A PATHWISE VERSION OF SPITZER'S LAW

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Abstract. Spitzer's law describes the long term asymptotic behaviour of the distribution of the winding number of a Brownian motion in the plane. The pathwise result of this paper shows that an analogous behaviour can be seen at every Brownian path by looking at the winding number of the path at a random time chosen according to the logarithmic laws of order three.

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1 Introduction and statement of the theorem

In the last ten years, following the discovery of pathwise, or almost sure, central limit theorems by Fisher (1987, 1990), Brosamler (1988) and Schatte (1988) there has been a great deal of research dealing with strong versions of the classical weak limit theorems. Some interesting examples can be found in the papers of Berkes and Dehling (1993), Csaki, Földes and Revesz (1993), Ibragimov and Lifshitz (1997) and the survey of Berkes (1998).

To formulate the problem, suppose that $\{X_t\}_{t\geq 0}$ with $X_t = X_t(\omega)$ is a stochastic process on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and we have a weak limit law

$$\lim_{t\to\infty} \mathbb{P} \circ X_t[\cdot]^{-1} = P\,, \tag{1}$$

for some limit distribution P. As a statement about the asymptotic distribution of the random variable X_t this does not give information about a single sample path of the process and we ask for a version of (1), which can be seen at (almost) every path. More formally, what we would like to find is a family P_t of laws on $[0,t]$ such that, for almost every sample path $\{X_t\}$ of the process, the distribution of $P_t \circ X.[\omega]^{-1}$ converges weakly to P. In other words, we are looking for distributions P_t of a random time such that, $\mathbb{P}$-almost surely, for every continuous bounded function $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{t\to\infty} \int f(X_s)\, P_t(ds) = \int f(x)\, P(dx)\,.$$

An interesting aspect in the study of path properties of Brownian motion is to ask whether pathwise versions can be obtained from the classical weak limit laws of a planar Brownian motion, namely the *Kallianpur Robbins law* describing the asymptotic distribution of the time spent in a fixed set by the Brownian motion and *Spitzer's law* describing the asymptotic winding number. Both these laws do not seem to be covered by the known results on pathwise limit

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theorems for sums of (dependent) random variables. The first question is treated extensively in Mörters (1998) and the second question, suggested by Jim Pitman, is the subject of the present paper. For the formulation of Spitzer's law let $\{B_t\}_{t\geq 0}$ be a complex Brownian motion started at time 0 from $B_0 = 1$, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. By $\{\theta(t)\}_{t\geq 0}$ denote the continuous specification of its argument with $\theta(0) = 0$.

Spitzer's law: *The distribution of $\theta(t)/\log\sqrt{t}$ is asymptotically a standard Cauchy distribution. In other words,*

$$\lim_{t\to\infty} \mathbb{P} \circ \Big(\frac{\theta(t)}{\log\sqrt{t}}\Big)^{-1} = \frac{dx}{\pi(1+x^2)}\,.$$

Canonical candidates for the distributions P_t in a pathwise version of this and other laws are the *logarithmic laws of order* n for some $n \geq 1$. Informally, choosing x according to a logarithmic law of order n means picking y uniformly from a large interval $[0, T]$ and letting $x = \exp^{(n-1)}(y)$, the $(n-1)$-th iterate of the exponential. More precisely, we define the logarithmic law of order n for $t > \exp^{(n-1)}(0)$ by

$$P_t^{(n)}(A) = \frac{1}{\log^{(n-1)}(t)} \int_{\exp^{(n-1)}(0)}^{t} \mathbf{1}_A(s)\, \frac{ds}{\prod_{j=0}^{n-2} \log^{(j)}(s)}\ , \text{ for } A \subset \mathbb{R} \text{ Borel.}$$

Fisher (1987, 1990) and Bedford and Fisher (1992) discuss logarithmic laws, the relationship to the logarithmic averaging methods of Hardy and Riesz and their universal character. An important feature is the hierarchy of the laws: a limit law with respect to the logarithmic laws of order n implies analogous limit laws with respect to all higher order logarithmic laws, but not conversely.

It turns out that in order to obtain a pathwise version of Spitzer's law we have to choose the random time according to a logarithmic law of order *three*. The following is the main result of this paper.

Theorem 1 $\mathbb{P}$*-almost surely, for every continuous bounded function $f : \mathbb{R} \to \mathbb{R}$,*

$$\lim_{T\to\infty} \frac{1}{\log\log T} \int_e^T f\Big(\frac{\theta(t)}{\log\sqrt{t}}\Big)\, \frac{dt}{t\log t} = \int \frac{f(x)}{\pi(1+x^2)}\, dx\ ,$$

i.e. for almost every Brownian path the distribution of $\theta(t)/\log\sqrt{t}$ at a random time t, chosen according to the logarithmic laws of order three, is asymptotically a standard Cauchy distribution.

