The Hausdorff dimension of the double points on the Brownian frontier

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Abstract: The frontier of a planar Brownian motion is the boundary of the unbounded component of the complement of its range. In this paper we find the Hausdorff dimension of the set of double points on the frontier.

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1. Introduction and statement of the results

Let $(W_s : 0 \leq s \leq \tau)$ be a standard planar Brownian motion running up to the first hitting time τ of the circle of unit radius around the origin, and consider the complement of its path, i.e.

$$\big\{x \in \mathbb{R}^2 : x \neq W_s \text{ for any } 0 \leq s \leq \tau\big\}.$$

This set is open and can be decomposed into connected components, exactly one of which is unbounded. We denote this component by U and define its boundary ∂U as the *frontier* of the Brownian path. Note that in this natural setup the frontier is a random closed curve enclosing the origin, which is contained in the unit disc and touches the unit circle in exactly one point. The frontier can be seen as the set of points on the Brownian path which are accessible from infinity and is therefore also called the *outer boundary* of Brownian motion.

Mandelbrot conjectured, based on a simulation and the analogy of the outer boundary and the self-avoiding walk, that the Brownian frontier has Hausdorff dimension 4/3, see [Ma82]. Rigorous confirmation of this conjecture, however, turned out to be a hard problem, which took a long time. In the late nineties Bishop, Jones, Pemantle and Peres [BP97] showed that the frontier has Hausdorff dimension strictly larger than one, and about the same time Lawler [La96] identified the Hausdorff dimension in terms of a (then) unknown constant, the disconnection exponent $\xi(2)$. A few years later, Lawler, Schramm and Werner, as one of the first applications of their SLE technique, found the explicit value of this constant and thus confirmed Mandelbrot's conjecture.

As a planar Brownian motion has points of any finite (and indeed infinite) multiplicity, and these points form a dense set of full dimension on the range, it is natural to ask whether there are multiple points also on the frontier.

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To begin with, it is easy to observe that the Brownian frontier must contain *double points* of the Brownian motion. The argument, which is due to Lévy [Le65], goes roughly like this: If there were no double points on the frontier, it would by construction contain a stretch of the original Brownian path. This would however imply that it had double points, which is a contradiction. Knowing that there are double points on the frontier, it is natural to ask, whether the frontier contains *triple points*. This problem was solved by Burdzy and Werner [BW96], who showed that, almost surely, there are *no* triple points on the frontier of a planar Brownian motion.

A second natural question that comes up is *how many* double points one can find on the Brownian frontier. Maybe surprisingly, it turns out that while the set

$$D = \big\{x \in \mathbb{R}^2 \colon x = W_s = W_t \text{ for distinct } s,t \in [0,\tau]\big\}$$

of double points has full Hausdorff dimension on the entire path, it does not have full dimension on the frontier. The following curious result is the main result of this paper.

Theorem 1.1. *Almost surely, the set of double points on the Brownian frontier satisfies*

$$\dim\big(D \cap \partial U\big) = \frac{\sqrt{97}+1}{24}\,.$$

Remark 1 By a variation of the proof one can see that the same formula holds when the Brownian motion is stopped at a fixed time, rather than the exit time τ from the unit disc.

Remark 2 Our contribution in the proof is to show that $\dim(D \cap \partial U) = 2 - \xi(4)$, where $\xi(4)$ is a disconnection exponent defined in Section 2. The full result follows from the actual value of $\xi(4)$, which was found by Lawler, Schramm and Werner, see [LSW01].

We would like to point out that our proof of Theorem 1.1 uses a technique different from that of [La96]. The latter paper works in the time domain and uses Kaufman's dimension doubling lemma, see e.g. [MP10, Theorem 9.28], to move to the spatial domain. We found this approach not suitable to deal with the lower bound in the case of double points, as it would require rather delicate estimates for Brownian motions which are constrained both in time and space. Instead, as in other problems related to multiple points, see e.g. [KM05], it is preferable to work directly in the plane. This allows us to work with Brownian motion without temporal constraint. In Section 2 we give an accessible sketch of the proof, and also state sharp estimates for disconnection probabilities, which are at the heart of our argument, and may be of independent interest, see Theorem 2.1. Full technical details of the proof are given in Section 3.

2. Proof of Theorem 1.1: Framework and ideas.

In the proof of this result, we consider Brownian motion up to the first exit time τ from a disc around the origin of fixed radius R, say larger than two, and adapt the definition of the set D of double points and the frontier U accordingly. We fix a compact square S_0 of unit sidelength contained in this disc, which does not contain the origin, and $\delta > 0$ smaller than half the distance of S_0 to the origin. We define a suitable collection $\mathcal{S}_n = \mathcal{S}_n(\delta)$ of nonoverlapping compact dyadic subsquares $S \subset S_0$ of sidelength 2^{-n}. Roughly speaking, the squares S in the collection are chosen to satisfy the following two conditions:

- the Brownian motion $(W_s \colon s \geq 0)$ hits the square S, moves a distance of order δ, and then hits S again before the killing time τ;
- the union of the paths outside the square S does not disconnect its boundary ∂S from infinity.

Clearly, these conditions are only heuristic and need to be made precise in the actual proof. In particular we will have to make sure that, for some integer $N(\delta) \in \mathbb{N}$, when $N(\delta) \le m \le n$ and $S \in \mathcal{S}_n(\delta)$, all dyadic squares $T \supset S$ with sidelength 2^{-m} are in $\mathcal{S}_m(\delta)$, and that

$$S_0 \cap D \cap \partial U = \bigcup_{\delta>0} \bigcap_{n=N(\delta)}^{\infty} \bigcup_{S \in \mathcal{S}_n(\delta)} S.$$

The Hausdorff dimension can then be determined with positive probability by verifying first and second moment criteria: Let $\xi > 0$ and assume that there exist constants $c_1, c_2, c_3 > 0$ such that

(i) for any dyadic subsquare $S \subset S_0$ of sidelength 2^{-n}, we have

$$c_1 2^{-\xi n} \le \mathbb{P}\big(S \in \mathcal{S}_n\big) \le c_2 2^{-\xi n};$$

(ii) for any pair of dyadic subsquares $S, T \subset S_0$ of sidelength 2^{-n} with distance of order 2^{-m}, $1 \le m \le n$, we have

$$\mathbb{P}\big(S, T \in \mathcal{S}_n\big) \le c_3\, 2^{-2\xi n} 2^{\xi m}.$$

These conditions imply that $\dim(S_0 \cap D \cap \partial U) \le 2 - \xi$ almost surely and $\dim(S_0 \cap D \cap \partial U) \ge 2 - \xi$ with positive probability, see [MP10, Theorem 10.43]. This is a standard technique (sometimes called *second moment method*) in fractal geometry and not too hard to verify, for example using the mass distribution principle.

