_1}(p,\varepsilon_1)$ is satisfied if Brownian motion starting from x leaves the window $\big(-k(n+p), k(n+p)\big)$ during $[T_n, T_{n+1}]$ only with a very small probability. This ensures that, for $p \geq N(\Gamma)$ the path W spends a sufficiently large time in a region where the stable medium Γ agrees with an $(n+p)$-perfect medium. On the other hand, $\mathbf{H_2}(p, l_{.}, \varepsilon_2)$ holds, if the probability that the collision local time accumulated in the periods $[T_n\,, T_{n+1}]$ in an $(n+p)$-perfect medium exceeds the threshold l_n is sufficiently high.

Here is the announced result on the survival probability.

Theorem 22 (Upper bound on survival probability). *Fix $0 < \varepsilon < 1$ and a probability measure μ supported by a compact interval I. Put $T_0 = 0$ and suppose that $T_n \uparrow T$ is a sequence of stopping times. Assume further there are nonnegative integers m and d and a sequence of thresholds $l_n > 0$ such that*

$$\sum_{n=0}^{\infty} \frac{2^{-m-n}}{l_n} \leq \frac{\varepsilon}{3}, \tag{93}$$

and the Hypotheses $\mathbf{H_1}\,(m+d, \varepsilon/3)$ and $\mathbf{H_2}(m+d, l.\,, \varepsilon/6)$ are satisfied. Then, for P–almost every Γ with $N(\Gamma) \leq m$ and for the stopped measure $Y_T = Y_T[\Gamma]$ for the process starting from $Y_0 = 2^{-m}\mu$, we have

$$\tilde{\mathbb{P}}^{\Gamma}_{2^{-m}\mu}\{Y_T \neq 0\} \leq \varepsilon. \tag{94}$$

This relatively abstract result provides, loosely speaking, the upper bound for the survival of Y at a Brownian stopping time T, which enters as a key tool in the proof of the crossing property (see Subsection 3.3) and the compound Poisson property (see Subsection 3.4). In these applications, checking Hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$ amounts to estimating from above the probability that a single particle is bad. To get a feeling for the versatility of the result we just mention here that the stopping time T we use is the first hitting time of the origin by a Brownian motion started at some $x \in (0, \infty)$ in the case of the crossing property, and T is a deterministic finite time in the case of the compound Poisson property.

The remainder of this subsection is devoted to the proof of Theorem 22. We work with a fixed catalyst sample Γ with characteristic $N = N(\Gamma) \leq m$ and use the notation from the theorem, additionally abbreviating $M_n := 2^{-m-n}$ for the fixed m. We start with a simple lemma taken from [11, Subsection 3.4].

Lemma 23 (Extinction by partitioning). *Define events $A_n := \big\{\|Y_{T_n}\| \leq M_n\big\}$ and $A := \bigcap_{n=1}^{\infty} A_n$. Then, for every $\nu \in \mathcal{M}_{\mathrm{tem}}(\mathbb{R})$, we have $\tilde{\mathbb{P}}^{\Gamma}_{\nu}$–almost surely on A that $Y_T = 0$.*

Proof. By Markov's inequality, for each $n \geq 1$ and arbitrary $\zeta > 0$,

$$\tilde{\mathbb{P}}^{\Gamma}_{\nu}\Big(\big\{\|Y_T\| > \zeta\big\} \cap A\Big) \leq \zeta^{-1}\, \tilde{\mathbb{E}}^{\Gamma}_{\nu}\big\{1_{A_n}\|Y_T\|\big\} \tag{95}$$

and, by the special Markov property for historical superprocesses and criticality,

$$\tilde{\mathbb{E}}^{\Gamma}_{\nu}\big\{1_{A_n}\|Y_T\|\big\} = \tilde{\mathbb{E}}^{\Gamma}_{\nu}\big\{1_{A_n}\tilde{\mathbb{E}}^{\Gamma}_{Y_{T_n}}\{\|Y_T\|\}\big\} = \tilde{\mathbb{E}}^{\Gamma}_{\nu}\big\{1_{A_n}\|Y_{T_n}\|\big\} \leq M_n\,. \tag{96}$$

As n was arbitrary, we infer that

$$\tilde{\mathbb{P}}^{\Gamma}_{\nu}\Big(\big\{\|Y_T\| > \zeta\big\} \cap A\Big) = 0, \tag{97}$$

and as ζ can be made arbitrarily small, we get the statement. □

Formally, define the set $E(n) = E(n)[\Gamma]$ of *good paths* on the interval $[T_n\,, T_{n+1}]$ with respect to the medium Γ as the set of paths $W \in \mathcal{W}(I, T_{n+1})$, which are not bad, i.e. where inequality (89) fails. For the good paths we use the comparison with the survival probability in Feller's branching diffusion from [11, Proposition 12].

Lemma 24 (Comparison with Feller's branching diffusion)**.** *For all $n \geq 0$, and every ν in $\mathcal{M}_{\text{tem}}(\mathbb{R})$, we have, for* P*–almost every* Γ,

$$\tilde{\mathbb{P}}_\nu^\Gamma\Big\{Y_{T_{n+1}}\big(E(n)\big) > 0 \;\Big|\; \mathcal{G}(T_n)\Big\} \;\leq\; \frac{\|Y_{T_n}\|}{l_n}\,. \tag{98}$$

This lemma takes care of the good paths; it is not too likely that good paths survive. It remains to verify that the probability of survival of bad paths is also sufficiently small. To show this we use Hypotheses $\mathbf{H_1}\big(m+d,\varepsilon/3\big)$ and $\mathbf{H_2}(m+d,l_\cdot,\varepsilon/6)$ as in the theorem. There are two possible reasons, why a path could be bad on $[T_n\,,T_{n+1}]$ for the medium Γ, namely the occurrence of one of the following two disjoint events.

Event $B_1(n)$**:** the set of paths $W \in \mathcal{W}(I,T_{n+1})$, that leave the space interval $\big[-k(d+n+m),\,k(d+n+m)\big]$ during $[T_n\,,T_{n+1}]$ and thus enter an area where we have no control over the catalytic atoms,

Event $B_2(n)$**:** the set of paths $W \in \mathcal{W}(I,T_{n+1})$, that stay inside the interval $\big[-k(d+n+m),\,k(d+n+m)\big]$ but for which the collision local time $L_{[\Gamma,W]}$ accumulated during $[T_n\,,T_{n+1}]$ is below the threshold l_n.

Note that in case of event $B_2(n)$, if $m \geq N(\Gamma)$, the path stays in a window, where the medium Γ coincides with an $(n+m+d)$–perfect medium Θ. It is clear that we have the decompositions

$$\operatorname{supp} Y_{T_{n+1}} \subseteq E(n)\cup B(n) \quad\text{and}\quad B(n)\subseteq B_1(n)\cup B_2(n), \tag{99}$$

where supp indicates the closed support of a measure. The following lemma provides estimates for the extinction probability of the bad paths. We still use notation from Theorem 22.

Lemma 25 (Mass of bad paths)**.** *Under the conditions of Theorem* 22*, for* P*–almost every* Γ *with* $N(\Gamma)\leq m$*, and* $\nu = 2^{-m}\mu$,

(i) $$\sum_{n=0}^{\infty}\tilde{\mathbb{P}}_\nu^\Gamma\Big\{Y_{T_{n+1}}\big(B_1(n)\big)\geq M_{n+1}\Big\} \;\leq\; \frac{\varepsilon}{3}\,,$$

(ii) $$\sum_{n=0}^{\infty}\tilde{\mathbb{P}}_\nu^\Gamma\Big\{Y_{T_{n+1}}\big(B_2(n)\big)\geq M_{n+1}\;\Big|\;\|Y_{T_n}\|\leq M_n\Big\} \;\leq\; \frac{\varepsilon}{3}\,.$$

Proof. (i) From Markov's inequality, the expectation formula for superprocesses, and (90), we infer,

$$\begin{aligned}\sum_{n=0}^{\infty}\tilde{\mathbb{P}}_\nu^\Gamma\Big\{Y_{T_{n+1}}\big(B_1(n)\big)\geq M_{n+1}\Big\} &\leq \sum_{n=0}^{\infty} M_{n+1}^{-1}\,\tilde{\mathbb{E}}_\nu^\Gamma\Big\{Y_{T_{n+1}}\big(B_1(n)\big)\Big\}\\ &= \sum_{n=0}^{\infty} M_{n+1}^{-1}2^{-m}\int_{\mathbb{R}}\mathcal{P}_{0,x}\Big\{\sup_{T_n\leq s\leq T_{n+1}<\infty}|W_s| > k(d+n+m)\Big\}\,\mu(dx) \leq \frac{\varepsilon}{3}\,,\end{aligned} \tag{100}$$

which is (i).

