How fast are the particles of super-Brownian motion?

Peter Mörters[1]
Universität Kaiserslautern, Fachbereich Mathematik
67663 Kaiserslautern, Germany. (E-Mail: peter@mathematik.uni-kl.de)

Abstract. In this paper we investigate fast particles in the range and support of super-Brownian motion in the historical setting. In this setting each particle of super-Brownian motion alive at time t is represented by a path $w : [0,t] \to \mathbb{R}^d$ and the state of historical super-Brownian motion is a measure on the set of paths. Typical particles have Brownian paths, however in the uncountable collection of particles in the range of a super-Brownian motion there are some which at exceptional times move faster than Brownian motion. We determine the maximal speed of all particles during a given time period E, which turns out to be a function of the packing dimension of E. A path w in the support of historical super-Brownian motion at time t is called *a-fast* if $\limsup_{h\downarrow 0} |w(t) - w(t-h)|/\sqrt{h \log(1/h)} \geq a$. We calculate the Hausdorff dimension of the set of a-fast paths in the support and the range of historical super-Brownian motion. A valuable tool in the proofs is a uniform dimension formula for the Brownian snake, which reduces dimension problems in the space of stopped paths to dimension problems on the line.

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1 Introduction and statement of the main results

Super-Brownian motion is the high density limit of a system of branching Brownian motions with a critical, finite variance branching mechanism. It has been an object of intensive research in the last fifteen years, leading to interesting results about its fractal geometry and its potential theoretic properties. Recent deep results emphasize its importance as the natural scaling limit of many interacting particle systems, see for example [2], [24] and [15] for surveys.

In the setting of historical processes, introduced by Dawson and Perkins in [3], super-Brownian motion is a continuous Markov process $\{\mathfrak{X}_t\}$ whose value at time t is a finite measure on the space

$$\mathcal{W} := \Big\{ (w,\zeta) \,:\, w : [0,\infty) \to \mathbb{R}^d \text{ continuous with } w(s) = w(\zeta) \text{ for all } s \geq \zeta \geq 0 \Big\}$$

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of *stopped paths* w *with total lifetime* ζ. Sometimes we write short w for (w, ζ). The state $\mathfrak{X}_t$ of super-Brownian motion is a random measure on $\mathcal{W}$ supported by a set of paths with lifetime $\zeta = t$. Intuitively, each path in $\operatorname{supp} \mathfrak{X}_t$ represents the trail of a particle alive at time t, so that the model describes the individual movements of all the particles. The historical setting also contains the complete description of the genealogical structure of super-Brownian motion: Any two particles $w_1, w_2 \in \operatorname{supp} \mathfrak{X}_t$ are obtained from the branching of the particle $w : [0, \infty) \to \mathbb{R}^d$ at time $\zeta = \sup\{r \,:\, w_1 = w_2 \text{ on } [0, r]\}$ with $w(s) = w_1(s) = w_2(s)$ for all $s \le \zeta$ and $w(s) = w(\zeta)$ for all $s \ge \zeta$.

In a super-Brownian motion, typical particles move according to Brownian motion, which means that the posterior distribution of a path picked according to a (normalised) sample measure $\mathfrak{X}_t$ is the Wiener measure. As is well known, the maximal speed of a Brownian particle is given by Paul Lévy's uniform modulus of continuity, which in our setting states that for every $t > 0$, almost surely, for $\mathfrak{X}_t$-almost every path w,

$$\sup_{s>0} \limsup_{h \downarrow 0} \frac{|w(s) - w(s-h)|}{\sqrt{h \log(1/h)}} = \sqrt{2}\,. \tag{1}$$

Recall that the *range* of a (historical) super-Brownian motion

$$\operatorname{range} \mathfrak{X} = \bigcup_{\varepsilon > 0} \overline{\bigcup_{t > \varepsilon} \operatorname{supp} \mathfrak{X}_t}$$

is an uncountable collection of paths, so that one might wonder whether for exceptional paths in the range of a super-Brownian motion a higher speed is possible. This question is answered affirmatively by a remarkable result of Dawson and Perkins [3, 8.7], who show that special particles can move faster than a Brownian motion. More precisely, almost surely,

$$\sup_{s>0} \sup_{w \in \operatorname{range} \mathfrak{X}} \limsup_{h \downarrow 0} \frac{|w(s) - w(s-h)|}{\sqrt{h \log(1/h)}} = 2\,. \tag{2}$$

The question of the maximal speed of all particles at a particular instant in time can be treated similarly, Verzani [27] points out that, at any fixed time $t > 0$, almost surely on $\{\mathfrak{X}_t \neq 0\}$,

$$\sup_{w \in \operatorname{range} \mathfrak{X}} \limsup_{h \downarrow 0} \frac{|w(t) - w(t-h)|}{\sqrt{h \log(1/h)}} = \sqrt{2}. \tag{3}$$

For a deeper understanding of the values appearing at the right hand sides in formulas (1), (2) and (3) we consider a Borel set $E \subset (0, \infty)$ of times and ask for the maximum speed of all particles at all times in E. It turns out that the maximum speed of all particles at all times in E is related to the packing dimension, denoted $\dim_P(E)$, of the set $E \subset \mathbb{R}$.

The most natural definition of *packing dimension* is based on the family of s-dimensional packing measures. A packing of the set E is a collection of disjoint balls centred in E and s-dimensional packing measure can be defined in essentially the same way as the s-dimensional Hausdorff measure replacing minimal covers by maximal packings of E. The packing dimension of E is the critical parameter s where the packing measure switches from infinity to 0. Readers unfamiliar with the concept are recommended to consult [17, 5.10] for more precise statements.

Theorem 1.1 (Speed in a given time period) *Suppose that $\{\mathfrak{X}_t\}$ is a super-Brownian motion in dimension $d \geq 1$ and $E \subset (0,t)$ a Borel set. Then, almost surely on $\{\mathfrak{X}_t \neq 0\}$,*

$$\sup_{s\in E}\sup_{w\in \mathrm{range}\,\mathfrak{X}}\limsup_{h\downarrow 0}\frac{|w(s)-w(s-h)|}{\sqrt{h\log(1/h)}} = \sqrt{2+2\dim_P(E)}\,.$$

Remark 1.2.

a) Observe that the theorem is independent of the choice of a finite starting mass for the super-Brownian motion.

b) Theorem 1.1 contains (3), by choosing E as a singleton, and (2), by exhausting $(0,\infty)$.

Theorem 1.1 should be compared to recent related results of Khoshnevisan, Peres and Xiao [11] on the modulus of continuity of classical Brownian paths in a given time set. Continuing work of Khoshnevisan and Shi [12], they showed that for each analytic set $E \subset (0,1)$, almost surely,

$$\sup_{t\in E}\limsup_{h\downarrow 0}\frac{|B(t)-B(t-h)|}{\sqrt{h\log(1/h)}} = \sqrt{2\dim_P(E)}\,. \tag{4}$$

Our proof adapts their technique to the superprocess setting, see Section 2 for an outline of the proof and a refined statement (Proposition 2.5).

We now assume that super-Brownian motion is started with a unit mass at the origin so that $w(0)=0$ for all $w \in \mathrm{range}\,\mathfrak{X}$. The second result of this paper determines a *Hausdorff-dimension spectrum* for fast paths in the range and support of historical super-Brownian motion, which is analogous to the result

$$\dim\Big\{t\in[0,1]\,:\,\limsup_{h\downarrow 0}\frac{|B(t)-B(t-h)|}{\sqrt{h\log(1/h)}}\geq a\Big\} = 1-\frac{a^2}{2},\ \text{ for } a\in[0,\sqrt{2}], \tag{5}$$

of Orey and Taylor [19] about the fast times of classical Brownian motion. Here and throughout the paper dim stands for the Hausdorff dimension of a metric space. For the formulation of our results we equip the space $\mathcal{W}$ of stopped paths with the metric

$$d\Big((w_1,\zeta_1),(w_2,\zeta_2)\Big) = \|w_1-w_2\|_\infty + \sqrt{|\zeta_1-\zeta_2|}\,. \tag{6}$$

Note that, comparing with the choice of the metric in [15] and most other papers on the subject, I have introduced a square root in the second summand, in order to sharpen the statements. See Remark 1.4(b) for other metrics.

Of course, the space $\mathcal{W}$ of stopped paths is of infinite Hausdorff dimension, but the range of historical super-Brownian motion has dimension no more than 4. This is intuitively plausible because for historical super-Brownian motion all particles alive at time t are obtained by branching from just a finite number of paths at time $t-\varepsilon$.

We consider a closed set $E \subset (0,\infty)$ of times and denote the *range of E* by

$$\mathcal{R}(E) = \overline{\bigcup_{t\in E}\mathrm{supp}\,\mathfrak{X}_t}\,.$$

Intuitively this is the collection of particles with total lifetime in the set E. The result of Theorem 1.1 also gives that

$$\sup_{w\in\mathcal{R}(E)} \limsup_{h\downarrow 0} \frac{|w(\zeta)-w(\zeta-h)|}{\sqrt{h\log(1/h)}} = \sqrt{2+2\dim_P(E)}\,.$$

We can give bounds for the dimension of the set of a-fast paths in the range of E, which are sharp for all closed sets E, for which Hausdorff- and packing dimension agree.

Theorem 1.3 (Dimension spectrum) *Suppose that $E \subset (0,t)$ is a fixed closed set and $\{\mathfrak{X}_t\}$ is a historical super-Brownian motion in dimension $d \geq 2$. Then, almost surely on $\{\mathfrak{X}_t \neq 0\}$, for all $a \in [0, \sqrt{2+2\dim_P(E)}]$,*

$$2+2\dim(E)-a^2 \leq \dim\Big\{(w,\zeta)\in\mathcal{R}(E) \,:\, \limsup_{h\downarrow 0}\frac{|w(\zeta)-w(\zeta-h)|}{\sqrt{h\log(1/h)}} \geq a\Big\} \leq 2+2\dim_P(E)-a^2\,.$$

Moreover, if $\dim_P(E) < a^2/2-1$ the set of a-fast paths in $\mathcal{R}(E)$ is empty. In particular, almost surely for all $a \in [0,2]$,

$$\dim\Big\{(w,\zeta)\in \mathrm{range}\,\mathfrak{X} \,:\, \limsup_{h\downarrow 0}\frac{|w(\zeta)-w(\zeta-h)|}{\sqrt{h\log(1/h)}} \geq a\Big\} = 4-a^2\,, \tag{7}$$

and, for all $t>0$, almost surely on $\{\mathfrak{X}_t \neq 0\}$, for all $a \in [0,\sqrt{2}]$,

$$\dim\Big\{(w,\zeta)\in \mathrm{supp}\,\mathfrak{X}_t \,:\, \limsup_{h\downarrow 0}\frac{|w(\zeta)-w(\zeta-h)|}{\sqrt{h\log(1/h)}} \geq a\Big\} = 2-a^2\,. \tag{8}$$

Remark 1.4.