Further interesting results about the almost sure behaviour of the winding numbers have been obtained, partly with closely related methods, for example by Bertoin and Werner (1994, 1996) and Shi (1998).

The proof of Theorem 1 is based on the skew-product representation of the Brownian motion, which describes $\theta(t)$ as a Brownian motion with an independent random time change. The crucial argument is an approximation of the clock of θ outside a set of vanishing logarithmic density by means of a stable subordinator of index $1/2$, which we infer from the precise approximations of Zhan Shi (1998). In Section 2 we take this approximation for granted and use the ergodic theorem to prove the approximative version of Theorem 1, and in Section 3 we discuss the details of the approximation. We finish the proof of the theorem in Section 4 and end this paper with some further remarks and open questions.

2 A heuristic argument towards the pathwise Spitzer's law

By the skew-product representation of Brownian motion we can write

$$\theta(t) = \beta(A(t)) \text{ for all } t \geq 0\,,$$

where $\{\beta(t)\}_{t\geq 0}$ is a linear Brownian started at $\beta(0)=0$, independent of $\{|B_t|\}$ and the increasing process

$$A(t) = \int_0^t |B_s|^{-2}\,ds$$

is the *clock of* θ, for a proof see Revuz and Yor (1994). By a transformation of variables it is easy to see that Theorem 1 is equivalent to the fact that, $\mathbb{P}$-almost surely, for every continuous bounded function $f:\mathbb{R}\to\mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T f\Big(\frac{\beta(A(e^{2t}))}{t}\Big)\,\frac{dt}{t} = \int \frac{f(x)}{\pi(1+x^2)}\,dx\,.$$

A crucial point in the work of Bertoin and Werner (1994) is the fact that $\{A(e^{2t})\}_{t\geq 0}$ looks to a certain extent like a stable subordinator $\{S(t)\}_{t\geq 0}$ of index $1/2$ and hence $\theta(e^{2t})$ like a symmetric Cauchy process. The extent of this likeness is limited — recall that the former process is continuous whereas the latter is not — but it will turn out in the next section that it is strong enough to infer a pathwise limit theorem for $\{\theta(t)\}$ from the corresponding statement for the symmetric Cauchy process. The limit theorem for the Cauchy process is the following (compare Theorem 7.1 in Berkes (1997)).

Lemma 2 *Suppose $\{\beta(t)\}_{t\geq 0}$ is a linear Brownian motion and $\{S(t)\}_{t\geq 0}$ is an independent stable subordinator of index $1/2$. Then, almost surely, for every continuous bounded function $f:\mathbb{R}\to\mathbb{R}$,*

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T f\Big(\frac{\beta(S(t))}{t}\Big)\,\frac{dt}{t} = \int \frac{f(x)}{\pi(1+x^2)}\,dx\,.$$

Proof. Recall that $S(t)$ and $t^2 S(1)$ coincide in distribution and so do $\beta(t^2)$ and $t\beta(1)$. Hence the distribution of the process $\beta\circ S$ is invariant under the scaling $f\mapsto f(\,\cdot\, t)/t$ and the Ergodic Theorem yields, for every $f:\mathbb{R}\to\mathbb{R}$ with $\mathbb{E}|f(\beta(S(1)))|<\infty$, almost surely,

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{\log T} \int_1^T f\Big(\frac{\beta(S(t))}{t}\Big)\,\frac{dt}{t} &= \lim_{T\to\infty} \frac{1}{T}\int_0^T f\Big(\frac{\beta(S(e^t))}{e^t}\Big)\,dt \\ &= \mathbb{E} f(\beta(S(1))) = \int \frac{f(x)}{\pi(1+x^2)}\,dx\,. \qquad (2)\end{aligned}$$

The last equality holds, because $\beta(S(1))$ is standard Cauchy distributed, see for example Revuz and Yor (1994), p.103. We now apply (2) to a countable family $\mathfrak{F}$ of continuous bounded functions, which is convergence determining for the weak convergence of probability measures on $\mathbb{R}$, in order to see that the statement of the Lemma holds simultaneously for all continuous bounded functions. ■

3 An approximation of the winding clock

In this section we provide an approximation of the winding clock outside a set of vanishing logarithmic density, which is (although by no means optimal) sufficiently strong to obtain Theorem 1 from Lemma 2. Recall that a set $A \subset (0,\infty)$ has *logarithmic density* $d \in [0,1]$ iff

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \mathbf{1}_{\{a\in A\}} \frac{da}{a} = d \, .$$

With this notation we can formulate the crucial approximation lemma.

