A construction of catalytic super-Brownian motion via collision local time

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Abstract

We give a direct construction of a random measure which is equal in law to the collision local time between a catalytic super-Brownian motion and its catalytic measure. Under a regularity assumption on the catalytic measure, we show that the catalytic super-Brownian motion can be constructed deterministically from this measure.

Key words: super-Brownian motion, collision local time, catalytic branching, singular medium, subordination, excursion theory.
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1 Introduction

Catalytic super-Brownian motion is a measure-valued branching process in which, loosely speaking, the branching activity of each individual particle is controlled by the amount of collision of the particle path with a catalyst modelled by a measure. This class of processes interacting with a (in many cases singular) catalytic medium was introduced in Dawson & Fleischmann (1991, 1992) and Dynkin (1991), and has attracted considerable interest since, for a recent survey see, e.g., Klenke (1999).

The aim of the present paper is to give a new construction of catalytic super-Brownian motion for a class of singular catalysts. Starting point for our construction is the *collision local time* between the measure-valued process and its catalytic measure. The collision local time is a measure in space and time,

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which measures the total amount of collisions between the particles of the process and the catalytic measure in a given space-time region. In Theorem 1 we construct a random measure — without reference to the catalytic super-Brownian motion — and show that it is equal in law to the collision local time. The random measure is constructed as the total occupation measure of a (non-catalytic) superprocess with the Markovian motion process $(W \circ A_t^{-1}, A_t^{-1})$, where W is Brownian motion and A the additive functional associated with the catalytic measure. This Markov process can be interpreted as running the particle motion in the branching time scale, and simultaneously keeping track of the process time.

Having constructed the collision local time, it is natural to ask whether the catalytic super-Brownian motion is already determined by this measure, i.e. whether the full catalytic super-Brownian motion can be recovered deterministically from the collision local time. We give an affirmative answer to this question under an additional regularity condition on the catalytic measure. The condition relates the catalytic measure to the capacitary measure of its support, and allows us to use last exit decompositions and thus the tool of probabilistic potential theory. In Theorem 2 we show that under this condition the catalytic super-Brownian motion can be represented as the solution to the heat equation in the complement of the support of the catalytic measure with noisy boundary conditions given by the collision local time. Our result gives an independent construction of the catalytic super-Brownian motion for a large class of catalysts.

An interesting application of our construction of collision local time is as a tool for the probabilistic representation of solutions of a nonlinear Dirichlet-Neumann problem. For this we refer the reader to Delmas & Vogt (2003).

Our results extend those of Fleischmann & Le Gall (1995) for the special case of a one-point catalyst. In this situation the space component of the collision local time degenerates and explicit calculations can be performed.

In a closely related work, Delmas (1996) shows that outside the closed support S of the catalytic measure, catalytic super-Brownian motion is always determined by the collision local time between the catalytic super-Brownian motion and a *different* measure, the capacitary measure of S. However, we do not know any natural construction of this collision local time, which does not refer to the catalytic super-Brownian motion. Therefore his result does not provide an independent construction of the catalytic super-Brownian motion. Fleischmann & Klenke (1999) contains a similar result for a model with a catalyst changing in time.

In Dawson, Fleischmann & Mörters (2002) an approach similar to ours is used to directly represent a special catalytic super-Brownian motion using an exit-measure construction and the (non-catalytic) superprocess we use in our representation of the collision local time. This method is presented for the case of a stable random measure acting as a catalyst, however it applies whenever the catalyst is dense, or more precisely, that the additive functional of Brownian motion induced by the catalytic measure is strictly increasing. The paper focuses on applications of this representation to the clumping behaviour of catalytic super-Brownian motion with a stable catalyst.

An interesting extension of the ideas of Dawson, Fleischmann & Mörters (2002) to the construction of a different class of catalytic superprocesses is given in Klenke (2003). His representation uses a superprocess with a path-valued Markov process as underlying motion, an approach which is quite different from ours.

2 Statement of the results

Throughout the paper we are looking at the class of catalytic super-Brownian motions described in Delmas (1996). In this class the catalyst κ is a nonzero sigma-finite Borel measure on $\mathbb{R}^d$, such that there is a $\beta \in (0,1)$ such that $d-2+2\beta \geq 0$ and

$$\sup_{x\in\mathbb{R}^d} \int_{B_1(x)} \frac{\kappa(dy)}{||x-y||^{d-2+2\beta}} < \infty, \tag{1}$$

where $B_1(x)$ denotes the open ball around x with radius 1. In particular, this potential condition holds for all finite measures in dimension $d = 1$ and surface measures on $(d-1)$-dimensional manifolds in $\mathbb{R}^d$, $d \geq 2$. Condition (1) implies that the support of κ has at least Hausdorff dimension $d-2+2\beta$, and that κ does *not* charge polar sets of Brownian motion, see e.g. Port & Stone (1978, Chapter 6). Each κ satisfying (1) induces a unique (homogeneous) continuous additive functional A of the d-dimensional Brownian motion $W = \{W_t : t \geq 0\}$ satisfying

$$\mathbb{E}_x \left\{ \int_0^\infty \varphi(W_s, s)\, dA_s \right\} = \int_0^\infty ds \int \kappa(dy)\, p(s, x-y)\varphi(y,s), \tag{2}$$

for all $x \in \mathbb{R}^d$ and all nonnegative, measurable and bounded functions $\varphi \in \mathcal{B}_+^b(\mathbb{R}^d \times [0,\infty))$, where p denotes the d-dimensional Brownian transition density. Conversely, to any continuous additive functional we can associate a sigma-finite measure satisfying (2), called the *Revuz-measure*, see e.g. Blumenthal & Getoor (1968, Chapter VI.3).