(a) In all the results of Theorem 1.3 one can get the same dimension spectra replacing the set of a-fast paths by

$$\Big\{(w,\zeta)\in\mathcal{R}(E) \,:\, \limsup_{h\downarrow 0}\frac{|w(\zeta)-w(\zeta-h)|}{\sqrt{h\log(1/h)}} = a\Big\}$$

of *strictly a-fast paths.*

(b) In $d \geq 4$ the results of Theorem 1.3 remain true if in the definition (6) of the metric d on $\mathcal{W}$ either the square root is removed from the second term or the term is completely removed. The results can also be stated in a non-historical setting by looking at *fast points* $w(\zeta) \in \mathbb{R}^d$, which are the endpoints of the fast paths (w,ζ). In $d \geq 4$ we obtain the same spectrum for the Hausdorff dimension of the fast points. The analogue of (8) even holds for the fast points in all dimensions $d \geq 2$. These variants of Theorem 1.3 follow by using the variants of Theorem 2.3, which are outlined in Remark 2.4 and can be proved with the same method.

In the following section we briefly review the construction of the historical super-Brownian motion by means of the Brownian snake and present a technique of reducing dimension problems in the space $\mathcal{W}$ to dimension problems on the line. We then review a recent technique for obtaining hitting estimates and dimension bounds for a class of random fractals on the line. Details of the proofs are carried out in Sections 3, 4, 5 and 6 and we finish the paper with some final remarks and a brief discussion of related work.

2 Outline and method of proof

2.1 Historical super-Brownian motion and a uniform dimension formula

In this subsection we briefly review the construction of historical super-Brownian motion by means of the Brownian snake. This construction is due to Le Gall, see for example [13], [14], [15]. The Brownian snake allows us to reduce dimension problems in path space to dimension problems on the line by means of a uniform dimension formula, see Theorem 2.3.

We restrict attention to super-Brownian motion started with a unit mass concentrated at the origin. With only a little more formal effort one can generalise the construction for super-Brownian motion with any finite starting mass. The Brownian snake is a continuous strong Markov process $\{W_s = (w_s, \zeta_s) : s \geq 0\}$ with values in the set $\mathcal{W}$ of stopped paths. Denote by $\Omega = \mathcal{C}([0,\infty), \mathcal{W})$ the space of continuous mappings from $[0,\infty)$ to $\mathcal{W}$ equipped with the Borel-σ-field $\mathcal{A}$ coming from the compact-open topology and by $W = \{W_s\}_{s\geq 0}$ the coordinate process on Ω. By $\mathbb{P}$ denote the law on $(\Omega, \mathcal{A})$ of the path-valued process associated with d-dimensional Brownian motion, or *Brownian snake*, starting in the constant path at the origin of lifetime 0. This law was introduced in [13], [14] and can be described as follows. First, the *lifetime process* $\{\zeta_s\}$ is a reflected Brownian motion. We denote the σ-field generated on Ω by the process $\{\zeta_s\}$ by $\mathcal{F}$. Second, the distribution of $\{w_s\}$ given $\mathcal{F}$ is that of an inhomogeneous Markov process whose transition kernel from $s > 0$ to $r > s$ is described by

(a) $w_r(u) = w_s(u)$ for all $u \leq m := \inf_{s\leq v\leq r} \zeta_v$.

(b) conditionally given m the process $\{w_r(m+t) - w_s(m) : 0 \leq t \leq \zeta_r - m\}$ is independent of w_s and distributed as a Brownian path started in the origin and stopped at time $\zeta_r - m$.

The intuitive picture is that $\{w_s\}$ grows like a Brownian motion in $\mathbb{R}^d$ when $\{\zeta_s\}$ is increasing and is erased, when $\{\zeta_s\}$ is decreasing (this is merely heuristical, as $\{\zeta_s\}$, of course, has neither points of increase, nor of decrease). We denote by $\{L_s^t : s \geq 0\}$ the continuous local time of $\{\zeta_s\}$ at level t, normalised to be a density of the occupation measure of $\{\zeta_s\}$, and by $\tau = \tau(\zeta) = \inf\{s \geq 0 : L_s^0 = 1\}$ the inverse local time at 1. Let $\tilde{W}_s = w_s(\zeta_s)$ be the endpoint of W_s. Then

$$X_t(A) = \int_0^\tau 1_A(\tilde{W}_s)\, L_\cdot^t(ds) \quad \text{, for } A \subseteq \mathbb{R}^d \text{ Borel,} \tag{9}$$

defines a continuous Markov process $\{X_t\}$ with values in the space $\mathcal{M}_F(\mathbb{R}^d)$ of finite measures on $\mathbb{R}^d$ with the weak topology. This process is a *super-Brownian motion started with a unit mass in the origin.* The corresponding *historical super-Brownian motion* can be defined by

$$\mathfrak{X}_t(A) = \int_0^\tau 1_A(W_s)\, L_\cdot^t(ds) \quad \text{, for } A \subseteq \mathcal{W} \text{ Borel.} \tag{10}$$

Observe that projection of paths onto their endpoints maps the process $\{\mathfrak{X}_t\}$ onto $\{X_t\}$. Conversely, in the historical setting we attach to each particle in $\operatorname{supp} X_t$ its path up to time $t > 0$. This richer model allows the formulation of more precise results than super-Brownian motion alone. Almost surely, for each $t > 0$, the support of $\mathfrak{X}_t$ agrees up to at most one path with $W\{s \in [0,\tau] : \zeta_s = t\}$. The exceptional paths arise, when $\zeta_s = t$ is a local maximum of ζ. As the isolated points of a fixed closed set $E \subset (0,\infty)$ are almost surely not local maxima of the reflected Brownian motion ζ, we obtain, almost surely,

$$\mathcal{R}(E) = W(\zeta^{-1}(E)) \text{ and, in particular, range}\, \mathfrak{X} = W[0,\tau]\,.$$

Observe at this point that

$$\sup_{t\in E}\ \sup_{w\in\mathrm{range}\,\mathfrak{X}}\ \limsup_{h\downarrow 0} \frac{|w(t)-w(t-h)|}{\sqrt{h\log(1/h)}} \quad = \quad \sup_{\substack{s\in[0,\tau]\cap\\ \zeta^{-1}(E)}}\ \limsup_{h\downarrow 0} \frac{|w_s(\zeta_s)-w_s(\zeta_s-h)|}{\sqrt{h\log(1/h)}}\,.$$

Define, for any $a\geq 0$, the set of *a-fast times* or simply *fast times* as

$$F(a) = \Big\{ s\in[0,\tau] \,:\, \limsup_{h\downarrow 0} \frac{|w_s(\zeta_s)-w_s(\zeta_s-h)|}{\sqrt{h\log(1/h)}} \geq a \Big\}\,,$$

and note that the fast times are not times for the super-Brownian motion, but for the Brownian snake. The statement of Theorem 1.1 is now reduced to the problem to identify those parameters a for which the set of a-fast times hits the preimage $\zeta^{-1}(E)$. To solve this, we prove a general result about hitting probabilities of the random set of a-fast times. Recall that $\mathcal{F}$ is the σ-field on Ω generated by the lifetime process $\{\zeta_s\}$.

Proposition 2.1 (Hitting Probabilities) *For the Brownian snake $W=\{(w_s,\zeta_s)\,:\, s\geq 0\}$ and any $a\geq 0$ let $F(a)$ be the set of a-fast times. Then, for every $\mathcal{F}$-measurable random Borel set $D\subset[0,\tau]$, almost surely,*

$$\mathbb{P}\Big\{F(a)\cap D\neq\emptyset\Big|\mathcal{F}\Big\} = \begin{cases} 1 & \text{if } \dim_P D > a^2/4,\\ 0 & \text{if } \dim_P D < a^2/4.\end{cases}$$

Observe that the set D is allowed to depend on the lifetime process ζ, so that it is feasible to apply it to $D=\zeta^{-1}(E)$. In order to carry this out we finally need a formula relating the packing dimension of $\zeta^{-1}(E)$ to the packing dimension of E. The necessary result is analogous to the result for Hausdorff dimension, which was proved by Serlet [23], see the appendix there.

Lemma 2.2 *Let $\{\zeta_s \,:\, 0\leq s\leq\tau\}$ be a reflected Brownian motion and $t>0$. Then, almost surely on $\sup\{\zeta_s\,:\,0\leq s\leq\tau\}>t$, we have for every nonempty Borel subset $E\subset(0,t)$,*

$$\dim_P \zeta^{-1}(E) = \frac{1}{2} + \frac{\dim_P E}{2}\,.$$

Now the proof of Theorem 1.1 follows easily from Proposition 2.1 and Lemma 2.2. Just observe that, almost surely on $\{\mathfrak{X}_t\neq\emptyset\}$,

$$\begin{aligned}
\sup_{\substack{w\in\mathrm{range}\,\mathfrak{X}\\ t\in E}}\ \limsup_{h\downarrow 0} \frac{|w(t)-w(t-h)|}{\sqrt{h\log(1/h)}} &= \sup_{s\in\zeta^{-1}(E)}\ \limsup_{h\downarrow 0} \frac{|w_s(\zeta_s)-w_s(\zeta_s-h)|}{\sqrt{h\log(1/h)}}\\
&= \sup\{a\,:\,F(a)\cap\zeta^{-1}(E)\neq\emptyset\} = 2\sqrt{\dim_P\zeta^{-1}(E)} = \sqrt{2+2\dim_P(E)}\,.
\end{aligned}$$

Proposition 2.1 also covers the classical Lévy modulus (1) and allows to shed some light on the equality of the right hand sides in (1) and (3). Fix $s > 0$. Then, conditional on $s < \tau$, let D be the associated set of record points

$$D = \{r \in [s, \tau] : \zeta_r = \min_{[s,r]} \zeta\}.$$

Then $W(D)$ consists of all restrictions of the path $w_s : [0, \zeta_s] \to \mathbb{R}^d$ on the intervals $[0, r]$ for times $r \in [0, \zeta_s]$ and the proposition yields that

$$\sup_{r\in[0,\zeta_s]} \limsup_{h\downarrow 0} \frac{|w_s(r) - w_s(r-h)|}{\sqrt{h \log(1/h)}} = \sup_{t\in D} \limsup_{h\downarrow 0} \frac{|w_t(\zeta_t) - w_t(\zeta_t - h)|}{\sqrt{h \log(1/h)}} = 2\sqrt{\dim_P(D)} = \sqrt{2}.$$

The equality of the right hand sides in (1) and (3) is hence a consequence of Lévy's famous distributional equivalence of the zero set and the record set of Brownian motion.

We now formulate the crucial tool in the proof of the Hausdorff dimension results. In order to reduce the study of sets in the range or support of $\mathfrak{X}_t$ to the study of subsets of the real line, we have to relate the dimension of a set $A \subset [0, \tau]$ and its image $W(A)$ under the Brownian snake.

Theorem 2.3 (Uniform dimension formula) *Suppose that W is a Brownian snake in dimension $d \geq 2$. Then, $\mathbb{P}$-almost surely, for every set $A \subset [0, \tau]$ such that $0 \notin \zeta(A)$, we have*

$$\dim W(A) = 4 \dim(A).$$

Remark 2.4. A closely related result was found by Serlet [23], who showed that, in the case $d \geq 4$, $\mathbb{P}$-almost surely, for every set $A \subset [0, \tau]$ such that $0 \notin \zeta(A)$, we have $\dim \tilde{W}(A) = 4 \dim(A)$. By the methods of Section 4, this result can be extended to dimension $d = 2, 3$ if the set A is a subset of an arbitrary levelset of ζ, see [18]. Note that, for every t, almost surely, the subsets of levelsets $\zeta^{-1}(t)$ are mapped by the Brownian snake onto the subsets of the support $\operatorname{supp} \mathfrak{X}_t$, of super-Brownian motion at time t.