(ii) The proof of (ii) is based on the expectation formula for the historical mass on a set B of stopped paths $\tilde{W}:[0,T_{n+1}]\to\mathbb{R}$ with $T_{n+1}[\tilde{W}]<\infty$. If B depends only on $\Big\{\tilde{W}(s):\,s\geq T_n\Big\}$, we have

$$\tilde{\mathbb{E}}_\nu^\Gamma\left\{Y_{T_{n+1}}(B)\,\middle|\,Y_{T_n}\right\} = \int_{\mathbb{R}}\mathcal{P}_{T_n\,,W(T_n)}\{\tilde{W}\in B\}\,Y_{T_n}(dW), \tag{101}$$

see e.g. [11, (37)]. Note that every path $\tilde{W}\notin B_1(n)$ spends the time $[T_n\,,T_{n+1}]$ inside a compact interval in which the medium Γ coincides with an $(n+m+d)$–perfect medium Θ_n. Hence we can use the bound in (92) together with Markov's inequality, the special Markov property and the expectation formula (101) to see that,

$$\sum_{n=0}^{\infty} \tilde{\mathbb{P}}_\nu^\Gamma\Big\{Y_{T_{n+1}}\big(B_2(n)\big) \geq M_{n+1} \,\Big|\, \|Y_{T_n}\| \leq M_n\Big\} \tag{102}$$

$$\leq \sum_{n=0}^{\infty} M_{n+1}^{-1}\, \tilde{\mathbb{E}}_\nu^\Gamma \Big\{Y_{T_{n+1}}\big(B_2(n)\big) \,\Big|\, \|Y_{T_n}\| \leq M_n\Big\}$$

$$\leq \sum_{n=0}^{\infty} M_{n+1}^{-1}\, \tilde{\mathbb{E}}_\nu^\Gamma \Big\{\tilde{\mathbb{E}}^{\Theta_n}\Big\{Y_{T_{n+1}}\big(B_2(n)\big) \,\Big|\, Y_{T_n}\Big\} \,\Big|\, \|Y_{T_n}\| \leq M_n\Big\}$$

$$\leq \sum_{n=0}^{\infty} M_{n+1}^{-1}\, \tilde{\mathbb{E}}_\nu^\Gamma \Big\{\int \mathcal{P}_{T_n, W(T_n)}\Big\{\tilde{W} \in B_2(n)\Big\}\, dY_{T_n}(W) \,\Big|\, \|Y_{T_n}\| \leq M_n\Big\}$$

$$\leq \sum_{n=0}^{\infty} M_n M_{n+1}^{-1} \sup_{W \in \mathcal{W}(I, T_n)} \mathcal{P}_{T_n, W(T_n)}\Big\{\tilde{W} \in B_2(n)\Big\} \;\leq\; 2\sum_{n=0}^{\infty} b_n \;\leq\; \frac{\varepsilon}{3},$$

which is (ii). This ends the proof of the lemma. □

Completion of the proof of Theorem 22. Recall Lemma 23 and in particular the definition of the sets A_n and A. Lemmas 24 and 25 provide the ingredients we need to bound $\tilde{\mathbb{P}}_\nu^\Gamma(A^c)$ by ε, for the starting measure $\nu = 2^{-m}\mu$. Note that the event A_0 has probability one. Hence, for P–almost all Γ with $N(\Gamma) \leq m$,

$$\tilde{\mathbb{P}}_\nu^\Gamma(A^c) = \sum_{n=0}^{\infty} \tilde{\mathbb{P}}_\nu^\Gamma\Big(A_0 \cap \cdots \cap A_n \cap A_{n+1}^c\Big) \tag{103a}$$

$$\leq \sum_{n=0}^{\infty} \tilde{\mathbb{P}}_\nu^\Gamma\Big\{Y_{T_{n+1}}\big(B_1(n)\big) \geq M_{n+1}\Big\} \tag{103b}$$

$$+ \sum_{n=0}^{\infty} \tilde{\mathbb{P}}_\nu^\Gamma\Big\{Y_{T_{n+1}}\big(B_2(n)\big) \geq M_{n+1} \,\Big|\, \|Y_{T_n}\| \leq M_n\Big\} \tag{103c}$$

$$+ \sum_{n=0}^{\infty} \tilde{\mathbb{P}}_\nu^\Gamma\Big\{Y_{T_{n+1}}\big(E(n)\big) \geq M_{n+1} \,\Big|\, \|Y_{T_n}\| \leq M_n\Big\}. \tag{103d}$$

Now, by Lemma 25, the series in (103b) and (103c) are each bounded by $\varepsilon/3$. By Lemma 24, we obtain for (103d), using (93),

$$\sum_{n=0}^{\infty} \tilde{\mathbb{P}}_\nu^\Gamma\Big\{Y_{T_{n+1}}\big(E(n)\big) \geq M_{n+1} \,\Big|\, \|Y_{T_n}\| \leq M_n\Big\} \;\leq\; \sum_{n=0}^{\infty} \frac{M_n}{l_n} \;\leq\; \frac{\varepsilon}{3}. \tag{104}$$

Hence $\tilde{\mathbb{P}}_\nu^\Gamma(A^c) \leq \varepsilon$ and, by Lemma 23, this implies the statement of Theorem 22.

3.3. **The crossing property.** In this section we prove Theorem 5. The idea is to apply the abstract upper bound of Theorem 22 for the survival of Y_T in the case of the stopping time T, which is the (almost surely finite) first hitting time of the origin by a Brownian motion. From this we get an upper bound for the rate of decay, as $x \uparrow \infty$, of the probability that the superprocess started with a fixed mass at $x > 0$ charges $(-\infty, 0)$. This result, Lemma 26, is the main step in the proof. From there it is quite simple, using the Borel-Cantelli lemma, to obtain the desired almost sure result.

We denote

$$d(x) := \Big[\log\Big(1+\frac{5}{2}\log x\Big)\Big]^2, \quad \text{for } x \geq 1. \tag{105}$$

Observe that $d(x)$ is growing slower than logarithmically as $x \uparrow \infty$. Thus, we can choose an $\underline{x} \geq 1$ such that

$$1 \leq d(x) \leq \frac{\log x}{4(\log 2+\beta)}, \text{ for all } x \geq \underline{x}. \tag{106}$$

Lemma 26 (Decay of crossing probability)**.** *There is an integer m depending only on the characteristic $N(\Gamma)$ of the catalytic medium Γ, such that for $x \geq \underline{x}$,*

$$\mathbb{P}^{\Gamma}_{2^{-m}\delta_x}\Big\{\int_0^\infty X_t(-\infty,0]\,dt > 0\Big\} \;\leq\; \frac{1}{x^{3/2}}. \tag{107}$$

The lemma is proved by choosing the right ingredients for the use of the survival probability bound of Theorem 22 with $\mu = \delta_x$, $I = \{x\}$, $\varepsilon = 1/x^{3/2}$, and $d = d(x)$, for fixed $x \geq \underline{x}$. As a preparation for the proof, define the Brownian stopping time T to be the first hitting time of level 0, then the event we are interested in is the survival of Y_T. To define the remaining quantities for an application of Theorem 22, we first leave the integer parameter $m \geq 0$ open and define T_n and l_n in terms of m. Recall that $\alpha < \beta < 2\alpha$ and define, for $n \geq 0$,

$$d_n := \frac{\mathrm{e}^{(\alpha-\beta)n}}{1-\mathrm{e}^{\alpha-\beta}} = \frac{1}{\varepsilon(m)}\,\mathrm{e}^{(\alpha-\beta)(n+m)}, \quad \text{where } \varepsilon(m) := \frac{\mathrm{e}^{(\alpha-\beta)m}}{1-\mathrm{e}^{\alpha-\beta}}. \tag{108}$$

Define barriers

$$x_0 = x \quad \text{and} \quad x_{n+1} = x_n - x d_n. \tag{109}$$

As $\sum_{n=0}^\infty d_n = 1$, we have $x_n \downarrow 0$, and we can define an increasing sequence of hitting times

$$T_n := \inf\big\{t > 0 : W(t) = x_n\big\}, \text{ for } n \geq 0, \tag{110}$$

so that $T_n \uparrow T$ as $n \uparrow \infty$. Finally define

$$l_n := x^{3/2}\, m\,(n+1)^2\, 2^{-n-m}. \tag{111}$$

Lemma 26 follows from Theorem 22 if we verify (93) and the Hypotheses $\mathbf{H_1}(m+d, \varepsilon/3)$ and $\mathbf{H_2}(m+d, l_\cdot, \varepsilon/6)$ for $\varepsilon = 1/x^{3/2}$ and a suitable m, which we choose when we complete the proof of Lemma 26 almost at the end of the subsection.

The next lemma prepares the verification of Hypothesis $\mathbf{H_1}$.