To get hold of the constant ξ we first recall the definition of the *disconnection exponents* of planar Brownian motion. Let $\mathcal{B}(z, r)$ the open disc of radius r with centre z and suppose $(W_s^{(i)} \colon s \ge 0)$, for $i \in \{1, \dots, k\}$, are independent Brownian motions started on the unit circle $\partial\mathcal{B}(0, 1)$, and stopped upon leaving the concentric disc $\mathcal{B}(0, e^n)$ of radius e^n. We denote by $\mathfrak{B}_n$ the union of their paths, and by V_n the event that the set $\mathfrak{B}_n$ does not disconnect the origin from infinity, i.e. the origin is in the unbounded connected component of the complement of $\mathfrak{B}_n$. The disconnection exponent $\xi(k)$ is then defined by the requirement that there exist positive constants c_1 and c_2 such that, for any $n \in \mathbb{N}$,

$$c_1 \, \exp\{-n\,\xi(k)\} \le \mathbb{P}(V_n) \le c_2 \, \exp\{-n\,\xi(k)\}\,. \tag{2.1}$$

Lawler [La96] showed that the disconnection exponents are well-defined by this requirement, and Lawler, Schramm and Werner [LSW01] found the explicit values

$$\xi(k) = \frac{(\sqrt{24k+1} - 1)^2 - 4}{48}.$$

The intuition behind our proof is that locally the paths of a Brownian motion seen from a typical double point look like *four* Brownian motions started at this point. Roughly speaking, each of the two segments of the path crossing in the double point, is split into a part prior to hitting the double point, and a part after hitting the double point, amounting to four paths altogether. Hence the probability that a disc or square of diameter 2^{-n} containing this double point is not disconnected from infinity by these paths should be of order $2^{-n\xi(4)}$. In reality, things are a bit more delicate and this observation is only correct up to a factor, which is polynomial in n. Indeed, when we place a small disc around a potential double point, and split a path, which is conditioned to hit this disc, at the first hitting time, the time-reversal of the path up to this instant spends somewhat less time in the critical area near the disc and therefore non-disconnection probabilities are slightly larger than for Brownian motion starting on the circle. Here is the rigorous statement behind our argument.

Theorem 2.1. *Suppose $(W_s^{(i)} \colon s \ge 0)$ for $i \in \{1, \dots, k\}$ are independent Brownian motions started uniformly on the circle $\partial\mathcal{B}(0, \frac{1}{2}\, e^n)$, and stopped upon leaving the disc $\mathcal{B}(0, e^n)$, $n \ge 2$, i.e. at times*

$$T_n^{(i)} = \inf\{s > 0 \colon |W_s^{(i)}| = e^n\}.$$

Denote by

$$\mathfrak{B}_n = \bigcup_{i=1}^{k} \{ W_s^{(i)} \colon 0 \le s \le T_n^{(i)} \}$$

the union of the paths, and by V_n the event that $\mathfrak{B}_n$ does not disconnect the unit disc $\mathcal{B}(0,1)$ from infinity. Then there exist constants $c_1, c_2 > 0$ independent of n and the starting positions, such that

$$c_1\, n^k\, e^{-n\xi(2k)} \le \mathbb{P}\big(V_n \,\big|\, T_0^{(i)} < T_n^{(i)} \text{ for all } 1 \le i \le k\ \big) \le c_2\, n^k\, e^{-n\xi(2k)}.$$

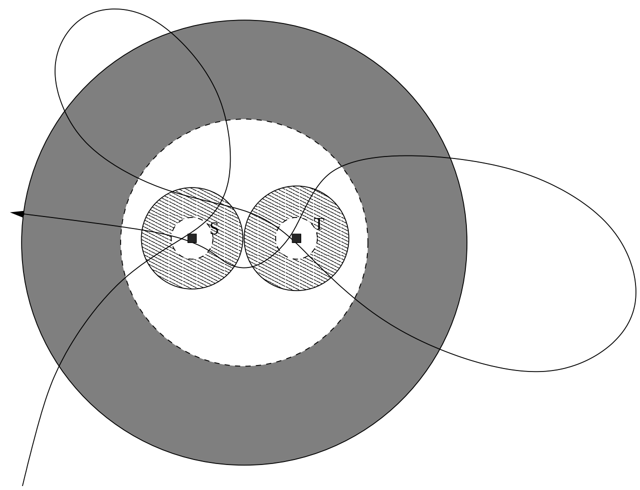

FIGURE 1. The second moment estimate. The small black squares S, T have sidelength 2^{-n} and distance of order 2^{-m}, for $n \ge m$. The dark grey annulus has outer radius of order δ and inner radius of order 2^{-m}, whereas the two light grey annuli have outer radius of order 2^{-m} and inner radius of order 2^{-n}.

Considering that $\mathbb{P}(T_0^{(i)} < T_n^{(i)})$ is a constant multiple of $\frac{1}{n}$, and applying Brownian scaling, we infer from this that (i) holds with $\xi = \xi(4)$. To explain that Theorem 2.1 also yields the second order estimate (ii) we refer to Figure 1. Observe that if both squares marked S and T are in $\mathcal{S}_n$ and have distance of order 2^{-m}, for some $m \le n$, then

- *twice* a square of sidelength 2^{-n} is hit from a distance of order δ, which has a probability of at most a constant multiple of $1/n^2$,
- independently, *twice* a square of sidelength 2^{-n} is hit from a distance of order 2^{-m}, which has a probability of at most a constant multiple of m^2/n^2.

Conditional on these hitting events,

- the large white disc is not disconnected from infinity by the paths in the dark grey annulus that surrounds it, which, by Theorem 2.1, has a probability of at most a constant multiple of $m^2 2^{-m\xi(4)}$,
- independently, the two small white discs inside are not disconnected from infinity by the paths in the light grey annuli that surround them, which has a probability of at most a constant multiple of $((n/m)^2 2^{-(n-m)\xi(4)})^2$. This is easily seen by removing the conditioning in Theorem 2.1 and then using Brownian scaling.

Altogether, the probability of the event $S, T \in \mathcal{S}_n$ is at most a constant multiple of $2^{-m\xi(4)}\, 2^{-2(n-m)\xi(4)}$, which readily implies (ii). These arguments show that, with positive probability,

$$\dim\big(D \cap \partial U\big) = 2 - \xi(4) = \frac{\sqrt{97}+1}{24}\,.$$

To verify that this holds almost surely, we observe that W_τ is always a point on the frontier. The previous arguments can be adapted to show that there is a positive probability that the double points on the frontier intersected with any small disc around this point have the given Hausdorff dimension. Then a variant of Blumenthal's zero-one law can be applied and yields the result with probability one.

Let us mention that our technique of proof can also be used to show that the dimension of the frontier itself is $2 - \xi(2) = 4/3$. However, in this case the original proof given by Lawler [La96] is easier. Also, the non-existence of triple points on the frontier follows rather easily from the fact that $\xi(6) > 2$, and indeed most of [BW96] is devoted to the derivation of this estimate, at a time when exact values of intersection exponents were not yet available.

We would further like to note that our method can be used to estimate the dimension of another type of sets. Burdzy and Werner conjectured [BW96] that there are no times t such that W_t is a triple point on the outer boundary of the set $\{W_s \colon s \in [0,t]\}$. These points may be called *pioneer-triple points*. Analogously defining pioneer-double and ordinary pioneer points of Brownian motion, our technique can be used to give the dimension of these sets as $2-\xi(5)$, $2-\xi(3)$ and $2-\xi(1)$, the latter being already known from [LSW01]. Unfortunately $2-\xi(5)$ is equal to zero, so that our result neither proves nor disproves the conjecture of Burdzy and Werner.

3. Proof of Theorem 1.1: Details.

For a Brownian motion $(W_t \colon t \geq 0)$ and sets $A_1, A_2, \ldots$ we define recursively

$$\tau(A_1) := \inf\big\{t > 0 \colon W_t \in A_1\big\},$$
$$\tau(A_1, \ldots, A_n) := \inf\big\{t > \tau(A_1, \ldots, A_{n-1}) \colon W_t \in A_n\big\}.$$

For any bounded set A we denote by $\mathcal{B}(A, r)$ the disc of radius r around the barycentre of A.

We keep the notation introduced in the previous section; recall in particular the meaning of the fixed parameters $\delta > 0$ and $R > 2$, on which all constants of this section may depend. Divide S_0 into its dyadic subsquares,

$$S_n^{i,j} := [x + i2^{-n}, x + (i+1)1^{-n}] \times [y + j2^{-n}, y + (j+1)2^{-n}], \qquad \text{for } i,j \in \{0, \ldots, 2^n - 1\},$$

where $(x,y) \in \mathbb{R}^2$ denotes the bottom left corner of S_0.