The catalytic super-Brownian motion $X = \{X_t : t \geq 0\}$ associated to the catalytic measure κ, or alternatively the additive functional A, is constructed for example in Delmas (1996), and is called the *κ-catalytic super-Brownian motion.* It is a continuous Markov process with values in the space $\mathcal{M}_f(\mathbb{R}^d)$ of finite Borel measures on $\mathbb{R}^d$ equipped with the weak topology.

To characterise X by its log-Laplace functional, note that from (1) and (2) one can derive, see (Delmas, 1996, Lemme 2.1), that

$$a_T := \sup_{x\in\mathbb{R}^d} \mathbb{E}_x\{A_T\} < \infty, \text{ for } T > 0. \tag{3}$$

For every $T > 0$ and $0 \le s \le t \le T$, $\eta \in \mathcal{M}_f(\mathbb{R}^d)$, and all $\varphi \in \mathcal{B}_+^b(\mathbb{R}^d)$ with $a_T\|\varphi\|_\infty < 1$,

$$\mathbb{E}\Big\{e^{-\langle X_t,\varphi\rangle}\,\Big|\,X_s = \eta\Big\} = \exp\left(-\langle \eta, v(t-s,\cdot)\rangle\right), \tag{4}$$

where $\{v(t,x) : 0 \le t \le T, x \in \mathbb{R}^d\}$ is the unique nonnegative solution of

$$v(t,x) + \tfrac{1}{2}\,\mathbb{E}_x\left\{\int_0^\infty \mathbf{1}_{\{s<t\}} v^2(t-s,W_s)\,dA_s\right\} = \mathbb{E}_x\varphi(W_t). \tag{5}$$

We use both notations $\langle\rho,\psi\rangle$ and $\int\rho(dx)\,\psi(x)$ to denote integration of a function ψ with respect to a measure ρ.

The first main aim of this paper is to give a representation of the *collision local time* of the catalytic super-Brownian motion X with its catalyst κ. Following Delmas (1996) let

$$H_b^T := \Big\{\psi \in \mathcal{B}_+^b(\mathbb{R}^d\times[0,\infty)) : \operatorname{supp}\psi \subseteq \mathbb{R}^d\times[0,T]\Big\}$$

and $H_b := \bigcup_{T\ge0} H_b^T$. Then there is a (random) measure $\mathcal{L}_{\kappa,X}$ called the *collision local time* of X with its catalyst κ, which is supported on $\operatorname{supp}\kappa\times[0,\infty)$, such that for all $\psi\in H_b$ almost surely

$$\langle\mathcal{L}_{\kappa,X},\psi\rangle = \lim_{\varepsilon\downarrow0}\int_0^\infty ds\int\kappa(dy)\int X_s(dz)\,p(\varepsilon,z-y)\,\psi(y,s). \tag{6}$$

Moreover, almost surely for all $T \ge 0$, we have $\mathcal{L}_{\kappa,X}(\mathbb{R}^d\times[0,T)) < \infty$. We represent $\mathcal{L}_{\kappa,X}$ as the *total occupation measure* of a suitable (*non-catalytic*) superprocess U, the definition of which we now prepare.

Define $S := \operatorname{supp}\kappa$ to be the closed support of the catalyst, and $\tau := \tau_S := \inf\{s > 0 : W_s \in S\}$ to be the first hitting time of S. Denote by $S^* \subseteq S$ the set of all *regular* points of the closed set S, i.e.,

$$S^* := \{x \in S \,:\, \mathbb{P}_x\{\tau_S = 0\} = 1\}.$$

Of course for $d = 1$ we have $S^* = S$. In higher dimensions this is not always the case, but $S\setminus S^*$ is always polar, see e.g. Port & Stone (1978, Theorem 2.6.3).

We define the (homogeneous) Markov process Γ with state space $E := S^* \times [0, \infty)$ started in $(x, s) \in E$ by

$$\Gamma_r := (W \circ A_r^{-1},\, A_r^{-1} + s), \text{ for } r \in [0, A_\infty), \tag{7}$$

where W is a Brownian motion started in x and A^{-1} is defined to be the right continuous inverse

$$A_r^{-1} := \inf\{t > 0 : A_t > r\}$$

of the continuous additive functional A associated to W. The process Γ is easily seen to be continuous in probability. Let $\eta \in \mathcal{M}_f(\mathbb{R}^d)$ and let $\nu_\eta \in \mathcal{M}_f(E)$ be the law of the first hitting place and time of $\operatorname{supp} \kappa$ by a Brownian motion distributed according to η at time 0, i.e., define

$$\langle \nu_\eta, \psi \rangle = \int \eta(dx)\, \mathbb{E}_x \Big\{ \psi(W_\tau, \tau)\, 1_{\{\tau < \infty\}} \Big\}.$$

Denote by $U = \{U_s : s \geq 0\}$ the measurable version of the superprocess with spatial motion Γ and starting measure $U_0 = \nu_\eta$. See Le Gall (1999, Chapter 3) for a proof of existence of U.