Lemma 27 (Escape probability I)**.** *With $x \geq \underline{x}$ fixed, x_n and T_n from* (109) *and* (110)*, respectively, for all integers $n, m \geq 0$, the events*

$$D_0(n) := \Big\{\sup_{t\in[T_n, T_{n+1}]} |W_t| > k(d+n+m)\Big\} \tag{112}$$

have probability

$$\mathcal{P}_{0,x}\big(D_0(n)\big) = \mathcal{P}_{T_n,x_n}\big(D_0(n)\big) \leq \frac{2}{x^{3/2}}\exp(-\exp\sqrt{n+m}\,). \tag{113}$$

Proof. To begin with, note that $d = d(x)$ is chosen in such a way that, for all $n \geq 0$,

$$x^{5/2} \leq \frac{\exp\exp\sqrt{d+n}}{\exp\exp\sqrt{n}} = \frac{k(d+n)}{k(n)} \tag{114}$$

Note that the inequality holds by definition (105) of $d(x)$ for $n = 0$, and the right hand side of the inequality is increasing in n. A Brownian path starting in x_n is in $D_0(n)$ if and only if it hits level $k(d+n+m)$ before x_{n+1}. Using first the formula for the exit probabilities of Brownian motion from an interval (see, for instance, [27, Proposition II.3.8]), then $d_n \leq 1$ and $x_{n+1} \leq k(d+n+m)/2$ (check for $n, m = 0$!), and finally (114), we get

$$\mathcal{P}_{T_n, x_n}\big(D_0(n)\big) = \frac{x\,d_n}{k(d+n+m) - x_{n+1}} \leq \frac{2x}{k(d+n+m)} \leq \frac{2}{x^{3/2}k(n+m)}\,. \tag{115}$$

□

We observe that Hypothesis $\mathbf{H_1}(m+d, \varepsilon/3)$ is verified if m is chosen such that

$$\sum_{n=0}^{\infty} 2^{n+2} \exp\big(-\exp\sqrt{n+m}\,\big) \leq \frac{1}{3}\,, \tag{116}$$

however, there will be other constraints on m coming from Hypothesis $\mathbf{H_2}$ and we turn to the verification of this hypothesis now.

Recall $\alpha < \beta < 2\alpha$ and fix $\beta < \delta < 2\alpha$. Further fix $0 < \theta_0 < 1/2$ such that $\delta(1-\theta_0) \leq \alpha$ and $(7/4)(1-\theta_0) \leq 1$, and $\theta_0 < \theta_1 < 1/2$ and put $\theta := 1 - 2\theta_1$. For a start, let m be arbitrary but large enough to satisfy

$$m^{1-\theta_0} \leq e^{\beta m \theta_0}. \tag{117}$$

Let $n \geq 0$ be an arbitrary integer and fix an $(m+d+n)$–perfect medium Θ. Let

$$a_n = a_n(x) := x^{7/4}\, \mathrm{e}^{\delta n}\, m\, \mathrm{e}^{\beta m}. \tag{118}$$

To estimate the probability from above that a path W is bad in $[T_n, T_{n+1}]$ it suffices to consider a special *hitting strategy*: As the medium Θ is $(m+d+n)$–perfect we can select atoms of mass in $[2^{-m-n-d}, 2^{-m-n-d+1})$ in such a way that all neighbouring atoms have distance in $[\Delta_{m+n+d}/2, 3\Delta_{m+n+d}/2]$. We call this set of atoms $\tilde{\pi}_{n+m} = \tilde{\pi}_{n+m}(x)$. During $[T_n, T_{n+1}]$ we count collisions with the atoms in $\tilde{\pi}_{n+m}$ in the following way: We define a sequence $\{\kappa_j\}$ of Brownian stopping times by $\kappa_0 := T_n$ and, for $j \geq 1$,

$$\kappa_j := \inf\Big\{t \geq \kappa_{j-1} : W_t \text{ hits } \tilde{\pi}_{n+m} \setminus \{W_{\kappa_{j-1}}\}\Big\} \tag{119}$$

and denote the *number of collisions* by

$$K_n := \max\{j : \kappa_j \leq T_{n+1}\}. \tag{120}$$

Further let $y_i = W(\kappa_i)$ the ith visited atom and

$$L_i := \int_{\kappa_i}^{\kappa_{i+1}} L^{y_i}(ds) \tag{121}$$

the local time accumulated during the ith visit. It is crucial to note that

$$\sum_{i=1}^{K_n - 1} L_i \;\leq\; L_{[\Theta,W]}(T_{n+1}) - L_{[\Theta,W]}(T_n). \tag{122}$$

The verification of Hypothesis $\mathbf{H_2}(m+d, l., \varepsilon/6)$ will be carried out with the help of the following lemma.

Lemma 28 (Upper bound for bad paths). *There are constants $C_0, C_1 > 0$ such that, for all integers $n \geq 0$, the events*

$$D(n) := \left\{ 2^{-d-n-m} \sum_{i=1}^{K_n - 1} L_i < l_n \right\} \tag{123}$$

have probability

$$\mathcal{P}_{T_n, W(T_n)}\big(D(n)\big) \;\leq\; C_0 \, \exp\big(- C_1 a_n^{\theta} \big), \; \textit{for all } W \in \mathcal{W}(I, T_n). \tag{124}$$

To get the upper bound of $\mathcal{P}_{T_n, W(T_n)}\big(D(n)\big)$, an easy coupling argument shows that we may assume from now on that any two neighbouring atoms in $\tilde{\pi}_{n+m}$ have minimal distance $(1/2)\Delta_{n+m+d}$ and the number of atoms in $\tilde{\pi}_{n+m} \cap [x_{n+1}, x_n)$ is also minimal possible, i.e. equal to

$$\Big\lfloor \frac{x d_n}{(3/2)\Delta_{d+n+m}} \Big\rfloor. \tag{125}$$

(Here $\lfloor \cdot \rfloor$ denotes the integer part.) If a path is in $D(n)$, then either the path has a small number K_n of collisions with the catalytic atoms of $\tilde{\pi}_{n+m}$, say less than a_n; or it takes more than a_n visits to the chosen atoms before the accumulated local time on the left hand side of inequality (122) exceeds the threshold l_n. More formally, let

$$D_1(n) := \{ K_n - 1 \leq a_n \} \tag{126}$$

and

$$D_2(n) := \left\{ \sum_{i=1}^{a_n} 2^{-(d+n+m)} L_i < l_n \right\} \tag{127}$$

then $B(n) \subseteq D(n) \subseteq D_1(n) \cup D_2(n)$ and hence we have to check the probability that a single Brownian path encounters one of the two events $D_1(n)$ and $D_2(n)$. This is accomplished in the next two lemmas, which are mainly based on large-deviation estimates. Note that Lemma 28 is an immediate consequence of these two lemmas.

Lemma 29 (Probability of Event $D_1(n)$). *There are constants $C_2, C_3 > 0$ such that, for all integers $n \geq 0$,*

$$\mathcal{P}_{T_n, W(T_n)}\big(D_1(n)\big) \;\leq\; C_2 \, \exp\big(- C_3 a_n^{\theta} \big), \; \textit{for all } W \in \mathcal{W}(I, T_n). \tag{128}$$

Proof. Observe that $W(T_n) = x_n$. Let $c = (2/3)(1 - \mathrm{e}^{\alpha - \beta})$. By our choice of θ_0 and by (117) we have

$$\frac{x d_n}{(3/2)\Delta_{d+n+m}} \;\geq\; \frac{2x}{3\varepsilon(m)} \, \mathrm{e}^{\alpha(n+m)} \;\geq\; c a_n^{1-\theta_0}. \tag{129}$$

The probability in our lemma is thus bounded above by the probability that a simple random walk S_n defined on a probability space $(\Omega, \mathcal{A}, P)$ needs less than $\lfloor a_n \rfloor = \lfloor x^{7/4} m \, \mathrm{e}^{\delta n + \beta m} \rfloor$ steps to cross the level $c a_n^{1-\theta_0}$. By the reflection principle,

$$P\Big\{ \max_{1 \leq k < \lfloor a_n \rfloor} S_k > c a_n^{1-\theta_0} \Big\} \;\leq\; 2 P\Big\{ S_{\lfloor a_n \rfloor - 1} \geq c a_n^{1-\theta_0} \Big\}. \tag{130}$$

By the refinement of Cramér's Theorem given in [14, (3.7.1)] for $\theta_0 < \theta_1 < 1/2$ there is a constant $C > 0$ such that, for all integers k,

$$P\Big\{ S_k \geq c k^{1-\theta_0} \Big\} \;\leq\; C \, \exp\Big(- k^{1-2\theta_1} (c^2/2) \Big). \tag{131}$$

Hence we can use (131) and let $C_0 := Ce^{c^2}$ to get

$$P\Big\{S_{\lfloor a_n \rfloor -1} > c a_n^{1-\theta_0}\Big\} \;\le\; C_0 \exp\Big(-a_n^{\theta}(c^2/2)\Big), \tag{132}$$

which is the required estimate with $C_1 := 2(1-\mathrm{e}^{\alpha-\beta})^2$. ☐

Lemma 30 (Probability of Event $D_2(n)$)**.** *There are constants $C_3 > 0$ and $C_4 > 0$ such that, for all integers $n \ge 0$,*

$$\mathcal{P}_{T_n, W(T_n)}\big(D_2(n)\big) \;\le\; C_2 \exp\big(-C_3 a_n\big), \text{ for all } W \in \mathcal{W}(I, T_n). \tag{133}$$