Because the exceptional nullset of paths in the previous theorem does not depend on A we may use it for sets A which depend on W in an arbitrary manner. It is due to this fact that the theorems can be used to reduce the study of random subsets of the range or support to the study of random subsets A of the real line.

Using Theorem 2.3 we can observe that, for closed sets $E \subset (0, t)$,

$$\begin{aligned} &\dim \Big\{ (w, \zeta) \in \mathcal{R}(E) : \limsup_{h\downarrow 0} \frac{|w(\zeta) - w(\zeta - h)|}{\sqrt{h \log(1/h)}} \geq a \Big\} \\ &= \; 4 \dim \{s \in [0, \tau] : \zeta_s \in E \text{ and } s \in F(a)\} \\ &= \; 4 \dim(F(a) \cap \zeta^{-1}(E)). \end{aligned}$$

Now recall that, almost surely on $\sup\{\zeta_s : 0 \leq s \leq \tau\} > t$, for every nonempty Borel subset $E \subset (0, t)$,

$$\dim \zeta^{-1}(E) = \frac{1}{2} + \frac{\dim E}{2} \text{ and } \dim_P \zeta^{-1}(E) = \frac{1}{2} + \frac{\dim_P E}{2}.$$

Hence, we have reduced the proof of Theorem 1.3 to the statement of the following proposition.

Proposition 2.5 (Dimension estimate) *Suppose that $D \subset [0,\tau]$ is an $\mathcal{F}$-measurable random closed set. Then, almost surely, for all $a \in [0, 2\sqrt{\dim_P E}]$,*

$$\dim(D) - \frac{a^2}{4} \le \dim(F(a) \cap D) \le \dim_P(D) - \frac{a^2}{4}\,. \tag{11}$$

If $\dim_P(D) < \frac{a^2}{4}$ we have $F(a) \cap D = \emptyset$.

Note that with the appropriate choice of D this also contains the result (5) on the fast points of Brownian motion. In the next subsection we review a technique to obtain results like Proposition 2.1 and Proposition 2.5.

2.2 Limsup random fractals and hitting probabilities

In their paper [11] Khoshnevisan, Peres and Xiao give a general framework for the discussion of so-called discrete limsup random fractals. In Section 3 we shall see that the set

$$W^{-1}\Big\{(w,\zeta) \in \mathcal{W} \,:\, \limsup_{h\downarrow 0} \frac{|w(\zeta) - w(\zeta-h)|}{\sqrt{h\log(1/h)}} \ge a\Big\}$$

contains a suitably defined discrete limsup random fractal. In this subsection we briefly review the framework and results of [11].

For every $n \in \mathbb{N}$ denote by $\mathcal{D}_n$ the collection of all dyadic intervals of the form $[k/2^n, (k+1)/2^n]$ for $k = 0, \ldots, 2^n - 1$. Suppose further that we have a collection $\{Z_n(I) \,:\, I \in \mathcal{D}_n\}$ of random variables taking values in $\{0,1\}$. The associated *discrete limsup random fractal* is defined as $A := \limsup_{n\to\infty} A_n$, where

$$A_n = \bigcup_{I \in \mathcal{D}_n, Z_n(I)=1} \operatorname{int}(I)\,,$$

and $\operatorname{int}(I)$ denotes the interior of I. The framework of Khoshnevisan, Peres and Xiao is based on the following two assumptions:

Index assumption: Suppose that for each $n \ge 1$ the distribution of $Z_n(I)$ is the same for each $I \in \mathcal{D}_n$ and that

$$\lim_{n\to\infty} \frac{\log_2 \mathbb{E}\{Z_n(I)\}}{n} = -\gamma$$

for some $\gamma > 0$, where $\log_2$ is the logarithm of base 2,

Correlation assumption: For each $\varepsilon > 0$ define

$$f(n,\varepsilon) = \max_{I\in\mathcal{D}_n} \#\Big\{J \in \mathcal{D}_n \,:\, \operatorname{Cov}\{Z_n(I), Z_n(J)\} \ge \varepsilon\, \mathbb{E}\{Z_n(I)\}\, \mathbb{E}\{Z_n(J)\}\Big\}.$$

Suppose that, for all $\varepsilon > 0$,

$$\limsup_{n\to\infty} \frac{\log_2 f(n,\varepsilon)}{n} = 0\,.$$

The result of [11] subsumes and refines earlier methods and approaches. For example the idea of limsup random fractals is present in the work of Orey and Taylor [19], Deheuvels and Mason [4] or Dembo, Peres, Rosen and Zeitouni [5] and examples for the idea of finding dimension bounds by means of intersections with independent random sets can be found in the work of Taylor [25], Hawkes [8], Perkins [21], Peres [20] or Khoshnevisan and Shi [12]. The **proof** of the following lemma can be found in [11, Theorem 3.1 and Corollary 3.3].

Lemma 2.6 *Suppose that A is a discrete limsup random fractal that fulfils the two assumptions above. Then, for every analytic set $D \subset [0,1]$,*

$$\mathbb{P}\Big\{A \cap D \neq \emptyset\Big\} = \begin{cases} 1 & \text{if } \dim_P(D) > \gamma, \\ 0 & \text{if } \dim_P(D) < \gamma. \end{cases}$$

Moreover,

$$\dim(D) - \gamma \leq \dim(A \cap D) \leq \dim_P(D) - \gamma\,.$$

We remark without going into details that [11] also offers a refined version of this result, which permits to deal with the problem of strictly fast points, see Remark 1.4(a).

In Section 3 we directly prove the upper bounds and use the method of limsup random fractals to prove the lower bounds of the hitting probabilities (Proposition 2.1) and the dimension estimate (Proposition 2.5). Section 4 contains the proof of Theorem 2.3. Section 5 provides two auxiliary lemmas and, finally, a brief proof of Lemma 2.2 is given in Section 6.

3 Proof of the hitting probabilities and dimension results

3.1 Preliminaries

Throughout the proofs we will use the following positive random quantities,

$$\Delta_1 := \sup\Big\{\delta < \frac{1}{4} \,:\, |\zeta_s - \zeta_t| \leq 2\sqrt{|s-t|\log(1/|s-t|)} \text{ for all } |s-t| \leq \delta\Big\}\,,$$

and

$$\Delta_2 := \sup\Big\{\delta < \frac{1}{4} \,:\, |w_s(t) - w_s(t-h)| \leq 4\sqrt{h\log(1/h)}\ \forall s \in [0,\tau], t \geq h \text{ and } 0 \leq h \leq \delta\Big\}.$$

These quantities are well-defined and positive by the results of Lévy resp. Dawson and Perkins (see [3, 8.7]) on the modulus of continuity of Brownian motion resp. super-Brownian motion. For later use we formulate the basic estimate as a lemma, for formal reasons put $w_s(-t) = w_s(0)$ for all $t \geq 0$.

Lemma 3.1 (Basic estimate) *Suppose that $|s-t| < \Delta_1$ and $2\sqrt{|s-t|\log(1/|s-t|)} < \Delta_2$, then*

$$|w_t(\zeta_t) - w_t(\zeta_t - h)| \geq |w_s(\zeta_s) - w_s(\zeta_s - h)| - 16|s-t|^{1/4}\log(1/|s-t|)\,.$$

Proof. We may assume that $s < t$. The triangle inequality gives

$$|w_t(\zeta_t) - w_t(\zeta_t - h)| \geq |w_s(\zeta_s) - w_s(\zeta_s - h)| - |w_s(\zeta_s) - w_t(\zeta_t)| - |w_s(\zeta_s - h) - w_t(\zeta_t - h)| .$$

By the construction of the Brownian snake one can estimate, choosing $m \in [s,t]$ such that $\zeta_m = \inf_{[s,t]} \zeta$,

$$\begin{aligned} |w_s(\zeta_s) - w_t(\zeta_t)| &\leq |w_s(\zeta_s) - w_s(\zeta_m)| + |w_t(\zeta_t) - w_t(\zeta_m)| \\ &\leq 4\sqrt{(\zeta_s - \zeta_m)\log(1/(\zeta_s - \zeta_m))} + 4\sqrt{(\zeta_t - \zeta_m)\log(1/(\zeta_t - \zeta_m))} \\ &\leq 8\sqrt{2\sqrt{(t-s)\log(1/(t-s))}\log(1/(2\sqrt{(t-s)\log(1/(t-s))}))} \\ &\leq 8(t-s)^{1/4}\log(1/(t-s)) . \end{aligned}$$

The same procedure works for the last term, and this yields the basic estimate. ∎

3.2 Upper bounds

In this subsection we show that, conditional on $\mathcal{F}$, $\dim_P(D) < a^2/4$ implies that $\mathbb{P}\{F(a) \cap D \neq \emptyset \,|\, \mathcal{F}\} = 0$ and prove the upper bound in Proposition 2.5. These parts of the proof follow standard routes and may be proved directly without reference to Lemma 2.6.

Suppose $a \leq 4$ and $\varepsilon > 0$ is held fixed. We choose $\gamma > 0$ so small that $e^\gamma - 1 \leq (\varepsilon/16)^2$ and define $h_n = e^{-\gamma n}$ and $s_n \downarrow 0$ such that, for all sufficiently large n,

$$h_n = \sqrt{s_n}\Big(\log\frac{1}{s_n}\Big)^2 .$$

We shall use intervals of length s_n to construct sets similar to limsup fractals, which contain $F(a)$. The choice of s_n guarantees that, for sufficiently large n,

$$16\sqrt[4]{s_n}\log(1/s_n) = 16\sqrt{h_n} \leq (\varepsilon/8)\sqrt{h_n\log(1/h_n)} , \tag{12}$$

and the choice of h_n gives, for all $h \in [h_n, h_{n-1}]$,

$$4\sqrt{(h - h_n)\log(1/(h-h_n))} \leq (\varepsilon/8)\sqrt{h_n\log(1/h_n)} , \tag{13}$$

together with

$$\sqrt{h\log(1/h)} \leq (1 + \varepsilon/32)\sqrt{h_n\log(1/h_n)} . \tag{14}$$

Let $\mathcal{I}_n$ be the collection of intervals $[ks_n, (k+1)s_n]$ for $0 \leq k < \tau/s_n$ integer and $\mathcal{J}_n$ be the subclass of intervals, such that for the left endpoint $t = ks_n$,

$$|w_t(\zeta_t) - w_t(\zeta_t - h_n)| \geq (a - \varepsilon)\sqrt{h_n\log(1/h_n)} . \tag{15}$$

We show that, almost surely,

$$F(a) \subset \bigcap_{j=1}^\infty \bigcup_{n=j}^\infty \bigcup_{I \in \mathcal{J}_n} I . \tag{16}$$

For this purpose, choose N so large that $s_n \le \Delta_1$, $2\sqrt{s_n \log(1/s_n)} \le \Delta_2$ and (12), (13), (14) hold for all $n \ge N$. Suppose that $s \in F(a)$. Then, there exists an arbitrarily large $n > N$ and an $h > 0$ with $h_n \le h \le h_{n-1}$ such that

$$|w_s(\zeta_s) - w_s(\zeta_s - h)| \ge (a - \varepsilon/8)\sqrt{h \log(1/h)}\,.$$

For the interval $I = [ks_n, (k+1)s_n] \in \mathcal{I}_n$ with $s \in I$ and left endpoint $t := ks_n$ we have to show that $I \in \mathcal{J}_n$. Using the basic estimate, we infer

$$|w_t(\zeta_t) - w_t(\zeta_t - h_n)| \ge |w_s(\zeta_s) - w_s(\zeta_s - h_n)| - 16\sqrt[4]{s_n}\log(1/s_n)\,.$$

For the first term we obtain

$$\begin{aligned}|w_s(\zeta_s) - w_s(\zeta_s - h_n)| &\ge |w_s(\zeta_s) - w_s(\zeta_s - h)| - |w_s(\zeta_s - h) - w_s(\zeta_s - h_n)| \\ &\ge (a - \varepsilon/8)\sqrt{h \log(1/h)} - 4\sqrt{(h - h_n)\log(1/(h - h_n))}\,.\end{aligned}$$

Putting this together with (12), (13) and (14) and recalling $a \le 4$ yields (15) and hence (16) is proved.

