BOCHNER LAPLACIAN AND BERGMAN KERNEL EXPANSION OF SEMI-POSITIVE LINE BUNDLES ON A RIEMANN SURFACE

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Abstract. We generalize the results of Montgomery [50] for the Bochner Laplacian on high tensor powers of a line bundle. When specialized to Riemann surfaces, this leads to the Bergman kernel expansion and geometric quantization results for semi-positive line bundles whose curvature vanishes at finite order. The proof exploits the relation of the Bochner Laplacian on tensor powers with the sub-Riemannian (sR) Laplacian.

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1. Introduction

The Bergman kernel of a holomorphic line bundle L on a complex manifold Y is the Schwartz kernel of the projector from smooth sections of L onto holomorphic ones. The analysis of the Bergman kernel, Kodaira Laplacian and holomorphic sections associated to tensor powers $L^k := L^{\otimes k}$ has important applications in complex geometry (see for e.g. [21, 44]). When L is positive, the diagonal asymptotic expansion for the Bergman kernel was first proved in [17, 61], motivated by [60], and subsequently by a different geometric method in [20, 44]. In the present article we prove the Bergman kernel expansion at points where the curvature of L is allowed to vanish at finite order, in complex dimension one.

A related problem is the asymptotics of the spectrum of the Bochner (magnetic) Laplacian on tensor powers. Besides geometric applications, the lowest eigenvalue (ground state) of

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the Bochner Laplacian and the Bergman kernel expansion are important in the mathematical physics of superconductors and the Quantum Hall effect [25, 40].

We now state our results more precisely. Let Y^{n-1} be a compact Riemannian manifold of dimension $n-1$ with complex Hermitian line bundle $\left(L, h^L\right)$ and vector bundle $\left(F, h^F\right)$. We equip these with unitary connections ∇^L, ∇^F to obtain the Bochner Laplacian

$$\Delta_k := \left(\nabla^{F\otimes L^k}\right)^* \nabla^{F\otimes L^k} : C^\infty\left(Y; F\otimes L^k\right) \to C^\infty\left(Y; F\otimes L^k\right), \tag{1.1}$$

on the tensor powers $F\otimes L^k$, where the adjoint above is taken with respect to the natural L^2 metric. As the above is elliptic, self-adjoint and positive, one has a complete orthonormal basis $\left\{\psi_j^k\right\}_{j=1}^\infty$ of $L^2\left(Y; F\otimes L^k\right)$ consisting of its eigenvectors $\Delta_k\psi_j^k = \lambda_j(k)\,\psi_j^k$, $0\le\lambda_0\le\lambda_1\ldots$. Denote by $R^L = \left(\nabla^L\right)^2 \in \Omega^2(Y; i\mathbb{R})$ the purely imaginary curvature form of the unitary connection ∇^L. The order of vanishing of R^L at a point $y\in Y$ is now defined[1]

$$r_y - 2 = \mathrm{ord}_y\left(R^L\right) := \min\left\{l | J^l\left(\Lambda^2 T^*Y\right) \ni j_y^l R^L \neq 0\right\}, \quad r_y\ge 2,$$

where $j^l R^L$ denotes the lth jet of the curvature. We shall assume that this order of vanishing is finite at any point of the manifold i.e.

$$r := \max_{y\in Y} r_y < \infty. \tag{1.2}$$

The function $y\mapsto r_y$ being upper semi-continuous then gives a decomposition of the manifold $Y = \bigcup_{j=2}^r Y_j$; $Y_j := \{y\in Y | r_y = j\}$ with each $Y_{\le j} := \bigcup_{j'=0}^j Y_{j'}$ being open. One first result is now the following.

Theorem 1. *Let $(L, h^L)\to\left(Y, g^{TY}\right)$, $(F, h^F)\to\left(Y, g^{TY}\right)$ be Hermitian line/vector bundles on a compact Riemannian manifold with unitary connections ∇^L, ∇^F . Assuming that the curvature R^L vanishes to finite order at any point, the first eigenvalue $\lambda_0(k)$ of the Bochner Laplacian 1.1 satisfies*

$$\lambda_0(k) \sim Ck^{2/r}, \tag{1.3}$$

for some constant C and with r being (1.2)*. Moreover the first eigenfunction concentrates on Y_r:*

$$\left|\psi_0^k(y)\right| = O\left(k^{-\infty}\right);\ y\in Y_{\le r-1}. \tag{1.4}$$

The proof of the above result is based on pointwise, diagonal heat kernel asymptotics for the rescaled Bochner Laplacian (see Theorem 16). The leading constant in (1.3) can be identified

$$C = \inf_{y\in Y_r} \lambda_0\left(\Delta_{g_y^{TY}, j^{r-2}R_y^L}\right) \tag{1.5}$$

in terms of the bottom of the spectrum of certain model Laplacians $\Delta_{g_y^{TY}, j^{r-2}R_y^L}$ defined on the tangent space, see Section A.

Without further hypotheses, the structure of the locus Y_r may be quite general (locally any closed subset of a hypersurface 3.0.2). To obtain further information on the small eigenvalues we introduce additional assumptions. First we assume $Y_r = \bigcup_{j=1}^N Y_{r,j}$ to be a union of embedded submanifolds with dimensions $d_j := \dim(Y_{r,j})$. Set $d_j^{\max} := \max\{d_j\}_{j=1}^N$ and let $NY_{r,j} := TY_{r,j}^\perp \subset TY$ denote the normal bundle of each $Y_{r,j}$. Note that there is a natural density on each $NY_{r,j}$ coming from the metric. At points $y\in Y_r$, the first non-vanishing jet of the curvature

[1]The reason for this normalization, besides a simplification of resulting formulas, is the significance of r_y as the degree of nonholonomy of a relevant sR distribution (see Proposition 11).

$j_y^{r-2}R^L \in S^{r-2}T_y^*Y \otimes \Lambda^2 T_y^*Y$ may be thought of as an element of the product with the $r-2$ symmetric power. We say that the curvature R^L vanishes non-degenerately along Y_r if

$$i_v^s\left(j_y^{r-2}R^L\right) = 0, \forall s \leq r-2 \implies v \in T_y Y_r, \tag{1.6}$$

where i^s above denotes the s-fold contraction of the symmetric part of $j_y^{r-2}R^L$. Denote by $N\left[c_1 k^{2/r}, c_2 k^{2/r}\right]$ the Weyl counting function for the number of eigenvalues of Δ_k and by $\chi_{[c_1,c_2]}$ the characteristic function for the given intervals. In Section 3.2 we show that under the non-degeneracy hypothesis (1.6), the Schwartz kernel of the model Laplacian on the tangent space

$$\chi_{[c_1,c_2]}\left(\Delta_{g_y^{TY}, j^{r-2}R_y^L}\right)(v,v) = O\left(|v|^{-\infty}\right), \;\; v \in NY_{r,j,y},$$

is rapidly decaying in the normal directions. One next result is now as follows.

Theorem 2. *Assuming $Y_r \subset Y$ to be a union of embedded submanifolds along which the curvature vanishes non-degenerately* (1.6)*, the counting function satisfies the asymptotics*

$$N\left[c_1 k^{2/r}, c_2 k^{2/r}\right] \sim k^{d_j^{max}} \sum_{d_j = d_j^{max}} \int_{NY_{r,j}} \chi_{[c_1,c_2]}\left(\Delta_{g_y^{TY}, j^{r-2}R_y^L}\right).$$

If further Y_r is a union of points, the smallest eigenvalue has a complete asymptotic expansion

$$\lambda_0(k) = k^{2/r}\left[\sum_{j=0}^{N} \lambda_{0,j} k^{-j/r} + O\left(k^{-(2N+1)/r}\right)\right].$$

Next, we consider the case when $\left(Y^{n-1}, h^{TY}\right)$ is a complex Hermitian manifold, of even dimension $n-1$. The bundles $\left(L, h^L\right)$, $\left(F, h^F\right)$ are then assumed to be holomorphic with L of rank one. Taking ∇^L, ∇^F to be the Chern connections one also has the associated Kodaira Laplacian

$$\Box_k^q : \Omega^{0,q}\left(Y; F \otimes L^k\right) \to \Omega^{0,q}\left(Y; F \otimes L^k\right), \quad 0 \leq q \leq m,$$

acting on tensor powers. The first eigenvalue of the above is typically 0 with $\ker \Box_k^q = H^q\left(X; F \otimes L^k\right)$ being cohomological and corresponding to holomorphic sections. The Bergman kernel $\Pi_k^q(y, y')$ is the Schwartz kernel of the orthogonal projector $\Pi_k^q : \Omega^{0,q}\left(Y; F \otimes L^k\right) \to \ker \Box_k^q$. Its value on the diagonal is

$$\Pi_k^q(y,y) = \sum_{j=1}^{N_k^q} |s_j(y)|^2, \quad N_k^q := \dim H^q\left(X; F \otimes L^k\right),$$

for an orthonormal basis $\{s_j\}_{j=1}^{N_k^q}$ of $H^q\left(X; F \otimes L^k\right)$ and thus controls pointwise norms of sections in $\ker \Box_k^q$ in the spirit of (1.4). To obtain the asymptotics for $\Pi_k^q(y,y)$, we specialize to the case of Riemann surface ($n-1=2$). Furthermore in addition to vanishing at finite order (1.2), the curvature is assumed to be semi-positive: $R^L(w,\bar{w}) \geq 0$, for all $w \in T^{1,0}Y$. Under these assumptions one has $H^1\left(X; F \otimes L^k\right) = 0$ for $k \gg 0$ with the asymptotics of the Bergman kernel $\Pi_k := \Pi_k^0$ being given by the following.

Theorem 3. *Let Y be a compact Riemann surface and $(L, h^L) \to Y$ a semi-positive line bundle whose curvature R^L vanishes to finite order at any point. Let $(F, h^F) \to Y$ be another Hermitian holomorphic vector bundle. Then for every $N \in \mathbb{N}$ the Bergman kernel has the pointwise asymptotic expansion on diagonal,*

$$\Pi_k(y,y) = k^{2/r_y}\left[\sum_{j=0}^{N} c_j(y) k^{-2j/r_y}\right] + O\left(k^{-2N/r_y}\right),, \tag{1.7}$$

where c_j are sections of $End(F)$, with the leading term $c_0(y) = \Pi^{g_y^{TY}, j_y^{r_y-2}R^L, J_y^{TY}}(0,0) > 0$ being given in terms of the Bergman kernel of the model Kodaira Laplacian on the tangent space at y (A.8).

Note that at points where R^L is positive one has $r_y = 2$ and the above expansion recovers the usual Bergman kernel expansion at these points. The presence of fractional exponents, at points where the curvature vanishes, given in terms of the order of vanishing represents a new feature. It would be desirable to have a more explicit formula for the leading term c_0 at vanishing points for the curvature. As an example 27 computes the leading term explicitly in the case of semi-positive line bundles obtained from branched coverings. Finally, we note that unlike (1.4) the Bergman kernel expansion (1.7) does not exhibit any concentration phenomenon.

The result of Theorem 1 was shown by Montgomery [50] in the case when Y is a Riemann surface with R^L vanishing to second order along a curve. It has since been actively explored in increasing generality and refinement [23, 30, 31, 52]. We remark that our non-degeneracy assumption (1.6) is less restrictive than in these earlier works. As was the motivation in [50], the proof here uses the relation of the Bochner Laplacian with the sub-Riemannian (sR) Laplacian on the unit circle bundle of L. This is a manifestation of the semiclassical/microlocal correspondence in this context. In particular the proof of Theorem 16 exploits a standard heat kernel expansion for the sR Laplacian [7, 16, 41, 58] on the circle bundle. A proof of the sR heat kernel expansion based on sR methods is the topic of recent research [3, 19].

The Bergman kernel expansion for positive line bundles on a compact complex manifold was first proved in [17, 61], motivated by [60], and subsequently in [20, 45] by a method similar to the one here. We refer to [44] for a detailed account of this technique and its applications. For semi-positive line bundles the expansion was previously only known on the positive part in some cases. In [8] the expansion is proved on the positive part, and furthermore away from the augmented base locus, assuming the line bundle to be ample. In [35] the expansion is proved on the positive part when one twists by the canonical bundle (i.e. $F = K_Y$). In [32] a similar expansion, on the positive part when twisted by the canonical bundle, is proved assuming h^L to be only bounded from above by a semi-positive metric. On a related note, [2] proves a weighted estimate for the Bergman kernels of a positive but singular Hermitian line bundle over a Riemann surface under the assumption that the curvature has singularities of Poincaré type at a finite set.

The paper is organized as follows. In Section 2 we begin with some standard preliminaries on sub-Riemannian geometry and the sR Laplacian. In particular 2.1.1 gives a proof of the on-diagonal expansion for the sR heat kernel. In Section 3 we specialize to the case of sR structures on unit circle bundles. Here 3.1 proves Theorem 1 based on an analogous heat kernel expansion for the Bochner Laplacian on tensor powers Theorem 16. Further 3.2 and 3.3 prove the Weyl law and expansion of the first value of Theorem 2 respectively. In 4 we come to the case of the Kodaira Laplacian on tensors powers of semi-positive line bundles on a Riemann surface. Here similar rescaling methods prove the Bergman kernel expansion Theorem 3 in 4.1. There are several applications of the Bergman kernel expansion to geometric quantization. In 4.2 we prove a version of Tian's approximation theorem for a semi-positive integral form by induced Fubini-Study metrics. In Section 4.3 we develop the theory of Toeplitz operators. The main tool is the expansion of their kernel, similar to that of the Bergman kernel, from which we deduce the semiclassical behavior of their norm and composition as well as their spectral density limit. In Section 4.4 we show the equidistribution of zeros of random sections to the semi-positive curvature form. In Section 4.5 we prove an asymptotic formula for the holomorphic torsion in the semi-positive case.

2. sub-Riemannian geometry

Sub-Riemannian (sR) geometry is the study of (metric-)distributions in smooth manifolds. More precisely, let X^n be an n-dimensional, compact, oriented differentiable manifold X. Let $E^m \subset X$ be a rank m subbundle of the tangent bundle which is assumed to be *bracket generating:* sections of E generate all sections of TX under the Lie bracket. The subbundle E is further equipped with a metric g^E. We refer to the triple $\left(X, E, g^E\right)$ as a sub-Riemannian (sR) structure. Riemannian geometry corresponds to $E = TX$.

The obvious length function $l\left(\gamma\right) \coloneqq \int_0^1 \left|\dot{\gamma}\right| dt$ maybe defined on the set of *horizontal* paths of Sobolev regularity one connecting the two points $x_0, x_1 \in X$ as

$$\Omega_E\left(x_0, x_1\right) \coloneqq \left\{\gamma \in H^1\left(\left[0,1\right]; X\right) | \gamma\left(0\right) = x_0,\, \gamma\left(1\right) = x_1,\, \dot{\gamma}\left(t\right) \in E_{\gamma(t)} \text{ a.e.}\right\}.$$

This allows for the definition of the sub-Riemannian distance function via

$$d^E\left(x_0, x_1\right) \coloneqq \inf_{\gamma \in \Omega_E(x_0,x_1)} l\left(\gamma\right). \tag{2.1}$$

A theorem of Chow [51, Thm 1.6.2], shows that this distance is finite (i.e. there exists a horizontal path connecting any two points) giving the manifold the structure of a metric space $\left(X, d^E\right)$.

The canonical flag

$$\underbrace{E_0\left(x\right)}_{=\{0\}} \subset \underbrace{E_1\left(x\right)}_{=E} \subset \ldots \subsetneq E_{r(x)}\left(x\right) = TX \tag{2.2}$$

of the distribution E at any point $x \in X$ is defined with $E_{j+1} \coloneqq E_j + \left[E_j, E_j\right],\ 0 \leq j \leq r\left(x\right) - 1$ denoting the span of the jth brackets. The dual-canonical flag is defined as

$$T^*X = \Sigma_0\left(x\right) \supset \Sigma_1\left(x\right) \supset \ldots \supset \underbrace{\Sigma_{r(x)}\left(x\right)}_{=\{0\}} \tag{2.3}$$

where $\Sigma_j\left(x\right) = E_j^\perp \coloneqq \ker\ \left[T^*X \to E_j^*\right],\ 0 \leq j \leq r\left(x\right)$, are the annihilators of the canonical flag. The number $r\left(x\right)$ is called the step or degree of nonholonomy of the distribution at x and in general depends on the point $x \in X$. Furthermore the ranks of the subspaces $E_j\left(x\right)$ might also might depend on $x \in X$ (E_j's need not be vector bundles). We define $m_j^E\left(x\right) \coloneqq \dim\ E_j\left(x\right)$ and

$$m^E\left(x\right) \coloneqq \left(\underbrace{m_0^E}_{:=0}, \underbrace{m_1^E}_{=m}, m_2^E, \ldots, \underbrace{m_r^E}_{=n}\right)$$

to be the growth vector of the distribution. It shall also be useful to define the weight vector

$$\left(w_1^E\left(x\right), \ldots, w_n^E\left(x\right)\right) \coloneqq \left(\underbrace{1, \ldots, 1}_{m_1 \text{ times}}, \underbrace{2, \ldots 2}_{m_2 - m_1 \text{ times}}, \ldots\ \underbrace{j, \ldots j}_{m_j - m_{j-1} \text{ times}}, \ldots, \underbrace{r, \ldots, r}_{m_r - m_{r-1} \text{ times}}\right)$$

of the distribution at x via $w_j^E = s, \quad \text{if } m_{s-1}^E < j \leq m_s^E$. Finally define

$$\begin{aligned} Q\left(x\right) &\coloneqq \sum_{j=1}^m j\left(m_j^E\left(x\right) - m_{j-1}^E\left(x\right)\right) \\ &= \sum_{j=1}^n w_j^E\left(x\right). \end{aligned}$$

A point is called regular if each distribution E_j is a locally trivial vector bundle near x (i.e. m_j^E's are locally constant functions near x). The significance of $Q(x)$ is given by Mitchell's measure theorem (cf. [51, Theorem 2.8.3]): $Q(x)$ is the Hausdorff dimension of $\left(X, d^E\right)$ as a metric space at any regular point $x \in X$. We call the distribution E *equiregular* if each point $x \in X$ is regular. Hence in the equiregular case each E_j is a subbundle of TX with $r(x)$, $m_j^E(x)$ and $Q(x)$ all being constants independent of x.