Definition 3.1. *Let $N(\delta)$ be the smallest integer satisfying $2^{-n} < \delta/16$. For $n \geq N(\delta)$ define the collection $\mathcal{S}_n$ of δ-good squares to be the set of all $S = S_n^{i,j}$ with the following properties:*

(1) *S is visited twice and between the visits the motion travels a distance close to δ; more precisely*

$$\tau(S, \partial\mathcal{B}(S, \delta - \tfrac{1}{\sqrt{2}}\, 2^{-n}), S) < \tau;$$

(2) *$\mathcal{B}(S, 2^{-n})$ is not disconnected from infinity by the path $\{W_s \colon s \in [0,\tau]\}$.*

We write S_n for the union of all $S \in \mathcal{S}_n$.

The difference of $(1/\sqrt{2})\,2^{-n}$ between the subtractive corrections for $n-1$ and n is exactly the distance between the centres of two dyadic squares $S_n^{i,j} \subset S_{n-1}^{k,l}$. It ensures that if $S_n^{i,j}$ is δ-good, then so is $S_{n-1}^{k,l}$. It is easy to see that a point x which is contained in every member of a decreasing sequence of δ-good squares is a double point, with two visits to x separated by an excursion reaching $\partial\mathcal{B}(x,\delta)$. We therefore have

$$S_0 \cap D \cap \partial U = \bigcup_{\delta>0} \bigcap_{n=N(\delta)}^{\infty} \bigcup_{S\in\mathcal{S}_n} S.$$

As explained in the previous section, the main step in the proof is to establish the following lemma, which implies that $\dim(D\cap\partial U) \le 2-\xi(4)$ almost surely, and $\dim(D\cap\partial U)\ge 2-\xi(4)$ with positive probability.

Lemma 3.2. *There exist constants $c_1, c_2, c_3 > 0$ such that for any $n \ge m \ge N(\delta)$ and any dyadic subsquare $S \subset S_0$ of sidelength 2^{-n}, we have*

$$c_1\, 2^{-\xi(4)n} \le \mathbb{P}\big(S \in \mathcal{S}_n\big) \le c_2\, 2^{-\xi(4)n},$$

and for any pair of dyadic subsquares $S, T \subset S_0$ of sidelength 2^{-n} with distance in $[\frac{1}{2}2^{-m}, 2^{-m}]$, we have

$$\mathbb{P}\big(S, T \in \mathcal{S}_n\big) \le c_3\, 2^{-2\xi(4)n}\, 2^{\xi(4)m}.$$

The following three sections are devoted to the proof of this lemma. In the course of the proof, we also provide the arguments needed to prove Theorem 2.1. The proof of Theorem 1.1 is then completed in Section 3.5 by means of a zero-one argument.

3.1 From Brownian paths to excursions. Given an annulus $A = \mathrm{cl}\,\mathcal{B}(x, r_1\vee r_2)\setminus\mathcal{B}(x, r_1\wedge r_2)$ we define an *excursion* from $\partial\mathcal{B}(x,r_1)$ to $\partial\mathcal{B}(x,r_2)$ as a continuous curve $\gamma\colon[0,\tau]\to A$ with

$$\gamma[0,\tau]\cap\partial\mathcal{B}(x,r_1) = \{\gamma(0)\}, \text{ and } \gamma[0,\tau]\cap\partial\mathcal{B}(x,r_2)=\{\gamma(\tau)\}.$$

To define a Brownian excursion, start a Brownian motion $\{W_t\colon t\ge 0)$ uniformly on $\partial\mathcal{B}(x,r_1)$ and define $\sigma = \sup\{t \le \tau(\partial\mathcal{B}(x,r_2))\colon W_t \in \partial\mathcal{B}(x,r_1)\}$. Then the random curve $(Y_t\colon 0\le t\le\tau)$ with $\tau = \tau(\partial\mathcal{B}(x,r_2)) - \sigma$ and $Y_t = W_{t+\sigma}$ defines a *Brownian excursion* from $\partial\mathcal{B}(x,r_1)$ to $\partial\mathcal{B}(x,r_2)$. As described, for example, in [LSW02] or [MS09], the time-reversal of a Brownian excursion from $\partial\mathcal{B}(x,r_1)$ to $\partial\mathcal{B}(x,r_2)$ is a Brownian excursion from $\partial\mathcal{B}(x,r_2)$ to $\partial\mathcal{B}(x,r_1)$.

Fix a square $S\subset S_0$ of sidelength $r_1 := 2^{-n}$ and radii $r_1 < r_2 < r_3$ sufficiently small to ensure $r_3<\delta$ and $r_2 < \frac{2}{3}\,r_3$. With such a configuration we associate natural curves and excursions embedded in a Brownian motion $(W_s\colon s\ge 0)$ as follows: Let

$$\begin{aligned} t_1^{(1)} &= \tau\big(\mathcal{B}(S,r_2)\big), & t_2^{(1)} &= \tau\big(\mathcal{B}(S,r_1)\big),\\ t_3^{(1)} &= \tau\big(S, \partial\mathcal{B}(S,r_1)\big), & t_4^{(1)} &= \tau\big(S,\partial\mathcal{B}(S,r_2)\big), \end{aligned}$$

and define the curves

$$\begin{aligned} W_1^{(1)}&\colon [0, t_2^{(1)} - t_1^{(1)}] \to \mathbb{R}^2\setminus\mathcal{B}\big(S,r_1\big), & W_1^{(1)}(t) &= W_{t_1^{(1)}+t},\\ W_2^{(1)}&\colon [0, t_4^{(1)} - t_3^{(1)}] \to \mathrm{cl}\,\mathcal{B}\big(S,r_2\big), & W_2^{(1)}(t) &= W_{t_3^{(1)}+t}. \end{aligned}$$

Similarly, we define curves associated with further visits to $\mathcal{B}(S,r_1)$. Indeed, for $i\ge 2$, let

$$t_5^{(i-1)} = \inf\big\{t\ge t_4^{(i-1)}\colon W_t\in\partial\mathcal{B}\big(S,r_3\big)\big\},$$

and let $W_1^{(i)}, W_2^{(i)}$ be defined as before, but for the Brownian motion started at time $t_5^{(i-1)}$. The next lemma states that these curves are almost independent.

Lemma 3.3. *Let $(X_t^{(i)}\colon 0 \le t \le \tau^{(i)})$, $1 \le i \le 2k$, be independent Brownian motions started uniformly on $\partial\mathcal{B}(S, r_2)$ and stopped upon reaching $\partial\mathcal{B}(S, r_1)$, if $1 \le i \le k$, and started uniformly on $\partial\mathcal{B}(S, r_1)$ and stopped upon reaching $\partial\mathcal{B}(S, r_2)$, if $k < i \le 2k$. Then the law of this family, and the joint law of the curves*

$$\big(W_1^{(i)}(t)\colon 0 \le t \le t_2^{(i)} - t_1^{(i)}\big), \quad i \le k;$$
$$\big(W_2^{(i)}(t)\colon 0 \le t \le t_4^{(i)} - t_3^{(i)}\big), \quad i \le k;$$

are mutually absolutely continuous with densities bounded by constants, which do not depend on the choice of the radii r_1, r_2, r_3, but may depend on the choice of δ.

Proof. By the Harnack principle, the laws of $W(t_1^{(1)})$ and $X_0^{(1)}$ are absolutely continuous with a bounded density. Moreover, conditional on these points, the curves $W_1^{(1)}$ and $X^{(1)}$ have the same law. Given their endpoints $W(t_2^{(1)})$ and $X_{\tau^{(1)}}^{(1)}$, using the Harnack principle again, the laws of $W(t_3^{(1)})$ and $X_0^{(k+1)}$, are absolutely continuous with a bounded density and, conditional on these points, the curves $W_2^{(1)}$ and $X^{(k+1)}$ have the same law. Together with the strong Markov property, this implies that the unconditional laws of the pairs $(W_1^{(1)}, W_2^{(1)})$ and $(X^{(1)}, X^{(k+1)})$ are mutually absolutely continuous with bounded densities. Iterating this argument further completes the proof. $\square$

For the lower bounds we need to study configurations of curves, which not only fail to disconnect, but do not even come close to doing so. To make this precise we introduce the notion of an α-nice configuration, which is a relaxation of the same notion in [LSW02].