Theorem* 1 *(Representation of the collision local time)
Let U the superprocess with spatial motion $\{\Gamma_r : r \geq 0\}$ and starting measure $U_0 = \nu_\eta$. Then the law of the total occupation measure of U defined by

$$V := \int_0^\infty ds\, U_s$$

agrees with the law of the collision local time $\mathcal{L}_{\kappa,X}$ of a κ-catalytic super-Brownian motion X started in $X_0 = \eta$ with its catalytic measure κ.

The case of a *one point catalyst* $\kappa = \delta_x$ in dimension $d = 1$ is treated in Fleischmann & Le Gall (1995). In this case the continuous additive functional A is the local time ℓ^x of the Brownian motion W at the catalytic point x, and A^{-1} is distributed as a (1/2)-stable subordinator, see e.g., Revuz & Yor (1999, p.240). Of course, $W \circ A^{-1} \equiv x$ and the spatial component is redundant in this case. As explained in Fleischmann & Le Gall (1995) the statement $\mathcal{L}_{\kappa,X} = V$ in law is a superprocess analogue of Lévy's well-known result that the local time measure has the same distribution as the occupation measure of a (1/2)-stable subordinator.

Note that V is constructed *without* any reference to X. This raises the question whether X can be constructed from V. The philosophy is that, because of the infinite branching intensity at the catalyst, V also describes the space-time points where branching occurs, hence where particles are born. At these points particles are released, and given V, they move independently according to Brownian excursions from S. By a law of large numbers effect, outside the catalyst one can only observe the average over these particles, i.e., a heat flow with boundary condition given by V. This philosophy turns out to be accurate under an additional condition on κ, which we now explain. To do this we need some tools from probabilistic potential theory, in particular the *last exit decomposition*, of which we would like to remind the reader first.

We use the concept of *exit systems*, following Maisonneuve (1975). Because the set $S \setminus S^*$ of irregular points is polar, we have

$$\mathcal{I} := \{t > 0 : W_t \in S\} = \{t > 0 : W_t \in S^*\}.$$

There is a *unique* continuous additive functional of the Brownian motion W denoted by $\{L_t : t \geq 0\}$ and called the *local time* on S^* which grows exactly on the set $\mathcal{I}$ such that its 1-potential fulfils,

$$\mathbb{E}_x \left\{ \int_0^\infty \mathrm{e}^{-t} \, dL_t \right\} = \mathbb{E}_x \{ \mathrm{e}^{-\tau_S} \},$$

for all $x \in \mathbb{R}^d$. The Revuz-measure μ associated to L by (2) is the *capacitary measure* of S, which may also be defined by

$$\mu = \int \mathbb{E}_x \left\{ \mathrm{e}^{-\tau_S}; W_{\tau_S} \in \cdot \right\} dx.$$

Furthermore, if S has vanishing Lebesgue measure, there exists a family of sigma-finite measures $\{H^x : x \in S^*\}$ on the space of excursions ω such that, for all $x \in \mathbb{R}^d$ and $\varphi \in \mathcal{B}_+(\mathbb{R}^d)$,

$$\mathbb{E}_x \{\varphi(W_t)\} = Q_t\varphi(x) + \mathbb{E}_x \left\{ \int_0^\infty \mathbf{1}_{\{s<t\}} H^{W_s}[\varphi(\omega(t-s))] \, dL_s \right\}, \tag{8}$$

where $\{Q_t : t \geq 0\}$ denotes the semigroup of Brownian motion killed on S, which is given by

$$Q_t\varphi(x) := \mathbb{E}_x \{\mathbf{1}_{\{t<\tau_S\}} \varphi(W_t)\}.$$

If we denote by $\rho(\omega) \in (0, \infty]$ the length of the excursion ω, then we let $\varphi(\omega(t)) = 0$ for all $t \geq \rho(\omega)$ on the right hand side of (8). Still following Maisonneuve (1975), the pair $\{L, \{H^x : x \in S^*\}\}$ is called the *exit system* associated to the set S and the Brownian motion W. The family $\{H^x : x \in S^*\}$ satisfies the *entrance law equation* for the semigroup $\{Q_t : t \geq 0\}$ of Brownian motion killed on S, i.e., for all $t \geq 0, s > 0$,

$$H^x[\varphi(\omega(t+s))] = \int H^x[\omega(s) \in dy] \int \varphi(z) \, Q_t(y, dz). \tag{9}$$

From (9) and the fact that $Q_t(y, \cdot)$ has a density one can see easily that each of the measure $H^x[\omega(s) \in dy]$ has a density.

Intuitively, equation (8) separates the computation of the transition probabilities of Brownian motion in two parts: the first arises from paths that do *not* hit S and the second from excursions starting at the *last-exit-time* from S which is distributed according to the measure dL.