In the equiregular case a canonical volume form on X (analogous to the Riemannian volume) can be defined starting from the sR structure. To define this, first note that any surjection $\pi : V \to W$ between two vector spaces allows one to pushforward a metric g^V on V to another $\pi_* g^V$ on W. This is simply the metric on W induced via the identification $W \cong (\ker \pi)^\perp \subset V$; with the metric on $(\ker \pi)^\perp$ being the restriction of g^V . Next for each $j \geq 1$ define the linear surjection

$$\begin{aligned} B_j &: E^{\otimes j} \to E_j / E_{j-1} \\ B_j\left(e_1, \ldots e_j\right) &\coloneqq \mathrm{ad}_{\tilde{e}_1} \mathrm{ad}_{\tilde{e}_2} \ldots \mathrm{ad}_{\tilde{e}_{j-1}} \tilde{e}_j \end{aligned}$$

with $\tilde{e}_j \in C^\infty(E)$ denoting local sections extending $e_j \in E$. The pushforward metrics $g_j^E \coloneqq \left(B_j\right)_* \left(g^E\right)^{\otimes j}$ are now well defined on E_j/E_{j-1} and hence define canonical volume elements $\det g_j^E \in \Lambda^*\left(E_j/E_{j-1}\right)^*$. The canonical isomorphism of determinant lines

$$\bigotimes_{j=1}^{m} \Lambda^*\left(E_j/E_{j-1}\right) = \Lambda^*\left(\bigoplus_{j=1}^{m} E_j/E_{j-1}\right) \cong \Lambda^* TX \tag{2.4}$$

along with its dual isomorphism to now gives a canonical smooth volume form

$$\mu_{\mathrm{Popp}} \coloneqq \bigotimes_{j=1}^{m} \det g_j^E \in \Lambda^*\left(T^* X\right) \tag{2.5}$$

known as the *Popp volume* form. We remark that although the definition makes sense in general it only leads to a smooth form in the equiregular case.

An important notion is the that of a privileged coordinate system at x. To define this, fix a set of local orthonormal generating vector fields $U_1, U_2, \ldots U_m$ near x. The $E-$order $\mathrm{ord}_{E,x}(f)$ of a function $f \in C^\infty(X)$ at a point $x \in X$ is the maximum integer $s \in \mathbb{N}_0$ for which $\sum_{j=1}^m s_j = s$ implies that $\left(U_1^{s_1} \ldots U_m^{s_m} f\right)(x) = 0$. Similarly the $E-$order $\mathrm{ord}_{E,x}(P)$ of a differential operator P at the point $x \in X$ is the maximum integer for which $\mathrm{ord}_{E,x}(Pf) \geq \mathrm{ord}_{E,x}(P) + \mathrm{ord}_{E,x}(f)$ holds for each function $f \in C^\infty(X)$. One then has the obvious relation $\mathrm{ord}_{E,x}(PQ) \geq \mathrm{ord}_{E,x}(P) + \mathrm{ord}_{E,x}(Q)$ for any pair of differential operators P, Q. A set of coordinates $(x_1, \ldots, x_n)$ near a point $x \in X$ is said to be *privileged* if: for all j the set

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{m_j^E}}$$

forms a basis for $E_j(x)$ of the canonical flag (2.2) and moreover each x_j has E-order $w_j^E(x)$ at x. A privileged coordinate system always exists near any point ([6] pg. 36). Furthermore the coordinate system may be chosen such that each $\frac{\partial}{\partial x_j}$ equals the value of some bracket monomial in the generating vector fields at x. The Popp measure (2.5) then equals the Euclidean measure dx at the point x in these coordinates. The $E-$order of the monomial x^α in privileged coordinates is clearly $w.\alpha$ while the defining vector fields U_j all have $E-$order -1. A basic vector field is one of the form $x^\alpha \partial_{x_j}$ for some j and has $E-$order $w.\alpha - w_j$. We may then use a Taylor expansion to write $U_j = \sum_{q=-1}^\infty \hat{U}_j^{(q)}$ with each vector field $\hat{U}_j^{(q)}$ being a sum of basic vector fields of E-order q. If one defines the rescaling/dilation $\delta_\varepsilon x = \left(\varepsilon^{w_1} x_1, \ldots, \varepsilon^{w_n} x_n\right)$ in privileged coordinates, the vector fields $\hat{U}_j^{(q)}$ are those appearing in the corresponding expansion $\left(\delta_\varepsilon\right)_* U_j = \sum_{q=-1}^\infty \varepsilon^q \hat{U}_j^{(q)}$

for the defining vector fields. A differential operator P on $\mathbb{R}^n$ is said to be $E-$homogeneous of $\text{ord}_E(P)$ iff $(\delta_\varepsilon)_* P = \varepsilon^{\text{ord}_E(P)} P$. It is clear that the product of two such homogeneous differential operators P_1, P_2 is homogeneous of $\text{ord}_E(P_1 P_2) = \text{ord}_E(P_1) + \text{ord}_E(P_2)$. The *nilpotentization* of the sR structure at an arbitrary $x \in X$ is the sR manifold given via $\hat{X} = \mathbb{R}^n, \hat{E} := \mathbb{R}\left[\hat{U}_1^{(-1)}, \ldots, \hat{U}_m^{(-1)}\right]$ with the metric $\hat{g}^E$ corresponding to the identification $\hat{U}_j^{(-1)} \mapsto (U_j)_x$. The nilpotentization $\hat{\mu}$ of a smooth measure μ at x is also defined as the leading part $\hat{\mu} = \hat{\mu}^{(0)}$ under the privileged coordinate expansion $(\delta_\varepsilon)_* \mu = \varepsilon^{Q(x)}\left[\sum_{q=0}^{\infty} \hat{\mu}^{(q)}\right]$. These nilpotentizations can be shown to be independent of the choice of privileged coordinates up to sR isometry ([6] Ch. 5).

An invariant definition of the nilpotentization can be given at regular points $x \in X$. First define the nilpotent Lie algebra

$$\mathfrak{g}_x := (E_1)_x \oplus (E_2/E_1)_x \oplus \ldots \oplus (E_m/E_{m-1})_x$$

with the Lie bracket $[.,.] : \mathfrak{g}_x \otimes \mathfrak{g}_x \to \mathfrak{g}_x$ given by the brackets of corresponding vector field extensions. The algebra is clearly graded with the j graded component $(\mathfrak{g}_x)_j := (E_j/E_{j-1})_x$ with the bracket preserving the grading $\left[(\mathfrak{g}_x)_i, (\mathfrak{g}_x)_j\right] \subset (\mathfrak{g}_x)_{i+j}$. Associated to the nilpotent Lie algebra $\mathfrak{g}_x$ is a unique simply connected Lie group G with the exponential map giving a diffeomorphism $\exp : \mathfrak{g}_x \to G$. The nilpotentization of the sR structure $\left(\hat{X}, \hat{E}, \hat{g}^E\right)$ at x can then be identified with $\hat{X} := G$ and the metric distribution $\hat{E}, \hat{g}^E$ obtained via left translation of $(E_1)_x, g_x$. The canonical identification $\Lambda^n \mathfrak{g}_x = \Lambda^n\left[(E_1)_x \oplus (E_2/E_1)_x \oplus \ldots \oplus \left(E_{r(x)}/E_{r(x)-1}\right)_x\right] \cong \Lambda^n T_x X$ gives the nilpotentization $\hat{\mu}$ of the measure μ on $\hat{X}$.

One may also profitably view sR geometry as a limit of a Riemannian geometry. Namely choose a complement $E' \subset TX$, $E \oplus E' = TX$, with a metric $g^{E'}$ giving rise to a family of Riemannian metrics

$$g_\epsilon^{TX} = g^E \oplus \frac{1}{\epsilon} g^{E'} \tag{2.6}$$

which converge $g_\epsilon^{TX} \to g^E$ as $\epsilon \to 0$. We call the above a family of Riemannian metrics *extending/taming* g^E.

We prove a first proposition in this regard.

Proposition 4. *Let d^ϵ denote the distance function for the Riemannian metrics* (2.6)*. Then for each $x_1, x_2 \in X$ one has*

$$d^\epsilon(x_1, x_2) \to d^E(x_1, x_2)$$

converges to the sR distance (2.1) *as $\epsilon \to 0$.*

Proof. Let γ_j be a sequence of (constant speed) geodesics connecting x_1, x_2 for a sequence of metrics $g_{\epsilon_j}^{TX}$; $\epsilon_j \to 0$. Since any g^E-geodesic also has $g_{\epsilon_j}^{TX}$-length $d^E(x_1, x_2)$ we have

$$l^{\epsilon_j}(\gamma_j) \leq d^E(x_1, x_2), \tag{2.7}$$

$\forall j$. For these constant speed geodesics their $g_{\epsilon_j}^{TX}$-energy

$$E^{\epsilon_j}(\gamma_j) := \left(\int_0^1 |\dot{\gamma}|^2_{g_{\epsilon_j}^{TX}}\right)^{1/2} = l^{\epsilon_j}(\gamma_j), \tag{2.8}$$

is bounded and hence so is their $g^{TX}_{\epsilon=1}$-energy. Thus by the Banach-Alauglou theorem a subsequence of $\gamma_{j_q} \rightharpoonup_{H^1_{g^{TX}_{\epsilon=1}}} \gamma_*$ converges weakly. Furthermore from (2.7), (2.8) this limit is horizontal. By weak convergence and Cauchy-Schwartz $\liminf l^{\epsilon_{j_q}}\left(\gamma_{j_q}\right) \geq \liminf l^{\epsilon=1}\left(\gamma_{j_q}\right) \geq \liminf l\left(\gamma_*\right) \geq d^E\left(x_1, x_2\right)$. From this and (2.7) the proposition follows. □

2.1. **sR Laplacian.** We shall be interested in the sub-Riemannian (sR) Laplacian. It shall be useful to define it as acting on sections of an auxiliary complex Hermitian vector bundle with connection $\left(F, h^F, \nabla^F\right)$ of rank l. To define this first define the sR-gradient $\nabla^{g^E,F}s \in C^\infty\left(X; E\otimes F\right)$ of a section $s \in C^\infty\left(X; F\right)$ by the equation

$$h^{E,F}\left(\nabla^{g^E,F}s, v\otimes s'\right) := h^F\left(\nabla^F_v s, s'\right), \quad \forall v\in C^\infty\left(X;E\right), s'\in C^\infty\left(X;F\right), \tag{2.9}$$

where $h^{E,F} := g^E \otimes h^F$. Next one defines the divergence (adjoint) of this gradient. In the sR context, the lack of canonical volume form presents a difficulty in doing so; hence we shall choose an auxiliary non-vanishing volume form μ. The divergence $\left(\nabla^{E,F}\right)^*_\mu \omega \in C^\infty\left(X;F\right)$ of a section $\omega \in C^\infty\left(X; E\otimes F\right)$ is now defined to be the formal adjoint satisfying

$$\int \left\langle \left(\nabla^{g^E,F}\right)^*_\mu \omega, s\right\rangle \mu = -\int \left\langle \omega, \nabla^{g^E,F}s\right\rangle \mu, \quad \forall s\in C^\infty\left(X;F\right). \tag{2.10}$$

The **sR-Laplacian** acting on sections of F is defined by the equation

$$\Delta_{g^E,F,\mu} := \left(\nabla^{g^E,F}\right)^*_\mu \circ \nabla^{g^E,F} : C^\infty\left(X;F\right) \to C^\infty\left(X;F\right).$$

In terms of a local orthonormal frame $\{U_j\}_{j=1}^m$ for E, we have the expression

$$\Delta_{g^E,F,\mu}s = \sum_{j=1}^m \left[-\nabla^2_{U_j}s + \left(\nabla^E U_j\right)^*_\mu \nabla_{U_j}s\right] \tag{2.11}$$

with $\left(\nabla^E U_j\right)^*_\mu$ being the divergence of the vector field U_j with respect to μ.

Remark 5. To remark on how the choice of the auxiliary form μ affects the Laplacian, let $\mu' = h\mu$ denote another non-vanishing volume form where h is a positive smooth function on X. From (2.11), it now follows easily that one has the relation

$$\Delta_{g^E,F,\mu'} = h^{-1/2}\Delta_{g^E,F,\mu}h^{1/2} + h^{-1/2}\left(\Delta_{g^E,\mu}h^{1/2}\right)\mathrm{Id}$$

with $\Delta_{g^E,\mu}$ denoting the sR Laplacian on functions (i.e. with $F=\mathbb{C}$). Thus the two corresponding Laplacians are conjugate modulo a zeroth-order term.

The sR Laplacian $\Delta_{g^E,F,\mu}$ is non-negative and self adjoint with respect to the obvious inner product $\langle s, s'\rangle = \int_X h^F\left(s,s'\right)\mu$, $s, s' \in C^\infty\left(X;F\right)$. Its principal symbol is easily computed to be the Hamiltonian

$$\sigma = \sigma\left(\Delta_{g^E,F,\mu}\right)(x,\xi) = H^E(x,\xi) = \left|\xi|_E\right|^2 \in C^\infty\left(T^*X\right) \tag{2.12}$$

while its sub-principal symbol is zero. The characteristic variety is

$$\Sigma_{\Delta_{g^E,F,\mu}} = \left\{(x,\xi)\in T^*X | \sigma\left(\Delta_{g^E,F,\mu}\right)(x,\xi) = 0\right\} = \left\{(x,\xi)\,|\,\xi|_E = 0\right\} =: E^\perp \tag{2.13}$$

is the annihilator of E. From the local expression (2.11) and the bracket generating condition on E, the Laplacian $\Delta_{g^E,F,\mu}$ is seen to be a sum of squares operator of Hörmander type [33]. It is then known to be hypoelliptic and satisfies the optimal sub-elliptic estimate [54] with a gain of $\frac{1}{r}$ derivatives

$$\|s\|^2_{H^{1/r}} \leq C\left[\left\langle \Delta_{g^E,F,\mu}s, s\right\rangle + \|s\|^2_{L^2}\right], \quad \forall s\in C^\infty\left(X;F\right) \tag{2.14}$$

where $r := \sup_{x\in X} r(x)$ is the total step of the distribution. For X non-compact, one also has the local subelliptic estimate

$$\|\psi s\|^2_{H^{1/r}} \leq C\left[\left\langle \Delta_{g^E,F,\mu}\varphi s, \varphi s\right\rangle + \|\varphi s\|^2_{L^2}\right], \quad \forall s \in C^\infty(X;F) \tag{2.15}$$

and $\varphi, \psi \in C_c^\infty(X)$ with $\varphi = 1$ on $\mathrm{spt}(\psi)$.

Thus the sR Laplacian has a compact resolvent, a discrete spectrum of non-negative eigenvalues $0 \leq \lambda_0 \leq \lambda_1 \leq \ldots$ approaching infinity and a well-defined heat kernel $e^{-t\Delta_{g^E,F,\mu}}$.

Remark 6. For each point $p \in E^\perp$ on the characteristic variety (2.13) the Hessian $\nabla^2\sigma$ of the symbol as well as the fundamental matrix $F_p : T_pM \to T_pM$, $\nabla^2\sigma(.,.) := \omega(., F_p.)$ are invariantly defined (here ω denotes the symplectic form on the cotangent space). Under the condition

$$\mathrm{tr}^+ F_p := \sum_{\mu\in\mathrm{Spec}^+(iF_p)} \mu > 0 \tag{2.16}$$

$\forall p \in E^\perp$, the Laplacian $\Delta_{g^E,F,\mu}$ is known to satisfy a better subelliptic estimate with loss of 1 derivative [34]. Furthermore, the heat kernel and trace asymptotics under this assumption were shown in [47, 48].