Proof. $\{L_i\}$ is a sequence of independent, identically distributed positive random variables. By scaling the distribution of L_i/Δ_{n+m+d} is independent of d, n and m. Hence,

$$\begin{aligned}\mathcal{P}_{T_n, W(T_n)}\big(D_2(n)\big) &\le \mathcal{P}_{T_n, W(T_n)}\Big\{\sum_{j=1}^{a_n} 2^{-n-m-d} L_j < l_n\Big\}\\ &\le \mathcal{P}_{T_n, W(T_n)}\Big\{\frac{1}{a_n}\sum_{j=1}^{a_n}\frac{L_j}{\Delta_{n+m+d}} < \frac{2^{n+m+d} l_n}{a_n \Delta_{n+m+d}}\Big\}. \end{aligned}\tag{134}$$

Note that, for all $n \ge 0$,

$$\frac{2^{n+m+d} l_n}{a_n \Delta_{n+m+d}} = \frac{(n+1)^2 2^d \mathrm{e}^{\beta d}}{\mathrm{e}^{(\delta-\beta)n} x^{1/4}} \le (n+1)^2 \mathrm{e}^{(\beta-\delta)n} \longrightarrow 0, \text{ as } n \uparrow \infty, \tag{135}$$

using that $2^d \mathrm{e}^{\beta d} \le x^{1/4}$ by (106). Hence, by Cramér's Theorem, see, for instance, [14, Theorem 2.2.3], the right hand side in (134) is bounded above by $C_2 \exp(-C_3 a_n)$, for suitable constants $C_2, C_3 > 0$. ☐

Completion of the proof of Lemma 26. It is now time to choose the value of m large enough such that $m \ge N(\Gamma)$, (117) holds and the following set of conditions is satisfied, for $\varepsilon = 1/x^{3/2}$:

$$\sum_{n=0}^{\infty} 2^{n+2} \exp\big(-\exp\sqrt{n+m}\,\big) \;\le\; \frac{1}{3}, \tag{136a}$$

$$\sum_{n=0}^{\infty} C_0 \exp\Big(-C_1\, a_n^{\theta}\Big) \;\le\; \frac{\varepsilon}{6}, \tag{136b}$$

$$\sum_{n=0}^{\infty} \frac{1}{m\,(n+1)^2} \;\le\; \frac{1}{3}. \tag{136c}$$

Note that $a_n = a_n(x)$ defined in (118) is a multiple of $x^{7/4}$ and hence m can be chosen independently of x. We have already seen in (116) that (136a) implies $\mathbf{H_1}\big(m+d, \varepsilon/3\big)$. Moreover, (136b) together with Lemma 28 implies Hypothesis $\mathbf{H_2}(m+d, l_., \varepsilon/6)$ and, finally (136c) is (93). Hence Lemma 26 follows from Theorem 22.

Completion of the proof of the crossing property, Theorem 5. We use the result of Lemma 26 to see that, for all Γ and m as in the lemma, for sufficiently large integers x,

$$\begin{aligned}\mathbb{P}^{\Gamma}_{2^{-m}\ell_{[x,x+1]}}\Big\{\int_0^{\infty} X_t(-\infty,0]\,dt = 0\Big\} &= \exp\Big(\int_x^{x+1} \log \mathbb{P}^{\Gamma}_{2^{-m}\delta_y}\Big\{\int_0^{\infty} X_t(-\infty,0]\,dt = 0\Big\}\,dy\Big)\\ &\ge \exp\Big(\int_x^{x+1} \log\big(1-y^{-3/2}\big)\,dy\Big) \ge 1 - \frac{1}{x^{3/2}}. \end{aligned}\tag{137}$$

Hence $\mathbb{P}^{\Gamma}_{2^{-m}\ell_{[x,x+1]}}\{\int_0^\infty X_t(-\infty,0]\,dt > 0\}$ is summable over all positive integers x. Therefore, by the Borel-Cantelli Lemma and path continuity only for finitely many positive integers x the process X with $X_0 = 2^{-m}\ell_{[x,x+1]}$ will ever attach mass to $(-\infty,0]$. By the branching property, repeating this argument 2^m times, the same is true for the processes X with $X_0 = \ell_{[x,x+1]}$. Thus it suffices to show

$$\sup_{t\geq 0} X_t(-\infty,0] < \infty, \quad \text{for } X_0 = \ell_{[0,K]}\,, \tag{138}$$

for all numbers $K>0$. Because the process X with finite starting measure with compact support has the finite time extinction property, by [11, Theorem 6], there is a random time $t_0 > 0$ such that $X_t = 0$ for all $t \geq t_0$. Finally, by path continuity, $X_t(-\infty,0]$ is bounded on $[0,t_0]$ and we are done.

3.4. **The compound Poisson property.** In this section we prove Theorem 1 (ii). Recall that we need only verify the finiteness property (28). We use the abstract bound of Theorem 22 for a deterministic time $T>0$ to obtain an upper bound for the survival probability of X_T in terms of $N(\Gamma)$ only. This leads to Corollary 31, which constitutes the main step in the proof. It allows to obtain (28) from the finiteness of an integral involving only the stable medium. This finiteness is easily obtained using the large gaps lemma, Lemma 19.

Corollary 31 (Upper bound on survival probability)**.** *For every time $t>0$ there is a constant $\theta=\theta(t)\in(0,1)$ such that for P–almost all Γ,*

$$\mathbb{P}^{\Gamma}_{\ell_{[0,1]}}\big\{X_t \neq 0\big\} \;\leq\; 1-\theta^{(2^{N(\Gamma)})}. \tag{139}$$

To prove Corollary 31 by application of Theorem 22 we proceed similarly as in the proof of finite time extinction carried out in [11]. However, we carefully keep track of the dependence on the medium Γ in terms of $N(\Gamma)$.

We fix t, leave the integer parameter m open for a while and define deterministic times T_n and thresholds l_n in terms of m. We first let

$$\varepsilon(m) := \Big(\frac{2}{t}\sum_{n=m}^{\infty} \mathrm{e}^{(\alpha-\beta)n}\Big)^{1/3}. \tag{140}$$

We then define $m_n := \lfloor \mathrm{e}^{\alpha(n+m)}/\varepsilon(m)\rfloor$ and $s_n := \mathrm{e}^{-\beta(n+m)}/\varepsilon(m)^2$. Put $I=[0,1]$,

$$T_0 := 0 \;\text{ and }\; T_{n+1} := T_n + 2m_n s_n\,. \tag{141}$$

Note that $t \geq T := \lim_{n\uparrow\infty} T_n$. Finally define

$$\bar{l}_n := m_n\sqrt{s_n}\,2^{-n}. \tag{142}$$

We later define l_n to be a constant multiple of $\bar{l}_n$. Corollary 31 follows from Theorem 22 if we verify (93) and the Hypotheses $\mathbf{H_1}(m+d,\varepsilon/3)$ and $\mathbf{H_2}(m+d,l_\cdot,\varepsilon/6)$, for $d=0$, $\varepsilon=1/2$ and a suitable integer m, which we choose at the end of the proof. Indeed, define $\theta := (1/2)^{(2^m)} > 0$. By the branching property and Theorem 22 we obtain, for $M = m+N(\Gamma)$,

$$\mathbb{P}^{\Gamma}_{\ell_{[0,1]}}\big\{X_t=0\big\} \geq \Big(\tilde{\mathbb{P}}^{\Gamma}_{\ell_{[0,1]}2^{-M}}\big\{Y_t=0\big\}\Big)^{(2^M)} \geq (1/2)^{(2^M)} = \theta^{(2^{N(\Gamma)})}, \tag{143}$$

which gives the statement of Corollary 31.

To prepare the verification of the Hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$ and hence the estimates for the bad paths, we formulate three lemmas. The constants $C_0,\dots,C_3$ in these lemmas depend only on the fixed values of α and β. The first lemma is the main ingredient in the verification of Hypothesis $\mathbf{H_1}$.

Lemma 32 (Escape probability II). *With the T_n from (141), there is a constant $C_0 > 0$ such that, for all starting points $x \in I = [0,1]$ and all integers $n, m \geq 0$, the events*

$$\overline{D_0}(n) := \Big\{ \sup_{s \in [T_n, T_{n+1}]} |W_s| > k(n+m) \Big\} \tag{144}$$

have probability

$$\mathcal{P}_{0,x}\big(\overline{D_0}(n)\big) \leq C_0 \exp\big(-\exp\sqrt{n+m}\,\big). \tag{145}$$

Proof. Recall that the random variable $\sup_{0 \leq s \leq t} |W_s|$ has finite first moment. Hence, using Markov's inequality, there is a constant $C_0 > 0$, depending only on t, such that

$$\mathcal{P}_{0,x}\Big\{ \sup_{T_n \leq s \leq T_{n+1}} |W_s| > k(n+m) \Big\} \leq \mathcal{P}_{0,x}\Big\{ \sup_{0 \leq s \leq t} |W_s| > k(n+m) \Big\} \leq C_0 \exp(-\exp\sqrt{n+m}\,)$$

for all $x \in [0,1]$ and $n, m \geq 0$. $\square$

We conclude that $\mathbf{H_1}(m+d, \varepsilon/3)$ holds if m is chosen such that

$$\sum_{n=0}^{\infty} C_0\, 2^{n+1} \exp\big(-\exp\sqrt{n+m}\,\big) \leq \frac{1}{6}. \tag{146}$$

Again further restrictions on m follow from the verification of $\mathbf{H_2}$. For this purpose let Θ be an $(n+m)$–perfect medium and let $\pi_{n+m} := \pi_{n+m}[\Theta]$.