In general the continuous additive functionals A and L do *not* agree. Assuming that L is absolutely continuous with respect to A and that S has zero Lebesgue measure, the following theorem provides the construction of the κ-catalytic process from the measure V.

Theorem 2 (Construction of κ-catalytic super-Brownian motion) *Suppose that S is a set of vanishing Lebesgue measure and that $\mu \ll \kappa$. Denote $f = d\mu/d\kappa$. Then the measure valued process $\{Z_t : t \geq 0\}$ defined by $Z_0 := \eta \in \mathcal{M}_f(\mathbb{R}^d)$ and*

$$\langle Z_t, \varphi \rangle = \langle \eta, Q_t \varphi \rangle + \iint V(dx\, ds)\, f(x)\, \mathbf{1}_{\{s<t\}}\, H^x[\varphi(\omega(t-s))], \qquad (10)$$

for any $\varphi \in \mathcal{B}_+^b(\mathbb{R}^d)$ and all $t > 0$, is a continuous version of the κ-catalytic super-Brownian motion X with start in $X_0 = \eta$.

For an *intuitive* interpretation of Theorem 2, note that the first term on the right hand side of (10) corresponds to the contribution of those particles that have *not* yet reached the catalyst at time t. To understand the second term, interpret $V(dx\, ds)$ as the reactant mass present at time s at the catalyst point x. The term $f(x)V(dx\, ds)$ now represents the mass of branching particles born at the space-time point (x, s), the smaller $f(x)$, the more likely it is that particles released from x are killed instantaneously. Particles born at time s in x move away from the catalyst according to the excursion laws H^x. As all these particles are independent we can only observe the average effect, and hence the only probabilistic input into the catalytic super-Brownian motion is the collision local time.

Remarks: The arguments from Delmas (1996, Theorème 8.1) confirm that the density field $\{z(t,x) : t > 0, x \in \mathbb{R}^d\}$ of Z is a solution of the *heat equation* on $(0,\infty) \times S^c$. Additionally to Theorem 2 it is possible to show by a second moment calculation that the boundary condition is given by V, i.e.,

$$\lim_{\varepsilon \downarrow 0} \int_0^\infty dt \int \kappa(dy) \int dx\, z(t,x) p(\varepsilon, x-y) \psi(y,t) = \iint V(dxds)\psi(x,s),$$

almost surely, for every $\psi \in H_b$. In other words, the collision local time $\mathcal{L}_{\kappa,Z}$ of Z and κ is given by V. For a more detailed proof we refer the reader to Vogt (2003).

The condition that there exists f with $\mu(dx) = f(x)\, \kappa(dx)$ is satisfied, for example, if κ is the surface measure on a smooth $d-1$ dimensional manifold, see Delmas & Vogt (2003). It may fail however in rougher cases, for example if κ is the Hausdorff measure on the middle-third Cantor set.

If S has positive Lebesgue measure, our representation holds only on the set $\mathbb{R}^d \setminus S$. However, if φ is supported by a subset of S a straightforward modification of the technique in Dawson, Fleischmann & Mörters (2002) yields a representation of $\langle X_t, \varphi\rangle$ in terms of the exit measures of U. This idea has been used in Fleischmann & Xiong (2003) to study an interface problem.

3 Proof of Theorem 1

Denote by $\mathcal{L}_{\kappa,X}$ the collision local time of a κ-catalytic super-Brownian motion $\{X_t : t \geq 0\}$ with start in $X_0 = \eta$ and its catalytic measure κ as defined in Section 2. We prove the theorem by checking that the Laplace functionals of $\mathcal{L}_{\kappa,X}$ and V agree.

Let $T > 0$, and fix $\varphi \in H_b^T$ such that

$$\|\varphi\|_\infty < 1/(2a_T^2). \tag{11}$$

Following Delmas (1996), we have the following expression for the moments of $\langle \mathcal{L}_{\kappa,X}, \varphi\rangle$,

$$\mathbb{E}_\eta\{\langle \mathcal{L}_{\kappa,X}, \varphi\rangle^p\} = p! \sum_{k=1}^{p} \frac{(-1)^{k+p}}{k!} \sum_{n_1+\cdots+n_k=p} \prod_{i=1}^{k} \langle \eta, c_{n_i}(\cdot, 0)\rangle,$$

where, for $x \in \mathbb{R}^d$, $s \geq 0$,

$$\begin{aligned} c_1(x,s) &:= \mathbb{E}_x\Big\{ \int_0^\infty dA_r\, \varphi(W_r, r+s)\Big\} \\ c_n(x,s) &:= -\tfrac{1}{2}\mathbb{E}_x\Big\{ \int_0^\infty dA_r \sum_{j=1}^{n-1} c_j(W_r, s+r) c_{n-j}(W_r, s+r)\Big\}. \end{aligned}$$