2.1.1. *sR heat kernel.* We are interested in the asymptotics of the heat kernel $e^{-t\Delta_{g^E,F,\mu}}$. As a first step we show the finite propagation speed for the corresponding wave equation.

Lemma 7 (Finite propagation speed)**.** *Let $f(x;t)$ be the unique solution to the initial value problem*

$$\left(i\partial_t + \sqrt{\Delta_{g^E,F,\mu}}\right) f = 0$$
$$f(x,0) = f_0 \in C_c^\infty(X;F).$$

Then the solution satisfies

$$sptf(x;t) \subset \left\{y | \exists x \in sptf_0;\ d^E(x,y) \leq |t|\right\}.$$

Proof. The result maybe restated in terms of the Schwartz kernel $K_t(x,y)$ of $e^{it\sqrt{\Delta_{g^E,F,\mu}}}$ as

$$\mathrm{spt}\ K_t \subset \left\{(x,y) | d^E(x,y) \leq |t|\right\}.$$

We choose a family of metrics g^{TX}_ϵ (2.6) extending g^E. The Riemannian Laplacian $\Delta_{g^{TX}_\epsilon,F,\mu}$ (still coupled to the form μ) is written

$$\Delta_{g^{TX}_\epsilon,F,\mu} = \Delta_{g^E,F,\mu} + \epsilon\Delta_{g^{E'},F,\mu} \tag{2.17}$$

where $\Delta_{g^{E'},F,\mu}$ is the sR Laplacian on the complementary distribution E'. Via the min-max principle for small eigenvalues this leads to the L^2 convergence $\Pi^{\Delta_{g^{TX}_\epsilon,F,\mu}}_{[0,L]} \to \Pi^{\Delta_{g^E,F,\mu}}_{[0,L]}$ of the corresponding spectral projectors onto any given interval $[0,L]$. Following this one has the weak convergence $K^\epsilon_t \rightharpoonup K_t$ with $K^\epsilon_t(x,y)$ denoting the Schwartz kernel of $e^{it\sqrt{\Delta_{g^{TX}_\epsilon,F,\mu}}}$. The proposition now follows from the finite propagation speed of $\Delta_{g^{TX}_\epsilon,F,\mu}$ along with $d^\epsilon \to d^E$ as $\epsilon \to 0$ by Proposition 4 . □

Next we show how the above finite propagation result leads to a localization for the heat kernel. To state this, fix a Riemannian metric g^{TX} and a privileged coordinate ball $B^{g^{TX}}_\varrho(x)$ centered at a point x of radius ϱ (depending on x). Let $U_1, \ldots, U_m$ be generating vector fields on this ball. Let $\chi \in C_c^\infty([-1,1];[0,1])$ with $\chi = 1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$. Define the modified measure and vector fields via

$$\tilde{\mu} = \hat{\mu} + \chi\left(\frac{d^{g^{TX}}(x,x')}{\varrho_x}\right)(\mu - \hat{\mu}),$$

$$\tilde{U}_j = U_j^{(-1)} + \chi\left(\frac{d^{g^{TX}}(x,x')}{\varrho_x}\right)\left(U_j - U_j^{(-1)}\right), \quad 1 \le j \le m,$$

on $\mathbb{R}^n$ in terms of the nilpotentization at x given by these privileged coordinates. These modified vector fields can be seen to be bracket generating for ϱ sufficiently small. The connection on F takes the form $\nabla^F = d + A$, $A \in \Omega^1\left(B_\varrho^{g^{TX}}(x);\mathfrak{u}(l)\right)$, $A(0) = 0$, in terms of an orthonormal trivialization for F over the ball. A modified connection on $\mathbb{R}^n$ is now defined as $\tilde{\nabla}^F = d + \chi\left(\frac{d^E(x,x')}{\varrho}\right)A$. A formula similar to (2.11) now gives an sR Laplacian on $\mathbb{R}^n$ via

$$\tilde{\Delta}_{g^E,F,\mu} s = \sum_{j=1}^m \left[-\tilde{\nabla}_{\tilde{U}_j}^2 s + \left(\nabla^E \tilde{U}_j\right)_{\tilde{\mu}}^* \nabla_{\tilde{U}_j} s\right].$$

Being semi-bounded from below, it is essentially self-adjoint and has a well defined heat kernel on $\mathbb{R}^n$ using functional calculus. Furthermore its resolvent maps $\left(\tilde{\Delta}_{g^E,F,\mu} - z\right)^{-1} : H^s_{\text{loc}} \to H^{s+\frac{1}{r}}_{\text{loc}}$, by the corresponding subelliptic estimate. We now have the localization lemma for the heat kernel.

Lemma 8. *The heat kernel satisfies*

$$e^{-t\Delta_{g^E,F,\mu}}(x,x') \le Ct^{-2nr-1}e^{-\frac{d^E(x,x')^2}{4t}} \tag{2.18}$$

uniformly $\forall x,x' \in X$ and $t \le 1$. Further there exists $\varrho_1 > 0$ (depending on x) such that

$$e^{-t\Delta_{g^E,F,\mu}}(x,x') - e^{-t\tilde{\Delta}_{g^E,F,\mu}}(x,x') \le C_x e^{-\frac{\varrho_1^2}{16t}} \tag{2.19}$$

for $d^E(x,x') \le \varrho_1$ as $t \to 0$.

Proof. Both claims are standard applications of finite propagation Lemma 7. With the cutoff χ as before, write the heat kernel in terms of the wave operator

$$\left[\Delta_{g^E,F,\mu}^q e^{-t\Delta_{g^E,F,\mu}}\right](x,x') = \frac{1}{2\pi}\int d\xi\, e^{i\xi\sqrt{\Delta_{g^E,F,\mu}}}(x,x')\, D_\xi^{2q}\frac{e^{-\frac{\xi^2}{4t}}}{\sqrt{4\pi t}} \tag{2.20}$$

$$= \frac{1}{2\pi}\int d\xi\, e^{i\xi\sqrt{\Delta_{g^E,F,\mu}}}(x,x')\,\chi\left(\frac{\xi}{\varrho_1}\right) D_\xi^{2q}\frac{e^{-\frac{\xi^2}{4t}}}{\sqrt{4\pi t}}$$

$$+\frac{1}{2\pi}\int d\xi\, e^{i\xi\sqrt{\Delta_{g^E,F,\mu}}}(x,x')\left[1-\chi\left(\frac{\xi}{\varrho_1}\right)\right] D_\xi^{2q}\frac{e^{-\frac{\xi^2}{4t}}}{\sqrt{4\pi t}}, \tag{2.21}$$

$\forall q \ge 1$. By finite propagation, the integral (2.20) maybe restricted to $|\xi| \ge d^E(x,x')$. The integral estimate

$$\left|\frac{1}{2\pi}\int_{|\xi|\ge d^E(x,x')} e^{i\xi s} D_\xi^{2q}\frac{e^{-\frac{\xi^2}{4t}}}{\sqrt{4\pi t}}d\xi\right| \le ct^{-2q-\frac{1}{2}}e^{-\frac{d^E(x,x')^2}{4t}},$$

now gives the bound

$$\left\|\Delta_{g,\mu}^{q}e^{-t\Delta_{g^{E},F,\mu}}\right\|_{L^{2}\to L^{2}}\leq ct^{-2q-\frac{1}{2}}e^{-\frac{d^{E}(x,x')^{2}}{8t}},$$

which combines with the sub-elliptic estimate (2.14) to give (2.18). For (2.19), note that the second summand of (2.21) is exponentially decaying $O\left(\exp(-\frac{\varrho_1^2}{16t})\right)$. Next for ϱ_1 sufficiently small, $B_{\varrho_1}^{g^E}(x)\subset B_{\varrho}^{g^{TX}}(x)$. Thus finite propagation and $\Delta_{g^E,F,\mu}=\tilde{\Delta}_{g^E,F,\mu}$ on $B_{\varrho_1}^{g^E}(x)$ give that the corresponding first summands for $\Delta_{g^E,F,\mu}$, $\tilde{\Delta}_{g^E,F,\mu}$ agree for $d^E(x,x')\leq\varrho_1$. □

We now give the on diagonal expansion for the sR heat kernel.

Theorem 9. *There exist smooth sections $A_j\in C^{\infty}(X;End(F))$such that*

$$\left[e^{-t\Delta_{g^{E},F,\mu}}\right]_{\mu}(x,x)=\frac{1}{t^{Q(x)/2}}\left[A_{0}(x)+A_{1}(x)t+\ldots+A_{N}(x)t^{N}+O\left(t^{N}\right)\right]\tag{2.22}$$

$\forall x\in X$, $N\in\mathbb{N}$. The leading term $A_0=\left[e^{-\hat{\Delta}_{\hat{g},\hat{\mu}}}\right]_{\hat{\mu}}(0,0)$ is a multiple of the identity given in terms of the scalar heat kernel on the nilpotent approximation.

Proof. By Lemma 8, it suffices to demonstrate the expansion for the localized heat kernel $e^{-t\tilde{\Delta}_{g^E,F,\mu}}(0,0)$ at the point x. Next, the heat kernel of the rescaled sR-Laplacian

$$\tilde{\Delta}_{g^{E},F,\mu}^{\varepsilon}\coloneqq\varepsilon^{2}\left(\delta_{\varepsilon}\right)_{*}\tilde{\Delta}_{g^{E},F,\mu}\tag{2.23}$$

under the privileged coordinate dilation satisfies

$$e^{-t\tilde{\Delta}_{g^{E},F,\mu}^{\varepsilon}}\left(x,x'\right)=\varepsilon^{Q(x)}e^{-t\varepsilon^{2}\tilde{\Delta}_{g^{E},F,\mu}}\left(\delta_{\varepsilon}x,\delta_{\varepsilon}x'\right).\tag{2.24}$$

Rearranging and setting $x=x'=0$, $t=1$; gives

$$\varepsilon^{-Q(x)}e^{-\tilde{\Delta}_{g^{E},F,\mu}^{\varepsilon}}(0,0)=e^{-\varepsilon^{2}\tilde{\Delta}_{g^{E},F,\mu}}(0,0)$$

and it suffices to compute the expansion of the left hand side above as the dilation $\varepsilon\to0$. To this end, first note that the rescaled Laplacian has an expansion under the privileged coordinate dilation

$$\tilde{\Delta}_{g^{E},F,\mu}^{\varepsilon}=\left(\sum_{j=0}^{N}\varepsilon^{j}\hat{\Delta}_{g^{E},F,\mu}^{(j)}\right)+\varepsilon^{N+1}R_{\varepsilon}^{(N)},\quad\forall N.\tag{2.25}$$

Here each $\hat{\Delta}_{g^E,F,\mu}^{(j)}$ is an ε-independent second order differential operator of homogeneous $E-$order $j-2$. While each $R_{\varepsilon}^{(N)}$ is an ε-dependent second order differential operators on $\mathbb{R}^n$ of E-order at least $N-1$. The coefficient functions of $\hat{\Delta}_{g^E,F,\mu}^{(j)}$ are polynomials (of degree at most $j+2r$) while those of $R_{\varepsilon}^{(N)}$ are uniformly (in ε) C^{∞}-bounded. The first term is a scalar operator given in terms of the nilpotent approximation

$$\hat{\Delta}_{g^{E},F,\mu}^{(0)}=\Delta_{\hat{g}^{E},\hat{\mu};x}=\sum_{j=1}^{m}\left(\hat{U}_{j}^{(-1)}\right)^{2}.\tag{2.26}$$

at the point x. This expansion (2.25) along with the subelliptic estimates now gives

$$\left(\tilde{\Delta}_{g^{E},F,\mu}^{\varepsilon}-z\right)^{-1}-\left(\hat{\Delta}_{g^{E},F,\mu}^{(0)}-z\right)^{-1}=O_{H_{\text{loc}}^{s}\to H_{\text{loc}}^{s+1/r-2}}\left(\varepsilon\left|\text{Im}z\right|^{-2}\right),$$

$\forall s \in \mathbb{R}$. More generally, we let $I_j := \{p = (p_0, p_1, \ldots) \,|p_\alpha \in \mathbb{N}, \sum p_\alpha = j\}$ denote the set of partitions of the integer j and define

$$\mathtt{C}_j^z := \sum_{p \in I_j} \left(\hat{\Delta}^{(0)}_{g^E,F,\mu} - z\right)^{-1} \left[\prod_\alpha \hat{\Delta}^{(p_\alpha)}_{g^E,F,\mu} \left(\hat{\Delta}^{(0)}_{g^E,F,\mu} - z\right)^{-1}\right]. \tag{2.27}$$

Then by repeated applications of the subelliptic estimate we have

$$\left(\tilde{\Delta}^{\varepsilon}_{g^E,F,\mu} - z\right)^{-1} - \sum_{j=0}^{N} \varepsilon^j \mathtt{C}_j^z = O_{H^s_{\mathrm{loc}} \to H^{s+N(1/r-2)}_{\mathrm{loc}}} \left(\varepsilon^{N+1} \left|\mathrm{Im} z\right|^{-2Nw_n - 2}\right),$$

$\forall s \in \mathbb{R}$. A similar expansion as (2.25) for the operator $\left(\tilde{\Delta}^{\varepsilon}_{g^E,F,\mu} + 1\right)^M \left(\tilde{\Delta}^{\varepsilon}_{g^E,F,\mu} - z\right)$, $M \in \mathbb{N}$, also gives

$$\left(\tilde{\Delta}^{\varepsilon}_{g^E,F,\mu} + 1\right)^{-M} \left(\tilde{\Delta}^{\varepsilon}_{g^E,F,\mu} - z\right)^{-1} - \sum_{j=0}^{N} \varepsilon^j \mathtt{C}_{j,M}^z = O_{H^s_{\mathrm{loc}} \to H^{s+N(1/r-2)+\frac{M}{r}}_{\mathrm{loc}}} \left(\varepsilon^{N+1} \left|\mathrm{Im} z\right|^{-2Nw_n - 2}\right) \tag{2.28}$$

for operators $\mathtt{C}_{j,M}^z = O_{H^s_{\mathrm{loc}} \to H^{s+N(1/r-2)+\frac{M}{r}}_{\mathrm{loc}}} \left(\varepsilon^{N+1} \left|\mathrm{Im} z\right|^{-2Nw_n - 2}\right)$, $j = 0, \ldots, N$, with

$$\mathtt{C}_{0,M}^z = \left(\hat{\Delta}^{(0)}_{g^E,F,\mu} + 1\right)^{-M} \left(\hat{\Delta}^{(0)}_{g^E,F,\mu} - z\right)^{-1}.$$

For $M \gg 0$ sufficiently large, Sobolev's inequality gives an expansion for the corresponding Schwartz kernels of (2.28) in $C^0(\mathbb{R}^n \times \mathbb{R}^n)$. The heat kernel expansion now follows by plugging the resolvent expansion into the Helffer-Sjöstrand formula. Finally, to see that the expansion only involves only even powers of ε (or that (2.22) has no half-integer powers of t), note that the operators $\hat{\Delta}^{(j)}_{g^E,F,\mu}$ in the expansion (2.25) change sign by $(-1)^j$ under the rescaling δ_{-1}. The integral expression (2.27) corresponding to $\mathtt{C}_j^z(0,0)$ then changes sign by $(-1)^j$ under this change of variables giving $\mathtt{C}_j^z(0,0) = 0$ for j odd. ☐

We note that one may similarly prove an on diagonal expansion

$$\left[\varphi\left(t\Delta_{g^E,F,\mu}\right)\right]_\mu (x,x) = \frac{1}{t^{Q(x)/2}} \left[A_0^\varphi(x) + A_1^\varphi(x)\, t + \ldots + A_N^\varphi(x)\, t^N + O\left(t^N\right)\right], \tag{2.29}$$

$\forall \varphi \in \mathcal{S}(\mathbb{R})$, by plugging the resolvent expansion (2.28) into the Helffer-Sjöstrand formula for φ, in the final step above. However both the above and the expansion Theorem 9 hold only pointwise along the diagonal. In particular the leading order $Q(x)$ is in general a function of the point x on the diagonal. This hence does not immediately give heat trace or spectral function asymptotics for the sR Laplacian as the expansion might not be uniform or integrable in x. However in the equiregular case, where $Q(x) = Q$ is constant, a uniform set of privileged coordinates (privileged at each point in a neighborhood of x) maybe chosen in the proof. This gives the uniformity of the expansion in x and one obtains the following.