Definition 3.4. *Suppose that $(\gamma_s^{(1)}\colon 0 \le s \le \tau^{(1)}), \ldots, (\gamma_s^{(k)}\colon 0 \le s \le \tau^{(k)})$ are planar curves started on the boundary of a fixed annulus A and stopped upon reaching the opposite boundary circle. This configuration of curves is called α-*nice *if*

(i) $\{\gamma_s^{(i)}\colon 0 \le s \le \tau^{(i)}\} \setminus A \subset \mathcal{B}(\gamma_0^{(i)}, \alpha\,|\gamma_0^{(i)}|)$, *and*

(ii) *the set*

$$\bigcup_{i=1}^k \{\gamma_t^{(i)}\colon 0 \le t \le \tau^{(i)}\} \cup \bigcup_{i=1}^k \mathcal{B}\big(\gamma_0^{(i)}, \alpha\,|\gamma_0^{(i)}|\big) \cup \bigcup_{i=1}^k \mathcal{B}\big(\gamma_{\tau^{(i)}}^{(i)}, \alpha\,|\gamma_{\tau^{(i)}}^{(i)}|\big)$$

does not disconnect the centre of the annulus from infinity.

Note that condition (i) is void if the curves are excursions between the bounding circles of the annulus.

As we often argue on an exponential scale, it is convenient to introduce the abbreviation $\mathcal{C}_a$ for $\partial\mathcal{B}(0, e^a)$ and to denote by $\mathcal{A}(a, b)$ the annulus between the circles $\mathcal{C}_a$ and $\mathcal{C}_b$. In several instances we will use that, for a planar Brownian motion started in x and $0 < e^a < |x| < e^b$,

$$\mathbb{P}\big(\tau(\mathcal{C}_a) < \tau(\mathcal{C}_b)\big) = \frac{b - \log|x|}{b - a}, \tag{3.1}$$

see [MP10, Theorem 3.18]. The following key lemma identifies the disconnection probabilities for Brownian excursions. Its proof is postponed to Section 3.3.

Lemma 3.5. *Fix a positive integer k and, for $n_1 < n_2$, suppose that*

$$(Y_t^{(i)}\colon 0 \le t \le \tau^{(i)}), \quad \text{for } i \in \{1, \ldots, k\},$$

are independent Brownian excursions from $\mathcal{C}_{n_1}$ to $\mathcal{C}_{n_2}$. Let $p(n_1, n_2, k)$ be the probability that the union of these excursions does not disconnect $\mathcal{C}_{n_1}$ from infinity and $p_\alpha(n_1, n_2, k)$ be the probability that they *form an α-nice configuration. Then there exist constants $C_1, C_2 > 0$, independent of n_1, n_2, and an $\alpha_0 > 0$ such that, for every $\alpha \in [0, \alpha_0]$,*

$$C_1\,(n_2-n_1)^k\,e^{(n_1-n_2)\,\xi(k)} \le p_\alpha(n_1,n_2,k) \le p(n_1,n_2,k) \le C_2\,(n_2-n_1)^k\,e^{(n_1-n_2)\,\xi(k)}.$$

The following result is the main tool from this section. It is derived from Lemma 3.5 by extracting suitable excursions from the curves.

Lemma 3.6. *Fix integers $0 \le \ell \le k$ and, for $n_1 < n_2$, suppose that*

$$(X_t^{(i)} \colon 0 \le t \le \tau^{(i)}), \qquad \text{for } i \in \{1, \dots, k\},$$

are independent Brownian motions, which in the case $1 \le i \le \ell$ are started uniformly in $\mathcal{C}_{n_1}$ and stopped upon reaching $\mathcal{C}_{n_2}$, and in the case $\ell < i \le k$ are started uniformly in $\mathcal{C}_{n_2}$ and stopped upon reaching $\mathcal{C}_{n_1}$. Let $q(n_1, n_2, k)$ be the probability that the union of the k paths does not disconnect $\mathcal{C}_{n_1}$ from infinity, and $q_\alpha(n_1, n_2, k)$ be the probability that the paths form an α-nice configuration. Then there exist constants $C_3, C_4 > 0$, independent of n_1, n_2, and an $\alpha_0 > 0$ such that, for every $\alpha \in [0, \alpha_0/2]$,

$$C_3\,\alpha^k\,e^{(n_1-n_2)\,\xi(k)} \le q_\alpha(n_1,n_2,k) \le q(n_1,n_2,k) \le C_4\,e^{(n_1-n_2)\,\xi(k)}.$$

Proof. We start with the upper bound. Let $\sigma_1^{(i)} = 0$ and, for $j \ge 1$, if $1 \le i \le \ell$ define stopping times

$$\tau_j^{(i)} = \inf\big\{t > \sigma_j^{(i)} \colon X_t^{(i)} \in \mathcal{C}_{n_1-1} \cup \mathcal{C}_{n_2}\big\}, \quad \sigma_{j+1}^{(i)} \;= \inf\big\{t > \tau_j^{(i)} \colon X_t^{(i)} \in \mathcal{C}_{n_1}\big\},$$

and similarly, if $\ell < i \le k$,

$$\tau_j^{(i)} = \inf\big\{t > \sigma_j^{(i)} \colon X_t^{(i)} \in \mathcal{C}_{n_1} \cup \mathcal{C}_{n_2+1}\big\}, \quad \sigma_{j+1}^{(i)} \;= \inf\big\{t > \tau_j^{(i)} \colon X_t^{(i)} \in \mathcal{C}_{n_2}\big\}.$$

By (3.1) the random variables $N^{(i)}$, defined by $\tau_{N^{(i)}}^{(i)} = \tau^{(i)}$, are geometric with success probability $1/(n_2 - n_1 + 1)$. Define the paths

$$X_j^{(i)} \colon [0, \tau_j^{(i)} - \sigma_j^{(i)}] \to \mathbb{R}^2, \qquad X_j^{(i)}(t) = X^{(i)}_{\sigma_j^{(i)}+t}.$$

In particular, the paths $X_{N^{(i)}}^{(i)}$ contain an excursion from $\mathcal{C}_{n_1}$ to $\mathcal{C}_{n_2}$, if $1 \le i \le \ell$, or from $\mathcal{C}_{n_2}$ to $\mathcal{C}_{n_1}$, if $\ell < i \le k$. Using this, together with the strong Markov property, the Harnack principle, and the time-reversibility of excursions, we obtain, for a suitable constant $C > 0$,

$$\begin{aligned} q(n_1,n_2,k) \le C\,p(n_1,n_2,k) &\sum_{\ell_1,\dots,\ell_k=1}^{\infty} \prod_{i=1}^{k} \mathbb{P}\big(N^{(i)} = \ell_i\big) \\ &\times \prod_{j=1}^{\ell_i-1} \mathbb{P}\big(X_j^{(i)} \text{ does not disconnect } \mathcal{C}_{n_1} \text{ from infinity } \big| \tau_j^{(i)} < \tau^{(i)}\big). \end{aligned}$$

As the factors in the second line are bounded from above by a constant $\rho < 1$, we obtain

$$\begin{aligned} q(n_1,n_2,k) &\le C\,p(n_1,n_2,k)\,\frac{1}{(n_2-n_1+1)^k} \sum_{\ell_1=1}^{\infty} \rho^{\ell_1-1} \cdots \sum_{\ell_k=1}^{\infty} \rho^{\ell_k-1} \\ &\le C\,C_2\,(1-\rho)^{-k}\,e^{(n_1-n_2)\,\xi(k)}. \end{aligned}$$