Lemma 3 *For every exponent $q > 0$ and every $\varepsilon > 0$ the sets*

$$A[\varepsilon, q] = \Big\{ t \in (0,\infty) \, : \, \Big| A(e^{2t}) - S(t) \Big| > \varepsilon t^q \Big\}$$

and

$$B[\varepsilon, q] = \Big\{ t \in (0,\infty) \, : \, \Big| \beta(A(e^{2t})) - \beta(S(t)) \Big| > \varepsilon t^q \Big\}$$

have logarithmic density zero. Also, there is a set Z of logarithmic density zero, such that

$$\lim_{\substack{t\to\infty \\ t\notin Z}} \Big| \frac{\beta(A(e^{2t}))}{t^q} - \frac{\beta(S(t))}{t^q} \Big| = \lim_{\substack{t\to\infty \\ t\notin Z}} \Big| \frac{A(e^{2t})}{t^q} - \frac{S(t)}{t^q} \Big| = 0 \, .$$

The main ingredient in the proof is the following strong approximation of the winding clock. By the skew-product represention of planar Brownian motion there is a linear Brownian motion γ, independent of β, such that $\log |B_t| = \gamma(A(t))$. The approximating stable subordinator S is defined by

$$S(t) = \inf\{ u > 0 \, : \, \gamma(u) > t \} \, .$$

The following lemma is due to Zhan Shi (1998), see his Lemma 2.1 for a proof.

Lemma 4 *For all positive numbers t and $s \geq 1$,*

$$\mathbb{P}\{A(t) \leq S(s)\} \leq 2\exp(-t\exp(-2s)) \text{ and } \mathbb{P}\{A(t) \geq S(s)\} \leq 4\exp(-\exp(2s)/16t) \, .$$

The following estimates follow from Shi's estimates and properties of the subordinator and the Brownian motion.

Lemma 5 *For every exponent $q > 0$ and for every $\varepsilon > 0$ there is an exponent $p > 0$ and a constant $C > 0$ such that, for all $t \geq 1$,*

$$\mathbb{P}\Big\{ \big| A(e^{2t}) - S(t) \big| > \varepsilon t^q \Big\} \leq \frac{C}{t^p} \text{ and } \mathbb{P}\Big\{ \big| \theta(e^{2t})) - \beta(S(t)) \big| > \varepsilon t^q \Big\} \leq \frac{C}{t^p} \, .$$

Proof. Fix $q > 0$ and $0 < \varepsilon < 1$ and let $h(t) = (1/2)\log\log(t)$. From Lemma 4 we infer that, for all sufficiently large t,

$$\mathbb{P}\{ A(e^{2t}) \leq S(t - h(t)) \} \leq 2\exp(-\exp(2h(t))) = \frac{2}{t} \, , \tag{3}$$

and

$$\mathbb{P}\{A(e^{2t}) \geq S(t+h(t))\} \leq 4\exp(-\exp(2h(t))/16) = \frac{4}{t^{1/16}}\,. \tag{4}$$

From the scaling properties of the stable subordinator of index 1/2 we infer that, for a suitable exponent $p = p(q) < 1/16$ and constants $C_0, C_1 > 0$,

$$\mathbb{P}\{S(t+h(t)) - S(s-h(t)) > \varepsilon t^q\} = \mathbb{P}\Big\{S\Big(2h(t)/\sqrt{\varepsilon t^q}\Big) > 1\Big\} \leq C_0\,\frac{h(t)}{\sqrt{\varepsilon t^q}} \leq \frac{C_1}{t^p}\,. \tag{5}$$

Hence the first statement of the lemma follows from (3), (4) and (5) upon recalling that the subordinator is monotonically increasing. We now choose $o > 2q$ and a new $p = p(o)$. Then there is $C_3 > 0$ such that

$$\mathbb{P}\Big\{\big|\beta(A(e^{2t})) - \beta(S(t))\big| > \varepsilon t^q\Big\} \leq \mathbb{P}\Big\{\big|\beta(t^o)\big| > \varepsilon t^q\Big\} + \frac{C_3}{t^p}\,.$$

The second statement now follows from the fact that, for some $C_4 > 0$,

$$\mathbb{P}\Big\{\big|\beta(t^o)\big| > \varepsilon t^q\Big\} \leq \mathbb{P}\Big\{\big|\beta(1)\big| > \varepsilon t^{q-o/2}\Big\} \leq \frac{C_4}{t^{q-o/2}}\,.$$

■

In order to obtain estimates in logarithmic density we use the following easy criterion.

Lemma 6 *Suppose that $\{A_t\}_{t\geq 1}$ is a measurable family of events and there is an exponent $p > 0$ and a constant $C > 0$ such that $\mathbb{P}(A_t) \leq C/t^p$ for all $t > 1$. Then, $\mathbb{P}$-almost surely, the set*

$$\Big\{t \in (1,\infty) \,:\, A_t \text{ holds}\Big\}$$

has zero logarithmic density.