Theorem 10. *In the equiregular case there is a trace expansion*

$$tr\, e^{-t\Delta_{g^E,F,\mu}} = \frac{1}{t^{Q/2}} \left[a_0 + a_1 t + \ldots + a_N t^N + O\left(t^N\right)\right], \quad \forall x \in X, N \in \mathbb{N},$$

with leading term given by

$$a_0 = \int_X \left[e^{-\hat{\Delta}_{\hat{g},\hat{\mu}}}\right]_{\hat{\mu}} (0,0)\,\mu$$
$$= \int_X \left[e^{-\hat{\Delta}_{\hat{g},\mu_{Popp}}}\right]_{\mu_{Popp}} (0,0)\,\mu_{Popp}$$

Thus, the Weyl counting functions satisfies

$$N(\lambda) := \#\mathit{Spec}\left(\Delta_{g^E,F,\mu}\right) \cap [0,\lambda]$$
$$= \frac{\lambda^{Q/2}(1+o(1))}{\Gamma(Q/2+1)} \int_X \left[e^{-\hat{\Delta}_{\hat{g},\hat{\mu}}}\right]_{\hat{\mu}} (0,0)\,\mu.$$

The above two theorems are by now well known cf. [49, 58, 7, 59, 41, 16]. The sR proof based on privileged coordinate dilations is from [3], [19].

3. Bochner Laplacian on tensor powers

A natural place where sub-Riemannian structures arise is on unit circle bundles. To precise, let us consider $\left(X, E, g^E\right)$ to be a corank 1 sR structure on an n-dimensional manifold X (i.e. rank$E = n-1$). We assume that there is a free S^1 action on X with respect to which the sR structure is invariant and transversal (i.e. the generator $e \in C^\infty(TX)$ of the action and E are transversal at each point). The quotient $Y := X/S^1$ is a then a manifold with a Riemannian metric g^{TY} induced from g^E. Equivalently, the natural projection $\pi : X \to Y$ is a principal S^1 bundle with connection given by the horizontal distribution E. Let $L := X \times_\rho S^1 \to Y$ be the Hermitian line associated to the standard one dimensional representation ρ of S^1 with induced connection ∇^L and curvature R^L. It is possible to restate the bracket generating condition in terms of the curvature R^L. First note that since the distribution is of corank 1, the growth vector at x is simply a function of the step $r(x)$ and given by

$$m^E(x) = \Big(0, \underbrace{n-1, n-1, \ldots, n-1}_{r(x)-1\,\text{times}}, n\Big).$$

Equivalently, the canonical flag (2.2) is given by

$$E_j(x) = \begin{cases} E; & 1 \le j \le r(x)-1 \\ TX; & j = r(x) \end{cases}.$$

Also note that the weight vector at x is

$$\Big(\underbrace{1, 1, \ldots, 1}_{n-1\ \text{times}}, r(x)\Big),$$

while the Hausdorff dimension is given by $Q(x) = n-1+r(x)$. On account of the S^1 invariance, each of $m^E(x)$,$r(x)$ and $Q(x)$ descend to functions on the base manifold Y. The degree of nonholonomy $r(x)$ at x is now characterized in terms of the order of vanishing of the curvature R^L.

Proposition 11. *The degree of nonholonomy*

$$r(x) - 2 = \mathit{ord}\left(R^L\right)$$
$$:= \min\left\{l | j^l_{\pi(x)}\left(R^L\right) \neq 0\right\} \tag{3.1}$$

where $j^l_{\pi(x)}\left(R^L\right)$ denotes the l-th jet of the curvature R^L.

Proof. In terms of local coordinates on Y and a local orthonormal section $\mathtt{1}$ for L, we may write $\nabla^L = d + ia^L$; $a^L \in \Omega^1(Y)$, while $E = \ker\left[d\theta + a^L\right]$ with θ being the induced coordinate on each fiber of X. The proposition now follows on noting $[U_i, U_j] = \left(da^L\right)_{ij}\partial_\theta = R^L_{ij}\partial_\theta$ for the local generating vector fields $U_j := \partial_{y_j} - a^L_j\partial_\theta$, $1 \leq j \leq n-1$. □

Thus we see that the bracket generating condition is equivalent to the curvature R^L having a finite order of vanishing at each point of Y. Horizontal curves on X can also be characterized in terms of the base Y. Namely, a horizontal curve $\tilde{\gamma}$ on X corresponds to a pair (γ, s) of a curve γ on the base Y along with an parallel orthonormal section s of L along it $|s| = 1, \nabla^L_{\dot{\gamma}}s = 0$, with the correspondence simply being given by projection/lift. The length of $\tilde{\gamma}$ is simply the length of γ on the base. It is clear that the horizontal lift of a Riemannian geodesic on Y is a sR geodesic on X. The sR distance function on X is now also characterized in terms of the holonomy distance on the base manifold Y

$$\begin{aligned} d^L_{\text{hol}}\left((y_1, l_{y_1}); (y_2, l_{y_2})\right) &= \inf_{\gamma \in \Omega_L\left((y_1, l_{y_1}); (y_2, l_{y_2})\right)} l(\gamma) \\ \Omega_L\left((y_1, l_{y_1}); (y_2, l_{y_2})\right) &= \Big\{(\gamma, s)\,|\gamma \in H^1([0,1]; Y), s \in H^1(\gamma; L), \\ &\qquad \gamma(0) = y_1,\ \gamma(1) = y_2, \\ &\qquad s(0) = l_{y_1},\ s(1) = l_{y_2}, \\ &\qquad |s| = 1, \nabla^L_{\dot{\gamma}}s = 0 \text{ a.e. } \Big\} \end{aligned}$$

$\forall y_j \in Y$, $l_{y_j} \in L_{y_j}$, $j = 1, 2$.

3.0.2. *Structure of* Y_r. As noted before, the function $y \mapsto r_y$ is upper semi-continuous and gives a decomposition of the manifold $Y = \bigcup_{j=2}^r Y_j$; $Y_j := \{y \in Y | r_y = j\}$ with each $Y_{\leq j} := \bigcup_{j'=0}^j Y_{j'}$ being open. We next address the structure of Y_r , the locus of highest vanishing order for the curvature.

Proposition 12. *The subset* $Y_r \subset Y$ *is locally the closed subset of a hypersurface.*

Proof. First express the curvature $R^L = R^L_{ij}dy_i \wedge dy_j$ in some coordinates centered at $y \in Y_r$. By definition one has

$$\partial^\alpha_y R^L_{ij} = 0, \quad \forall i, j = 1, 2, \ldots, n-1,\ \alpha \in \mathbb{N}_0^{n-1}, |\alpha| \leq r-3, \quad \text{while} \tag{3.2}$$

$$\partial^{\alpha_0}_y R^L_{i_0 j_0} \neq 0, \quad \text{for some } i_0, j_0 = 1, 2, \ldots, n-1,\ \alpha_0 \in \mathbb{N}_0^{n-1}, |\alpha_0| = r-2, \tag{3.3}$$

with Y_r being described by the above equations near y. The second equation (3.3) implies that one of the functions $\partial^\alpha_y R^L_{ij}$, $|\alpha| = r-3$, has a non-zero differential and cuts out a hypersurface. □

The following examples show that this is the most that can be said about Y_r in general.

Example 13. Let $0 \in S \subset \mathbb{R}_{y_2}$ be any arbitrary (non-empty) closed subset. By the Whitney extension theorem, there exists $f \in C^\infty(\mathbb{R}_{y_2})$ such that $S = f^{-1}(0)$. The (closed) two form $R^L = \left(y_1^2 + f^2\right)dy_1dy_2$ on $\mathbb{R}^2_y$ is the curvature form of some connection on trivial line bundle on $\mathbb{R}^2_y$. We clearly have $r = 4$ with $Y_4 = \{y | y_1 = f = 0\} = \{0\} \times S \subset \mathbb{R}^2_{y_1, y_2}$ in this example.

Example 14. Let $0 \in S \subset \mathbb{R}^2_{y_2, y_3}$ be any arbitrary (non-empty) closed subset. By the Whitney extension theorem, there exists $f \in C^\infty\left(\mathbb{R}^2_{y_2, y_3}\right)$ such that $S = f^{-1}(0)$. The (closed) two form $R^L = y_1dy_1dy_2 + f(y_2, y_3)\,dy_2dy_3$ on $\mathbb{R}^3_y$ is the curvature form of some connection on trivial line bundle on $\mathbb{R}^3_y$. We clearly have $r = 3$ with $Y_3 = \{y | y_1 = f(y_2, y_3) = 0\} = \{0\} \times S \subset \mathbb{R}^3_{y_1, y_2, y_3}$ in this example.

3.1. **Smallest eigenvalue.** The unit circle bundle of L being X, the pullback $\mathbb{C} \cong \pi^* L \to X$ is canonically trivial with the identification $\pi^* L \ni (x, l) \mapsto x^{-1} l \in \mathbb{C}$. Pulling back sections then gives the identification

$$C^{\infty}(X) \otimes \mathbb{C} = \oplus_{k\in\mathbb{Z}} C^{\infty}\left(Y; L^k\right). \tag{3.4}$$

Each summand on the right hand side above corresponds to an eigenspace of e with eigenvalue $-ik$. While horizontal differentiation d^H on the left corresponds to differentiation with respect to the tensor product connection∇^{L^k} on the right hand side above. Pick μ_X an invariant density on X (e.g. the pullback of the Riemannian density) inducing a density μ_Y on Y. Pick an auxiliary complex Hermitian vector bundle with connection $\left(F, h^F, \nabla^F\right)$ on Y and we denote by the same notation its pullback to X. This now defines the sR Laplacian Δ_{g^E,F,μ_X} : $C^{\infty}(X; F) \to C^{\infty}(X; F)$ acting on sections of F. By invariance, the sR Laplacian commutes $\left[\Delta_{g^E,F,\mu_X}, e\right] = 0$, preserves the decomposition (3.4) and acts via

$$\Delta_{g^E,F,\mu_X} = \oplus_{k\in\mathbb{Z}} \Delta_k \tag{3.5}$$

on each component where $\Delta_k \coloneqq \left(\nabla^{F\otimes L^k}\right)^* \nabla^{F\otimes L^k} : C^{\infty}\left(Y; F \otimes L^k\right) \to C^{\infty}\left(Y; F \otimes L^k\right)$ is the Bochner Laplacian (1.1) on the tensor powers $F \otimes L^k$, with adjoint being taken with respect to μ_Y. As a first result we show how the subelliptic estimate (2.14) immediately gives a general spectral gap property for the Bochner Laplacian.

Proposition 15. *There exist constants $c_1, c_2 > 0$, such that one has Spec $(\Delta_k) \subset \left[c_1 k^{2/r} - c_2, \infty\right)$ for each k.*

Proof. The subelliptic estimate (2.14) on the circle bundle is

$$\left\|\partial_\theta^{1/r} s\right\|^2 \leq \|s\|^2_{H^{1/r}} \leq C\left[\left\langle \Delta_{g^E,F,\mu_X} s, s\right\rangle + \|s\|^2_{L^2}\right], \ \forall s \in C^{\infty}(X; F).$$

Letting $s = \pi^* s'$ be the pullback of an orthonormal eigenfunction s' of Δ_k with eigenvalue λ on the base gives $k^{2/r} \leq C(\lambda + 1)$as required. □

The spectral gap above was earlier shown for symplectic curvature [28, 43] (where $r = 2$) and for surfaces with curvature vanishing to first order along a curve [50] (where $r = 3$).

3.1.1. *Heat kernels.* Next we would like to investigate the heat kernel of the Bochner Laplacian Δ_k. From (3.5), we have the relation

$$e^{-T\Delta_k}(y_1, y_2) = \left[\int d\theta \, e^{-T\Delta_{g^E,F,\mu_X}}\left(l_{y_1}, l_{y_2} e^{i\theta}\right) e^{-ik\theta}\right] l_{y_1} \otimes l^*_{y_2} \tag{3.6}$$

between the heat kernels with l_{y_1}, l_{y_2} denoting two unit elements in the fibers of L above y_1, y_2 respectively. We again note that the kernels are computed with respect to the densities μ_X, μ_Y chosen before. The above relation together with (2.18) first gives

$$e^{-\frac{1}{k^{2/r}}\Delta_k}(y_1, y_2) = c_{\varepsilon,N} k^{-N}, \quad \forall N \in \mathbb{N}, \tag{3.7}$$

when $d(y_1, y_2) > \varepsilon > 0$. If we choose a coordinate system centered at a point $y \in Y$ and a trivialization $\mathtt{1}$ of L that is parallel along coordinate rays starting at the origin, the connection form can be expressed in terms of the curvature as $\nabla^L = d + ia^L$,

$$a^L_p(x) = \int_0^1 d\rho \rho x^q R^L_{pq}(\rho x). \tag{3.8}$$

It is now easy to see that the induced coordinate system on the unit circle bundle X is privileged at each point on the fiber above y.

Next, using (3.6) with $T=\varepsilon^2 t$ and $y_1=y_2$ belonging to this coordinate chart one has

$$e^{-\varepsilon^2 t\Delta_k}\left(y_1,y_1\right)=\left[\int d\delta_\varepsilon\theta'\, e^{-\varepsilon^2 t\Delta_{g^{E,F,\mu_X}}}\left(\mathtt{1}\left(y_1\right),\mathtt{1}\left(y_1\right)e^{i\delta_\varepsilon\theta'}\right)e^{-ik\delta_\varepsilon\theta'}\right]$$

where δ_ε denotes the privileged coordinate dilation as before. Now setting $y_1=\varepsilon\mathsf{y}=\delta_\varepsilon\mathsf{y}$, the equations (2.24), (2.25) as in the proof of Theorem 9 give that the right hand side above has an expansion

$$e^{-\varepsilon^2 t\Delta_k}\left(\delta_\varepsilon\mathsf{y},\delta_\varepsilon\mathsf{y}\right)=\int d\delta_\varepsilon\theta' e^{-ik\delta_\varepsilon\theta'}\,\varepsilon^{-Q(y)}\left[\sum_{j=0}^{N}a_{2j}\left(\mathsf{y},\theta';t\right)\varepsilon^{2j}+\frac{\varepsilon^{2N+1}}{t^{Q(y)/2}}R_{N+1}\left(\mathsf{y},\theta';t\right)\right]\tag{3.9}$$

uniformly in $k\in\mathbb{N}$, $t\leq 1$ and $\mathsf{y}\in B_R\left(0\right)$, $\forall R>0$. A slight difference above being that the coefficients $a_j\left(\mathsf{y},\theta';t\right)$ above are computed with respect to the nilpotent sR Laplacian $\hat{\Delta}_{g_y^{TY},j^{r-2}R_y^L}$ (A.5) on the product $S_\theta^1\times\mathbb{R}^{n-1}$ rather than (2.26) on Euclidean space. In particular the leading term is $a_0\left(\mathsf{y},\theta';t\right)=e^{-t\hat{\Delta}_{g^{TY},j^{r-2}R^L}}\left(\mathsf{y},0;\mathsf{y},\theta'\right)$. Now let $\varepsilon=k^{-\frac{1}{r}}$, and set $r_1\left(y\right)\coloneqq 1-\frac{r(y)}{r}$ to get

$$\begin{aligned}
e^{-\frac{t}{k^{2/r}}\Delta_k}\left(k^{-\frac{1}{r}}\mathsf{y},k^{-\frac{1}{r}}\mathsf{y}\right)&=\int d\delta_{k^{-1/r}}\theta' e^{-ik^{r_1(y)}\theta'}\,k^{Q(y)/r}\left[\sum_{j=0}^{N}a_{2j}\left(\mathsf{y},\theta';t\right)k^{-2j/r}+O\left(k^{-(2N+1)/r}\right)\right]\\
&=\begin{cases}k^{(n-1)/r}\left[\sum_{j=0}^{N}a_{2j}\left(\mathsf{y};t\right)k^{-2j/r}+O\left(k^{-(2N+1)/r}\right)\right]; & y\in Y_r\\ O\left(k^{-\infty}\right); & y\in Y_{\leq r-1}\end{cases}
\end{aligned}\tag{3.10}$$

following a stationary phase expansion in θ'. Above we again note that the remainders are uniform for $\mathsf{y}\in B_R\left(0\right),\forall R>0$. The first coefficient is $a_0\left(\mathsf{y};t\right)=\int d\theta' e^{-i\theta'}e^{-t\hat{\Delta}_{g^{TY},j^{r-2}R^L}}\left(\mathsf{y},0;\mathsf{y},\theta'\right)=e^{-t\Delta_{g_y^{TY},j^{r-2}R_y^L}}\left(\mathsf{y},\mathsf{y}\right)$ while the general coefficient has the form

$$\begin{aligned}
a_{2j}\left(\mathsf{y};t\right)&=-\frac{1}{\pi}\int_{\mathbb{C}}\bar{\partial}\tilde{\rho}\left(z\right)\mathsf{C}_{2j}^{z}\left(\mathsf{y},\mathsf{y}\right)dzd\bar{z}\\
\mathsf{C}_{2j}^{z}&=\sum_{p\in I_{2j}}\left(\Delta_{g_y^{TY},j^{r-2}R_y^L}-z\right)^{-1}\left[\prod_{\alpha}\triangle_{p_\alpha}\left(\Delta_{g_y^{TY},j^{r-2}R_y^L}-z\right)^{-1}\right]
\end{aligned}\tag{3.11}$$

as in (2.27), for some set of second-order differential operators $\triangle_j$, $j=1,2,\ldots$, (see also (3.9) below). Further $\tilde{\rho}$ denotes an almost analytic continuation of ρ satisfying $\rho\left(x\right)=e^{-tx}$, $x\geq 0$. Finally, setting $y=y_1=\mathsf{y}$ in (3.10), we have arrived at the following.