For the lower bound we consider last exit times $\sigma^{(i)}$ defined by

$$\sigma^{(i)} = \sup\big\{t < \tau^{(i)} \colon |X_t^{(i)}| = |X_0^{(i)}|\big\}.$$

Observe that the intersection of the events

(1) $\big\{X_t^{(i)}\colon 0\le t\le\sigma^{(i)}\big\}\subset\mathcal{B}\big(X_0^{(i)},\alpha\,|X_0^{(i)}|\big)$ for all $1\le i\le k$,

(2) the set
$$\bigcup_{i=1}^k\big\{X_t^{(i)}\colon \sigma^{(i)}\le t\le\tau^{(i)}\big\}\cup\bigcup_{i=1}^k\mathcal{B}\big(X_{\sigma^{(i)}}^{(i)},2\alpha\,|X_0^{(i)}|\big)\cup\bigcup_{i=1}^k\mathcal{B}\big(X_{\tau^{(i)}}^{(i)},2\alpha\,|X_{\tau^{(i)}}^{(i)}|\big)$$
does not disconnect $\mathcal{C}_{n_1}$ from infinity,

imply that the configuration is α-nice. By (3.1) the probability of (1) is bounded from below by a constant multiple of $(\alpha/(n_2-n_1))^k$. Conditional on (1) the paths $\{X_t^{(i)}\colon\sigma^{(i)}\le t\le\tau^{(i)}\}$ are independent Brownian excursions and hence the probability of (2) is bounded from below by a constant multiple of $p_{2\alpha}(n_1,n_2,k)$. Combining these two estimates and using Lemma 3.5 implies the result. □

Remark 3 Lemma 3.6 implies that in the definition of Brownian disconnection exponents we can allow that any of the paths, instead of starting in $\partial\mathcal{B}(0,1)$ and being stopped on leaving $\mathcal{B}(0,e^n)$, start on $\partial\mathcal{B}(0,e^n)$ and are stopped on hitting $\mathcal{B}(0,1)$. Observe also that Theorem 2.1 is an immediate consequence of Lemma 3.6.

3.2 Proof of Lemma 3.2. We now complete the proof of Lemma 3.2 using the framework provided in the previous section. We start with the easiest part.

Lemma 3.7. *There exists a constant $c_2>0$ such that, for any $n\ge N(\delta)$, and any dyadic subsquare $S\subset S_0$ of sidelength 2^{-n}, we have*
$$\mathbb{P}\big(S\in\mathcal{S}_n\big)\le c_2\,2^{-\xi(4)n}.$$

Proof. The event $\{S\in\mathcal{S}_n\}$ implies that, for $r_1=2^{-n}$, $r_2=\frac{\delta}{2}$ and $r_3=\delta-2^{-n-\frac12}$, the embedded paths $W_1^{(1)},W_2^{(1)},W_1^{(2)},W_2^{(2)}$ do not disconnect the disc $\mathcal{B}(S,2^{-n})$ from infinity. Combining Lemma 3.3 and Lemma 3.6, for $\ell=2$ and $k=4$, gives the result. □

The idea of the corresponding lower bound is to describe an explicit event, which implies $S\in\mathcal{S}_n$. The crucial part of this event is that $W_1^{(1)},W_2^{(1)},W_1^{(2)},W_2^{(2)}$ form a configuration of α-nice curves.

Lemma 3.8. *There exists a constant $c_1>0$ such that, for any $n\ge N(\delta)$, and any dyadic subsquare $S\subset S_0$ of sidelength 2^{-n}, we have*
$$c_1\,2^{-\xi(4)n}\le\mathbb{P}\big(S\in\mathcal{S}_n\big).$$

Proof. We keep the choice of $r_1=2^{-n}$, $r_2=\frac{\delta}{2}$ and $r_3=\delta-2^{-n-\frac12}$ as in the proof of the upper bound, and fix $0<\alpha<\alpha_0$. Define two strips S_1 and S_2 as the set of all points of distance at most $\alpha\frac{\delta}{2}$ to the straight line

- connecting the origin with the nearest point in $\partial\mathcal{B}(S,\frac{\delta}{2})$, respectively,
- connecting $W_{t_4^{(2)}}$ with the nearest point in $\partial\mathcal{B}(S,2R)$.

Now look at the five events

(1) the path $\{W_s\colon 0\le s\le t_1^{(1)}\}$ remains in the strip S_1,

(2) the path $\{W_s\colon t_2^{(1)}\le s\le t_3^{(1)}\}$ remains in the set $\mathcal{B}(S,2^{-n})\cup\mathcal{B}(W_{t_2^{(1)}},\alpha2^{-n})$,

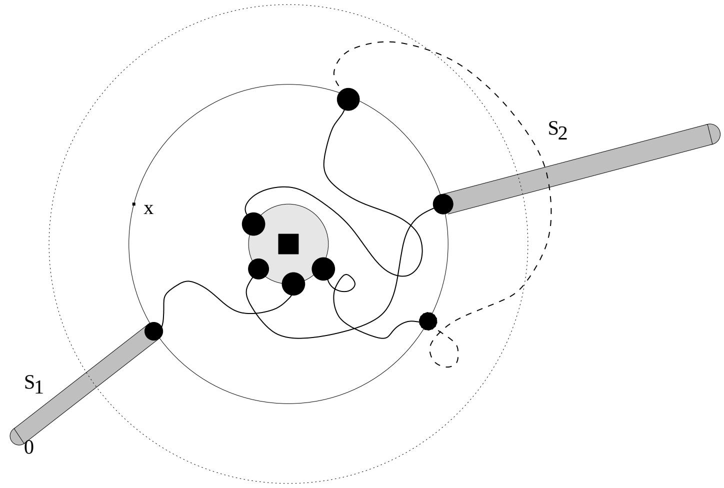

FIGURE 2. Illustration of the strategy for the Brownian path explained in Lemma 3.8. The three circles around the solid square S have radii $r_1 < r_2 < r_3$. The indicated configuration of curves is α-nice as the shaded disc is not disconnected from infinity by the union of the paths and the small solid discs. The initial and final parts of the path have to remain in the shaded strips, and the dashed path does not disconnect the point x from infinity.

(3) the path $\{W_s\colon t_2^{(2)} \leq s \leq t_3^{(2)}\}$ remains in the set $\mathcal{B}(S, 2^{-n}) \cup \mathcal{B}(W_{t_2^{(2)}}, \alpha 2^{-n})$,
(4) the path $\{W_s\colon t_4^{(2)} \leq s \leq \tau\}$ remains in the strip S_2,
(5) the four curves $W_1^{(1)}, W_2^{(1)}, W_1^{(2)}, W_2^{(2)}$ are an α-nice configuration.

Using the strong Markov property, Lemma 3.3, and Lemma 3.6, we see that the probability of the intersection of these five events is bounded from below by a constant multiple of $2^{-n\xi(4)}$. Given curve segments $(W_s\colon s \in [0, t_4^{(1)}] \cup [t_1^{(2)}, \tau])$ satisfying these events, we may identify a point $x \in \partial\mathcal{B}(S, \frac{\delta}{2})$ which is not disconnected from $\mathcal{B}(S, 2^{-n})$ by the set

$$\bigcup_{i,j=1}^{2} \{W_t\colon t_{2j-1}^{(i)} \leq t \leq t_{2j}^{(i)}\} \cup \bigcup_{i=1}^{2} \bigcup_{j=1}^{4} \mathcal{B}\big(W_{t_j^{(i)}}, \alpha\, |W_{t_j^{(i)}}|\big).$$

We then additionally require

(6) the path $\{W_s\colon t_4^{(1)} \leq s \leq t_1^{(2)}\}$ stays in $\mathcal{B}(t_4^{(1)}, \alpha\frac{\delta}{2}) \cup (\mathcal{B}(0, R) \setminus \mathcal{B}(S, \frac{\delta}{2}))$ and also does not disconnect the point x from infinity.