Note that the neighbouring pairs of atoms in π_{n+m} are no further than Δ_{n+m} apart. On the interval $[T_n, T_{n+1}]$ we consider only the collisions with the atoms of π_{n+m}. In fact, we can even restrict our view to a selection of collisions chosen according to a special strategy of [11], which is based on our choice of the sequences $s_n > 0$ of small times and m_n of positive integers. Heuristically, on the interval $[T_n, T_{n+1}]$ the *strategy* suggests to wait until the Brownian particle hits the first atom of π_{n+m}, count the collision local time with this particular atom for s_n time units and then wait for the next collision with π_{n+m}. This procedure is iterated for m_n periods. A visualisation of this procedure can be found in [18, Figure 5]. If the path W is bad on $[T_n, T_{n+1}]$ either the m_nth period does not finish before time T_{n+1} or the accumulated collision local time during m_n periods is below the threshold l_n.

Formally, define a sequence $\{\kappa_n\}$ of Brownian stopping times by $\kappa_0 := T_n$ and, for $j \geq 1$,

$$\kappa_j := \bar\kappa_j + s_n\,, \text{ where } \bar\kappa_j := \inf\big\{s \geq \kappa_{j-1} : W_s \text{ hits } \pi_{n+m}\big\}, \tag{147}$$

and denote waiting times by $H_m := \bar\kappa_m - \kappa_{m-1}$. Define the events

$$\overline{D_1}(n) := \Big\{ \sum_{j=1}^{m_n} H_j \geq m_n s_n \Big\} \tag{148}$$

and

$$\overline{D_2}(n) := \Big\{ \int_{T_n}^{\kappa_{m_n}} L_{[W,\Gamma]}(dr) < l_n \Big\}. \tag{149}$$

Clearly, we have the decomposition

$$B(n) \subseteq \overline{D_1}(n) \cup \overline{D_2}(n). \tag{150}$$

We now formulate estimates for the probabilities for these two events, the second one also providing the final definition of the threshold values l_n.

Lemma 33 (Probability of Event $\overline{D_1}(n)$)**.** *There is a constant $C_1 > 0$ such that, for all starting points $x \in \mathbb{R}$ at time T_n, and integers $n \geq 0$,*

$$\mathcal{P}_{T_n,x}\left(\overline{D_1}(n)\right) \leq C_1^{-1} m_n \exp\Big(- \frac{C_1 s_n}{\Delta_{n+m}^2}\Big). \tag{151}$$

Proof. This is estimate (93) in [11]. The proof is based on an explicit eigenfunction representation for the distribution of the exit times of Brownian motion from a bounded interval. □

Lemma 34 (Probability of Event $\overline{D_2}(n)$)**.** *There are constants $C_2 > 0$ and $C_3 > 0$ with the property that, with $l_n = C_2 \bar{l}_n$, for all starting points $x \in \mathbb{R}$ at time T_n, and integers $n \geq 0$,*

$$\mathcal{P}_{T_n,x}\big(\overline{D_2}(n)\big) \leq \exp\big(- C_3 m_n\big). \tag{152}$$

Proof. This is estimate (99) in [11]. The idea here is to estimate $\int_{T_n}^{\kappa_{m_n}} L_{[W,\Gamma]}(dr)$ from below by the scaled sum of the local times of W during different time windows and at the levels given by the atoms in the strategy described above. These local times are independent and the desired estimate can be derived using large-deviation theory. □

Completion of the proof of Corollary 31. Having provided the estimates for the probability of a path being bad, it is now time to make precise the value of m. We choose m large enough such that the following set of conditions is satisfied

$$\sum_{n=0}^{\infty} C_0\, 2^{n+1} \exp\big(-\exp\sqrt{n+m}\,\big) \leq \frac{1}{6}, \tag{153a}$$

$$\sum_{n=m}^{\infty} \frac{1}{C_1} \frac{e^{\alpha n}}{\varepsilon(m)} \exp\Big(- C_1 \frac{e^{\beta n}}{\varepsilon(m)^2}\Big) \leq \frac{1}{12}, \tag{153b}$$

$$\sum_{n=m}^{\infty} \exp\Big(- C_3 \Big\lfloor \frac{e^{\alpha n}}{\varepsilon(m)} \Big\rfloor\Big) \leq \frac{1}{12}, \tag{153c}$$

$$\frac{1}{C_2}\, 2^{-m}\, \varepsilon(m) \sum_{n=0}^{\infty} \frac{e^{\beta/2(n+m)}}{\lfloor e^{\alpha(n+m)}/\varepsilon(m)\rfloor} \leq \frac{1}{6}. \tag{153d}$$

Note that it is possible to find such an m: For (153a) this is trivial, for (153c) this is due to the fact that $\varepsilon(m) \downarrow 0$ and for (153d) note that $\alpha > \beta/2$. For (153b) it suffices to check that, for $a, b \geq 1$, the function $x \mapsto (a/x)\exp(-b/x^2)$ is increasing on the interval $(0,1)$. Hence Hypothesis $\mathbf{H_1}(m, 1/6)$ holds by (153a), see (146), and $\mathbf{H_2}(m, l_\cdot, 1/12)$ holds by (153b), (153c) together with Lemmas 33 and 34 and (150). Finally, (153d) is (93). This finishes the proof.

Completion of the proof of the compound Poisson property, Theorem 1 (ii). Fixing $t > 0$ and a starting measure $\ell_{(a,b)}$, for (a,b) an interval of unit length, we have to show that the measure

$$\pi^\infty(dw\, dx) = \int_a^b \int_{\mathcal{M}_{\text{tem}}} \mathbb{N}^{\Upsilon}_{(0,0)}(dw) \otimes \delta_y(dx)\, \mathsf{P}(d\Gamma)\, dy \tag{154}$$

is finite on the set

$$S = \Big\{(w,x) \in \mathfrak{W} \times (a,b) : L_\sigma^t[w] > 0\Big\}. \tag{155}$$

Then the snake representation Theorem 6 (ii) of the limit process describes a compound Poisson point field on (a,b) with underlying Poisson intensity $\lambda(t) := \pi^\infty(S)$. To prove finiteness of $\lambda(t)$ we have to show that the following expression is finite

$$\lambda(t) = \mathsf{E}\Big\{\mathbb{N}^{\Upsilon}_{(0,0)}\{w : L^t_\sigma[w] > 0\}\Big\} = \mathsf{E}\Big\{\int_0^1 \mathbb{N}^{\Upsilon}_{(0,x)}\{w : L^t_\sigma[w] > 0\}\, dx\Big\}, \tag{156}$$

where we have used the fact that the law of $\mathbb{N}^{\Upsilon}_{(0,x)}\{w : L^t_\sigma[w] > 0\}$ under P is independent of x. To interpret the integrand on the right hand side of (156) recall the snake representation in Theorem 6 (i). The process X started in $X_0 = \ell_{[0,1]}$ has become extinct at time t if and only if a Poisson point field with intensity $\int_0^1 \mathbb{N}^{\Gamma}_{(0,x)}\,\{w : L^t[w] > 0\}\, dx$ has the value 0, the probability of this event is

$$\exp\Big(-\int_0^1 \mathbb{N}^{\Gamma}_{(0,x)}\{w : L^t_\sigma > 0\}\, dx\Big). \tag{157}$$

This reduces our task to showing that

$$\mathsf{E}\Big\{-\log \mathbb{P}^{\Gamma}_{\ell_{[0,1]}}\{X_t = 0\}\Big\} < \infty. \tag{158}$$

We now apply Fubini's Theorem to rewrite

$$\begin{aligned} \mathsf{E}\Big\{-\log \mathbb{P}^{\Gamma}_{\ell_{[0,1]}}\{X_t = 0\}\Big\} &= \int_0^\infty \mathsf{P}\Big\{-\log \mathbb{P}^{\Gamma}_{\ell_{[0,1]}}\{X_t = 0\} > a\Big\}\, da \\ &= \int_0^\infty \mathsf{P}\Big\{\mathbb{P}^{\Gamma}_{\ell_{[0,1]}}\{X_t = 0\} < \mathrm{e}^{-a}\Big\}\, da. \end{aligned} \tag{159}$$

Hence our problem can be formulated as

$$\int_0^\infty \mathsf{P}\Big\{\mathbb{P}^{\Gamma}_{\ell_{[0,1]}}\{X_t = 0\} < \mathrm{e}^{-a}\Big\}\, da < \infty. \tag{160}$$