Proof. We use Markov's inequality to see that, for every $\varepsilon > 0$, there is $C(\varepsilon) > 0$ such that

$$\mathbb{P}\Big\{\frac{1}{\log T}\int_1^T \mathbf{1}_{A_t}\,\frac{dt}{t} > \varepsilon\Big\} \leq \frac{1}{\varepsilon \log T}\int_1^T \mathbb{P}(A_t)\,\frac{dt}{t} \leq \frac{C}{\varepsilon\log T}\int_1^T t^{-p}\,\frac{dt}{t} \leq \frac{C(\varepsilon)}{\log T}\,.$$

The Borel Cantelli Lemma thus yields that, for $a_n = \exp(n^2)$, almost surely,

$$\lim_{n\to\infty}\frac{1}{\log a_n}\int_1^{a_n}\mathbf{1}_{A_t}\,\frac{dt}{t} = 0\,.$$

For every T there is n such that $a_n \leq T \leq a_{n+1}$ and hence

$$\frac{1}{\log T}\int_1^T \mathbf{1}_{A_t}\,\frac{dt}{t} \leq \Big(\frac{\log a_{n+1}}{\log a_n}\Big)\frac{1}{\log a_{n+1}}\int_1^{a_{n+1}}\mathbf{1}_{A_t}\,\frac{dt}{t}.$$

The desired result follows, as $\log a_n/\log a_{n+1}$ converges to 1. ■

We can now plug the estimates obtained in Lemma 5 into Lemma 6 and get that $A[\varepsilon, q]$ and $B[\varepsilon, q]$ have zero logarithmic density. The last statement of Lemma 3, which we do not use explicitly in the sequel, follows from the first and Lemma 4.9 in Fisher (1990), see also Lemma 2 of Berkes and Dehling (1993).

4 Completion of the proof

To complete the proof of Theorem 1, we choose a countable family $\mathfrak{F}$ of uniformly continuous, bounded functions, which is convergence determining for the weak convergence of probabilty measures on $\mathbb{R}$. Let $\varepsilon > 0$ and $f \in \mathfrak{F}$. Then there are $\delta > 0$ and $\Delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$, and $|f| \le \Delta$, and we find, for T sufficiently large,

$$\frac{1}{\log T} \int_1^T f\Big(\frac{\beta(A(e^{2t})}{t}\Big) \frac{dt}{t} \le \frac{1}{\log T} \int_1^T f\Big(\frac{\beta(S(t))}{t}\Big) \frac{dt}{t} + \frac{\Delta}{\log T} \int_1^T \mathbf{1}_{B[\delta,1]} \frac{dt}{t} + \varepsilon\,.$$

As $T \to \infty$ we obtain, using Lemma 2 and Lemma 3,

$$\begin{aligned}\limsup_{T\to\infty} \frac{1}{\log\log T} \int_e^T f\Big(\frac{\theta(t)}{\log\sqrt{t}}\Big) \frac{dt}{t\log t} &= \limsup_{T\to\infty} \frac{1}{\log T} \int_1^T f\Big(\frac{\theta(e^{2t})}{t}\Big) \frac{dt}{t} \\ &= \limsup_{T\to\infty} \frac{1}{\log T} \int_1^T f\Big(\frac{\beta(A(e^{2t}))}{t}\Big) \frac{dt}{t} \le \int \frac{f(x)}{\pi(1+x^2)}\, dx + \varepsilon\,,\end{aligned}$$

and, analogously,

$$\liminf_{T\to\infty} \frac{1}{\log\log T} \int_e^T f\Big(\frac{\theta(t)}{\log\sqrt{t}}\Big) \frac{dt}{t\log t} \ge \int \frac{f(x)}{\pi(1+x^2)}\, dx - \varepsilon\,.$$

Theorem 1 follows, as $\varepsilon > 0$ and $f \in \mathfrak{F}$ were arbitrary.

5 Further remarks and open questions

In Mörters (1998) the question of pathwise versions of the Kallianpur Robbins law for a random time is discussed. By completely different methods, a pathwise Kallianpur Robbins law for a random time distributed according to the logarithmic laws of order three is obtained. The paper also discusses the possibility of obtaining a pathwise Kallianpur Robbins law for the random choice of a scale parameter and a ratio ergodic theorem for small scales outside an exceptional set of vanishing logarithmic density (of order three). In Dembo, Peres, Rosen and Zeitouni (1999) a pathwise Kallianpur Robbins law for the random choice of a spatial shift was investigated (see Section 10.3 there). In a similar spirit it would be very intersting to see whether there is a family P_t of laws on the plane such that a pathwise version of Spitzer's law can be obtained for the windings around a randomly chosen point in space. Finally, the work of Pitman and Yor (1986 and 1989) offers a lot of interesting material about joint distributions of winding numbers around more than one point and occupation times, which has not been studied from a pathwise point of view so far.

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