Theorem 16. *The heat kernel of the Bochner Laplacian Δ_k has the pointwise expansion*

$$e^{-\frac{t}{k^{2/r}}\Delta_k}\left(y,y\right)=\begin{cases}k^{(n-1)/r}\left[\sum_{j=0}^{N}a_{2j}\left(y;t\right)k^{-2j/r}+O\left(k^{-(N+1)/r}\right)\right]; & y\in Y_r\\ O\left(k^{-\infty}\right); & y\in Y_{\leq r-1}\end{cases}\tag{3.12}$$

with leading coefficient $a_0\left(y;t\right)=e^{-t\Delta_{g_y^{TY},j^{r-2}R_y^L}}\left(0,0\right)$ being the heat kernel of the model operator (A.3) *on the tangent space.*

We note that the above heat kernel expansion like Theorem 9 is in general again not uniform in the point y on the diagonal and does not immediately give heat trace asymptotics. We shall explore the heat trace asymptotics in the next subsection 3.2.

We end this subsection by showing how the heat kernel expansion of Theorem 16 immediately gives the estimates on the first positive eigenvalue/eigenfunction of the Bochner Laplacian of

Theorem 1. First for any $0 < t_1 < t_2$, $y \in Y_r$ and $R > 0$ one has the following estimate at leading order

$$\frac{\lambda_0(k)}{k^{2/r}} \leq \frac{1}{(t_2-t_1)} \ln\left(\frac{\int_{B_R(0)} d\left(k^{-\frac{1}{r}}\mathsf{y}\right) e^{-\frac{t_1}{k^{2/r}}\Delta_k}\left(k^{-\frac{1}{r}}\mathsf{y}, k^{-\frac{1}{r}}\mathsf{y}\right)}{\int_{B_R(0)} d\left(k^{-\frac{1}{r}}\mathsf{y}\right) e^{-\frac{t_2}{k^{2/r}}\Delta_k}\left(k^{-\frac{1}{r}}\mathsf{y}, k^{-\frac{1}{r}}\mathsf{y}\right)}\right)$$

$$= \frac{1}{(t_2-t_1)} \ln\left(\frac{\int_{B_R(0)} d\mathsf{y} e^{-t_1\Delta_{g_y^{TY}, j^{r-2}R_y^L}}(\mathsf{y},\mathsf{y}) + O\left(k^{-1/r}\right)}{\int_{B_R(0)} d\mathsf{y} e^{-t_2\Delta_{g_y^{TY}, j^{r-2}R_y^L}}(\mathsf{y},\mathsf{y}) + O\left(k^{-1/r}\right)}\right)$$

$$= \frac{1}{(t_2-t_1)} \ln\left(\frac{\int_{B_R(0)} d\mathsf{y} e^{-t_1\Delta_{g_y^{TY}, j^{r-2}R_y^L}}(\mathsf{y},\mathsf{y})}{\int_{B_R(0)} d\mathsf{y} e^{-t_1\Delta_{g_y^{TY}, j^{r-2}R_y^L}}(\mathsf{y},\mathsf{y})}\right) + O\left(k^{-1/r}\right). \tag{3.13}$$

This already gives an upper bound on the first eigenvalue. To identify the constant (1.5) one takes the limit as $t_1 \to t_2$ to obtain

$$\frac{\lambda_0(k)}{k^{2/r}} \leq \frac{\int_{B_R(0)} d\mathsf{y}\left[\Delta_{g_y^{TY}, j^{r-2}R_y^L} e^{-t_1\Delta_{g_y^{TY}, j^{r-2}R_y^L}}\right](\mathsf{y},\mathsf{y})}{\int_{B_R(0)} d\mathsf{y} e^{-t_1\Delta_{g_y^{TY}, j^{r-2}R_y^L}}(\mathsf{y},\mathsf{y})} + O\left(k^{-1/r}\right),$$

$\forall t_1 > 0$. Using Proposition 39 of Section A this now gives $\limsup_{k\to\infty}\frac{\lambda_0(k)}{k^{2/r}} \leq \lambda_0\left(\Delta_{g_y^{TY}, j^{r-2}R_y^L}\right) + \varepsilon$, $\forall \varepsilon > 0$, $y \in Y_r$, and hence

$$\limsup_{k\to\infty}\frac{\lambda_0(k)}{k^{2/r}} \leq \inf_{y\in Y_r}\lambda_0\left(\Delta_{g_y^{TY}, j^{r-2}R_y^L}\right). \tag{3.14}$$

For the lower bound on $\lambda_0(k)$, first note that as in (2.29) one may prove an on diagonal expansion

$$\varphi\left(\frac{1}{k^{2/r}}\Delta_k\right)(y,y) = k^{(n-1)/r}\left[a_0^{\varphi}(x) + a_1^{\varphi}(x)k^{-1/r} + \ldots + a_N^{\varphi}(x)k^{-N/r} + O\left(k^{-(N+1)/r}\right)\right]$$

$\forall \varphi \in \mathcal{S}(\mathbb{R})$, and where the coefficient a_j^{φ} has the form (3.11) with $\tilde{\rho}$ replaced with an analytic continuation of φ. Next note that each of the terms C_{2j}^{z} (3.11) is holomorphic in z for $\mathrm{Re}z < \inf_{y\in Y_r}\lambda_0\left(\Delta_{g_y^{TY}, j^{r-2}R_y^L}\right)$. This gives $\varphi\left(\frac{1}{k^{2/r}}\Delta_k\right)(y,y) = O\left(k^{-N}\right)$, $\forall N \in \mathbb{N}$, uniformly in $y \in Y$. Thus

$$\varphi\left(\frac{\lambda_0(k)}{k^{2/r}}\right) \leq \mathrm{tr}\ \varphi\left(\frac{1}{k^{2/r}}\Delta_k\right) = O\left(k^{-N}\right)$$

$\forall N \in \mathbb{N}$, $\varphi \in C_c^{\infty}(-\infty, C)$, and hence

$$\inf_{y\in Y_r}\lambda_0\left(\Delta_{g_y^{TY}, j^{r-2}R_y^L}\right) \leq \liminf_{k\to\infty}\frac{\lambda_0(k)}{k^{2/r}}. \tag{3.15}$$

From (3.14), (3.15) we have (1.3). The estimate on the eigenfunction (1.4) then follows from $\left|\psi_0^k(y)\right|^2 \leq e^{\frac{\lambda_0(k)}{k^{2/r}}} e^{-\frac{1}{k^{2/r}}\Delta_k}(y,y)$ on using (3.12) and (3.13).

3.2. **Weyl law.** We now prove the first part of Theorem 2 on the asymptotics of the Weyl counting function $N\left(c_1k^{2/r}, c_2k^{2/r}\right)$; under the assumption that $Y_r = \bigcup_{j=1}^{N} Y_{r,j}$ is a union of embedded submanifolds, of dimensions $d_j := \dim(Y_{r,j})$, along which the curvature R^L vanishes non-degenerately (1.6). By a standard Tauberian argument, this shall follow from the following heat trace expansion.

Theorem 17. *Given $f \in C^{\infty}(Y)$, the heat trace of the Bochner Laplacian satisfies the asymptotics*

$$tr\left[fe^{-\frac{t}{k^{2/r}}\Delta_k}\right] = \sum_{j=1}^{N}\left\{\sum_{s=0}^{M} k^{(d_j-2s)/r}\left[\int_{NY_{r,j}} a_{j,s}(f;t)\right] + O\left(k^{(d_j-2M-1)/r}\right)\right\} \tag{3.16}$$

$\forall M \in \mathbb{N}$, $t \leq 1$. *Moreover, the leading terms above are given by*

$$a_{j,0}(f;t) = f|_{Y_{r,j}}\, e^{-t\Delta_{g_y^{TY},j^{r-2}R_y^L}}(v,v), \quad v \in N_y Y_{r,j},$$

in terms of the pullback to normal bundle of $f|_{Y_{r,j}}$.

Proof. By Theorem 16 it suffices to consider f supported in a sufficiently small neighborhood of a given point $y \in Y_{r,j}$. We then choose a coordinate system

$$\Big(\underbrace{y_1,\ldots,y_{d_j}}_{=y'};\underbrace{y_{d_j+1},\ldots,y_{n-1}}_{=y''}\Big)$$

near y in which $Y_{r,j} = \{y''=0\}$ is given by the vanishing of the last $n-1-d_j$ of these coordinates. Further we may assume $\left\{\partial_{y_j}\right\}_{j=1}^{n-1}$ to be orthonormal at y. A trivialization for L is chosen which is parallel with respect to coordinate rays starting at the origin. The curvature may then be written

$$R^L = \underbrace{\sum_{|\alpha|=r-2} R_{pq,\alpha}(y'')^{\alpha}\, dy_p dy_q}_{=R_0^L} + O\left((y'')^{r-1}\right).$$

The non-degeneracy condition (1.6) is now equivalent to

$$\left(\partial^{\beta}R_0^L\right)(y) = 0,\ \forall\,|\beta| < r-2 \iff y''=0, \tag{3.17}$$

i.e. the $(r-2)$-order vanishing locus is locally the same for R^L and its leading part R_0^L. The model operator (A.4) on the tangent space

$$\Delta_{g_y^{TY},j^{r-2}R_y^L} = -\sum_{q=1}^{n}\left(\partial_{y_p} + \frac{i}{r}y^q(y'')^{\alpha}R_{pq,\alpha}\right)^2,$$

is given in terms of this leading part of the curvature. We may also similarly define

$$\Delta_{g_y^{TY},j^{r-2}R_y^L;\mathsf{k}} := -\sum_{q=1}^{n}\left(\partial_{y_p} + \frac{i\mathsf{k}}{r}y^q(y'')^{\alpha}R_{pq,\alpha}\right)^2. \tag{3.18}$$

to be the model k-Bochner Laplacian for each $\mathsf{k} > 0$.

Now firstly, from (3.10) one has

$$e^{-\frac{1}{k^{2/r}}\Delta_k}(\delta_\varepsilon \mathsf{y},\delta_\varepsilon \mathsf{y}) = k^{(n-1)/r}\left[\sum_{j=0}^{N} a_{2j}\left(\varepsilon k^{1/r}\mathsf{y};t\right)k^{-2j/r} + O\left(k^{-(2N+1)/r}\right)\right],$$

$$a_0\left(\varepsilon k^{1/r}\mathsf{y};t\right) = e^{-t\Delta_{g_y^{TY},j^{r-2}R_y^L}}\left(\varepsilon k^{1/r}\mathsf{y},\varepsilon k^{1/r}\mathsf{y}\right) \tag{3.19}$$

uniformly for $k^{-1/r}\geq\varepsilon$ and $\mathsf{y}\in B_1(0)$. Furthermore, substituting $t=\frac{1}{\varepsilon^2 k^{2/r}}$ in (3.9) we obtain

$$e^{-\frac{1}{k^{2/r}}\Delta_k}\left(\delta_\varepsilon\mathsf{y},\delta_\varepsilon\mathsf{y}\right)=\varepsilon^{-(n-1)}\int d\theta' e^{-ik\varepsilon^r\theta'}\left[\sum_{j=0}^{N}a_{2j}\left(\mathsf{y},\theta';\frac{1}{\varepsilon^2 k^{2/r}}\right)\varepsilon^{2j}\right.$$
$$\left.+\frac{\varepsilon^{2N+1}}{\left(\varepsilon k^{1/r}\right)^{n-1+r}}R_{2N+1}\left(\mathsf{y},\theta';\frac{1}{\varepsilon^2 k^{2/r}}\right)\right],$$

$$a_0\left(\mathsf{y},\theta';\frac{1}{\varepsilon^2 k^{2/r}}\right)=e^{-\frac{1}{\varepsilon^2 k^{2/r}}\hat{\Delta}_{g^{TY},j^{r-2}R^L}}\left(\mathsf{y},0;\mathsf{y},\theta'\right)\tag{3.20}$$

uniformly for $k\in\mathbb{N}$, $k^{-1/r}\leq\varepsilon$ and $\mathsf{y}\in B_1(0)$. The leading term above is identified with the heat kernel

$$e^{-\frac{1}{\mathsf{k}^{2/r}}\Delta_{g_y^{TY},j^{r-2}R_y^L;\mathsf{k}}}\left(\mathsf{y},\mathsf{y}\right)=\int d\theta' e^{-ik\varepsilon^r\theta'}a_0\left(\mathsf{y},\theta';\frac{1}{\varepsilon^2 k^{2/r}}\right),$$

of the model k-Bochner Laplacian (3.18) for $\mathsf{k}:=k\varepsilon^r$. One next chooses

$$\mathsf{y}=\Big(\underbrace{0,\ldots,0}_{=y'};\underbrace{y_{d_j+1},\ldots,y_{n-1}}_{=y''}\Big),\ |y''|=1,$$

of the given form so that $\mathrm{ord}_{\mathsf{y}}\left(R_0^L\right)<r-2$ by (3.17). Then

$$e^{-\frac{1}{\mathsf{k}^{2/r}}\Delta_{g_y^{TY},j^{r-2}R_y^L;\mathsf{k}}}\left(\mathsf{y},\mathsf{y}\right)=e^{-\Delta_{g_y^{TY},j^{r-2}R_y^L}}\left(\mathsf{k}^{1/r}\mathsf{y},\mathsf{k}^{1/r}\mathsf{y}\right)=O\left(\mathsf{k}^{-\infty}\right),\tag{3.21}$$

follows by a stationary phase type argument as in Theorem 16. A similar argument applied to the subsequent terms in (3.20), which are given by convolution integrals with the leading part, shows that $\int d\theta' e^{-ik\varepsilon^r\theta'}a_{2j}\left(\mathsf{y},\theta';\frac{1}{\varepsilon^2 k^{2/r}}\right)=O\left(\mathsf{k}^{-\infty}\right)$, $\forall j$. In particular the terms of (3.20) are integrable in ε for fixed k. Finally (3.19), (3.20), (3.21) and a Taylor expansion for f near $\mathsf{y}=0$ combine to give (3.16). □

3.3. **Expansion for the ground state.** Next we show the expansion for the first eigenvalue under the non-degeneracy assumption (1.6) and when Y_r is a union of points. The technique below borrows from [10] Ch. 9. Firstly we now choose the trivialization for L to be parallel with respect to geodesics starting at the origin in some geodesic ball $B_{2\varrho}(y)$ centered at any $y\in Y_r$. The curvature may then be written

$$R^L=\underbrace{\sum_{|\alpha|=r-2}R_{pq,\alpha}y^\alpha dy_p dy_q}_{=R_0^L}+O\left(y^{r-1}\right).\tag{3.22}$$

in these coordinates/trivialization with $R_{pq;\alpha}$ being identified with the components of the non-vanishing jet $j^{r-2}R^L$. The non-degeneracy condition (1.6) is now equivalent to

$$\left(\partial^\beta R_0^L\right)(y)=0,\ \forall\,|\beta|<r-2\implies y=0.$$

Having $\mathrm{EssSpec}\left(\Delta_{g_y^{TY},j^{r-2}R_y^L}\right)=\emptyset$ under the non-degeneracy assumption (see Proposition 38) we may set $\lambda_{0,y}<\lambda_{1,y}$ to be the two smallest eigenvalues of $\Delta_{g_y^{TY},j^{r-2}R_y^L}$ and $\bar{\lambda}_0:=\min_{y\in Y_r}\lambda_{0,y}$. Further set $\bar{Y}_r:=\left\{y\in Y_r|\lambda_{0,y}=\bar{\lambda}_0\right\}\subset Y_r$ and $\bar{\lambda}_1:=\min\left\{\lambda_{1,y}|y\in\bar{Y}_r\right\}\cup\left\{\lambda_{0,y}|y\in Y_r\setminus\bar{Y}_r\right\}>\bar{\lambda}_0$.