Here comes the *key idea* of our proof: With respect to the random medium Γ the event

$$\Big\{\mathbb{P}^{\Gamma}_{\ell_{[0,1]}}\{X_t = 0\} < \mathrm{e}^{-a}\Big\} \tag{161}$$

can only occur if Γ has unusually low density, or equivalently, if the points in the Poisson point fields π_n introduced before (78) are unusually far apart. This can be expressed in terms of the quantity $N(\Gamma)$. In fact, by Corollary 31,

$$\mathbb{P}^{\Gamma}_{\ell_{[0,1]}}\{X_t = 0\} \;\geq\; \exp\Big((\log\theta)2^{N(\Gamma)}\Big), \tag{162}$$

and hence, the latter event implies

$$(\log\theta)2^{N(\Gamma)} < -a, \;\text{ which implies }\; N(\Gamma) > \frac{1}{\log 2}\,\log\big(-a/\log\theta\big). \tag{163}$$

We can now use the estimate (79) obtained in the large gaps lemma, Lemma 19, for the quantity $N(\Gamma)$,

$$\begin{aligned}
(164) \qquad & \int_0^\infty \mathsf{P}\Big\{\mathbb{P}^\Gamma_{\ell_{[0,1]}}\{X_t = 0\} < \mathrm{e}^{-a}\Big\}\, da \\
& \leq \int_0^\infty \mathsf{P}\Big\{N(\Gamma) > \frac{1}{\log 2}\, \log\big(-a/\log\theta\big)\Big\}\, da \\
& \leq c \int_0^\infty \exp\Big(- d\exp\Big[(\varrho/\log 2)\log\big(-a/\log\theta\big)\Big]\Big)\, da \\
& \leq c_0 + c_1 \int_0^\infty \exp\big(- c_2 a^{c_3}\big)\, da < \infty,
\end{aligned}$$

using suitable constants $c_0, c_1, c_2, c_3 > 0$. This proves (160), and Theorem 1 (ii) is established.

4. Properties of the macroscopic clumps

In this section we prove the various parts of Theorem 11. Part (i) and, perhaps surprisingly, part (ii) can be obtained by soft arguments, whereas part (iii) requires a new approach based on a Feynman-Kac formula for the solutions of (6).

4.1. The extinction probability of the clumps.

In this subsection we prove Theorem 11 (i) and (ii). From the definition of the renormalised processes (9) we infer, for all $k, l > 0$ and $t \geq 0$,

$$(165) \qquad X_t^{kl}(B) = k^{-\eta} X^l_{(kt)}(k^\eta B), \qquad \text{for } B \subseteq \mathbb{R} \text{ Borel}.$$

Choose any continuity set $B \subseteq \mathbb{R}$, that is, any Borel set with $\ell(\partial B) = 0$. Letting $l \uparrow \infty$ we obtain the self-similarity property

$$(166) \qquad X_t^\infty(B) = k^{-\eta} X_{kt}^\infty(k^\eta B) \quad \text{in distribution},$$

first for all continuity sets $B \subseteq \mathbb{R}$ and then, by approximation, for all Borel sets $B \subseteq \mathbb{R}$. This proves Theorem 11 (i) and is also the key to part (ii). By the compound Poisson structure of X_t^∞ the Laplace functionals have the form

$$(167) \qquad \mathbf{E}_\ell\Big\{\exp\big(-\theta X_t^\infty(0,a)\big)\Big\} = \exp\Big(- \lambda(t)\, a\big(1 - \Lambda^t(\theta)\big)\Big), \text{ for } a > 0,\ \theta \geq 0,$$

where $\lambda(t)$ is the intensity of the Poisson point field underlying the compound Poisson point field and Λ^t is the Laplace functional of the weights of an atom. Using (166) one obtains

$$\begin{aligned}
(168) \quad \exp\Big(- \lambda(t)\, a\big(1 - \Lambda^t(\theta)\big)\Big) &= \mathbf{E}_\ell\Big\{\exp\big(-\theta X_t^\infty(0,a)\big)\Big\} = \mathbf{E}_\ell\Big\{\exp\big(-\theta k^{-\eta} X_{kt}^\infty(0, k^\eta a)\big)\Big\} \\
&= \exp\Big(- \lambda(kt) k^\eta a\big(1 - \Lambda^{kt}(\theta k^{-\eta})\big)\Big).
\end{aligned}$$

Letting $\theta \uparrow \infty$, we infer that $\lambda(t) = k^\eta \lambda(kt)$, hence $\Lambda^t(\theta) = \Lambda^{kt}(\theta k^{-\eta})$. The former expression gives us the decay of the intensity $\lambda(t) = t^{-\eta}\lambda(1)$ of the Poisson point field, the latter yields the equality in distribution of $\gimel_t(t)$ and $(t/s)^\eta \gimel_s(s)$. Using $\lambda(s)\mathbf{P}_\ell\big\{\gimel_s(t) > 0\big\} = \lambda(t)$ for $t > s$, we infer that the survival probabilities of the clumps satisfy

$$(169) \qquad \mathbf{P}_\ell\big\{\gimel_s(t) > 0\big\} = \Big(\frac{s}{t}\Big)^\eta \text{ and } \mathbf{P}_\ell\big\{X_t^\infty(0,a) > 0\big\} \sim \frac{\lambda(1)\, a}{t^\eta}, \text{ as } a \uparrow \infty,$$

where the latter form is obtained by conditioning on the number of clumps in an interval.

4.2. **The tail behaviour of the clump size.** This subsection is devoted to the proof of Theorem 11 (iii). We first note that it suffices to give the proof for a fixed value of t, because the particular dependence on t, which is claimed in Theorem 11 (iii), already follows from the self-similarity of the process $\{\beth_t(t) : t > 0\}$ proved in Subsection 4.1.

Recall (12) for the case of a constant function $\varphi = \theta > 0$. This equation can also be written probabilistically as

$$U_t\theta\,(y) = \theta - 2\,\mathcal{E}_y\Big\{\int_0^t \big[U_{t-s}\theta\,(W_s)\big]^2\; L_{[\Gamma,W]}(ds)\Big\}, \tag{170}$$

where $\mathcal{E}_{s,y}$ is used to indicate expectation with respect to a Brownian motion W started at time s in y, $\mathcal{E}_y := \mathcal{E}_{0,y}$, and $L_{[\Gamma,W]}$ is the collision local time between Γ and W, as defined in (15). We use the Feynman-Kac representation of the solutions $U\theta := U^{\Gamma}\theta$ of (170) in order to obtain the tail asymptotics of the mass clumps.

Lemma 35 (Feynman-Kac-representation)**.** *For each fixed Γ the family $U = \{U_t\theta\,(y) : t \geq 0,\ y \in \mathbb{R}\}$ is a solution of* (170) *if and only if it is a solution of*

$$U_t\theta\,(y) = \theta\,\mathcal{E}_y\Big\{\exp\Big(-2\int_0^t U_{t-s}\theta\,(W_s)\,L_{[\Gamma,W]}(ds)\Big)\Big\}, \quad \textit{for } t \geq 0, \quad y \in \mathbb{R}. \tag{171}$$

Proof. Fix Γ. For both parts of the proof it suffices to consider $\theta = 1$, as a simple scaling argument shows that $U^{\Gamma}\theta$ is a solution of (170) respectively (171) for arbitrary $\theta > 0$ in the medium Γ if and only if

$$V = \{V_t(y) : t \geq 0,\ y \in \mathbb{R}\} \ \text{ with } V_t(y) := \frac{1}{\theta}\,U_{\theta^{-1/\eta}t}\theta\,(\theta^{-1/2\eta}y) \tag{172}$$

is a solution of (170) respectively (171) for $\theta = 1$ in the rescaled medium $\tilde{\Gamma}$ given by

$$\tilde{\Gamma}(A) \;:=\; \theta^{\frac{1}{\gamma+1}}\,\Gamma\big(\theta^{\frac{-\gamma}{\gamma+1}}A\big), \ \text{for } A \subseteq \mathbb{R} \text{ Borel}. \tag{173}$$

We first show that every solution of (171) solves (170). Let $U = \{U_t(y) : t \geq 0,\ y \in \mathbb{R}\}$ be a bounded solution of equation (171), which is continuous in t. In the sense of distributions we calculate the derivative

$$\begin{aligned}\frac{d}{ds}\exp\Big[-2\int_s^t U_{t-r}(W_r)\,dL_{[\Gamma,W]}(r)\Big]&\\ = 2U_{t-s}(W_s)\exp\Big[-2\int_s^t U_{t-r}(W_r)\,dL_{[\Gamma,W]}(r)\Big]&\,L_{[\Gamma,W]}(ds).\end{aligned} \tag{174}$$

Integrating this over $[0,t]$ yields

$$\begin{aligned}1-\exp\Big[-2\int_0^t U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big]&\\ = \;2\int_0^t U_{t-s}(W_s)\exp\Big[-2\int_s^t U_{t-r}(W_r)\,dL_{[\Gamma,W]}(r)\Big]&\,L_{[\Gamma,W]}(ds).\end{aligned} \tag{175}$$

Taking expectations,

$$U_t(y) - 1 \;=\; -2\,\mathcal{E}_y\Big\{\int_0^t U_{t-s}(W_s)\exp\Big[-2\int_s^t U_{t-r}(W_r)\,dL_{[\Gamma,W]}(r)\Big]\,L_{[\Gamma,W]}(ds)\Big\}. \tag{176}$$

The Markov property (and a glance at the definition of Stieltjes integrals) allows us to continue this with

$$
\begin{aligned}
(177)\qquad &= -2\,\mathcal{E}_y\Big\{\int_0^t U_{t-s}(W_s)\,\mathcal{E}_{W_s}\Big\{\exp\Big(-2\int_0^{t-s} U_{t-s-r}(\tilde{W}_r)\,dL_{[\tilde{W},\Gamma]}(r)\Big)\Big\}\,L_{[\Gamma,W]}(ds)\Big\}\\
&= -2\,\mathcal{E}_y\Big\{\int_0^t \{U_{t-s}(W_s)\}^2\,L_{[\Gamma,W]}(ds)\Big\} = -2\int_{\mathbb{R}}\Gamma(dx)\int_0^t p_s(y-x)\,\big[U_{t-s}(x)\big]^2\,ds,
\end{aligned}
$$

where $\tilde{W}$ is a Brownian motion started in W_s. This is the formula we had to prove.