First, we show that for all $x \in \mathbb{R}^d$, $s \geq 0$ and all $\lambda \in (0,1)$ the sequence

$$\sum_{n=1}^{\infty} \lambda^n c_n(x,s)$$

is absolutely convergent. We proceed similarly to the proof of Lemma 4.2 in Delmas (1996). The function $g : (0,1) \to \mathbb{R}$ defined by $g(\lambda) := a_T^{-1}[1 - \sqrt{1-\lambda}]$ can be expanded in a power series $g(\lambda) = \sum_{n=0}^{\infty} \beta_n \lambda^n$ for all $\lambda \in (0,1)$. It is easy to check using its definition, that g also satisfies

$$g(\lambda)^2 = \tfrac{1}{a_T}\Big(2g(\lambda) - \tfrac{\lambda}{a_T}\Big).$$

Moreover, $\beta_0 = 0$ and $\beta_1 = \frac{1}{2a_T}$. By plugging the power series representation for g in the last equation we get, comparing the coefficients, for all $n \geq 2$,

$$\beta_n = \tfrac{1}{2} a_T \sum_{j=1}^{n-1} \beta_{n-j} \beta_j \;\geq\; 0.$$

Note that

$$\|c_1\| = \sup_{x \in \mathbb{R}^d, s \geq 0} \mathbb{E}_x \int_0^\infty dA_r \, \varphi(W_r, r+s) \;\leq\; a_T \|\varphi\|_\infty \;\leq\; \tfrac{1}{2a_T} \;=\; \beta_1.$$

Moreover, we get by induction on n that for all $n \geq 2$,

$$\|c_n\| \leq \tfrac{1}{2} a_T \sum_{j=1}^{n-1} \|c_{n-j}\| \, \|c_j\| \leq \tfrac{1}{2} a_T \sum_{j=1}^{n-1} \beta_{n-j} \, \beta_j \;=\; \beta_n. \tag{12}$$

Hence, the sequence $\sum_{n=1}^\infty \lambda^n \|c_n\|$ converges absolutely for all $\lambda \in (0,1)$.

Fix $\lambda \in (0,1)$ for the moment. Expanding the exponential series and using the formulas for the moments, we get

$$\begin{aligned}
\mathbb{E}_\eta \{\exp -\lambda \langle \mathcal{L}_{\kappa,X}, \varphi \rangle\} &= \sum_{p=0}^\infty \tfrac{(-\lambda)^p}{p!} \, \mathbb{E}_\eta \{\langle \mathcal{L}_{\kappa,X}, \varphi \rangle^p\} \\
&= 1 + \sum_{p=1}^\infty \sum_{k=1}^p \lambda^p \, \tfrac{(-1)^k}{k!} \sum_{n_1+\cdots+n_k=p} \prod_{i=1}^k \langle \eta, c_{n_i}(\cdot,0) \rangle \\
&= \sum_{k=0}^\infty \tfrac{(-1)^k}{k!} \Big[\sum_{n=1}^\infty \langle \eta, \lambda^n c_n(\cdot,0) \rangle \Big]^k \;=\; \exp - \Big\langle \eta, \sum_{n=1}^\infty \lambda^n c_n(\cdot,0) \Big\rangle.
\end{aligned}$$

As a special case of the *weighted occupation times formula* we compute the Laplace functionals of the occupation measure V as

$$\mathbb{E}_{\nu_\eta} \{\exp -\lambda \langle V, \varphi \rangle\} = \exp - \langle \nu_\eta, w_\lambda \rangle, \tag{13}$$

where $\{w_\lambda(x,s) \,:\, x \in S^*,\ s \geq 0\}$ is a nonnegative solution of

$$w_\lambda(x,s) + \tfrac{1}{2} \mathbb{E}_{(x,s)} \left\{ \int_0^\infty dr \, w_\lambda^2(\Gamma_r) \right\} = \lambda \, \mathbb{E}_{(x,s)} \left\{ \int_0^\infty dr \, \varphi(\Gamma_r) \right\}. \tag{14}$$

This formula is shown for example in Le Gall (1999, Corollary II.3.9). The solution w_λ can be extended consistently to $\mathbb{R}^d \times [0,\infty)$ by letting $w_\lambda(x,s) = \mathbb{E}_x \{ w_\lambda(W_\tau, \tau + s) \mathbf{1}_{\{\tau < \infty\}} \}$. Note that we do not claim uniqueness in (14).

The first step to complete the proof is to show that the function defined by $\tilde{w}_\lambda(x,s) := \sum_{n=1}^\infty \lambda^n c_n(s,x)$ solves (14) for $x \in S^*$, $s \geq 0$. First using the definition (7) of Γ, substitution and (2) we obtain,

$$\begin{aligned}
\mathbb{E}_{(x,s)} \left\{ \int_0^\infty dr \, \varphi(\Gamma_r) \right\} &= \mathbb{E}_x \left\{ \int_0^\infty dr \, \varphi(W \circ A_r^{-1}, A_r^{-1} + s) \right\} \\
&= \mathbb{E}_x \Big\{ \int_0^\infty dA_r \, \varphi(W_r, r+s) \Big\} = c_1(x,s).
\end{aligned}$$