We shall begin by constructing quasimodes localized at any $y \in Y_r$. The Bochner Laplacian $\Delta_k = \left(\nabla^{F\otimes L^k}\right)^* \nabla^{F\otimes L^k}$ can be written in this local frame/coordinates where

$$\begin{aligned}\nabla^{F\otimes L^k} =&d + a^F + ka^L\\ a_p^L =& \int_0^1 d\rho \left(\rho y^q R_{pq}^L(\rho x)\right),\end{aligned}$$

as in (3.8) and with a^F being the Christoffel symbol for F in some orthonormal trivialization. Further, the adjoint is taken with respect to the metric $g_{pq}^{TY} = \delta_{pq} + O(y^2)$ when expressed in geodesic coordinates. With $\chi \in C_c^\infty([-1,1];[0,1])$ with $\chi = 1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$, we define the modified connections on $\mathbb{R}^{n-1}$ via

$$\begin{aligned}\tilde{\nabla}^F &= d + \chi\left(\frac{|y|}{2\varrho}\right) a^F\\ \tilde{\nabla}^L &= d + \left[\underbrace{\int_0^1 d\rho\, \rho y^k \left(\tilde{R}^L\right)_{jk}(\rho y)}_{=\tilde{a}_j^L}\right] dy_j, \quad \text{where}\\ \tilde{R}^L &= \chi\left(\frac{|y|}{2\varrho}\right) R^L + \left[1 - \chi\left(\frac{|y|}{2\varrho}\right)\right] R_0^L. \qquad (3.23)\end{aligned}$$

Further we choose a modified metric $\tilde{g}^{TY}$ which is Euclidean outside $B_{2\varrho}(y)$ and agrees with g^{TY} on $B_\varrho(y)$. This defines the modified Bochner Laplacian

$$\tilde{\Delta}_k := \left(\tilde{\nabla}^{F\otimes L^k}\right)^* \tilde{\nabla}^{F\otimes L^k} \qquad (3.24)$$

agreeing with $\Delta_k = \tilde{\Delta}_k$ on the geodesic ball $B_\varrho(y)$.

A rescaling/dilation is now again defined via $\delta_{k^{-1/r}} y := \left(k^{-1/r} y_1, \ldots, k^{-1/r} y_{n-1}\right)$ and we consider the rescaled Bochner Laplacian

$$\triangle := k^{-2/r} \left(\delta_{k^{-1/r}}\right)_* \tilde{\Delta}_k. \qquad (3.25)$$

Using a Taylor expansion and (3.23), (3.24), the rescaled Bochner Laplacian has an expansion

$$\triangle \;=\; \left(\sum_{j=0}^N k^{-j/r} \triangle_j\right) + k^{-2(N+1)/r} \mathrm{E}_{N+1},\ \forall N. \qquad (3.26)$$

Here each

$$\triangle_j = a_{j;pq}(y)\, \partial_{y_p}\partial_{y_q} + b_{j;p}(y)\, \partial_{y_p} + c_j(y) \qquad (3.27)$$

is a (k-independent) self-adjoint, second-order differential operator while each

$$\mathrm{E}_j = \sum_{|\alpha|=N+1} y^\alpha \left[a_{j;pq}^\alpha(y;k)\, \partial_{y_p}\partial_{y_q} + b_{j;p}^\alpha(y;k)\, \partial_{y_p} + c_j^\alpha(y;k)\right] \qquad (3.28)$$

is a k-dependent self-adjoint, second-order differential operator on $\mathbb{R}^{n-1}$. Furthermore the functions appearing in (3.27) are polynomials with degrees satisfying

$$\begin{aligned}\deg a_j = j,\ \deg b_j \leq j + r - 1, \deg c_j \leq j + 2r - 2\\ \deg b_j - (j-1) = \deg c_j - j = 0 \pmod 2\end{aligned}$$

and whose coefficients involve

$$\begin{aligned}
a_j &: \leq j-2 \text{ derivatives of } R^{TY} \\
b_j &: \leq j-2 \text{ derivatives of } R^F, R^{TY} \\
&\quad \leq j+r-2 \text{ derivatives of } R^L \\
c_j &: \leq j-2 \text{ derivatives of } R^F, R^{TY} \\
&\quad \leq j+r-2 \text{ derivatives of } R^L
\end{aligned}$$

while the coefficients $a^{\alpha}_{j;pq}(y;k), b^{\alpha}_{j;p}(y;k), c^{\alpha}_j(y;k)$ of (3.28) are uniformly (in k) C^∞ bounded. The leading term of (4.23) is computed

$$\triangle_0 = \Delta_{g_y^{TY}, j^{r-2}R_y^L} \tag{3.29}$$

in terms of the the model Bochner Laplacian on the tangent space TY (A.3). We note that these operators (3.27) are the same as those appearing in (3.11).

We choose $0 < \varrho \ll 1$ such that the $B_\varrho(y)$'s, $y \in Y_r$, are disjoint. Next with $E_{0,y} \coloneqq \ker\left[\Delta_{g_y^{TY}, j^{r-2}R_y^L} - \lambda_{0,y}\right]$ the smallest eigenspace of $\triangle_0$, any normalized $\tilde{\psi} \in E_{0,y}$ defines a quasi-mode

$$\begin{aligned}
\widetilde{\psi}_k(y) \coloneqq& \chi\left(\frac{2|y|}{\varrho}\right)\underbrace{k^{(n-1)/2r}\tilde{\psi}\left(k^{1/r}y\right)}_{=k^{(n-1)/2r}\delta^*_{k^{-1/r}}\tilde{\psi}} \in C^\infty\left(Y; F\otimes L^k\right), \qquad \text{satisfying} \\
\left\|\widetilde{\psi}_k\right\| =& 1 + o(1) \\
\Delta_k\widetilde{\psi}_k =& k^{2/r}\lambda_{0,y}\widetilde{\psi}_k + O_{L^2}\left(k^{1/r}\right).
\end{aligned} \tag{3.30}$$

We then define $\tilde{E}_{0,y}$ to be the span of the quasimodes corresponding to an orthonormal basis of $E_{0,y}$. Set $\tilde{E}_0 \coloneqq \oplus_{y\in\bar{Y}_r}\tilde{E}_{0,y} \subset C^\infty\left(Y; F\otimes L^k\right)$ with $\tilde{E}_0^\perp$ being its L^2 orthogonal complement. We now have the following proposition.

Proposition 18. *There exists $c>0$, $k_0 \in \mathbb{N}$ such that*

$$\left|\left\langle \Delta_k\tilde{\psi}, \tilde{\psi}\right\rangle - \bar{\lambda}_0 k^{2/r}\right| \leq ck^{1/r} \tag{3.31}$$

$$\left\langle \Delta_k\psi, \psi\right\rangle \geq \frac{1}{2}\left(\bar{\lambda}_0 + \bar{\lambda}_1\right)k^{2/r} \tag{3.32}$$

$\forall k > k_0$ and $\tilde{\psi} \in \tilde{E}_0$, $\psi \in C^\infty\left(Y; F\otimes L^k\right) \cap \tilde{E}_0^\perp$ of unit norm.

Proof. The first equation (3.31) follows easily from construction (3.30).

For (3.32), we first set $\chi_y\psi \coloneqq \chi\left(\frac{d(.,y)}{\varrho}\right)\psi$ for each $y \in Y_r$ and split

$$\psi = \underbrace{\left(\sum_{y\in Y_r}\chi_y\right)\psi}_{=\psi_1} + \underbrace{\left(1 - \sum_{y\in Y_r}\chi_y\right)\psi}_{=\psi_2}.$$

Now since the ψ_2 is compactly supported away from Y_r, an argument similar to Proposition 15 gives

$$\left\langle \Delta_k\psi_2, \psi_2\right\rangle \geq \left[c_1k^{2/(r-1)} - c_2\right]\left\|\psi_2\right\|^2 \tag{3.33}$$

for some constants $c_1, c_2 > 0$ depending only on ϱ. Next each $\chi_y \psi$, $y \in Y_r$, having compact support in $B_\varrho(y)$, we may decompose

$$k^{-(n-1)/2r} \left(\delta^{-1}_{k^{-1/r}}\right)^* \chi_y \psi = \underbrace{\psi^0_y}_{\in \ker\left[\triangle_0 - \bar{\lambda}_0\right]} + \underbrace{\psi^+_y}_{\in \ker\left[\triangle_0 - \bar{\lambda}_0\right]^\perp} .$$

Clearly ψ^0_y is orthogonal to ψ^+_y, $\triangle_0 \psi^+_y$ while $\left\langle \triangle_0 \psi^+_y, \psi^+_y \right\rangle \geq \bar{\lambda}_1 \left\|\psi^+_y\right\|^2$ by definition. Furthermore, since $\chi_y \psi \perp \tilde{E}_{0,y}$ by hypothesis, we may compute

$$\begin{aligned}
\left\langle \chi_y \psi, k^{(n-1)/2r} \delta^*_{k^{-1/r}} \tilde{\psi} \right\rangle &= \left\langle \chi_y \psi, (1-\chi)\, k^{(n-1)/2r} \delta^*_{k^{-1/r}} \tilde{\psi} \right\rangle \\
&= \left\langle k^{-(n-1)/2r} \left(\delta^{-1}_{k^{-1/r}}\right)^* \chi_y \psi, \left[1 - \chi\left(k^{-1/r} y\right)\right] \tilde{\psi} \right\rangle = o(1) \left\|\chi_y \psi\right\|
\end{aligned}$$

for any normalized $\tilde{\psi} \in E_{0,y}$. This in turn gives $\left\|\psi^0_y\right\| = o(1) \left\|\chi_y \psi\right\|$, $\left\|\psi^+_y\right\| = [1 - o(1)] \left\|\chi_y \psi\right\|$ and hence

$$\begin{aligned}
\left\langle \triangle_0 k^{-n/2r} \left(\delta^{-1}_{k^{-1/r}}\right)^* \chi_y \psi, k^{-n/2r} \left(\delta^{-1}_{k^{-1/r}}\right)^* \chi_y \psi \right\rangle &= \left\langle \triangle_0 \psi^0_y, \psi^0_y \right\rangle + \left\langle \triangle_0 \psi^+_y, \psi^+_y \right\rangle \\
&\geq \bar{\lambda}_1 \left\|\psi^+_y\right\|^2 \\
&\geq \left[\bar{\lambda}_1 - o(1)\right] \left\|\chi_y \psi\right\|^2 .
\end{aligned}$$

On account of the rescaling (3.25), (3.26), (3.29) we then have

$$\left\langle \Delta_k \chi_y \psi, \chi_y \psi \right\rangle \geq k^{2/r} \left[\bar{\lambda}_1 - o(1)\right] \left\|\chi_y \psi\right\|^2 . \tag{3.34}$$

Finally, with $\chi_1 = \sum_{y \in Y_r} \chi_y$ and $\rho \in (0,1)$ we estimate

$$\begin{aligned}
\left\langle \Delta_k \psi, \psi \right\rangle^{1/2} = \left\|\nabla^{F \otimes L^k} \psi\right\| &\geq \rho \left\|\chi_1 \nabla^{F \otimes L^k} \psi\right\| + (1-\rho) \left\|(1-\chi_1) \nabla^{F \otimes L^k} \psi\right\| \\
&= \rho \left\| -d\chi_1 \psi + \nabla^{F \otimes L^k} \chi_1 \psi \right\| \\
&\quad + (1-\rho) \left\| d\chi_1 \psi + \nabla^{F \otimes L^k} (1-\chi_1) \psi \right\| \\
&= \rho \left\|\nabla^{F \otimes L^k} \chi_1 \psi\right\| + (1-\rho) \left\|\nabla^{F \otimes L^k} (1-\chi_1) \psi\right\| \\
&\qquad - O(1) \left\|\psi\right\| \\
&\geq \rho k^{1/r} \left[\bar{\lambda}_1 - o(1)\right]^{1/2} \left\|\chi_1 \psi\right\| \\
&\quad + (1-\rho) \left[c_1 k^{2/(r-1)} - c_2\right]^{1/2} \left\|(1-\chi_1) \psi\right\| - O(1) \left\|\psi\right\| \\
&\geq \frac{1}{2} \left(\bar{\lambda}_0 + \bar{\lambda}_1\right)^{1/2} k^{1/r} \left\|\psi\right\|
\end{aligned}$$

from (3.33) and (3.34) on choosing $0 < \rho \ll 1$ and $k \gg 0$ respectively. □

Following the above proposition, the min-max principle for eigenvalues immediately gives

$$\operatorname{Spec}(\Delta_k) \subset \left[\bar{\lambda}_0 k^{2/r} - c k^{1/r}, \bar{\lambda}_0 k^{2/r} + c k^{1/r}\right] \cup \left[\frac{1}{2}\left(\bar{\lambda}_0 + \bar{\lambda}_1\right) k^{2/r}, \infty\right) . \tag{3.35}$$

Let $\Gamma = \left\{|z| = \frac{2}{3}\bar{\lambda}_0 + \frac{1}{3}\bar{\lambda}_1\right\}$ denote the circular contour in the complex plane and $\varphi \in C^\infty_c\left(0, \frac{2}{3}\bar{\lambda}_0 + \frac{1}{3}\bar{\lambda}_1\right)$ with $\varphi = 1$ near $\bar{\lambda}_0$. The resolvent $\left(\frac{1}{k^{2/r}} \Delta_k - z\right)^{-1}$ then exists for $z \in \Gamma$, $k \gg 0$ and one may define via

$$P_0 := \frac{1}{2\pi i} \int_\Gamma \left(\frac{1}{k^{2/r}} \Delta_k - z\right)^{-1} = \varphi\left(\frac{1}{k^{2/r}} \Delta_k\right)$$

the spectral projection onto the span of the Δ_k-eigenspaces with eigenvalue in the first interval

$$\left[\bar{\lambda}_0 k^{2/r} - ck^{1/r}, \bar{\lambda}_0 k^{2/r} + ck^{1/r}\right]$$

of (3.35). Finally, (3.31) and (3.32) imply that

$$P_0 : \tilde{E}_0 \xrightarrow{\sim} E_0 := \bigoplus \left\{ \ker\left(\Delta_k - \lambda\right) : \lambda \in \left[\bar{\lambda}_0 k^{2/r} - ck^{1/r}, \bar{\lambda}_0 k^{2/r} + ck^{1/r}\right] \right\} \tag{3.36}$$

is an isomorphism for $k \gg 0$.

We now have the following.

Theorem 19. *Given $\widetilde{\psi}_k, \widetilde{\psi}'_k \in \tilde{E}_0$ two quasimodes* (3.30), *the inner product*

$$\left\langle \widetilde{\psi}_k, \Delta_k P_0 \widetilde{\psi}'_k \right\rangle = k^{2/r} \sum_{j=0}^{N} \tilde{c}_j k^{-j/r} + O\left(k^{(1-N)/r}\right) \tag{3.37}$$

has an asymptotic expansion for some $\tilde{c}_j \in \mathbb{R}$, $j = 0, 1, \ldots$.

Proof. For two quasimodes $\widetilde{\psi}_k, \widetilde{\psi}'_k$ localized at different points of $\bar{Y}_r$ one has $\left\langle \widetilde{\psi}_k, \Delta_k P_0 \widetilde{\psi}'_k \right\rangle = O\left(k^{-\infty}\right)$ following a similar off-diagonal decay for the kernel of $\varphi\left(\frac{1}{k^{2/r}}\Delta_k\right)$ as (3.7). We now consider $\widetilde{\psi}_k, \widetilde{\psi}'_k \in \tilde{E}_{0,y}$ of the form (3.30) localized at some $y \in \bar{Y}_r$. To this end (and as in the proof of Theorem 9) note first that by a finite propagation argument we have

$$\left\langle \widetilde{\psi}_k, \Delta_k P_0 \widetilde{\psi}'_k \right\rangle = \left\langle \widetilde{\psi}_k, \tilde{\Delta}_k \varphi\left(\frac{1}{k^{2/r}}\tilde{\Delta}_k\right) \widetilde{\psi}'_k \right\rangle, \quad \text{while}$$

$$\frac{1}{k^{2/r}} \tilde{\Delta}_k \varphi\left(\frac{1}{k^{2/r}}\tilde{\Delta}_k\right)(y, y') = k^{(n-1)/r} \triangle \varphi\left(\triangle\right)\left(k^{1/r} y, k^{1/r} y'\right). \tag{3.38}$$

Next, the expansion (3.26) along with local elliptic estimates gives

$$\left(\triangle - z\right)^{-1} - \left(\triangle_0 - z\right)^{-1} = O_{H^s_{\text{loc}} \to H^{s+2}_{\text{loc}}}\left(k^{-1/r} \left|\text{Im} z\right|^{-2}\right)$$

for each $s \in \mathbb{R}$. More generally, we let $I_j := \left\{p = (p_0, p_1, \ldots) \,|\, p_\alpha \in \mathbb{N}, \sum p_\alpha = j\right\}$ denote the set of partitions of the integer j and define

$$\mathtt{C}^z_j := \sum_{p \in I_j} \left(\triangle_0 - z\right)^{-1} \left[\prod_\alpha \triangle_{p_\alpha} \left(\triangle_0 - z\right)^{-1}\right].$$