To show conversely that every solution $U_t(y)$ of (170) solves (171), we start with the formula

$$
\begin{aligned}
(178)\qquad & 2\int_0^t \exp\Big(-2\int_0^s U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)\Big[U_{t-s}(W_s)-1\Big]U_{t-s}(W_s)\,L_{[\Gamma,W]}(ds)\\
&= -\int_0^t \big[U_{t-s}(W_s)-1\big]\;d_s\Big(\exp\Big(-2\int_0^s U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)\Big).
\end{aligned}
$$

We take the expectation, use (170), apply the Markov property as before, and finally use Fubini's Theorem to see

$$
\begin{aligned}
&\mathcal{E}_y\Big\{2\int_0^t \exp\Big(-2\int_0^s U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)\,\big[U_{t-s}(W_s)\big]^2\,L_{[\Gamma,W]}(ds)\Big\}\\
&\qquad -2\,\mathcal{E}_y\Big\{\int_0^t \exp\Big(-2\int_0^s U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)U_{t-s}(W_s)\,L_{[\Gamma,W]}(ds)\Big\}\\
&= -\mathcal{E}_y\Big\{\int_0^t \mathcal{E}_{s,W_s}\Big\{-2\int_s^t \Big[U_{t-v}(\tilde{W}_v)\Big]^2 L_{[\Gamma,\tilde{W}]}(dv)\Big\}d_s\Big(\exp\Big(-2\int_0^s U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)\Big)\Big\}\\
&= \mathcal{E}_y\Big\{2\int_0^t\int_s^t \big[U_{t-v}(W_v)\big]^2 L_{[\Gamma,W]}(dv)d_s\Big(\exp\Big(-2\int_0^s U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)\Big)\Big\}\\
&= \mathcal{E}_y\Big\{2\int_0^t\int_0^v d_s\Big(\exp\Big(-2\int_0^s U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)\Big)\,\big[U_{t-v}(W_v)\big]^2 L_{[\Gamma,W]}(dv)\Big\}\\
&= \mathcal{E}_y\Big\{2\int_0^t \Big[\exp\Big(-2\int_0^v U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)-1\Big]\,\big[U_{t-v}(W_v)\big]^2 L_{[\Gamma,W]}(dv)\Big\}.
\end{aligned}
$$

From this and (170) we infer that

$$
\begin{aligned}
(179)\qquad & \mathcal{E}_y\Big\{2\int_0^t \exp\Big(-2\int_0^s U_{t-v}(W_v)\,L_{[\Gamma,W]}(dv)\Big)U_{t-s}(W_s)\,L_{[\Gamma,W]}(ds)\Big\}\\
&= \mathcal{E}_y\Big\{2\int_0^t \big[U_{t-r}(W_r)\big]^2 L_{[\Gamma,W]}(dr)\Big\} = 1-U_t(y).
\end{aligned}
$$

Writing the integrand on the left hand side of (179) as a distributional derivative, as before, we see that the left hand side of (179) equals

$$
(180)\qquad 1+\mathcal{E}_y\Big\{-\exp\Big(-2\int_0^t U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)\Big\},
$$

from which (171) follows. □

We now aim for upper and lower bounds of $\mathsf{E}U_t\theta\,(0)$, which give the tail asymptotic of the clump sizes by means of a Tauberian theorem. From (170) one immediately sees that $U_t\theta\,(y)\le\theta$ and hence $\mathsf{E}U_t\theta\,(0)\le\theta$. Now plugging $U_t\theta\,(y)\le\theta$ into (170) leads only to a trivial lower bound for $\mathsf{E}U_t\theta\,(y)$. A better lower bound is obtained by means of the Feynman-Kac representation (171):

Lemma 36 (Asymptotic behaviour of $\mathsf{E}U_t\theta$)**.** *For every $t>0$ there are positive, finite constants $C_1 = C_1(t)$ and $C_2 = C_2(t)$, such that*

$$\theta - C_1(t)\,\theta^{\gamma+1} \le \mathsf{E}U_t\theta\,(0) \le \theta - C_2(t)\,\theta^{\gamma+1}, \quad \textit{for all } \theta \in (0,1). \tag{181}$$

Proof. Fix $t>0$ and let $0<\theta<1$. Plugging $U_t\theta\,(y) \le \theta$ into the Feynman-Kac-representation (171) yields,

$$U_t\theta\,(y) \ge \theta\,\mathcal{E}_y\Big\{\exp\big(-2\theta L_{[\Gamma,W]}(t)\big)\Big\}, \ \text{ for all } y \in \mathbb{R}. \tag{182}$$

Taking expectation with respect to the medium, using the Laplace functional formula (3) for stable random measures, and the definition (15) of collision local time, gives

$$\begin{aligned}\mathsf{E}U_t\theta\,(0) \;&\ge\; \theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\big(-2\,\theta L_{[\Gamma,W]}(t)\big)\Big\} &&=\; \theta\,\mathcal{E}_0\exp\Big(-2^\gamma\theta^\gamma\int_{\mathbb{R}}\left(L^x(t)\right)^\gamma\,dx\Big)\\ &\ge\; \theta\Big(1-2^\gamma\theta^\gamma\,\mathcal{E}_0\int_{\mathbb{R}}\left(L^x(t)\right)^\gamma\,dx\Big) &&=\; \theta-\theta^{\gamma+1}2^\gamma\int_{\mathbb{R}}\mathcal{E}_0\left\{\left(L^x(t)\right)^\gamma\right\}\,dx.\end{aligned} \tag{183}$$

We now show that

$$C_1(t) \;:=\; 2^\gamma\int_{\mathbb{R}}\mathcal{E}_0\{\left(L^x(t)\right)^\gamma\}\,dx \;=\; t^{\gamma\eta}\,\frac{\gamma}{\gamma+1}\,\frac{2^{\gamma(\eta+1)}}{\sqrt{\pi}}\,G\Big(\frac{\gamma}{2}\Big) < \infty, \tag{184}$$

where G denotes the Gamma function (usually denoted by Γ, which would be in conflict with our catalyst notation). Indeed, for $x,y>0$ the density function of $L^x(t)$ at y with respect to $\mathcal{P}_0$ is given by $2p_t(x+y)$, see e.g. [4, 1.3.4]. Hence

$$\int_{\mathbb{R}}\mathcal{E}_0\{\left(L^x(t)\right)^\gamma\}\,dx \;=\; 2\int_0^\infty y^\gamma\int_y^\infty 2p_t(z)\,dz\,dy \;=\; \sqrt{\frac{8}{\pi t}}\int_0^\infty\frac{y^{\gamma+1}}{\gamma+1}\exp(-y^2/2t)\,dy, \tag{185}$$

using integration by parts. One can get the result by substituting $x=y^2/2t$ and recalling the definition of the Gamma function G.