Observe with the help of Figure 2, that under the intersection of these six events we have $\{S \in \mathcal{S}_n\}$. The conditional probability of the sixth event is bounded from zero, with a bound depending on the choice of δ and α. This proves the lower bound of the lemma. □

In the last part we derive the second moment estimate in Lemma 3.2 by looking at the path at two scales, roughly speaking, the size and the distance of the two squares S, T.

Lemma 3.9. *There exists a constant $c_3 > 0$ such that for any $n \geq m \geq N(\delta)$ and any pair of dyadic subsquares $S, T \subset S_0$ of sidelength 2^{-n} with distance in $[\frac{1}{2}2^{-m}, 2^{-m}]$, we have*

$$\mathbb{P}\big(S, T \in \mathcal{S}_n\big) \leq c_3 \cdot 2^{-m\xi(4)} \cdot 2^{-2(n-m)\xi(4)}.$$

Proof. We denote by z be the middle point between the centres of S and T. If $S,T\in\mathcal{S}_n$ we know that the Brownian path visits the sets $S,\partial\mathcal{B}(z,\frac{\delta}{2}),S$ and $T,\partial\mathcal{B}(z,\frac{\delta}{2}),T$, both in that order, before exiting $\mathcal{B}(0,R)$, but there are eight possible combinations of these events, not counting possible additional visits of $\partial\mathcal{B}(z,\frac{\delta}{2})$. These can be described symbolically as follows:

$$\begin{aligned}
E_1:&\quad S\rightsquigarrow\partial\mathcal{B}(z,\tfrac{\delta}{2})\rightsquigarrow S\rightsquigarrow T\rightsquigarrow\partial\mathcal{B}(z,\tfrac{\delta}{2})\rightsquigarrow T\\
E_2:&\quad S\rightsquigarrow\partial\mathcal{B}(z,\tfrac{\delta}{2})\rightsquigarrow T\rightsquigarrow S\rightsquigarrow\partial\mathcal{B}(z,\tfrac{\delta}{2})\rightsquigarrow T\\
E_3:&\quad S\rightsquigarrow T\rightsquigarrow\partial\mathcal{B}(z,\tfrac{\delta}{2})\rightsquigarrow S\rightsquigarrow T\\
E_4:&\quad S\rightsquigarrow T\rightsquigarrow\partial\mathcal{B}(z,\tfrac{\delta}{2})\rightsquigarrow T\rightsquigarrow S\\
E_5:&\quad T\rightsquigarrow\partial\mathcal{B}(z,\tfrac{\delta}{2})\rightsquigarrow T\rightsquigarrow S\rightsquigarrow\partial\mathcal{B}(z,\tfrac{\delta}{2})\rightsquigarrow S\\
E_6:&\quad T\rightsquigarrow\partial\mathcal{B}(z,\tfrac{\delta}{2})\rightsquigarrow S\rightsquigarrow T\rightsquigarrow\partial\mathcal{B}(z,\tfrac{\delta}{2})\rightsquigarrow S\\
E_7:&\quad T\rightsquigarrow S\rightsquigarrow\partial\mathcal{B}(z,\tfrac{\delta}{2})\rightsquigarrow T\rightsquigarrow S\\
E_8:&\quad T\rightsquigarrow S\rightsquigarrow\partial\mathcal{B}(z,\tfrac{\delta}{2})\rightsquigarrow S\rightsquigarrow T
\end{aligned}$$

Note that, owing to possible additional visits, these events are not disjoint. Each of the events allows a similar estimate, and for notational convenience we focus here on the event E_4, which is satisfied in the case sketched in Figure 1. We first assume that $n\geq m+4$. In this case we define an increasing sequence of twenty-four stopping times:

$$\begin{aligned}
t_1&=\tau\big(\mathcal{B}(z,\tfrac{\delta}{2})\big), & t_2&=\tau\big(\mathcal{B}(z,2^{-m+2})\big),\\
t_3&=\tau\big(\mathcal{B}(S,2^{-m-3})\big), & t_4&=\tau\big(\mathcal{B}(S,2^{-n})\big),\\
t_5&=\tau\big(S,\partial\mathcal{B}(S,2^{-n})\big), & t_6&=\tau\big(S,\partial\mathcal{B}(S,2^{-m-3})\big),\\
t_7&=\tau\big(S,\mathcal{B}(T,2^{-m-3})\big), & t_8&=\tau\big(S,\mathcal{B}(T,2^{-n})\big),\\
t_9&=\tau\big(S,T,\partial\mathcal{B}(T,2^{-n})\big), & t_{10}&=\tau\big(S,T,\partial\mathcal{B}(T,2^{-m-3})\big),\\
t_{11}&=\tau\big(S,T,\partial\mathcal{B}(z,2^{-m+2})\big), & t_{12}&=\tau\big(S,T,\partial\mathcal{B}(z,\tfrac{\delta}{3})\big),\\
t_{13}&=\tau\big(S,T,\partial\mathcal{B}(z,\tfrac{\delta}{2})\big), & t_{14}&=\tau\big(S,T,\partial\mathcal{B}(z,\tfrac{\delta}{2}),\mathcal{B}(z,2^{-m+2})\big),\\
t_{15}&=\tau\big(S,T,\partial\mathcal{B}(z,\tfrac{\delta}{2}),\mathcal{B}(T,2^{-m-3})\big), & t_{16}&=\tau\big(S,T,\partial\mathcal{B}(z,\tfrac{\delta}{2}),\mathcal{B}(T,2^{-n})\big),\\
t_{17}&=\tau\big(S,T,\partial\mathcal{B}(z,\tfrac{\delta}{2}),T,\partial\mathcal{B}(T,2^{-n})\big), & t_{18}&=\tau\big(S,T,\partial\mathcal{B}(z,\tfrac{\delta}{2}),T,\partial\mathcal{B}(T,2^{-m-3})\big),\\
t_{19}&=\tau\big(S,T,\partial\mathcal{B}(z,\tfrac{\delta}{2}),T,\mathcal{B}(S,2^{-m-3})\big), & t_{20}&=\tau\big(S,T,\partial\mathcal{B}(z,\tfrac{\delta}{2}),T,\mathcal{B}(S,2^{-n})\big),\\
t_{21}&=\tau\big(S,T,\partial\mathcal{B}(z,\tfrac{\delta}{2}),T,S,\partial\mathcal{B}(S,2^{-n})\big), & t_{22}&=\tau\big(S,T,\partial\mathcal{B}(z,\tfrac{\delta}{2}),T,S,\partial\mathcal{B}(S,2^{-m-3})\big),\\
t_{23}&=\tau\big(S,T,\partial\mathcal{B}(z,\tfrac{\delta}{2}),T,S,\partial\mathcal{B}(z,2^{-m+2})\big), & t_{24}&=\tau\big(S,T,\partial\mathcal{B}(z,\tfrac{\delta}{2}),T,S,\partial\mathcal{B}(z,\tfrac{\delta}{2})\big).
\end{aligned}$$