Let us now look at the hitting probabilities. We recall the definition of the upper Minkowski dimension in the form most convenient for our purpose. Supposing that D is a nonempty subset of $[0, \tau]$, we let $N(D, n)$ be the number of intervals in $\mathcal{I}_n$, which hit the set D. The *upper Minkowski dimension of* D is defined as

$$\overline{\dim}_M(D) = \inf\Big\{ s \ge 0 \,:\, \limsup_{n\uparrow\infty} N(D,n) s_n^s = 0 \Big\}.$$

Alternative definitions and properties of the upper Minkowski dimension can be found in [17, 5.3]. The crucial fact for our purpose is that *packing* dimension can be characterised as a regularisation of upper Minkowski dimension, more precisely

$$\dim_P(A) = \inf\Big\{ \sup_n \overline{\dim}_M(D_n) \,:\, A \subset \bigcup_{n=1}^{\infty} D_n, D_n \text{ a bounded set}\Big\}, \tag{17}$$

see [17, 5.9]. Hence it suffices to show that, whenever $\overline{\dim}_M(D) < a^2/4$, then $F(a) \cap D = \emptyset$. Suppose that $\vartheta > 0$ is such that $\overline{\dim}_M(D) < \vartheta < (a - \varepsilon)^2/4$. By the definition of the upper Minkowski dimension, we have, for all sufficiently large n,

$$\#\Big\{ I \in \mathcal{I}_n \,:\, I \cap D \ne \emptyset \Big\} = N(D, n) \le s_n^{-\vartheta}\,. \tag{18}$$

Almost surely, for all left endpoints $t = ks_n$, w_t is a Brownian path on $[0, \zeta_t]$. Hence, we know that, for every $I \in \mathcal{I}_n$,

$$\mathbb{P}\Big\{ I \in \mathcal{J}_n \,\Big|\, \mathcal{F} \Big\} = \mathbb{P}\Big\{ |w_t(\zeta_t) - w_t(\zeta_t - h_n)| \ge (a - \varepsilon)\sqrt{h_n \log(1/h_n)} \Big| \mathcal{F} \Big\} = h_n^{(a-\varepsilon)^2/2 + o(1)}\,. \tag{19}$$

Denote by S_n the number of $I \in \mathcal{J}_n$ with $I \cap D \ne \emptyset$. From (18) and (19) we infer that

$$\begin{aligned}\mathbb{P}\Big\{ S_n > 0 \,\Big|\, \mathcal{F} \Big\} &\le s_n^{-\vartheta} \max_{I \in \mathcal{I}_n} \mathbb{P}\Big\{ I \in \mathcal{J}_n \,\Big|\, \mathcal{F} \Big\} \\ &\le s_n^{-\vartheta}\left(\sqrt{s_n}\Big(\log\frac{1}{s_n}\Big)^2 \right)^{(a-\varepsilon)^2/2 + o(1)}\,.\end{aligned} \tag{20}$$

As $(a-\varepsilon)^2/4 - \vartheta > 0$ the latter quantity is summable and we obtain, using (16) and the Borel-Cantelli lemma,

$$\mathbb{P}\Big\{D \cap F(a) \neq \emptyset \,\Big|\, \mathcal{F}\Big\} \leq \mathbb{P}\Big\{\forall_{j=1}^{\infty} \exists_{n=j}^{\infty} \exists I \in \mathcal{J}_n \text{ with } I \cap D \neq \emptyset \Big| \mathcal{F}\Big\} = 0\,.$$

This proves the first part of the hitting estimates. To obtain the upper bound on the Hausdorff dimension in (11), we first observe that, using again regularisation (17), it suffices to prove

$$\dim(F(a) \cap D) \leq \overline{\dim}_M(D) - a^2/4 \text{ almost surely.}$$

Choose any $\vartheta > \overline{\dim}_M(D)$ and $\eta < (a-\varepsilon)^2/4$ and recall that S_n denotes the number of intervals $I \in \mathcal{J}_n$ with $D \cap I \neq \emptyset$. Then, using (19) again,

$$\mathbb{E}\Big\{\sum_{n=1}^{\infty} S_n s_n^{\vartheta-\eta} \Big| \mathcal{F}\Big\} \leq \sum_{n=1}^{\infty} s_n^{-\eta} \Big(\sqrt{s_n}\Big(\log\frac{1}{s_n}\Big)^2\Big)^{(a-\varepsilon)^2/2+o(1)} < \infty\,.$$

Hence, almost surely, $\sum_{n=1}^{\infty} S_n s_n^{\vartheta-\eta}$ is finite. As, for arbitrarily large N, the set $D \cap F(a)$ can be covered by the union over $n \geq N$ of the collection of S_n intervals of length s_n, we infer that $\dim(D \cap F(a)) \leq \vartheta - \eta$ and as this holds for every small $\varepsilon > 0$, $\eta < (a-\varepsilon)^2/4$ and $\vartheta > \overline{\dim}_M(D)$, this yields the desired result.

3.3 Lower bounds

In this section we prove the lower bounds in Proposition 2.1 and 2.5 with the help of Lemma 2.6. We start by showing that the set

$$F(a) = \Big\{s \in [0,\tau] \,:\, \limsup_{h \downarrow 0} \frac{|w_s(\zeta_s) - w_s(\zeta_s - h)|}{\sqrt{h \log(1/h)}} \geq a\Big\}\,,$$

contains a suitably defined discrete limsup random fractal.

Lemma 3.2 *Let $\varepsilon > 0$ be given and define $h_n = 2^{-n/2} n^4$. For $I = [k2^{-n}, (k+1)2^{-n}] \in \mathcal{D}_n$ with left endpoint $t = k2^{-n}$ define the random variables $\{Z_n(I)\}$ to be the indicator function of the event*

$$\frac{|w_t(\zeta_t) - w_t(\zeta_t - h_n)|}{\sqrt{h_n \log(1/h_n)}} \geq a + \varepsilon\,.$$

Then the discrete limsup random fractal A defined through the random variables $\{Z_n(I)\}$ is contained in $F(a)$.

Proof. Suppose that $s \in I$ and $Z_n(I) = 1$. Then, by the basic estimate,

$$|w_s(\zeta_s) - w_s(\zeta_s - h_n)| \geq (a+\varepsilon)\sqrt{h_n \log(1/h_n)} - 16(\log 2) 2^{-n/4} n \geq a\sqrt{h_n \log(1/h_n)}\,,$$

provided n is sufficiently large. ∎

Note that the lower bounds in Proposition 2.1 and Proposition 2.5 follow if we show that the limsup random fractal in the lemma fulfils the index assumption with index $\gamma = (a+\varepsilon)^2/4$ and the correlation assumption. The former is easy to see.

Lemma 3.3 *The conditional distribution of $Z_n(I)$ given $\mathcal{F}$ is the same for each $I \in \mathcal{D}_n$ and*

$$\lim_{n\to\infty} \frac{\log_2 \mathbb{E}\{Z_n(I)\,|\,\mathcal{F}\}}{n} = -(a+\varepsilon)^2/4\,.$$

Proof. We have

$$\begin{aligned}\mathbb{E}\{Z_n(I)\,|\,\mathcal{F}\} &= \mathbb{P}\Big\{|w_t(\zeta_t) - w_t(\zeta_t - h_n)| \geq (a+\varepsilon)\sqrt{h_n \log(1/h_n)}\,\Big|\,\mathcal{F}\Big\} \\ &= h_n^{(a+\varepsilon)^2/2+o(1)} = 2^{-n(a+\varepsilon)^2/4+o(1)}\,.\end{aligned}$$

This implies the statement. ∎

It remains to check that our limsup random fractals fulfils the *correlation assumption.* Here is a significant difference to the example cases discussed in [11], in which the correlation assumption was verified by means of tail correlation bounds that guaranteed a small correlation of intervals, which are far apart relative to their (common) size. Here suppose that s_1, s_2 are the beginning and end of an excursion of the reflected Brownian motion $\{\zeta_s\}$ above level t. For large n let $I, J \in \mathcal{D}_n$ be the two intervals containing s_1 resp. s_2. Then the correlation of $Z_n(I)$ and $Z_n(J)$ is large although the distance of I and J relative to their length is large. The next lemma shows that this phenomenon of long range dependence is sufficiently rare, so that the correlation assumption still holds true.