Moreover, as $\varphi \in H_b^T$,

$$
\begin{aligned}
-\tfrac{1}{2}\mathbb{E}_{(x,s)}\Big\{\int_0^\infty dr\, \tilde{w}_\lambda^2(\Gamma_r)\Big\} &= -\tfrac{1}{2}\mathbb{E}_{(x,s)}\Big\{\int_0^\infty dr\,\Big(\sum_{n=1}^\infty \lambda^n c_n(\Gamma_r)\Big)^2\Big\} \\
&= -\tfrac{1}{2}\mathbb{E}_{(x,s)}\Big\{\int_0^\infty dr\, \sum_{n=1}^\infty\sum_{j=1}^n \Big(\lambda^{n+1-j}c_{n+1-j}(\Gamma_r)\lambda^j c_j(\Gamma_r)\Big)\Big\} \\
&= -\tfrac{1}{2}\mathbb{E}_{x}\Big\{\int_0^\infty dA_r\, \mathbf{1}_{\{r<T\}}\sum_{n=1}^\infty\sum_{j=1}^n \Big(\lambda^{n+1}c_{n+1-j}(W_r, r+s)c_j(W_r,r+s)\Big)\Big\} \\
&= \sum_{n=1}^\infty -\tfrac{1}{2}\lambda^{n+1}\mathbb{E}_x\Big\{\int_0^\infty dA_r\, \mathbf{1}_{\{r<T\}}\sum_{j=1}^n c_{n+1-j}(W_r,r+s)c_j(W_r,r+s)\Big\} \\
&= \sum_{n=1}^\infty \lambda^{n+1}c_{n+1}(x,s) = \sum_{n=2}^\infty \lambda^n c_n(x,s).
\end{aligned}
$$

Note, that we use bounded convergence for the fourth equality, which is allowed since, by (12), the modulus of the double sum in the third line is bounded by $(2/a_T)\sum_{n=1}^\infty \lambda^n\beta_n < \infty$.

Hence, $\tilde{w}_\lambda$ solves the integral equation (14). As w_λ is nonnegative, it is easy to see from (14) that $w_\lambda(x,s) = 0$ for all $s \geq T$. Moreover, $c_n(x,s) = 0$ for all $s \geq T$, and hence we also have $\tilde{w}_\lambda(x,s) = 0$ for all $s \geq T$. Thus,

$$
\begin{aligned}
\|w_\lambda - \tilde{w}_\lambda\|_\infty &\leq \sup_{x\in\mathbb{R}^d, s\in[0,T]} \mathbb{E}_x\Big\{\mathbf{1}_{\{\tau<\infty\}}|w_\lambda(W_\tau,\tau+s) - \tilde{w}_\lambda(W_\tau,\tau+s)|\Big\} \\
&\leq \tfrac{1}{2}\sup_{x\in\mathbb{R}^d, s\in[0,T]} \mathbb{E}_x\Big\{\int_0^T dA_r\, |w_\lambda^2(W_r,r+s) - \tilde{w}_\lambda^2(W_r,r+s)|\Big\} \\
&\leq a_T^2\|\varphi\|_\infty\, \|w_\lambda - \tilde{w}_\lambda\|_\infty,
\end{aligned}
\tag{15}
$$

which implies $\tilde{w}_\lambda = w_\lambda$, by (11).

Using the definition of ν_η,

$$
\langle \nu_\eta, w_\lambda\rangle = \int \eta(dy)\,\mathbb{E}_y\Big\{w_\lambda(W_\tau,\tau)\mathbf{1}_{\{\tau<\infty\}}\Big\} = \langle \eta, w_\lambda(\cdot,0)\rangle.
$$

We get that, for *every* $\varphi \in H_b$, and all sufficiently small $\lambda > 0$,

$$
\begin{aligned}
\mathbb{E}_\eta\Big\{\exp -\lambda\langle \mathcal{L}_{\kappa,X},\varphi\rangle\Big\} &= \exp -\langle \eta, w_\lambda(\cdot,0)\rangle \\
&= \exp -\langle \nu_\eta, w_\lambda\rangle = \mathbb{E}_{\nu_\eta}\Big\{\exp -\lambda\langle V,\varphi\rangle\Big\},
\end{aligned}
$$

hence all moments of $\langle \mathcal{L}_{\kappa,X},\varphi\rangle$ and $\langle V,\varphi\rangle$ agree. The power series built from these moments has a positive radius of convergence, and by analytic continuation the characteristic functions also agree. Hence both random measures have the same distribution. $\square$

4 Proof of Theorem 2

We first show that the measure valued process $\{Z_t : t \geq 0\}$ defined by (10) has the same finite-dimensional distributions as a κ-catalytic super-Brownian motion. Let $n \in \mathbb{N}$ and $T > 0$. Fix $0 < t_1 < \ldots < t_n \leq T$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{B}^b_+(\mathbb{R}^d)$ such that

$$2a_T \sum_{i=1}^{n} \|\varphi_i\|_\infty < 1. \tag{16}$$

Using (10) we can rewrite the Laplace functional of Z as

$$\mathbb{E}\Big\{ \exp - \sum_{i=1}^{n} \langle Z_{t_i}, \varphi_i \rangle \,\Big|\, Z_0 = \eta \Big\} = \mathbb{E}\Big\{ \exp - \Big(\sum_{i=1}^{n} \langle \eta, Q_{t_i}\varphi_i \rangle + \langle V, \phi \rangle \Big) \Big\}, \tag{17}$$

where

$$\phi(x, s) := \sum_{i=1}^{n} 1_{\{s<t_i\}} f(x)\, H^x[\varphi_i(\omega(t_i - s))].$$