For $1\leq j\leq 12$ we define the curves

$$W_j\colon[0,t_{2j}-t_{2j-1}]\to\mathbb{R}^2,\qquad W_j(t)=W_{t_{2j-1}+t}.$$

Although the twelve curves we have now defined are not independent, arguing with the Harnack principle as in Lemma 3.3 shows that this may be assumed at the expense of a constant multiplicative factor. The event $S,T\in\mathcal{S}_n$ implies that, on the large scale, the curves W_1,W_6,W_7,W_{12} do not disconnect $\mathcal{B}(z,2^{-m+2})$ from infinity, and, on the small scale, the curves W_2,W_3,W_{10},W_{11} do not disconnect $\mathcal{B}(S,2^{-n})$ from infinity, and the curves W_4,W_5,W_8,W_9 do not disconnect $\mathcal{B}(T,2^{-n})$ from infinity. Using Lemma 3.6 now shows that, for a suitable constant $C > 0$,

$$\mathbb{P}\big(\{S,T \in \mathcal{S}_n\} \cap E_4\big) \le C \ \cdot 2^{-m\xi(4)} \cdot 2^{-2(n-m)\xi(4)}.$$

This also holds in the degenerate case $n < m+4$, in which we can neglect the small scale and apply the first moment bound to a square with sidelength of order 2^{-m}, which contains both S and T. The final result follows now by summing over all estimates for $E_1, \dots, E_8$, in which we only have to define the stopping times suitably, in order to avoid an overlap of the intervals defining the twelve curves. $\square$

3.3 Proof of Lemma 3.5 We follow the lines of [LSW02]. Recall that $\mathcal{A}(0,n)$ denotes the annulus between the circles $\mathcal{C}_0$ and $\mathcal{C}_n$ and suppose that $(Y_t^{(i)} \colon 0 \le t \le \tau^{(i)})$ are independent Brownian excursions across $\mathcal{A}(0,n)$ for $i \in \{1,\dots,k\}$. The set

$$\mathcal{A}(0,n) \setminus \bigcup_{i=1}^{k} \{Y_t^{(i)} \colon 0 \le t \le \tau^{(i)}\}$$

contains at most k connected components joining $\mathcal{C}_0$ and $\mathcal{C}_n$. As defined in [LSW02] we let $L_n(j)$ be π times the extremal distance between $\mathcal{C}_0$ and $\mathcal{C}_n$ in the j-th of these components and L_n^k be the minimum of these extremal distances over all $1 \le j \le k$. The crucial fact about extremal distances used here is that L_n^k is finite if and only if the excursions do not disconnect $\mathcal{C}_0$ from infinity.

The following lemma is Theorem 3.1 in [LSW02].

Lemma 3.10. *For any $\lambda_0 > 0$ and any $k \in \mathbb{N}$, there exist $\xi(k,\lambda)$ such that*

$$\mathbb{E}\left[e^{-\lambda L_n^k}\right] \asymp n^k e^{-n\xi(k,\lambda)},$$

where $\asymp$ means that the ratio of the two sides is bounded away from zero and infinity by constants not depending on the choice of $\lambda \in (0,\lambda_0]$ and $n \ge 1$.

Letting $\lambda \downarrow 0$ in the previous lemma shows that the probability that the excursions do not disconnect $\mathcal{C}_0$ from infinity equals

$$\mathbb{P}\big(L_n^k < \infty\big) = \lim_{\lambda \downarrow 0} \mathbb{E}\left[e^{-\lambda L_n^k}\right] \asymp n^k e^{-n\xi(k)},$$

where $\xi(k) = \lim_{\lambda\downarrow 0} \xi(k,\lambda)$. In [LSW02] it is shown, using a similar argument for Brownian motion in place of Brownian excursion, that this $\xi(k)$ is also the disconnection exponent for Brownian motion as defined in our framework.

We now adapt Lemma 4.1 in [LSW02] for our purpose, remembering that our notion of an α-nice configuration is relaxed compared to the notion in [LSW02].

Lemma 3.11. *There exists an $\alpha_0 > 0$ such that, for any $\lambda_0 > 0$,*

$$\mathbb{E}\left[e^{-\lambda L_n^k} \mathbb{1}_{\alpha\text{-nice}}\right] \asymp n^k e^{-n\xi(k,\lambda)},$$

where the implied constants are not depending on the choice of $\lambda \in (0,\lambda_0]$, $\alpha \in (0,\alpha_0]$ and $n \ge 1$.

Proof. We assume that $n \ge 3$, and recall that Lemma 4.1 in [LSW02] implies that for any $j \le k$, $\varepsilon > 0$ there is $\alpha_0 > 0$ such that, for all $0 < \alpha \le \alpha_0$,

$$\mathbb{E}\left[e^{-\lambda L_n(j)}\right] - \mathbb{E}\left[e^{-\lambda L_n(j)} \mathbb{1}_{\alpha\text{-nice}}\right] \le \varepsilon\, \mathbb{E}\left[e^{-\lambda L_{n-2}^k}\right].$$

Hence, using Lemma 3.10 in the last step,

$$\begin{aligned}\mathbb{E}\left[e^{-\lambda L_n^k}\mathbb{1}_{\text{not }\alpha\text{-nice}}\right] &\le \sum_{j=1}^{k}\Big(\mathbb{E}\big[e^{-\lambda L_n(j)}\big]-\mathbb{E}\big[e^{-\lambda L_n(j)}\mathbb{1}_{\alpha\text{-nice}}\big]\Big)\\ &\le k\,\varepsilon\,\mathbb{E}\left[e^{-\lambda L_{n-2}^k}\right]\\ &\le n^k e^{-n\xi(k,\lambda)}\times\Big(\varepsilon\,k\,c_2\,\big(1-\tfrac{2}{n}\big)^k e^{2\xi(k,\lambda)}\Big).\end{aligned}$$

The bracket can be made arbitrarily small by choice of $\varepsilon>0$. The result now follows by combining this inequality with Lemma 3.10. ☐

The proof of Lemma 3.5 is completed by the following lemma together with Brownian scaling.

Lemma 3.12. *Suppose $(Y_t^{(i)}\colon 0\le t\le\tau^{(i)})$ are independent Brownian excursions across $\mathcal{A}(0,n)$ for $i\in\{1,\dots,k\}$. Then there exists an $\alpha_0>0$ such that, for every $\alpha\in[0,\alpha_0]$,*

$$\begin{aligned}\mathbb{P}\big(L_n^k<\infty\big) &\asymp n^k e^{-n\xi(k)}, \qquad\qquad \textit{and}\\ \mathbb{P}\big(\{L_n^k<\infty\}\cap\{\alpha\textit{-nice}\}\big) &\asymp n^k e^{-n\xi(k)}.\end{aligned}$$

Proof. We use monotone convergence and Lemma 3.10 to see

$$\mathbb{P}\big(L_n^k<\infty\big) = \lim_{\lambda\downarrow 0}\mathbb{E}\left[e^{-\lambda L_n^k}\right]\asymp\lim_{\lambda\downarrow 0} n^k e^{-n\xi(k,\lambda)} = n^k e^{-n\xi(k)},$$

and the second estimate is proved the same way using Lemma 3.11. ☐

3.4 The zero-one law. It remains to show that $\dim(D\cap\partial U)\ge 2-\xi(4)$ not only with positive probability, but actually with probability one. To do this, we need to identify a point on the frontier and establish a variant of Blumenthal's zero-one law for the germ-$\sigma-$algebra of Brownian motion around that point. As explained in [La96], one can use the point on the path which has the minimal coordinate in a suitable direction and derive the result from a result of Burdzy and San Martin [BS89]. This technique can be used to prove the result for Brownian motion running for unit time (recall Remark 1), and also to show in our context that any disc B is, almost surely, either disconnected from infinity (and hence does not contain any points on the frontier) or has $\dim(D\cap\partial U\cap B)=\frac{1}{24}(\sqrt{97}+1)$.