Lemma 3.4 *Define*

$$f(n) = \max_{I\in\mathcal{D}_n} \#\Big\{J \in \mathcal{D}_n\,:\,\mathrm{Cov}\{Z_n(I), Z_n(J)\,|\,\mathcal{F}\} > 0\Big\}.$$

Then, almost surely,

$$\limsup_{n\to\infty} \frac{\log_2 f(n)}{n} = 0\,.$$

Proof. Look at intervals $I = [k/2^n, (k+1)/2^n]$ and $J = [l/2^n, (l+1)/2^n]$ with $k < l$. Recalling the construction of the Brownian snake we observe that, conditional on $\mathcal{F}$, $Z_n(I)$ and $Z_n(J)$ are independent if either

$$\zeta_{l/2^n} - \zeta_{k/2^n} \geq h_n$$

or

$$\zeta_{k/2^n} - \min\Big\{\zeta_s\,:\,s \in [k/2^n, l/2^n]\Big\} \geq h_n\,.$$

We therefore have to show that, for all $\delta > 0$, almost surely, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$ and $k = 1, \ldots, 2^n$, $A(n,k) \leq 2^{n\delta}$, where

$$A(n,k) := \#\Big\{l \in \{k, \ldots, 2^n\}\,:\,\zeta_{l/2^n} - \zeta_{k/2^n} \leq h_n \text{ and } \max_{s\in[k/2^n, l/2^n]} \Big(\zeta_{k/2^n} - \zeta_s\Big) \leq h_n\Big\}.$$

Now fix n sufficiently large to satisfy $2^{-n} < \Delta_1$ and fix $1 \leq k \leq 2^n$. We may assume that $\zeta(k/2^n) \geq h_n$, the opposite case is easier. Look at the stopped Brownian motion $B(t) = \zeta(k/2^n + t) - \zeta(k/2^n)$ stopped at time

$$\tau := \inf\Big\{l \in \{k, \ldots, 2^n\}\,:\,\min\{B(t)\,:\,0 \leq t \leq (l-k)/2^n\} \leq -h_n\Big\}\,.$$

Then,
$$A(n,k) = \#\Big\{ l \in \{k,\dots,\tau\} \,:\, B((l-k)/2^n) \le h_n \Big\}\,.$$
Now suppose that $A(n,k) > 2^{n\delta}$. If $B((l-k)/2^n) \le h_n$, then, recalling that $2^{-n} < \Delta_1$, we have $B(s) \le 2h_n$ for all $s \in [(l-k)/2^n, (l-k+1)/2^n]$. We let
$$T_n := \inf\Big\{ t \ge 0 \,:\, B(t) \le -2h_n \Big\}\,,$$
and note that $T_n \ge \tau$. Our assumption implies
$$\int_0^{T_n} \mathbf{1}_{\{|B(s)| \le 2h_n\}}\, ds \ge 2^{n(\delta-1)}\,.$$
We thus get, for all N,
$$\begin{aligned}
&\mathbb{P}\Big\{ \text{there is } n \ge N \text{ with } 2^{-n} < \Delta_1 \text{ and } A(n,k) \ge 2^{n\delta} \text{ for some } k \Big\} \\
&\le \sum_{n=N}^{\infty} 2^n \mathbb{P}\Big\{ \int_0^{T_n} \mathbf{1}_{\{|B_s| \le 2h_n\}}\, ds \ge 2^{n(\delta-1)} \Big\} \\
&= \sum_{n=N}^{\infty} 2^n \mathbb{P}\Big\{ (2h_n)^2 \int_0^{T} \mathbf{1}_{\{|B_s| \le 1\}}\, ds \ge 2^{n(\delta-1)} \Big\}\,. \qquad (21)
\end{aligned}$$
In the last step we use Brownian scaling and define the stopping time T as the first hitting time of level -1 by $\{B_s\}$. We now use the following auxiliary lemma, which will be proved in Section 5 together with a similar lemma. Observe that the stopping time T does not even have finite first moment, so that the statement is not trivial.

Lemma 3.5 *Let $\{B_s\}$ be a linear Brownian motion and T the first hitting time of level -1 by $\{B_s\}$. Then, for all $0 \le \lambda < 1/4e$, we have*
$$\mathbb{E}\Big\{ \exp\Big(\lambda \int_0^T \mathbf{1}_{\{|B_s| \le 1\}}\, ds \Big) \Big\} \le \frac{e^{4e\lambda} - e^{4e\lambda - 1}}{1 - e^{4e\lambda - 1}}\,.$$
Using Chebyshev's inequality we now obtain,
$$\begin{aligned}
\mathbb{P}\Big\{ (2h_n)^2 \int_0^T \mathbf{1}_{\{|B_s| \le 1\}}\, ds \ge 2^{n(\delta-1)} \Big\} &= \mathbb{P}\Big\{ 4n^8 \int_0^T \mathbf{1}_{\{|B_s| \le 1\}}\, ds \ge 2^{n\delta} \Big\} \\
&\le \exp\Big(-\lambda 2^{n\delta}/4n^8 \Big) \mathbb{E}\Big\{ \exp\Big(\lambda \int_0^T \mathbf{1}_{\{|B_s| \le 1\}}\, ds \Big) \Big\}\,.
\end{aligned}$$
From Lemma 3.5 we infer that, for a suitably small $\lambda > 0$, the series (21) converges and hence, almost surely, there is a random N such that $A(n,k) \le 2^{n\delta}$ for all $k = 1, \dots, 2^n$ and $n \ge N$. This finishes the proof. ■

4 A uniform dimension formula for the Brownian snake

The proof of the upper bound is easy. We recall from [15, IV.4, Prop. 5] that, for every $\eta > 0$, there is an $\mathcal{F}$-measurable random variable $C_\eta(\zeta) > 0$ such that, for all integers $p > 4$ and $s, t \in [0, \tau]$,

$$\mathbb{E}\Big\{ d(W_s, W_t)^p \Big| \mathcal{F} \Big\} \le C_\eta(\zeta) |s - t|^{\frac{p}{2}(\frac{1}{2} - \eta)} ,$$

where we note that the different definition of the metric on $\mathcal{W}$ does not affect the calculation. Hence, by Kolmogorov's Theorem, for every $\delta > 0$, the mapping $s \mapsto W_s$ is locally Hölder continuous with exponent $(4 + \delta)^{-1}$ almost surely and we get the upper bound $\dim W(A) \le 4 \dim A$ without any effort.

It remains to prove the lower bound. For this purpose we have to show that, on a set of full $\mathbb{P}$-measure, for every $H \subset \mathcal{W}$ and every $t > 0$, we have

$$\dim\{s \in (0, \tau) : W_s \in H\} \le \frac{\dim H}{4} . \tag{22}$$

The lower bound then follows by choosing $H = \{W_s : s \in A\}$. Without loss of generality we may assume that there is a unit cube $C = a + [0, 1)^d \subset \mathbb{R}^d$ at positive distance from the origin and a unit interval $I = b + [0, 1]$ with $b > 0$ such that, for all stopped paths $(w, \zeta) \in H$, we have $\zeta \in I$ and $w(\zeta) \in C$.

We first define a set of full measure such that the dimension property (22) holds for all W in the set. The basic setup will be as in Serlet's proof of the uniform dimension result for $\tilde{W}$, but we have to add one refined property to the set.

For this purpose let $\{x_m : m \ge 1\}$ be the set of dyadic points in the cube C indexed such that, for every n,

$$\Big\{ x_m : 1 \le m \le 2^{nd} \Big\} = \Big\{ a + \Big(\frac{k_1}{2^n}, \dots, \frac{k_d}{2^n} \Big) : 0 \le k_i \le 2^n - 1 \Big\} .$$

Next, let $\{a_m : m \ge 0\}$ be the set of dyadic points in the interval I indexed such that, for every n, $\{a_m : 0 \le m \le 2^n\} = \{b + \frac{k}{2^n} : 0 \le k \le 2^n\}$. Define $I_m(\varepsilon) = [a_m - \varepsilon, a_m + \varepsilon]$. The *occupation measure* Z for the Brownian snake is defined by

$$Z(J, A) = \int_0^\tau \mathbf{1}_J(\zeta_s) \mathbf{1}_A(\tilde{W}_s) \, ds , \quad \text{for all } A \subset \mathbb{R}^d \text{ and } J \subset (0, \infty) \text{ Borel.}$$

Also, we have to define the successive entrance times in the ball $B(x, \varepsilon)$ and exit times from the ball $B(x, 2\varepsilon)$ for the process $\{\tilde{W}_s : s \ge 0\}$ by

$$T_1(x, \varepsilon) = \inf\{s \ge 0 : \tilde{W}_s \in B(x, \varepsilon)\} ,$$

and, for $k \ge 1$,

$$\begin{aligned} S_k(x, \varepsilon) &= \inf\{s > T_k(x, \varepsilon) : \tilde{W}_s \notin B(x, 2\varepsilon)\} , \\ T_{k+1}(x, \varepsilon) &= \inf\{s > S_k(x, \varepsilon) : \tilde{W}_s \in B(x, \varepsilon)\} . \end{aligned}$$

Finally, throughout the paper, denote $L(x) = 1 + |\log(x)|$.

Lemma 4.1 *For every $\delta > 0$ there is a set $\Omega(\delta) \subset \Omega$ of full $\mathbb{P}$-measure, a constant $c > 1$ and a random variable Q, which is finite on $\Omega(\delta)$, such that, for every path $W = \{(w_s, \zeta_s) : s \geq 0\} \in \Omega(\delta)$, every $\varepsilon \in \{2^{-q} : q \geq Q\}$ and all integers $m, n \geq 1$,*

(i) $Z\Big(I_n(\varepsilon^2), B(x_m, 2\varepsilon)\Big) \leq cL(n)^2 L(m)^2 L(\varepsilon)^6 \varepsilon^4$,

(ii) $S_k(x_m, \varepsilon) - T_k(x_m, \varepsilon) \leq cL(m)^3 L(k)^3 L(\varepsilon)^3 \varepsilon^4$, *for every k such that $T_k(x_m, \varepsilon) < \tau$,*

(iii) $|\tilde{W}_s - \tilde{W}_t| \leq |s-t|^{1/(4+\delta)}$, *for all $0 \leq s, t \leq \tau$ such that $|s-t| \leq 2^{-Q}$,*

(iv) $|\zeta_s - \zeta_t| \leq |s-t|^{1/(2+\delta)}$, *for all $0 \leq s, t \leq \tau$ such that $|s-t| \leq 2^{-Q}$.*

Proof. It is clear from the fact that $s \mapsto \tilde{W}_s$ is locally $1/(4+\delta)$-Hölder continuous and that $s \mapsto \zeta_s$ is locally $1/(2+\delta)$-Hölder continuous that properties (iii) and (iv) can be realized on a set of full measure. From Serlet [23, Lemma 9] we know that property (ii) can be realized on a set of full measure. The novel part is the proof of property (i). We formulate two lemmas to prepare the proof.

We first need the following statement about reflected Brownian motion stopped when its local time at level 0 first hits 1. Recall that the local time of $\{|B_s|\}$ at level 0 is twice the local time of $\{B_s\}$ at level 0.

Lemma 4.2 *Let $\{B_s\}$ be a linear Brownian motion, L_s^0 its local time at level 0 and $\tau = \tau(B) = \inf\{s : L_s^0 = 1/2\}$. Then, for all $0 \leq \lambda < 1/3e$, we have*

$$\mathbb{E}\Big\{ \exp\Big(\lambda \int_0^\tau \mathbf{1}_{\{|B_s| \leq 1\}}\, ds\Big)\Big\} \leq \frac{e^{3e\lambda} - e^{3e\lambda - 1}}{1 - e^{3e\lambda - 1}}.$$

A similar lemma was used in the proof of the hitting estimates and they are proved together in Section 5. We also need the existence of exponential moments for the random measure of balls. The following lemma is well known, see [16, Lemma 3.1] for an easy proof. Denote by $p(x,s) = (1/\sqrt{2\pi s})^d \exp(-\|x\|^2/2s)$ the standard heat kernel.

Lemma 4.3 *Denote $G(r,t) = \int_0^t [\int_{\|y\| \leq r} p(y,s)\, dy]\, ds$. Then, for all $t > 0, r > 0$ and $x \in C$,*

$$\mathbb{P}\Big\{\exp\Big(\lambda X_t(B(x,r))\Big)\Big\} \leq \exp\Big(2\lambda \int_{\|y\| \leq r} p(y,t)\, dy\Big),$$

for all $0 \leq \lambda \leq (4G(r,t))^{-1}$.