We turn to the upper bound. We use (182) in (171) and obtain

$$U_t\theta\,(y) \le \theta\,\mathcal{E}_y\Big\{\exp\Big(-2\,\theta\int_0^t\mathcal{E}_{W_s}\Big\{\exp\big(-2\,\theta L_{[\tilde W,\Gamma]}(t-s)\big)\Big\}\,L_{[\Gamma,W]}(ds)\Big)\Big\} \tag{186}$$

for a Brownian motion $\tilde W$ started in W_s. By the definition of collision local time and Jensen's inequality,

$$\begin{aligned}&\mathsf{E}U_t\theta\,(0) \le \theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\int_0^t L_{[\Gamma,W]}(ds)\,\mathcal{E}_{W_s}\{\exp\big(-2\,\theta L_{[\tilde W,\Gamma]}(t-s)\big)\}\Big)\Big\}\\ &\le \theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\int_{\mathbb{R}}\Gamma(dx)\int_0^t L^x(ds)\,\mathcal{E}_x\{\exp\big(-2\,\theta\int_{\mathbb{R}}\Gamma(dy)\,\tilde L^y(t-s)\big)\}\Big)\Big\}\\ &\le \theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\int_{\mathbb{R}}\Gamma(dx)\int_0^t L^x(ds)\,\Big[\exp\big(-2\,\theta\int_{\mathbb{R}}\Gamma(dy)\,\mathcal{E}_x\{\tilde L^y(t-s)\}\big)\Big]\Big)\Big\}\end{aligned} \tag{187}$$

where $\tilde{L}$ refers to local time built with $\tilde{W}$. Using monotonicity we can continue the estimate with

$$\begin{aligned} &\theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\int_{\mathbb{R}}\Gamma(dx)\int_0^t L^x(ds)\Big[\exp\Big(-2\,\theta\int_{\mathbb{R}}\Gamma(dy)\,\mathcal{E}_x\{\tilde{L}^y(t-s)\}\Big)\Big]\Big)\Big\} \\ &\quad\le \theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\int_{\mathbb{R}}\Gamma(dx)\,L^x(t)\Big[\exp\Big(-2\,\theta\int_{\mathbb{R}}\Gamma(dy)\,\mathcal{E}_x\{\tilde{L}^y(t)\}\Big)\Big]\Big)\Big\} \\ &\quad\le \theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\int_{|x|\le 1}\Gamma(dx)\,L^x(t)\Big[\exp\Big(-2\,\theta\int_{\mathbb{R}}\Gamma(dy)\,\mathcal{E}_x\{\tilde{L}^y(t)\}\Big)\Big]\Big)\Big\}. \end{aligned} \tag{188}$$

Now we split the integral in the innermost exponential into the integral over $|y| \le 1$ and $|y| > 1$, respectively. Recall that

$$\mathcal{E}_x\big\{\tilde{L}^y(t)\big\} = q_t(x-y) \quad \text{for} \quad q_t(x) := \int_0^t p_s(x)\,ds. \tag{189}$$

For $|y| \le 1$ we use the estimate

$$\mathcal{E}_x\big\{\tilde{L}^y(t)\big\} \le c_0 := \sqrt{\frac{2t}{\pi}}\,, \tag{190}$$

which gives

$$\begin{aligned} &\theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\int_{|x|\le 1}\Gamma(dx)\,L^x(t)\Big[\exp\Big(-2\,\theta\int_{\mathbb{R}}\Gamma(dy)\,\mathcal{E}_x\{\tilde{L}^y(t)\}\Big)\Big]\Big)\Big\} \\ &\quad\le \theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\int_{|x|\le 1}\Gamma(dx)\,L^x(t)\exp(-2\,\theta c_0\Gamma_1)\exp(-2\,\theta\Gamma_2)\Big)\Big\} \end{aligned} \tag{191}$$

with

$$\Gamma_1 := \Gamma(-1,1) \qquad \text{and} \qquad \Gamma_2 := \int_{|y|>1}\Gamma(dy)\,q_t\big(|y|-1\big)\,. \tag{192}$$

Under P the random variables Γ_1 and Γ_2 are independent, almost surely finite, and stable of index γ. We infer that, for arbitrary fixed $c > 0$, the event

$$A := \Big\{\inf_{|x|\le 1} L^x(t) \ge c\Big\} \cap \Big\{\Gamma_2 \le \frac{1}{\theta}\Big\} \cap \Big\{\frac{1}{\theta} \le \Gamma_1 \le \frac{2}{\theta}\Big\} \tag{193}$$

has (annealed) probability

$$\begin{aligned} \mathsf{E}\,\mathcal{P}_0(A) &= \mathcal{P}_0\Big\{\inf_{|x|\le 1} L^x(t) \ge c\Big\}\,\mathsf{P}\Big\{\Gamma_2 \le \frac{1}{\theta}\Big\}\,\mathsf{P}\Big\{\frac{1}{\theta} \le \Gamma_1 \le \frac{2}{\theta}\Big\} \\ &\ge \mathcal{P}_0\Big\{\inf_{|x|\le 1} L^x(t) \ge c\Big\}\,\big(1 - c_1\theta^\gamma\big)\big(c_2\theta^\gamma\big) \ge c_3\theta^\gamma\,, \end{aligned} \tag{194}$$

for a suitable choice of constants $c_1, c_2, c_3 > 0$ where $c_3 = c_3(t)$, recall that $0 < \theta < 1$. On A we have

$$\exp\Big(-2\,\theta c\Gamma_1\exp\big(-2\,\theta c_0\Gamma_1\big)\exp\big(-2\,\theta\Gamma_2\big)\Big) \le \exp\Big(-2c\,\mathrm{e}^{-4c_0}\,\mathrm{e}^{-2}\Big) =: c_4 < 1\,. \tag{195}$$

We can thus continue the estimate (191) with

$$
\begin{aligned}
&\theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\int_{|x|\le 1}\Gamma(dx)\,L^x(t)\exp(-2\,\theta c_0\Gamma_1)\exp(-2\,\theta\Gamma_2)\Big)\Big\}\\
(196)\qquad &\le \theta-\theta\,\mathsf{E}\,\mathcal{E}_0\Big\{1_A\Big(1-\exp\Big[-2\,\theta c\Gamma_1\exp\big(-2\,\theta c_0\Gamma_1\big)\exp\big(-2\,\theta\Gamma_2\big)\Big]\Big)\Big\}\\
&\le \theta-\theta\,\mathsf{E}\,\mathcal{P}_0(A)\,(1-c_4)\;=\;\theta-C_2\theta^{\gamma+1},
\end{aligned}
$$

by (194) for a suitable choice of $C_2=C_2(t)>0$. This finishes the proof. $\square$

The bounds for $\mathsf{E}U_t\theta$ in Lemma 36 translate easily into bounds for the Laplace transform

$$
(197)\qquad \Lambda^t(\theta)=\mathbf{E}_\ell\big\{\exp\big(-\theta\,\daleth_t(t)\big)\big\},\ \text{ for }\ \theta\ge 0,
$$

of the mass of a clump alive at a macroscopic time t.

Lemma 37 (Asymptotic behaviour of Λ^t)**.** *For all* $t>0$*, as* $\theta\downarrow 0$*,*

$$
(198)\qquad 1-\frac{1}{\lambda(t)}\,\theta+\frac{C_2(t)}{\lambda(t)}\,\theta^{\gamma+1}\;\le\;\Lambda^t(\theta)\;=\;1-\frac{1}{\lambda(t)}\,\theta+\frac{C_1(t)}{\lambda(t)}\,\theta^{\gamma+1}.
$$

Proof. Using the Laplace transform of a general compound Poisson point field,

$$
(199)\qquad \mathsf{E}U_t\theta\,(0)=-\log\mathbf{E}_\ell\big\{-\theta X_t^\infty[0,1]\big\}=\lambda(t)\big(1-\Lambda^t(\theta)\big),
$$

hence the statement follows by applying Lemma 36 and the scaling relation of $\lambda(t)$. $\square$

Finally, to get the tail behaviour of the clump sizes we observe that Theorem 11 (iii) follows directly from the previous lemma together with the following version of the Tauberian Theorem of Bingham and Doney, see [2, Theorem 8.1.6] for the original statement. Here we can apply the definition of the relation $\approx$ occurring in Theorem 11 (iii) in a t–independent situation.

Lemma 38 (Tauberian Theorem)**.** *Suppose* ξ *is a nonnegative random variable defined on a probability space* $(\Omega,\mathcal{A},P)$ *with positive and finite mean* m *and Laplace transform* Λ*. Then*

$$
(200)\qquad \Lambda(\theta)-\big(1-m\theta\big)\;\approx\;\theta^{\gamma+1},\ \text{ as }\ \theta\downarrow 0,
$$

implies

$$
(201)\qquad P\{\xi>x\}\;\approx\;\frac{1}{x^{\gamma+1}}\,,\ \text{ as }\ x\uparrow\infty.
$$

Proof. We denote $F(x):=P\{\xi\le x\}$ and

$$
(202)\qquad h(x):=\frac{1}{m}\int_x^\infty\big(1-F(y)\big)\,dy\,.
$$

Then, using integration by parts twice and then plugging in the assumption,

$$
\begin{aligned}
(203)\qquad \int_0^\infty \mathrm{e}^{-\theta x}h(x)\,dx\;&=\;\frac{1}{\theta}-\frac{1}{\theta m}\int_0^\infty \mathrm{e}^{-\theta x}\big(1-F(x)\big)\,dx\\
&=\;\frac{1}{\theta}-\frac{1}{\theta m}\,\frac{1-\Lambda(\theta)}{\theta}\;\approx\;\theta^{\gamma-1}.
\end{aligned}
$$

We now apply the Tauberian Theorem of de Haan and Stadtmüller [2, Theorem 2.10.2] to infer

$$
(204)\qquad \int_0^x h(y)\,dy\;\approx\;x^{1-\gamma}\,,\ \text{ as }\ x\uparrow\infty.
$$

Next we use the O-version of the Monotone Density Theorem [2, Proposition 2.10.3] twice to conclude

$$\text{first } h(x) \approx x^{-\gamma} \text{ and then } 1 - F(x) \approx \frac{1}{x^{\gamma+1}}, \text{ as } x \uparrow \infty, \tag{205}$$

as claimed. $\square$

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