In order to prove the statement of Theorem 3.2 we can use a slightly easier argument, and use the endpoint W_τ of the path, which is easily seen to be always on the frontier. The time reversal of $(W_t\colon t\in[0,\tau])$ for which, by rotational invariance, we may assume that $W_\tau=1$, is the image under the analytic map

$$f\colon\mathbb{H}\to\mathcal{B}(0,1),\quad f(z)=\exp\{\mathrm{i}z\}$$

of a *half-plane excursion* started at zero. Half-plane excursions, as discussed in [La05], can be written as $(Y_t\colon t\ge 0)$ with $Y_t:=B_t+\mathrm{i}\hat{B}_t$ where $(B_t\colon t\ge 0)$ is a real Brownian motion and $(\hat{B}_t\colon t\ge 0)$ an independent three-dimensional Bessel process both started at zero. Note that the restriction of f to a subset of $\mathcal{B}(0,\pi)\cap\mathbb{H}$ is a conformal map and that any small neighbourhood of 1 in $\mathcal{B}(0,1)$ is the image of some small neighbourhood of zero in $\mathbb{H}$. By the conformal invariance of Brownian excursions (up to time change) and the fact that conformal mappings preserve the Hausdorff-dimension of sets, it will hence be sufficient to consider the lower bound on the dimension of the double points on the frontier for a half-plane excursion in neighbourhoods of zero. We therefore now denote by D the set of double points of a half-plane excursion $(Y_t\colon t\ge 0)$.

Let $I(a) := \{z \in \mathbb{C} \colon \Im(z) \in [0,a)\}$ and $J(a) := \{z \in \mathbb{C} \colon \Im(z) \in [a,\infty)\}$ and we denote by T_a the first hitting time of $J(a)$, i.e. $T_a := \inf\{s > 0 : Y_s \in J(a)\}$. For a set $A \subset \mathbb{H}$, we write $U(A)$ for the union of the unbounded connected components of $\mathbb{H} \setminus A$. We initially focus on the half-plane excursion up to time T_{3b}, where $b > 0$ is some small constant, and require the following variant of our main result so far.

Lemma 3.13. *For every $b > 0$ there is a positive probability that*

$$\dim \big(\mathcal{B}(0,b) \cap D \cap \partial U(\{Y_s \colon s \in [0,T_{3b}]\} \cup J(2b))\big) \geq 2 - \xi(4).$$

Let us first argue how to complete the proof using Lemma 3.13. Note first that the probability in Lemma 3.13 is independent of b, which is clear by scaling invariance. Also, by the transience of the excursion, there is a positive probability, independent of b, that once the excursion has reached $J(3b)$ it never visits $I(2b)$ again. This implies that, for every b, the event

$$A_b := \big\{\dim \big(\mathcal{B}(0,b) \cap D \cap \partial U(\{Y_s \colon s \in [0,T_\infty]\})\big) \geq 2 - \xi(4)\big\}$$

has a positive probability p, independent of b. It is easy to see that, for $b' < b$, we have $A_{b'} \subset A_b$ and therefore $\bigcap_{b>0} A_b$ is an event of the germ-σ-algebra of the half-plane excursion, which is trivial by Blumenthal's zero-one law. Hence this intersection and all events A_b must have probability one, because

$$\mathbb{P}\Big(\bigcap_{b>0} A_b\Big) = \inf_{b>0} \mathbb{P}\left(A_b\right) = p > 0.$$

So it remains to show Lemma 3.13, and by scaling invariance it suffices to discuss the case $b = 2$. The following lemma is a variant of the lower bound proved in the previous sections.

Lemma 3.14. *Let $(W_t \colon t \geq 0)$ be a planar Brownian motion started in some point z and stopped at the first hitting time T of the circle $\partial\mathcal{B}(z,\frac{1}{2})$. Fix $\gamma = \frac{1}{10}$ and an arbitrary square S_0 of sidelength $\frac{1}{8}$ in the upper half of $\mathcal{B}(z,\frac{1}{2})$ with distance more than 2γ from both the horizontal line through z and the circle $\partial\mathcal{B}(z,\frac{1}{2})$. Let*

$$\Gamma := \{W_s \colon 0 \leq s \leq T\} \cup \mathcal{B}(W_T,\gamma) \cup \big\{x \in \mathcal{B}(z,\tfrac{1}{2}) \colon \Im(x) \leq \Im(z)\big\},$$

and define $\partial U(\Gamma)$ to be the boundary of the unbounded component of the complement of Γ. Then, with positive probability,

$$\dim\left(S_0 \cap D \cap \partial U(\Gamma)\right) \geq 2 - \xi(4).$$

We now divide the strong Markov process $\{Y_t \colon t \in [0,T_6]\}$ into three parts: First the part up to the first hitting time T_1 of $J(1)$, second the part from T_1 up to the time T when the process has moved a distance of $1/2$ from its starting point Y_{T_1}, and third the remaining part starting from T up to the first hitting time T_6 of $J(6)$. For the three parts we require the following events:

(1) The first part remains in a small vertical strip around its starting point, more precisely

$$\{Y_t \colon t \in [0,T_1]\} \subset \mathsf{S} := \{z \in \mathbb{H} \colon \Re(z) \in [-\gamma,\gamma]\}.$$

(2) The second part satisfies $\dim\left(S_0 \cap D \cap \partial U(\Gamma)\right) \geq 2 - \xi(4)$ where the implied sets are defined as in Lemma 3.14 for the process $(Y_{T_1+t} \colon t \geq 0)$ in place of the Brownian motion.

(3) The third part intersects neither the strip S nor the disc $\mathcal{B}(Y_{T_1},\frac{1}{2})$ except possibly inside the ball $\mathcal{B}(Y_T,\gamma)$.

Observe, possibly with the help of Figure 3, that under the intersection of these three events we have

$$\dim(S_0 \cap D \cap \partial U(\{Y_t \colon t \in [0,T_6]\} \cup J(4))) \geq 2 - \xi(4).$$

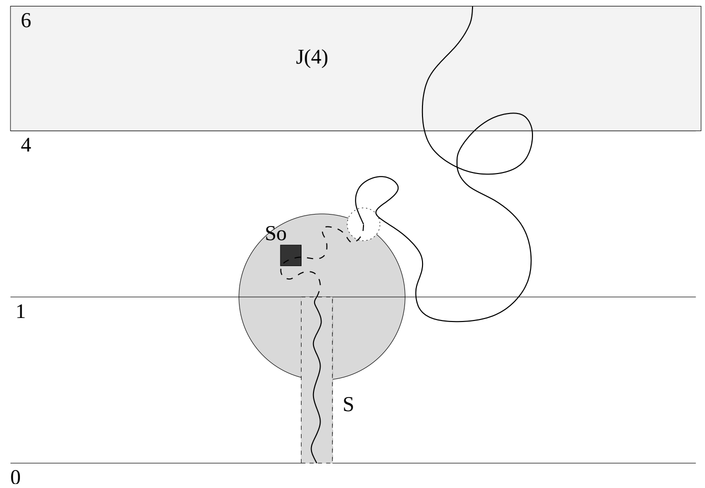

Figure 3. Any point in $S_0 \cap \partial U(\Gamma)$ is also in $U(\{Y_t \colon t \in [0, T_6]\} \cup J(4)))$, as the third part avoids the set $\{Y_t \colon t \in [0, T_1]\}$ completely and hits the set $\{Y_t \colon t \in [T_1, T]\}$, i.e. the dashed second part of the path, only inside $\mathcal{B}(Y_T, \gamma)$.

Moreover, the three events, and by the strong Markov property also their intersection, have positive probability. Indeed, for events (1) and (3) this is obvious. For event (2) recall from [La05, 5.3] that, given Y_{T_1}, the process $(Y_{T_1+s} \colon s \in [0, T_2 - T_1])$ is distributed like an ordinary Brownian motion conditioned to hit $J(2)$ before the real line. It is therefore absolutely continuous with respect to Brownian motion and the claim follows from Lemma 3.14. This completes the proof of Lemma 3.13.

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