Then by repeated applications of the local elliptic estimate we have

$$\left(\triangle - z\right)^{-1} - \sum_{j=0}^{N} k^{-j/r} \mathtt{C}^z_j = O_{H^s_{\text{loc}} \to H^{s+2}_{\text{loc}}}\left(k^{-(N+1)/r} \left|\text{Im} z\right|^{-2rN-2}\right),$$

for each $N \in \mathbb{N}$, $s \in \mathbb{R}$. A similar expansion as (3.26) for the operator $\left(\triangle + 1\right)^M \left(\triangle - z\right)$, $M \in \mathbb{N}$, also gives

$$\left(\triangle + 1\right)^{-M} \left(\triangle - z\right)^{-1} - \sum_{j=0}^{N} k^{-j/r} \mathtt{C}^z_{j,M} = O_{H^s_{\text{loc}} \to H^{s+2}_{\text{loc}}}\left(k^{-(N+1)/r} \left|\text{Im} z\right|^{-2rN-2}\right) \tag{3.39}$$

for operators $\mathtt{C}^z_{j,M} = O_{H^s_{\text{loc}} \to H^{s+2+2M}_{\text{loc}}}\left(k^{-(N+1)/r} \left|\text{Im} z\right|^{-2rN-2}\right)$, $j = 0, \ldots, N$, with

$$\mathtt{C}^z_{0,M} = \left(\hat{\Delta}^{(0)}_{g^E, F, \mu} + 1\right)^{-M} \left(\hat{\Delta}^{(0)}_{g^E, F, \mu} - z\right)^{-1}.$$

For $M \gg 0$ sufficiently large, Sobolev's inequality gives an expansion for the corresponding Schwartz kernels in (3.39) in $C^0(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. Plugging the resulting expansion into the Helffer-Sjöstrand formula then gives

$$\left|\triangle\varphi(\triangle)-\sum_{j=0}^{N}k^{-j/r}\mathtt{C}_j^{\varphi}\right|_{C^0(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})}=O\left(k^{-(N+1)/r}\right)$$

$\forall N\in\mathbb{N}$ and for some (k-independent) $\mathtt{C}_j^{\varphi}\in C^0(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, $j=0,1,\ldots$. Finally, the last equation with (3.30), (3.38) gives

$$\left\langle\widetilde{\psi}_k,\Delta_k P_0\widetilde{\psi}'_k\right\rangle-k^{2/r}\left(\sum_{j=0}^{N}c_jk^{-j/r}\right)=O\left(k^{-(N-1)/r}\right)\quad\text{where}$$
$$\tilde{c}_j:=\left\langle\tilde{\psi},\mathtt{C}_j^{\varphi}\tilde{\psi}'\right\rangle,$$

as required. □

Following this we note from (3.36) that the low lying eigenvalues of Δ_k are given by Spec $\left(\Delta_k|_{E_0}\right)=$ Spec $\left(\Delta_k|_{P_0\tilde{E}_0}\right)$ for $k\gg 0$. However, since the matrix coefficients of $\Delta_k|_{P_0\tilde{E}_0}$ were just shown to have an expansion, the expansion for the smallest eigenvalue now follows from an application of standard perturbation theory for self-adjoint matrices (see [39] Ch. 2).

4. Kodaira Laplacian on tensor powers

We now specialize to the case when Y is a complex Hermitian manifold with integrable complex structure J. For the arguments of this section, we shall further need to restrict to the two dimensional case, i.e. Y is a Riemann surface (see Remark 22). The metric g^{TY} is induced from the Hermitian metric on the complex tangent space $T_{\mathbb{C}}Y=T^{1,0}Y$. Further (L,h^L), (F,h^F) are chosen to be a Hermitian, holomorphic bundles where L is of rank one. We denote by ∇^L, ∇^F the corresponding Chern connections. The curvature R^L of ∇^L is a $(1,1)$ form which is further assumed to be semi-positive

$$iR^L(v,Jv)\geq 0,\quad\forall v\in TY\quad\text{or equivalently}$$
$$R^L(w,\bar{w})\geq 0,\quad\forall w\in T^{1,0}Y. \tag{4.1}$$

We also assume as before that the curvature R^L vanishes at finite order at any point of Y. We note that semipositivity implies that the order of vanishing $r_y-2\in 2\mathbb{N}_0$ of the curvature R^L at any point y is even. Semipositivity and finite order of vanishing imply that there are points where the curvature is positive (the set where the curvature is positive is in fact an open dense set). Hence

$$\deg L=\int_Y c_1(L)=\int_Y\frac{i}{2\pi}R^L>0,$$

so that L is ample.

Denote by $\left(\Omega^{0,*}\left(X;F\otimes L^k\right);\bar{\partial}_k\right)$ the Dolbeault complex and define the Kodaira Laplace and Dirac operators acting on $\Omega^{0,*}\left(X;F\otimes L^k\right)$

$$\Box_k:=\frac{1}{2}\left(D_k\right)^2=\bar{\partial}_k\bar{\partial}_k^*+\bar{\partial}_k^*\bar{\partial}_k \tag{4.2}$$

$$D_k:=\sqrt{2}\left(\bar{\partial}_k+\bar{\partial}_k^*\right). \tag{4.3}$$

Clearly, D_k interchanges while $\Box_k$ preserves $\Omega^{0,0/1}$. We denote $D_k^{\pm}=D_k|_{\Omega^{0,0/1}}$ and $\Box_k^{0/1}=\Box_k|_{\Omega^{0,0/1}}$. The Clifford multiplication endomorphism $c:TY\to\mathrm{End}\left(\Lambda^{0,*}\right)$ is defined via $c(v):=$

$\sqrt{2}\left(v^{1,0}\wedge -i_{v^{0,1}}\right)$, $v\in TY$, and extended to the entire exterior algebra $\Lambda^{*}TY$ via $c\left(1\right)=1$, $c\left(v_{1}\wedge v_{2}\right)\coloneqq c\left(v_{1}\right)c\left(v_{2}\right)$, $v_{1},v_{2}\in TY$.

Denote by $\nabla^{TY},\nabla^{T^{1,0}Y}$ the Levi-Civita and Chern connections on the real and holomorphic tangent spaces as well as by $\nabla^{T^{0,1}Y}$ the induced connection on the anti-holomorphic tangent space. Denote by Θ the real $(1,1)$ form defined by contraction of the complex structure with the metric $\Theta\left(.,.\right)=g^{TY}\left(J.,.\right)$. This is clearly closed $d\Theta=0$ (or Y is Kähler) and the complex structure is parallel $\nabla^{TY}J=0$ or $\nabla^{TY}=\nabla^{T^{1,0}Y}\oplus\nabla^{T^{1,0}Y}$.

With the induced tensor product connection on $\Lambda^{0,*}\otimes F\otimes L^{k}$ being denoted via $\nabla^{\Lambda^{0,*}\otimes F\otimes L^{k}}$, the Kodaira Dirac operator (4.3) is now given by the formula

$$D_{k}=c\circ\nabla^{\Lambda^{0,*}\otimes F\otimes L^{k}}.$$

Next we denote by R^{F} the curvature of ∇^{F} and by κ the scalar curvature of g^{TY}. Define the following endomorphisms of $\Lambda^{0,*}$

$$\begin{aligned}\omega\left(R^{F}\right)&\coloneqq R^{F}\left(w,\bar{w}\right)\bar{w}i_{\bar{w}}\\ \omega\left(R^{L}\right)&\coloneqq R^{L}\left(w,\bar{w}\right)\bar{w}i_{\bar{w}}\\ \omega\left(\kappa\right)&\coloneqq\kappa\bar{w}i_{\bar{w}}\\ \tau^{F}&\coloneqq R^{F}\left(w,\bar{w}\right)\\ \tau^{L}&\coloneqq R^{L}\left(w,\bar{w}\right)\end{aligned}\tag{4.4}$$

in terms of an orthonormal section w of $T^{1,0}Y$. The Lichnerowicz formula for the above Dirac operator ([44] Thm 1.4.7) simplifies for a Riemann surface and is given by

$$2\Box_{k}=D_{k}^{2}=\left(\nabla^{\Lambda^{0,*}\otimes F\otimes L^{k}}\right)^{*}\nabla^{\Lambda^{0,*}\otimes F\otimes L^{k}}+k\left[2\omega\left(R^{L}\right)-\tau^{L}\right]+\left[2\omega\left(R^{F}\right)-\tau^{F}\right]+\frac{1}{2}\omega\left(\kappa\right).\tag{4.5}$$

We now have the following.

Proposition 20. *Let Y be a compact Riemann surface, $(L,h^{L})\rightarrow Y$ a semi-positive line bundle whose curvature R^{L} vanishes to finite order at any point. Let $(F,h^{F})\rightarrow Y$ be a Hermitian holomorphic vector bundle. Then there exist constants $c_{1},c_{2}>0$, such that*

$$\left\Vert D_{k}s\right\Vert^{2}\geq\left(c_{1}k^{2/r}-c_{2}\right)\left\Vert s\right\Vert^{2}$$

for all $s\in\Omega^{0,1}\left(Y;F\otimes L^{k}\right)$.

Proof. Writing $s=\left|s\right|\bar{w}\in\Omega^{0,1}\left(Y;F\otimes L^{k}\right)$ in terms of a local orthonormal section $\bar{w}$ gives

$$\left\langle\left[2\omega\left(R^{L}\right)-\tau^{L}\right]s,s\right\rangle=R^{L}\left(w,\bar{w}\right)\left|s\right|^{2}\geq0\tag{4.6}$$

from (4.1), (4.4). This gives

$$\begin{aligned}\left\Vert D_{k}s\right\Vert^{2}&=\left\langle D_{k}^{2}s,s\right\rangle\\ &=\left\langle\left[\left(\nabla^{\Lambda^{0,*}\otimes F\otimes L^{k}}\right)^{*}\nabla^{\Lambda^{0,*}\otimes F\otimes L^{k}}+k\left[2\omega\left(R^{L}\right)-\tau^{L}\right]\right.\right.\\ &\qquad\left.\left.+\left[2\omega\left(R^{F}\right)-\tau^{F}\right]+\frac{1}{2}\omega\left(\kappa\right)\right]s,s\right\rangle\\ &\geq\left\langle\left(\nabla^{\Lambda^{0,*}\otimes F\otimes L^{k}}\right)^{*}\nabla^{\Lambda^{0,*}\otimes F\otimes L^{k}}s,s\right\rangle-c_{0}\left\Vert s\right\Vert^{2}\\ &\geq\left(c_{1}k^{2/r}-c_{2}\right)\left\Vert s\right\Vert^{2}\end{aligned}$$

from Proposition 15, (4.5) and (4.6). □

We now derive as a corollary a spectral gap property for Kodaira Dirac/Laplace operators D_k, $\Box_k$ corresponding to Proposition 15.

Corollary 21. *Under the hypotheses of Proposition 20 there exist constants $c_1, c_2 > 0$, such that Spec* $(\Box_k) \subset \{0\} \cup \left[c_1 k^{2/r} - c_2, \infty\right)$ *for each k. Moreover,* $\ker D_k^- = 0$ *and* $H^1\left(Y; F \otimes L^k\right) = 0$ *for k sufficiently large.*

Proof. From Proposition 20, it is clear that

$$\operatorname{Spec}\left(\Box_k^1\right) \subset \left[c_1 k^{2/r} - c_2, \infty\right) \tag{4.7}$$

for some $c_1, c_2 > 0$ giving the second part of the corollary. Moreover, the eigenspaces of $D_k^2|_{\Omega^{0,0/1}}$ with non-zero eigenvalue being isomorphic by Mckean-Singer, the first part also follows. $\square$

Since L is ample, we know also by the Kodaira-Serre vanishing theorem that $H^1\left(Y; F \otimes L^k\right)$ vanishes for k sufficiently large. If F is also a line bundle this follows from the well known fact that for a line bundle E on Y we have $H^1(Y; E) = 0$ whenever $\deg E > 2g - 2$. It is however interesting to have a direct analytic proof. Of course, the vanishing theorem for a semi-positive line bundle works only in dimension one, see Remark 22 below.

The vanishing $H^1\left(Y; F \otimes L^k\right) = 0$ for k sufficiently large gives

$$\begin{aligned}\dim H^0\left(Y; F \otimes L^k\right) &= \chi\left(Y; F \otimes L^k\right)\\ &= \int_Y ch\left(F \otimes L^k\right) \operatorname{Td}(Y)\\ &= k\left[\operatorname{rk}(F)\int_Y c_1(L)\right] + \int_Y c_1(F) + 1 - g,\end{aligned} \tag{4.8}$$

by Riemann-Roch, with $\chi\left(Y; F \otimes L^k\right)$, $ch\left(F \otimes L^k\right)$, $\operatorname{Td}(Y)$, g denoting the holomorphic Euler characteristic, Chern character, Todd genus and genus of Y respectively.

Remark 22. The Corollary 21 also gives a corresponding spectral gap for the 'renormalized Laplacian' ([28, 43]),

$$\Delta_k^{\#} \coloneqq \Delta_k - k\tau^L = D_k^2 + \tau^F,$$

acting on functions : there exist $c_0, c_1, c_2 > 0$ such that for any k,

$$\operatorname{Spec}\left(\Delta_k^{\#}\right) \subset [-c_0, c_0] \cup \left[c_1 k^{2/r} - c_2, \infty\right),$$

with the number of eigenvalues in the interval $[-c_0, c_0]$ again being the holomorphic Euler characteristic (4.8). The argument for Proposition 20 breaks down in higher dimensions since there are more components to $\left[2\omega\left(R^L\right) - \tau^L\right]$ in the Lichnerowicz formula (4.5) which semi-positivity is insufficient to control. Indeed, there is a known counterexample to the existence of a spectral gap for semi-positive line bundles in higher dimensions [24].

4.1. **Bergman kernel expansion.** We now investigate the asymptotics of the Bergman kernel on the Riemann surface Y. This is the Schwartz kernel $\Pi_k(y_1, y_2)$ of the projector onto the nullspace of $\Box_k$

$$\Pi_k : C^\infty\left(Y; F \otimes L^k\right) \to \ker\left(\Box_k|_{C^\infty\left(Y; F \otimes L^k\right)}\right), \tag{4.9}$$

with respect to the L^2 inner product given by the metrics g^{TY}, h^F and h^L. Alternately, if $s_1, s_2, \ldots, s_{N_k}$ denotes an orthonormal basis of eigensections of $H^0\left(X; F \otimes L^k\right)$ then

$$\Pi_k(y_1, y_2) = \sum_{j=1}^{N_k} s_j(y_1) \otimes s_j(y_2)^*. \tag{4.10}$$

We wish to describe the asymptotics of Π_k along the diagonal in $Y \times Y$.

Consider $y \in Y$, and fix orthonormal bases $\{e_1, e_2 (= Je_1)\}$, $\{l\}$, $\{f_j\}_{j=1}^{\mathrm{rk}(F)}$ for T_yY, L_y , F respectively and let $\left\{w := \frac{1}{\sqrt{2}}(e_1 - ie_2)\right\}$ be the corresponding orthonormal frame for $T_y^{1,0}Y$. Using the exponential map from this basis obtain a geodesic coordinate system on a geodesic ball $B_{2\varrho}(y)$. Further parallel transport these bases along geodesic rays using the connections $\nabla^{T^{1,0}Y}$, ∇^L, ∇^F to obtain orthonormal frames for $T^{1,0}Y$, L, F on $B_{2\varrho}(y)$. In this frame and coordinate system, the connection on the tensor product again has the expression

$$\nabla^{\Lambda^{0,*}\otimes F\otimes L^k} = d + a^{\Lambda^{0,*}} + a^F + ka^L$$

$$a_j^{\Lambda^{0,*}} = \int_0^1 d\rho\left(\rho y^k R_{jk}^{\Lambda^{0,*}}(\rho x)\right)$$

$$a_j^F = \int_0^1 d\rho\left(\rho y^k R_{jk}^F(\rho x)\right)$$

$$a_j^L = \int_0^1 d\rho\left(\rho y^k R_{jk}^L(\rho x)\right) \tag{4.11}$$

in terms of the curvatures of the respective connections similar to (3.8). We now define a modified frame $\{\tilde{e}_1, \tilde{e}_2\}$ on $\mathbb{R}^2$ which agrees with $\{e_1, e_2\}$ on $B_\varrho(y)$ and with $\{\partial_{x_1}, \partial_{x_2}\}$ outside $B_{2\varrho}(y)$. Also define the modified metric $\tilde{g}^{TY}$ and almost complex structure $\tilde{J}$ on $\mathbb{R}^2$ to be standard in this frame and hence agreeing with g^{TY}, J on $B_\varrho(y)$. The Christoffel symbol of the corresponding modified induced connection on $\Lambda^{0,*}$now satisfies

$$\tilde{a}^{\Lambda^{0,*}} = 0 \quad \text{outside } B_{2\varrho}(y).$$

With $r_y - 2 \in 2\mathbb{N}_0$ being the order of vanishing of the curvature R^L as before, we may Taylor expand the curvature as in (3.22) with

$$iR_0^L(e_1, e_2) \geq 0. \tag{4.12}$$

Further we may as before define the modified connections $\tilde{\nabla}^F$, $\tilde{\nabla}^L$ (3.23) as well as the corresponding tensor product connection $\tilde{\nabla}^{\Lambda^{0,*}\otimes F\otimes L^k}$ which agrees with $\nabla^{\Lambda^{0,*}\otimes F\otimes L^k}$ on $B_\varrho(y)$. Clearly the curvature of the modified connection $\tilde{\nabla}^L$ is given by $\tilde{R}^L$(3.23) and is semi-positive by (4.12). Equation (3.23) also gives $\tilde{R}^L = R_0^L + O(\varrho^{r_y - 1})$ and that the $(r_y - 2)$-th derivative/jet of $\tilde{R}^L$ is non-vanishing at all points on $\mathbb{R}^2$ for

$$0 < \varrho < c\left|j^{r_y-2}R^L(y)\right|. \tag{4.13}$$

Here c is a uniform constant depending on the C^{r-2} norm of R^L. We now define the modified Kodaira Dirac operator on $\mathbb{R}^2$ by the similar formula

$$\tilde{D}_k = c \circ \tilde{\nabla}^{\Lambda^{0,*}\otimes F\otimes L^k}, \tag{4.14}$$

agreeing with D_k on $B_\varrho(y)$. This has a similar Lichnerowicz formula

$$\tilde{D}_k^2 = 2\tilde{\Box}_k := \left(\tilde{\nabla}^{\Lambda^{0,*}\otimes F\otimes L^k}\right)^* \tilde{\nabla}^{\Lambda^{0,*}\otimes F\otimes L^k} + k\left[2\omega\left(\tilde{R}^L\right) - \tilde{\tau}^L\right] \tag{4.15}$$

$$+ \left[2\omega\left(\tilde{R}^F\right) - \tilde{\tau}^F\right] + \frac{1}{2}\omega(\tilde{\kappa}) \tag{4.16}$$

the adjoint being taken with respect to the metric $\tilde{g}^{TY}$ and corresponding volume form. Also the endomorphisms $\tilde{R}^F, \tilde{\tau}^F, \tilde{\tau}^L$ and $\omega(\tilde{\kappa})$ are the obvious modifications of (4.4) defined using the curvatures of $\tilde{\nabla}^F, \tilde{\nabla}^L$ and $\tilde{g}^{TY}$ respectively. The above (4.15) again agrees with $\Box_k$ on