We can now formulate the main lemma behind the dimension formula. Recall the definition of Δ_2 from the beginning of Section 3. The idea of the lemma is the following: For $0 < \varepsilon < e^{-36}$, if a particle at time $t + \varepsilon^2$ is outside $B(x, \varepsilon L(\varepsilon))$ at time t and $\Delta_2(W) > \varepsilon^2$, then at time t the particle has a distance to x, which is at least

$$\varepsilon L(\varepsilon) - 4\sqrt{\varepsilon^2 \log(1/\varepsilon^2)} \geq \varepsilon + \varepsilon \log(1/\varepsilon) - 6\varepsilon\sqrt{\log(1/\varepsilon)} \geq \varepsilon.$$

In other words, the offspring of a particle outside $B(x, \varepsilon L(\varepsilon))$ at time t cannot move into $B(x, \varepsilon)$ in less than ε^2 time units. Heuristically this means that, if $\int_t^{t+\varepsilon^2} X_s(B(x,\varepsilon))\, ds$ is large and

$\Delta_2(W) > \varepsilon^2$, then either $X_t(B(x, \varepsilon L(\varepsilon)))$ is large or the particles which are in $B(x, \varepsilon L(\varepsilon))$ at time t are unusually fertile. Both these events are shown to be unlikely.

As the next proof uses the Markov property of super-Brownian motion, we suppose that $\{X_t\}$ under $\mathbb{P}_\mu$ is super-Brownian motion started at time $t = 0$ with mass μ and abbreviate $\mathbb{P}_x := \mathbb{P}_{\delta_x}$.

Lemma 4.4 *For every bounded interval $I \subset (0, \infty)$ there exist positive constants c_1 and c_2 such that, for all $t \in I$, $x \in C$, all $\lambda > 0$ and all $\varepsilon = 2^{-q}$, $q \geq 72$,*

$$\mathbb{P}_0\Big\{ \Delta_2(W) > \varepsilon^2 \,, \int_t^{t+\varepsilon^2} X_s(B(x,\varepsilon))\, ds \geq \lambda \varepsilon^4 L(\varepsilon)^4 \Big\} \leq c_1 \exp(-c_2 \sqrt{\lambda}) \,.$$

Proof. Denote by

$$M = \Big\{ \mu \in \mathcal{M}_F(\mathbb{R}^d) \,:\, \mu(\mathbb{R}^d) \leq \sqrt{\lambda} \varepsilon^2 L(\varepsilon)^4 \Big\} \,.$$

We have, by the Markov property of super-Brownian motion, and our initial consideration,

$$\begin{aligned} &\mathbb{P}_0\Big\{ \Delta_2(W) > \varepsilon^2, \int_t^{t+\varepsilon^2} X_s(B(x,\varepsilon))\, ds \geq \lambda \varepsilon^4 L(\varepsilon)^4 \Big\} \\ &\leq \;\; \mathbb{P}_0\Big\{ X_t(B(x, \varepsilon L(\varepsilon))) \geq \sqrt{\lambda} \varepsilon^2 L(\varepsilon)^4 \Big\} + \sup_{\mu \in M} \mathbb{P}_\mu \Big\{ \int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\, dt \geq \lambda \varepsilon^4 L(\varepsilon)^4 \Big\} \,. \end{aligned}$$

We give separate estimates of these two probabilities. To investigate the first probability we use Lemma 4.3. We have

$$G(r,t) = \int_{\|y\| \leq r} \Big[\int_0^t p(y,s)\, ds \Big]\, dy = r^2 \int_{\|y\| \leq 1} \Big[\int_0^{t/r^2} p(y,s)\, ds \Big]\, dy \,,$$

which, for all $t \in I$ and $0 < r < 1/2$, is bounded from above by a constant multiple of $r^2 \log(1/r)$ in dimension $d = 2$ and by a constant multiple of r^2 in dimension $d \geq 3$. In all cases we find a positive constant k_1 such that

$$t \int_{B(0, \varepsilon L(\varepsilon))} p(y,t)\, dy \leq G(\varepsilon L(\varepsilon), t) \leq k_1 \varepsilon^2 L(\varepsilon)^4 \,.$$

Choose $k_2 = 1/(4k_1)$ and k_3 such that $\exp(1/(2t)) < k_3$ for all $t \in I$. Then, using Chebyshev and Lemma 4.3,

$$\begin{aligned} &\mathbb{P}_0\Big\{ X_t(B(x, \varepsilon L(\varepsilon))) \geq \sqrt{\lambda} \varepsilon^2 L(\varepsilon)^4 \Big\} \\ &\leq \;\; \exp\Big(- \frac{\sqrt{\lambda} \varepsilon^2 L(\varepsilon)^4}{4G(\varepsilon L(\varepsilon), t)} \Big)\, \mathbb{P}_0\Big\{ \exp\Big(\frac{X_t(B(x, \varepsilon L(\varepsilon)))}{4G(\varepsilon L(\varepsilon), t)} \Big) \Big\} \\ &\leq \;\; \exp\Big(- \sqrt{\lambda}/(4k_1) \Big) \cdot \exp\Big(2 \frac{\int_{\|y\| \leq \varepsilon L(\varepsilon)} p(y,t)\, dy}{4G(\varepsilon L(\varepsilon), t)} \Big) \\ &\leq \;\; k_3 \exp(-k_2 \sqrt{\lambda}) \,. \end{aligned} \tag{23}$$

This is the required estimate for the first term. To investigate the second probability we use Lemma 4.2. We choose a positive constant $k_4 < 1/3e$. Then, by Chebyshev,

$$\mathbb{P}_\mu \Big\{ \int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\, dt \geq \lambda \varepsilon^4 L(\varepsilon)^4 \Big\} \leq e^{-k_4 \lambda}\, \mathbb{P}_\mu \Big\{ \exp\Big(k_4 \frac{\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\, dt}{\varepsilon^4 L(\varepsilon)^4} \Big) \Big\} \,. \tag{24}$$

Now let $a = k_4/L(\varepsilon)^4$ so that $0 < a < 1/3e$. We have, by the branching property of super-Brownian motion, for all $\mu \in M$,

$$\mathbb{P}_\mu\Big\{ \exp\Big(\frac{a}{\varepsilon^4}\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt\Big)\Big\} = \exp\Big(\int \log \mathbb{P}_x\Big\{ \exp\Big(\frac{a}{\varepsilon^4}\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt\Big)\Big\}\,d\mu(x)\Big). \qquad (25)$$

Now, recalling either the snake representation or the Ray-Knight Theorem,

$$\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt = \int_0^\tau \mathbf{1}_{\{\zeta_s \le \varepsilon^2\}}\,ds\,,$$

and we infer that

$$\begin{aligned} &\int \log \mathbb{P}_x\Big\{ \exp\Big(\frac{a}{\varepsilon^4}\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt\Big)\Big\}\,d\mu(x) \\ &= \int \log P_0\Big\{ \exp\Big(\frac{a}{\varepsilon^4}\int_0^{\tau} \mathbf{1}_{\{|B_s|\le\varepsilon^2\}}\,ds\Big)\Big\}\,d\mu(x) \\ &\le \sqrt{\lambda}\varepsilon^2 L(\varepsilon)^4 \cdot \log P_0\Big\{ \exp\Big(\frac{a}{\varepsilon^4}\int_0^{\tau} \mathbf{1}_{\{|B_s|\le\varepsilon^2\}}\,ds\Big)\Big\}\,. \end{aligned} \qquad (26)$$

We now use a scaling argument. Define $\tau_0 = 0$ and $\{B_s^0\} = \{B_s\}$. Proceed inductively by putting, for $j \ge 0$, $\tau_{j+1} = \inf\{s : L_s^0(B_s^j) = 1/2\}$ and $\{B_s^{j+1}\} = \{B_{\tau_{j+1}+s}\}$. Finally, let $\tilde\tau = \inf\{s : L_s^0 = 1/(2\varepsilon^2)\}$. Then, using Lemma 4.2 in the final step,

$$\begin{aligned} &\log P_0\Big\{ \exp\Big(\frac{a}{\varepsilon^4}\int_0^{\tau} \mathbf{1}_{\{|B_s|\le\varepsilon^2\}}\,ds\Big)\Big\} = \log P_0\Big\{ \exp\Big(\frac{a}{\varepsilon^4}\int_0^{\varepsilon^4\tilde\tau} \mathbf{1}_{\{|B_{s/\varepsilon^4}|\le 1\}}\,ds\Big)\Big\} \\ &= \log P_0\Big\{ \exp\Big(a\sum_{j=1}^{1/\varepsilon^2}\int_0^{\tau_j} \mathbf{1}_{\{|B_s^{j-1}|\le 1\}} ds\Big)\Big\} = \sum_{j=1}^{1/\varepsilon^2} \log P_0\Big\{ \exp\Big(a\int_0^{\tau_j} \mathbf{1}_{\{|B_s^{j-1}|\le 1\}} ds\Big)\Big\} \\ &\le \frac{1}{\varepsilon^2}\cdot \log\Big(\frac{e^{3ea}-e^{3ea-1}}{1-e^{3ea-1}}\Big)\,. \end{aligned} \qquad (27)$$

Putting together (24), (25), (26), (27) and using that, for some constant $k_5 > 0$, $\log((e^{3ea} - e^{3ea-1})/(1-e^{3ea-1})) \le k_5 a$, we infer

$$\begin{aligned} \mathbb{P}_\mu\Big\{ \int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt \ge \sqrt{\lambda}\varepsilon^4 L(\varepsilon)^4\Big\} &\le e^{-k_4\lambda}\cdot \exp\Big(\sqrt{\lambda}L(\varepsilon)^4 \log\Big(\frac{e^{3ea}-e^{3ea-1}}{1-e^{3ea-1}}\Big)\Big) \\ &\le \exp\Big(-\lambda k_4(1-k_5/\sqrt{\lambda})\Big)\,. \end{aligned} \qquad (28)$$

The required result follows from (23) and (28). ■

We can now add the probabilities and obtain, for some $c > 0$, the convergence of the series

$$\sum_{q=1}^\infty \mathbb{P}_0\Big\{ \Delta_2(W) > 2^{-2q}, \int_{I_n(2^{-2q})} X_t(B(x_m, 2^{-q}))\,dt \ge c2^{-4q}L(n)^2L(m)^2L(2^q)^6 \text{ for some } n, m\Big\}.$$

Hence Borel-Cantelli yields a random integer Q_1 such that, whenever $\Delta_2(W) > 2^{-2Q_1}$, we have

$$\int_{I_n(2^{-2q})} X_t(B(x_m, 2^{-q}))\,dt \le c2^{-4q}L(n)^2L(m)^2L(2^q)^6\,,$$

for all integers n, m and $q \geq Q_1$. By the modulus of continuity of Dawson and Perkins (i) can be realized on a set of full measure and hence we obtain the statement of Lemma 4.1. ■