Recall from (13), (14) that

$$\mathbb{E}\{\exp - \langle V, \phi \rangle \,|\, U_0 = \nu_\eta\} = \exp - \langle \nu_\eta, w \rangle,$$

where $\{w(x, s) : x \in S^*,\, s \geq 0\}$ is a nonnegative solution of

$$w(x, s) + \tfrac{1}{2}\, \mathbb{E}_{(x,s)} \left\{ \int_0^\infty dr\, w^2(\Gamma_r) \right\} = \mathbb{E}_{(x,s)} \left\{ \int_0^\infty dr\, \phi(\Gamma_r) \right\}. \tag{18}$$

Using the definition of Γ and substitution we get, for all $x \in S^*$ and $s \geq 0$,

$$\begin{aligned}
\mathbb{E}_{(x,s)} \left\{ \int_0^\infty dr\, w^2(\Gamma_r) \right\} &= \mathbb{E}_x \left\{ \int_0^\infty dr\, w^2(W_{A_r^{-1}}, A_r^{-1} + s) \right\} \\
&= \mathbb{E}_x \left\{ \int_0^\infty dA_r\, w^2(W_r, r + s) \right\}.
\end{aligned}$$

Similarly, we get by substitution, the definition of ϕ and f, and the last exit decomposition (8),

$$\begin{aligned}
&\mathbb{E}_{(x,s)}\Big\{ \int_0^\infty dr\, \phi(\Gamma_r) \Big\} = \mathbb{E}_x\Big\{ \int_0^\infty \phi(W_r, r + s)\, dA_r \Big\} \\
&\quad = \sum_{i=1}^{n} \mathbb{E}_x\Big\{ \int_0^\infty dA_r\, 1_{\{r+s<t_i\}} H^{W_r}[\varphi_i(\omega(t_i - r - s))] f(W_r) \Big\} \\
&\quad = \sum_{i=1}^{n} \mathbb{E}_x\Big\{ \int_0^\infty dL_r\, 1_{\{r<t_i-s\}} H^{W_r}[\varphi_i(\omega(t_i - r - s))] \Big\} \\
&\quad = \sum_{i=1}^{n} 1_{\{s<t_i\}} \Big(\mathbb{E}_x\{\varphi_i(W_{t_i-s})\} - Q_{t_i-s}\varphi_i(x) \Big).
\end{aligned}$$

Note that, for $x \in S^*$, we have $Q_{t_i - s}\varphi_i(x) = 0$. Hence we have just shown that $\{w(x,s) : x \in S^*,\ s \geq 0\}$ solves

$$w(x,s) \,+\, \tfrac{1}{2}\mathbb{E}_x\Big\{ \int_0^\infty dA_r\, w^2(W_r, s+r)\Big\} = \sum_{i=1}^n \mathbf{1}_{\{s<t_i\}} \mathbb{E}_x\{\varphi_i(W_{t_i - s})\}. \qquad (19)$$

We can now define $w(x,s)$ consistently for all $x \in \mathbb{R}^d$, $s \geq 0$, by

$$w(x,s) := \sum_{i=1}^n \mathbf{1}_{\{s<t_i\}}\, \mathbb{E}_x\, \varphi_i(W_{t_i - s}) - \tfrac{1}{2}\mathbb{E}_x\Big\{ \int_0^\infty w^2(W_r, r+s)\, dA_r\Big\}.$$

Let X be a κ-catalytic super-Brownian motion with start in $X_0 = \eta$. From (5) using induction on n and the Markov property of X, see e.g. Le Gall (1999, Proposition II.3.7), we can rewrite the dual equation of X in its backward form, i.e.,

$$\mathbb{E}_\eta \Big\{ \exp - \sum_{i=1}^n \langle X_{t_i}, \varphi_i\rangle \Big\} = \exp - \langle \eta, u(\cdot, 0)\rangle,$$

where $\{u(x,s) : x \in \mathbb{R}^d,\ s \geq 0\}$ is the *unique* nonnegative solution of

$$u(x,s) \,+\, \tfrac{1}{2}\mathbb{E}_x\Big\{ \int_0^\infty dA_r\, u^2(W_r, r+s)\,\Big\} = \sum_{i=1}^n \mathbf{1}_{\{s<t_i\}}\, \mathbb{E}_x\, \varphi_i(W_{t_i - s}). \qquad (20)$$

Note that the uniqueness follows similarly to (15) using assumption (16). By (19) we have that $\{w(x,s) : x \in \mathbb{R}^d,\ s \geq 0\}$ is a nonnegative solution of (20) and hence $w(x,s) = u(x,s)$ for all $x \in \mathbb{R}^d$, $s \geq 0$.