We finish the proof of Theorem 2.3 by showing that the dimension property holds for all W which are in the intersection of the sets $\Omega(\delta)$. Pick a set $H \subset \mathcal{W}$ with $\zeta \in I$ and $w(\zeta) \in C$ for all $(w, \zeta) \in H$. We may assume that $\dim H < 4$, and let $4 > \gamma > \dim H$ and $0 < \delta < \gamma - \dim H$. We further fix an integer K with $2^K \geq \sqrt{d} + 1$. We choose a covering of H by a sequence $\{B(w_j, r_j) : j \geq 1\}$ of balls whose radii satisfy $r_j \leq 2^{-(Q+K+1)}$, such that

$$\sum_{j=1}^{\infty} r_j^{\gamma} \, L(2^K r_j)^{28} \leq 2^{-(K+1)\gamma}, \tag{29}$$

and $\zeta_j = \zeta(w_j) \in I$ and $y_j = w_j(\zeta_j) \in C$. Then

$$\Big\{ B(y_j, r_j) \times [\zeta_j - r_j^2, \zeta_j + r_j^2] \, : \, j \geq 1 \Big\}$$

is a covering of the set $\{(w(\zeta), \zeta) \, : \, w \in H\}$. This covering can be replaced by a covering consisting of sets $B(x_{m_j}, \varepsilon_j) \times I_{n_j}(\varepsilon_j^2)$ such that $\varepsilon_j \in \{2^{-q} \, : \, q \geq Q\}$ with $2^{-(K+1)}\varepsilon_j \leq r_j \leq 2^{-K}\varepsilon_j$ and

$$|y_j - x_{m_j}| \leq \sqrt{d}\frac{\varepsilon_j}{2^K} \text{ for some } m_j \leq \left(\frac{2^K}{\varepsilon_j}\right)^d,$$

$$|\zeta_j - a_{n_j}| \leq \frac{\varepsilon_j^2}{2^K} \text{ for some } n_j \leq \frac{2^K}{\varepsilon_j^2}.$$

We write T_k^j, S_k^j for $T_k(x_{m_j}, \varepsilon_j)$, $S_k(x_{m_j}, \varepsilon_j)$, and denote by

$$\mathcal{N}_j = \Big\{ k \geq 1 \, : \, T_k^j < \tau \text{ and } \zeta[T_k^j, S_k^j] \cap I_{n_j}(\varepsilon_j^2) \neq \emptyset \Big\},$$

and

$$\mathcal{M}_j = \Big\{ k \geq 1 \, : \, T_k^j < \tau \text{ and } \zeta[T_k^j, S_k^j] \subset \bigcup_{m=0}^{2^K \varepsilon_j^{-2}} I_m(\varepsilon_j^2) \Big\}.$$

Let $N_j = \# \mathcal{N}_j$ and $M_j = \# \mathcal{M}_j$. Because

$$B(y_j, r_j) \times [\zeta_j - r_j^2, \zeta_j + r_j^2] \subset B(x_{m_j}, \varepsilon_j) \times I_{n_j}(\varepsilon_j^2),$$

we have that

$$\Big\{ s \in [0, \tau] \, : \, W_s \in H \Big\} \subset \bigcup_{j=1}^{\infty} \bigcup_{k \in \mathcal{N}_j} [T_k^j, S_k^j].$$

This is the covering which will give a good upper bound of the Hausdorff measure. Property (ii) ensures that we have control over the length of the individual covering intervals $[T_k^j, S_k^j]$. We now argue that also the cardinality of the sets $\mathcal{N}_j$ is not too large. We begin by observing that $|\tilde{W}_{S_k^j} - \tilde{W}_{T_k^j}| \geq \varepsilon_j$. As $\varepsilon_j \leq 2^{-Q}$ we infer from property (iii) that

$$S_k^j - T_k^j \geq \varepsilon_j^{4+\delta}. \tag{30}$$

From (i) applied to the family of intervals $I_m(\varepsilon_j^2)$, $m=0,\ldots,2^K\varepsilon_j^{-2}$, we infer that

$$\begin{aligned} M_j\varepsilon_j^{4+\delta} &\le \sum_{k\in\mathcal{M}_j} S_k^j - T_k^j \le Z\Big(\bigcup_{m=0}^{2^K\varepsilon_j^{-2}} I_m(\varepsilon_j^2), B(x_{m_j},2\varepsilon_j)\Big) \\ &\le cL(2^K\varepsilon_j^{-2})^2 L(m_j)^2 L(\varepsilon_j)^6\varepsilon_j^4(2^K\varepsilon_j^{-2}+1), \end{aligned}$$

and hence, for some positive constant c_3,

$$M_j \le c_3 L(\varepsilon_j)^{10}\varepsilon_j^{-2-\delta}\,.$$

Properties (iv) and (ii) imply that, for some positive constant c_4, for every k,

$$\operatorname{diam}\zeta[T_k^j,S_k^j] \le (S_k^j-T_k^j)^{1/(2+\delta)} \le cL(m_j)^3L(M_j)^3L(\varepsilon_j)^3\varepsilon_j^{4/(2+\delta)} \le c_4L(\varepsilon_j)^9\varepsilon_j^{4/(2+\delta)}\,. \tag{31}$$

We infer from this and the numbering of the points that there is $m\le 2^n$ such that for all k,

$$I_{n_j}(\varepsilon_j^2)\cap\zeta[T_k^j,S_k^j]\ne\emptyset \text{ implies } \zeta[T_k^j,S_k^j]\subset[a_m-2^{-n},a_m+2^{-n}]\,,$$

where n satisfies

$$2^{-(n+1)} \le c_4L(\varepsilon_j)^9\varepsilon_j^{4/(2+\delta)} < 2^{-n}\,.$$

On the one hand we then have

$$Z\Big([a_m-2^{-n},a_m+2^{-n}], B(x_{m_j},2\varepsilon_j)\Big) \ge \sum_{k\in\mathcal{N}_j}(S_k^j-T_k^j) \ge N_j\cdot\varepsilon_j^{4+\delta}\,,$$

and on the other hand, $[a_m-2^{-n},a_m+2^{-n}]$ can be covered by at most $2c_4L(\varepsilon_j)^9\varepsilon_j^{-\delta}$ intervals from the collection $\{I_k(\varepsilon_j^2): 0\le k\le 1/\varepsilon_j^2\}$. From (i), we hence know that

$$Z\Big([a_m-2^n,a_m+2^n], B(x_{m_j},2\varepsilon_j)\Big) \le 2cc_4L(m)^2L(m_j)^2L(\varepsilon_j)^{15}\varepsilon_j^{4-\delta}\,,$$

which together gives, for some positive constant c_5,

$$N_j \le c_5L(\varepsilon_j)^{19}\varepsilon_j^{-2\delta}\,. \tag{32}$$

Using again (ii) and (32), we get, for every $\alpha\in[0,1]$, and a suitable positive constant c_6,

$$\sum_{j=1}^\infty\sum_{k\in\mathcal{N}_j}(S_k^j-T_k^j)^\alpha \le c\sum_{j=1}^\infty N_j\Big(L(m_j)^3L(M_j)^3L(\varepsilon_j)^3\varepsilon_j^4\Big)^\alpha \le c_6\sum_{j=1}^\infty\varepsilon_j^{4\alpha-2\delta}L(\varepsilon_j)^{28}\,.$$

We choose $\alpha=(\gamma+2\delta)/4$, and get, upon recalling (29) and $2^{-K}\varepsilon_j\ge r_j\ge 2^{-(K+1)}\varepsilon_j$,

$$\sum_{j=1}^\infty\sum_{k\in N_j}(S_k^j-T_k^j)^{(\gamma+2\delta)/4} \le c_6\sum_{j=1}^\infty\varepsilon_j^\gamma L(\varepsilon_j)^{28} \le c_6\sum_{j=1}^\infty(2^{K+1}r_j)^\gamma L(2^Kr_j)^{28} \le c_6<\infty\,.$$

This implies that $\dim\{s: W_s\in H\}\le(\gamma+2\delta)/4$ and, as this holds for all $\delta>0$, the required result follows and Theorem 2.3 is proved. ∎

5 Proof of two auxiliary lemmas

In this section we prove Lemmas 3.5 and 4.2, which were used in the previous proofs. The proofs follow the same approach, we do the proof of Lemma 4.2 in full detail and indicate the necessary changes to obtain Lemma 3.5.

Let $\{B_s\}$ be a linear Brownian motion, L_s^0 its local time at level 0 and $\tau = \tau(B) = \inf\{s : L_s^0 = 1/2\}$. We intend to show that, for all $0 \le \lambda < 1/3e$, we have

$$\mathbb{E}\Big\{ \exp\Big(\lambda \int_0^\tau \mathbf{1}_{\{|B_s|\le 1\}}\, ds\Big)\Big\} \le \frac{e^{3e\lambda+1} - e^{3e\lambda}}{e - e^{3e\lambda}}\,.$$

We first check that, for the process $\{X_t\}$ defined as

$$X_t = \int_0^{t\wedge\tau} \mathbf{1}_{\{|B_s|\le 1\}}\, ds\,,$$

we have, for all stopping times S and associated σ-algebras $\mathcal{F}_S$, that $\mathbb{E}\{|X_\infty - X_S|\,|\mathcal{F}_S\} \le 1$ almost surely. Indeed,

$$\mathbb{E}\{X_\infty - X_S|\mathcal{F}_S\} \le \sup_{|a|\le 1} P_a\Big\{ \int_0^{\tau(B)} \mathbf{1}_{\{|B_s|\le 1\}}\, ds \Big\} = \sup_{|a|\le 1} \int_0^\infty P_a\{|B_s|\le 1, L_s^0 < 1/2\}\, ds,$$

where P_a refers to a Brownian motion B starting in $B_0 = a$. Recall the explicit density of the joint distribution of (B_s, L_s^0), see for example [1, (1.3.8)], which yields

$$\begin{aligned}
&\sup_{|a|\le 1} \int_0^\infty P_a\{|B_s|\le 1, L_s^0 < 1/2\}\, ds \\
&= \sup_{a\in[0,1]} \int_0^\infty \frac{ds}{\sqrt{2\pi s^3}} \int_{-1}^1 dz \int_0^{1/2} dy \Big\{ (|z|+y+a)\exp(-(|z|+y+a)^2/2s)\Big\} \\
&\quad + \int_0^\infty \frac{ds}{\sqrt{2\pi s}} \int_{-1}^1 dz \Big\{ \exp(-(z-a)^2/2s) - \exp(-(-|z|+a)^2/2s)\Big\} \\
&= \sup_{a\in[0,1]} \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-t/2}}{\sqrt{t}}\, dt + \int_0^1 (|x+a| - |x-a|)\, dx = 3\,.
\end{aligned}$$

With this property of $\{X_t\}$ at hand, we can see that, for every $a > 0$ and $S = \inf\{t : X_t = a\}$,

$$\begin{aligned}
P_0\{X_\infty > a + 3e\} &= \mathbb{E}\Big\{ P_0\{X_\infty - X_S > 3e \,|\, \mathcal{F}_S\} \Big| S < \infty \Big\} \cdot P\{S < \infty\} \\
&\le \frac{1}{3e}\mathbb{E}\Big\{ \mathbb{E}\{X_\infty - X_S \,|\, \mathcal{F}_S\} \Big| S < \infty \Big\} \cdot P\{S < \infty\} \\
&\le P_0\{X_\infty \ge a\}/e\,.
\end{aligned}$$

From this it is easy to see that $\mathbb{E}\{\exp(\lambda X_\infty)\} < \infty$ for all $0 \le \lambda < 1/3e$ and, similarly, we see that

$$\mathbb{E}\{\exp(\lambda X_\infty)\} = \int_1^\infty P_0\Big\{X_\infty > \log a/\lambda\Big\}\, da + 1$$

$$
\begin{aligned}
&= e^{3e\lambda} \cdot \int_{e^{-3e\lambda}}^{\infty} P_0\{X_\infty > 3e + \log a/\lambda\}\, da + 1 \\
&= e^{3e\lambda} \cdot \int_{1}^{\infty} P_0\{X_\infty > 3e + \log a/\lambda\}\, da + e^{3e\lambda} \\
&\le e^{3e\lambda-1} \cdot \int_{1}^{\infty} P_0\{X_\infty \ge \log a/\lambda\}\, da + e^{3e\lambda} \\
&= e^{3e\lambda-1}\, \mathbb{E}\{\exp(\lambda X_\infty)\} + e^{3e\lambda} - e^{3e\lambda-1}\,,
\end{aligned}
$$

from which we infer that $\mathbb{E}\{\exp(\lambda X_\infty)\} \le (e^{3e\lambda} - e^{3e\lambda-1})/(1 - e^{3e\lambda-1})$, as required to finish the proof of Lemma 4.2.

We proceed similarly to obtain Lemma 3.5. Replacing τ by T in the definition of the process $\{X_t\}$ we get, for every stopping time S,

$$\mathbb{E}\{X_\infty - X_S | \mathcal{F}_S\} \le \sup_{|a|\le 1} \int_0^\infty P_a\{|B_s| \le 1, \min_{0\le t\le s} B_t > -1\}\, ds\,.$$

From the joint distribution of Brownian motion and its minimum up to date, see e.g. [1, 1.2.8], we get that the value of the integral on the right is $3 - a^2 + 2a$, so that the supremum is 4. Hence,

$$P_0\{X_\infty > a + 4e\} \le \frac{1}{e} P_0\{X_\infty \ge a\}\,,$$

and we can proceed as before to get the result of Lemma 3.5. ∎

6 A packing dimension result for linear Brownian motion

In this section we prove Lemma 2.2. For the lower bound recall from Joyce and Preiss [9] that there is a Frostman type lemma for packing dimension: $\dim_P(E) > a$ is equivalent to the existence of a positive, finite measure μ on E such that,

$$\limsup_{r\downarrow 0} \frac{\log \mu(B(x,r))}{\log r} > a \text{ for } \mu\text{-almost all } x \in E\,.$$

Let ν be such a measure on E. For each $t \in E$ let $L^t(s)$ be the local time at level t and $\mu_t(ds) = L^t(ds)$ be the associated local time measure on the level set $\zeta^{-1}(\{t\})$. Recall that, given $\varepsilon > 0$, there is a random constant $c > 0$ such that

$$|L^t(u) - L^t(v)| \le c|u - v|^{1/2-\varepsilon}, \text{ for all } t > 0 \text{ and } u, v \in [0, \tau]. \tag{33}$$

Define a positive measure on $\zeta^{-1}(E)$ by

$$\mu = \int_E \mu_t\, \nu(dt)\,.$$

Fix $x \in E$ such that there is a sequence $r_n \downarrow 0$ with $\nu(B(x, r_n)) \le r_n^{\dim_P E - \varepsilon}$ for all n. Pick a sequence $s_n \downarrow 0$ such that $r_n = 3s_n\sqrt{\log(1/s_n)}$ and fix $y \in \zeta^{-1}(E)$ with $\zeta(y) = x$. By the property (33) of local time, there is, for every $\varepsilon > 0$, an N such that

$$\mu_t(B(y, s_n^2)) \le 2cs_n^{1-2\varepsilon} \text{ for all } t > 0 \text{ and } n \ge N\,.$$

Hence we infer that, for sufficiently large n,

$$\mu(B(y,s_n^2)) = \int_{B(\zeta(y),r_n)} \mu_t(B(y,s_n^2))\,\nu(dt) \le 2cr_n^{1+\dim_P(E)-3\varepsilon}\,.$$

Therefore the packing dimension of $\zeta^{-1}(E)$ is bounded below by $1/2+\dim_P(E)/2$. The corresponding upper bound can be obtained in a similar manner as for the Hausdorff case in [23]. Recall for this purpose that, for a Brownian motion $\{B_t\}$ in dimension $d\ge 2$, almost surely for all $A\subset[0,\infty)$,

$$\dim_P(B(A)) = 2\dim_P(A)\,.$$

This result was proved by Kaufman [10] for Hausdorff dimension, but the result extends without problems to the case of packing dimension. Now, for a planar Brownian motion $B_t=(\zeta_t,\xi_t)$, we have

$$\dim_P\{t\,:\,\zeta_t\in E\} = \dim_P\{t\,:\,B_t\in E\times\mathbb{R}\} \le \frac{1}{2}\dim_P\{E\times\mathbb{R}\} \le \frac{1}{2}+\frac{\dim_P(E)}{2}\,,$$

using $\dim_P(A\times B)\le\dim_P(A)+\dim_P(B)$. This completes the proof. ■

7 Final remarks

- Obvious changes to the final part of the proof show that the uniform dimension formula for the Brownian snake also holds for packing dimension instead of Hausdorff dimension. However, one cannot expect an interesting packing dimension spectrum for fast paths, because, in general, the packing dimension of nonempty limsup-random fractals is full. For example, one can check that for all dimensions $d\ge 2$, for all $t>0$, almost surely on $\{\mathfrak{X}_t\ne 0\}$, for every $0\le a\le\sqrt{2}$ the set of a-fast paths in $\operatorname{supp}\mathfrak{X}_t$ has packing dimension 2. This interesting feature of limsup-random fractals is discussed for example in [5] and [11].

- Combining our approach with the arguments of [12, Theorem 1.1] and [11, Remark 1.5] one can obtain the following variant of (3). Almost surely on $\{\mathfrak{X}_t\ne 0\}$,

$$\lim_{h\downarrow 0}\sup_{w\in\operatorname{supp}\mathfrak{X}_t}\frac{|w(t)-w(t-h)|}{\sqrt{h\log(1/h)}} = \sqrt{2}\,.$$

  This result on the *past* movements of all particles at time t can be compared with an analogous result of Tribe [26] on the *future* movements of their offspring. Almost surely on $\{\mathfrak{X}_t\ne 0\}$,

$$\lim_{h\downarrow 0}\sup_{w\in\operatorname{range}\mathfrak{X}}\frac{|w(t+h)-w(t)|}{\sqrt{h\log(1/h)}} = \sqrt{2}\,.$$

  The methods of [26] and later refinements of Dhersin and Le Gall [6], [7] are quite different from our methods and, generally, questions of past and future movements of particles seem to be of different nature.

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References

[1] A.N. Borodin and P. Salminen. Handbook of Brownian motion. Birkhäuser, Basel (1996).

[2] T. Cox, R. Durrett, and E.A. Perkins. Rescaled particle systems converging to super-Brownian motion. In: Perplexing problems in probability, Eds. M. Bramson, R. Durrett, pp. 269–284. Birkhäuser, Basel (1999).

[3] D.A. Dawson and E.A. Perkins. Historical Processes. *Memoirs Amer. Math. Soc.* **454** (1991).

[4] P. Deheuvels and D.M. Mason. Random fractal functional laws of the iterated logarithm. *Studia Sci. Math. Hungar.* **34**, 89–106 (1998).

[5] A. Dembo, Y. Peres, J. Rosen and O. Zeitouni. Thick points for spatial Brownian motion: multifractal analysis of occupation measure. *Ann. Probab.* **28(1)**, 1–35 (2000).

[6] J.S. Dhersin and J.F. Le Gall. Kolmogoroff's test for super-Brownian motion. *Ann. Probab.* **26**, 1041–1056 (1998).

[7] J.S. Dhersin. Lower functions for the support of super-Brownian motion. *Stoch. Proc. Appl.* **78**, 145–154 (1998).

[8] J. Hawkes. Trees generated by a simple branching process. *J. London Math. Soc.*, **24**, 373–384 (1981).

[9] H. Joyce and D. Preiss. On the existence of subsets of finite positive packing measure. *Mathematika* **42**, 15–24 (1995).

[10] R. Kaufman. Une propriété métrique du mouvement Brownien. *C. R. Acad. Sci. Paris* **268**, 727-728 (1969).

[11] D. Khoshnevisan, Y. Peres and Y. Xiao. Limsup random fractals. *El. J. Probab.* **5**, Paper 4, pp.1–24 (2000).

[12] D. Khoshnevisan and Z. Shi. Fast sets and points for fractional Brownian motion. To appear in Séminaire de Probabilités, XXXIV, Springer LNM (2000).

[13] J.F. Le Gall. Brownian excursions, trees and measure-valued branching processes. *Ann. Probab.* **19**, 1399–1439 (1991).

[14] J.F. Le Gall. A class of path-valued Markov processes and its applications to superprocesses. *Probab. Theory Related Fields* **95**, 25–46 (1993).

[15] J.F. Le Gall. Spatial branching processes, random snakes and partial differential equations. Birkhäuser, Basel, (1999).

[16] J.F. Le Gall and E.A. Perkins. The Hausdorff measure of the support of two-dimensional super-Brownian motion. *Ann. Probab.* **23**, 1719–1747 (1995).

[17] P. Mattila, *The Geometry of Sets and Measures in Euclidean Spaces.* Cambridge University Press, Cambridge, (1995).

[18] P. Mörters. A uniform dimension result for the Brownian snake. *Preprint series of WIAS Berlin, Preprint No.* **478**, (1999).

[19] S. Orey and S.J. Taylor. How often on a Brownian path does the law of the iterated logarithm fail? *Proc. London Math. Soc.(3)* **28**, 174–192 (1974).

[20] Y. Peres. Remarks on intersection-equivalence and capacity equivalence. *Ann. Inst. H. Poincaré (Physique Théorique)* **64**, 339–347 (1996).

[21] E.A. Perkins. On the Hausdorff dimension of the Brownian slow points. *Zeitschr. Wahr. verw. Gebiete* **64**, 369–399 (1983).

[22] L. Serlet. Quelques propriétés du super-mouvement brownien. *Thèse, Université Paris VI*, (1993).

[23] L. Serlet. Some dimension results for super-Brownian motion. *Probab. Theory Related Fields* **101**, 371-391 (1995).

[24] G. Slade. Lattice trees, percolation and super-Brownian motion. In: Perplexing problems in probability, Eds. M. Bramson, R. Durrett, pp. 35–53. Birkhäuser, Basel, (1999).

[25] S.J. Taylor. Multiple points for the sample paths of the symmetric stable process. *Zeitschr. Wahr. verw. Gebiete* **5**, 247–264 (1966).

[26] R. Tribe. Path properties of superprocesses. *Ph.D.-thesis, Univ. of British Columbia*, (1989).

[27] J. Verzani. The slow points in the support of historical super-Brownian motion. *Ann. Probab.* **23**, 56–70 (1995).