Using the definition of ν_η, the fact that w and u agree for all $x \in S^*$, equation (20) twice, and the strong Markov property,

$$\begin{aligned}
\langle \nu_\eta, w\rangle &= \int \eta(dy)\, \mathbb{E}_y \Big\{ w(W_\tau, \tau)\mathbf{1}_{\{\tau<\infty\}}\Big\} = \int \eta(dy)\, \mathbb{E}_y \Big\{ u(W_\tau, \tau)\mathbf{1}_{\{\tau<\infty\}}\Big\} \\
&= \int \eta(dy) \Big(\sum_{i=1}^n \mathbb{E}_y \{\mathbf{1}_{\{\tau<t_i\}}\varphi_i(W_{t_i})\} - \tfrac{1}{2}\mathbb{E}_y\Big\{ \int_0^\infty u^2(W_r, r)\, dA_r\Big\}\Big) \\
&= \int \eta(dy) \Big(- \sum_{i=1}^n \mathbb{E}_y \{\mathbf{1}_{\{\tau\geq t_i\}}\varphi_i(W_{t_i})\} + u(y,0)\Big),
\end{aligned}$$

and therefore, by definition of Q_t,

$$\sum_{i=1}^n \langle \eta, Q_{t_i}\varphi_i\rangle + \langle \nu_\eta, w\rangle = \langle \eta, u(\cdot,0)\rangle.$$

Hence,

$$\begin{aligned}
\mathbb{E}\Big\{ \exp- \sum_{i=1}^n \langle Z_{t_i}, \varphi_i\rangle \,\Big|\, U_0 = \nu_\eta\Big\} &= \exp - \Big(\sum_{i=1}^n \langle \eta, Q_{t_i}\varphi_i\rangle + \langle \nu_\eta, w\rangle\Big) \\
&= \exp - \langle \eta, u(\cdot,0)\rangle = \mathbb{E}\Big\{ \exp - \sum_{i=1}^n \langle X_{t_i}, \varphi_i\rangle \,\Big|\, X_0 = \eta\Big\},
\end{aligned}$$

and we infer that Z has the same finite-dimensional marginals as a κ-catalytic super-Brownian motion X.

There is a continuous modification $\tilde{Z}$ of Z which is a κ-catalytic super-Brownian motion. To show that $\tilde{Z}$ and Z are indistinguishable, observe that, by definition, $Z_s(S) = 0$ for all $s > 0$. Hence it is enough to show that $t \mapsto \langle Z_t, \varphi\rangle$ is right-continuous, for every continuous, nonnegative φ with compact support disjoint from S.

Fix $t > 0$ and φ as above. First observe that $t \mapsto \langle \eta, Q_t\varphi\rangle$ is continuous, and it suffices to look at the second term on the right hand side of (10). To prepare an application of dominated convergence note that,

$$\begin{aligned}
\mathbb{E}_{\nu_\eta} \iint V(dx\,dr)\,\mathbf{1}_{\{r<t\}} f(x) &= \mathbb{E}_{\nu_\eta} \int_0^\infty dr \iint U_r(dx\,ds)\,\mathbf{1}_{\{s<t\}} f(x)\\
&= \int \nu_\eta(dz)\,\mathbb{E}_z \int_0^\infty dA_s\, f(W_s)\,\mathbf{1}_{\{s<t\}} = \int \eta(dz)\,\mathbb{E}_z\mathbb{E}_{W_\tau} \int_0^\infty dL_s\,\mathbf{1}_{\{s+\tau<t\}}\\
&= \int \eta(dz) \int \mu(dy) \int_0^t ds\, p_s(z-y).
\end{aligned}$$

This is finite, as it is bounded by

$$\int \eta(dz) \int \mu(dy) \int_0^\infty ds\; \mathrm{e}^{t-s}\, p_s(z-y) = \mathrm{e}^t \int \eta(dz)\,\mathbb{E}_z\{\mathrm{e}^{-\tau}\} < \infty.$$

As S is closed and the support of φ is compact, there is an $\varepsilon > 0$ such that the distance $d(\operatorname{supp}\varphi, S) > \varepsilon$. Denote by $T_\varepsilon = T_\varepsilon(\omega)$ the first hitting time of $S_\varepsilon^c := \{x \in \mathbb{R}^d : d(x,S) > \varepsilon\}$ by the excursion ω. Recall from Delmas (1996, Lemme 8.3) that $\sup_{x\in S^*} H^x[T_\varepsilon < \infty] < \infty$. Then, using dominated convergence in the second, and continuity of $t \mapsto H^x[\varphi(\omega(t))]$ in the last step,

$$\begin{aligned}
\lim_{s\downarrow t} &\iint V(dx\,dr)\mathbf{1}_{\{r<s\}} f(x) H^x[\varphi(\omega(s-r))]\\
&= \lim_{s\downarrow t} \iint V(dx\,dr)\mathbf{1}_{\{r<s\}} f(x) H^x[\mathbf{1}_{\{T_\varepsilon<\infty\}}\,\varphi(\omega(s-r))]\\
&= \iint V(dx\,dr) \lim_{s\downarrow t} \mathbf{1}_{\{r<s\}} f(x) H^x[\mathbf{1}_{\{T_\varepsilon<\infty\}}\,\varphi(\omega(s-r))]\\
&= \iint V(dx\,dr)\mathbf{1}_{\{r<t\}} f(x) H^x[\mathbf{1}_{\{T_\varepsilon<\infty\}}\,\varphi(\omega(t-r))].
\end{aligned}$$

This shows that, almost surely, $s \mapsto \langle Z_s, \varphi\rangle$ is right-continuous, which completes the proof. □

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