$B_\varrho(y)$ while the endomorphisms $\tilde{R}^F, \tilde{\tau}^F, \omega(\tilde{\kappa})$ all vanish outside $B_\varrho(y)$. Being semi-bounded below (4.15) is essentially self-adjoint. A similar argument as Corollary 21 gives a spectral gap

$$\text{Spec}\left(\tilde{\Box}_k\right) \subset \{0\} \cup \left[c_1 k^{2/r_y} - c_2, \infty\right). \tag{4.17}$$

Thus for $k \gg 0$, the resolvent $\left(\tilde{\Box}_k - z\right)^{-1}$ is well-defined in a neighborhood of the origin in the complex plane. On account on the sub-elliptic estimate (2.14), the projector $\tilde{\Pi}_k$ from $L^2\left(\mathbb{R}^2; \Lambda_y^{0,*} \otimes F_y \otimes L_y^{\otimes k}\right)$ onto $\ker\left(\tilde{\Box}_k\right)$ then has a smooth Schwartz kernel with respect to the Riemannian volume of $\tilde{g}^{TY}$.

We are now ready to prove the Bergman kernel expansion Theorem 3.

Proof of Theorem 3. First choose $\varphi \in \mathcal{S}(\mathbb{R}_s)$ even satisfying $\hat{\varphi} \in C_c\left(-\frac{\varrho}{2}, \frac{\varrho}{2}\right)$ and $\varphi(0) = 1$. For $c > 0$, set $\varphi_1(s) = 1_{[c,\infty)}(s)\varphi(s)$. On account of the spectral gap Corollary 21, and as φ_1 decays at infinity, we have

$$\begin{aligned} \varphi(D_k) - \Pi_k = \varphi_1(D_k) \quad &\text{with} \\ \left\|D_k^a \varphi_1(D_k)\right\|_{L^2 \to L^2} = O\left(k^{-\infty}\right) \end{aligned} \tag{4.18}$$

for $a \in \mathbb{N}$. Combining the above with semiclassical Sobolev and elliptic estimates gives

$$\left|\varphi(D_k) - \Pi_k\right|_{C^l(Y\times Y)} = O\left(k^{-\infty}\right), \tag{4.19}$$

$\forall l \in \mathbb{N}_0$. Next we may write $\varphi(D_k) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi D_k}\hat{\varphi}(\xi)\, d\xi$ via Fourier inversion. Since $D_k = \tilde{D}_k$ on $B_\varrho(y)$ and $\hat{\varphi} \in C_c\left(-\frac{\varrho}{2}, \frac{\varrho}{2}\right)$, we may use a finite propagation argument to conclude

$$\varphi(D_k)(., y) = \varphi\left(\tilde{D}_k\right)(., 0).$$

By similar estimates as (4.18) for $\tilde{D}_k$ we now have a localization of the Bergman kernel

$$\begin{aligned} \Pi_k(., y) &= O\left(k^{-\infty}\right), \quad \text{on } B_\varrho(y)^c \\ \Pi_k(., y) - \tilde{\Pi}_k(., 0) &= O\left(k^{-\infty}\right), \quad \text{on } B_\varrho(y). \end{aligned} \tag{4.20}$$

It thus suffices to consider the Bergman kernel of the model Kodaira Laplacian (4.15) on $\mathbb{R}^2$.

Next with the rescaling/dilation $\delta_{k^{-1/r}} y = \left(k^{-1/r}y_1, \ldots, k^{-1/r}y_{n-1}\right)$ as in 3.3, the rescaled Kodaira Laplacian

$$\boxdot := k^{-2/r_y}\left(\delta_{k^{-1/r}}\right)_* \tilde{\Box}_k \tag{4.21}$$

satisfies

$$\varphi\left(\frac{\tilde{\Box}_k}{k^{2/r_y}}\right)(y, y') = k^{2/r_y}\varphi(\boxdot)\left(yk^{1/r_y}, y'k^{1/r_y}\right) \tag{4.22}$$

for $\varphi \in \mathcal{S}(\mathbb{R})$. Using a Taylor expansion via (3.23), (4.14) the rescaled Dirac operator has an expansion

$$\boxdot \;=\; \left(\sum_{j=0}^{N} k^{-j/r_y}\boxdot_j\right) + k^{-2(N+1)/r_y}\mathrm{E}_{N+1}, \ \forall N. \tag{4.23}$$

Here each

$$\boxdot_j = a_{j;pq}(y)\,\partial_{y_p}\partial_{y_q} + b_{j;p}(y)\,\partial_{y_p} + c_j(y) \tag{4.24}$$

is a (k-independent) self-adjoint, second-order differential operator while each

$$\mathrm{E}_j = \sum_{|\alpha|=N+1} y^\alpha\left[a^\alpha_{j;pq}(y;k)\,\partial_{y_p}\partial_{y_q} + b^\alpha_{j;p}(y;k)\,\partial_{y_p} + c^\alpha_j(y;k)\right] \tag{4.25}$$

is a k-dependent self-adjoint, second-order differential operator on $\mathbb{R}^2$. Furthermore the functions appearing in (4.24) are polynomials with degrees satisfying

$$\deg a_j = j,\ \deg b_j \leq j + r_y - 1, \deg c_j \leq j + 2r_y - 2$$
$$\deg b_j - (j-1) = \deg c_j - j = 0 \pmod 2$$

and whose coefficients involve

$$\begin{aligned} a_j &: \leq j-2 \text{ derivatives of } R^{TY} \\ b_j &: \leq j-2 \text{ derivatives of } R^F, R^{\Lambda^{0,*}} \\ &\ \ \leq j+r-2 \text{ derivatives of } R^L \\ c_j &: \leq j-2 \text{ derivatives of } R^F, R^{\Lambda^{0,*}} \\ &\ \ \leq j+r-2 \text{ derivatives of } R^L \end{aligned}$$

while the coefficients $a^{\alpha}_{j;pq}(y;k), b^{\alpha}_{j;p}(y;k), c^{\alpha}_j(y;k)$ of (4.25) are uniformly (in k) C^{∞} bounded. Using (4.11), (A.4), (A.8) and (A.9) the leading term of (4.23) is computed

$$\boxdot_0 = \boxdot_{g^{TY}, j_y^{r_y-2}R^L, J^{TY}} \tag{4.26}$$

in terms of the the model Kodaira Laplacian on the tangent space TY (A.8).

It is now clear from (4.21) that for φ supported and equal to one near 0. In light of the spectral gap (4.17), the equation (4.22) specializes to

$$\tilde{\Pi}_k(y', y) = k^{2/r_y} \Pi^{\boxdot}\left(y' k^{1/r_y}, y k^{1/r_y}\right) \tag{4.27}$$

as a relation between the Bergman kernels of $\tilde{\Box}_k$, $\boxdot$. Next, the expansion (4.23) along with local elliptic estimates gives

$$(\boxdot - z)^{-1} - (\boxdot_0 - z)^{-1} = O_{H^s_{\mathrm{loc}} \to H^{s+2}_{\mathrm{loc}}}\left(k^{-1/r_y} |\mathrm{Im} z|^{-2}\right)$$

for each $s \in \mathbb{R}$. More generally, we let $I_j \coloneqq \{p = (p_0, p_1, \ldots) | p_\alpha \in \mathbb{N}, \sum p_\alpha = j\}$denote the set of partitions of the integer j and define

$$\mathtt{C}^z_j = \sum_{p \in I_j} (z - \boxdot_0)^{-1} \left[\Pi_\alpha \left[\boxdot_{p_\alpha}(z - \boxdot_0)^{-1}\right]\right]. \tag{4.28}$$

Then by repeated applications of the local elliptic estimate using (4.23) we have

$$(z - \boxdot)^{-1} - \left(\sum_{j=0}^{N} k^{-j/r_y} \mathtt{C}^z_j\right) = O_{H^s_{\mathrm{loc}} \to H^{s+2}_{\mathrm{loc}}}\left(k^{-(N+1)/r_y} |\mathrm{Im} z|^{-2Nr_y - 2}\right), \tag{4.29}$$

for each $N \in \mathbb{N}$, $s \in \mathbb{R}$. A similar expansion as (4.23) for the operator $(\boxdot + 1)^M (\boxdot - z)$, $M \in \mathbb{N}$, also gives

$$(\boxdot + 1)^{-M} (\boxdot - z)^{-1} - \sum_{j=0}^{N} k^{-j/r_y} \mathtt{C}^z_{j,M} = O_{H^s_{\mathrm{loc}} \to H^{s+2+2M}_{\mathrm{loc}}}\left(k^{-(N+1)/r_y} |\mathrm{Im} z|^{-2Nr_y - 2}\right) \tag{4.30}$$

for operators $\mathtt{C}^z_{j,M} = O_{H^s_{\mathrm{loc}} \to H^{s+2+2M}_{\mathrm{loc}}}\left(k^{-(N+1)/r_y} |\mathrm{Im} z|^{-2Nr_y - 2}\right)$, $j = 0, \ldots, N$, with

$$\mathtt{C}^z_{0,M} = \left(\hat{\Delta}^{(0)}_{g^E, F, \mu} + 1\right)^{-M} \left(\hat{\Delta}^{(0)}_{g^E, F, \mu} - z\right)^{-1}.$$

For $M \gg 0$ sufficiently large, Sobolev's inequality gives an expansion for the corresponding Schwartz kernels in (4.30) in $C^l(\mathbb{R}^2 \times \mathbb{R}^2)$, $\forall l \in \mathbb{N}_0$. Next, plugging the above resolvent expansion into the Helffer-Sjöstrand formula as before gives

$$\left| \varphi(\boxdot) - \sum_{j=0}^{N} k^{-j/r_y} \mathtt{C}_j^{\varphi} \right|_{C^l(\mathbb{R}^2 \times \mathbb{R}^2)} = O\left(k^{-(N+1)/r_y}\right)$$

$\forall l, N \in \mathbb{N}_0$ and for some (k-independent) $\mathtt{C}_j^{\varphi} \in C^{\infty}(\mathbb{R}^2 \times \mathbb{R}^2)$, $j = 0, 1, \ldots$, with leading term $\mathtt{C}_0^{\varphi} = \varphi(\boxdot_0) = \varphi\left(\boxdot_{g^{TY}, j_y^{r_y-2}R^L, J^{TY}}\right)$. As φ was chosen supported near 0, the spectral gap properties (4.17), 38 give

$$\left| \Pi^{\boxdot} - \sum_{j=0}^{N} k^{-j/r_y} \mathtt{C}_j \right|_{C^l(\mathbb{R}^2 \times \mathbb{R}^2)} = O\left(k^{-(N+1)/r_y}\right) \tag{4.31}$$

for some $\mathtt{C}_j \in C^{\infty}(\mathbb{R}^2 \times \mathbb{R}^2)$, $j = 0, 1, \ldots$, with leading term $\mathtt{C}_0 = \Pi^{\boxdot}{}_{g^{TY}, j_y^{r_y-2}R^L, J^{TY}}$. The expansion is now a consequence of (4.19), (4.20) and (4.27). Finally, in order to show that there are no odd powers of k^{-j/r_y}, one again notes that the operators $\boxdot_j$ (4.24) change sign by $(-1)^j$ under $\delta_{-1}x := -x$. Thus the integral expression (4.28) corresponding to $\mathtt{C}_j^z(0,0)$ changes sign by $(-1)^j$ under this change of variables and must vanish for j odd. □

Next we show that a pointwise expansion on the diagonal also exists for derivatives of the Bergman kernel. In what follows we denote by $j^l s / j^{l-1} s \in S^l T^* Y \otimes E$ the component of the l-th jet of a section $s \in C^{\infty}(E)$ of a Hermitian vector bundle E that lies in the kernel of the natural surjection $J^l(E) \to J^{l-1}(E)$.

Theorem 23. *For each $l \in \mathbb{N}_0$, the l-th jet of the on-diagonal Bergman kernel has a pointwise expansion*

$$j^l [\Pi_k(y,y)] / j^{l-1} [\Pi_k(y,y)] = k^{(2+l)/r_y} \left[\sum_{j=0}^{N} c_j(y) k^{-2j/r_y} \right] + O\left(k^{-(2N-l-1)/r_y}\right), \tag{4.32}$$

$\forall N \in \mathbb{N}$, in $j^l End(F) / j^{l-1} End(F) = S^l T^* Y \otimes End(F)$, with the leading term

$$c_0(y) = j^l \left[\Pi^{g_y^{TY}, j_y^{r_y-2}R^L, J_y^{TY}}(0,0) \right] / j^{l-1} \left[\Pi^{g_y^{TY}, j_y^{r_y-2}R^L, J_y^{TY}}(0,0) \right]$$

being given in terms of the l-th jet of the Bergman kernel of the Kodaira Laplacian (A.8) *on the tangent space at y.*

Proof. The proof is a modification of the previous. First note that a similar localization

$$\Pi_k(\mathsf{y},\mathsf{y}) - \tilde{\Pi}_k(\mathsf{y},\mathsf{y}) = O\left(k^{-\infty}\right), \tag{4.33}$$

to (4.20) is valid in C^l, $\forall l \in \mathbb{N}_0$, and for y in a uniform neighborhood of y. Next differentiating (4.27) with $y = y'$ gives

$$\partial_{\mathsf{y}}^{\alpha} \tilde{\Pi}_k(\mathsf{y},\mathsf{y}) = k^{(2+|\alpha|)/r_y} \partial_{\mathsf{y}}^{\alpha} \Pi^{\boxdot}\left(\mathsf{y} k^{1/r_y}, \mathsf{y} k^{1/r_y}\right), \tag{4.34}$$

$\forall \alpha \in \mathbb{N}_0^2$. Finally, the expansion (4.31) being valid in C^l, $\forall l \in \mathbb{N}_0$, maybe differentiated and plugged into the above with $\mathsf{y} = 0$ to give the theorem. □

Remark 24. The expansion (1.7) is the same as the positive case on Y_2 (points where $r_y = 2$) and furthermore uniform in any C^l-topology on compact subsets of Y_2 cf. [44, Theorem 4.1.1]. In particular the first two coefficients for $y \in Y_2$ are given by

$$c_0(y) = \Pi^{g_y^{TY}, j_y^0 R^L, J_y^{TY}}(0,0) = \frac{1}{2\pi}\tau^L$$
$$c_1(y) = \frac{1}{16\pi}\tau^L\left[\kappa - \Delta \ln \tau^L + 4\tau^F\right].$$

The derivative expansion on Y_2 is also known to satisfy $c_0 = c_1 = \ldots = c_{\left[\frac{l-1}{2}\right]} = 0$ (i.e. begins at the same leading order k) with the leading term given by

$$c_{\left[\frac{l+1}{2}\right]}(y) = \frac{1}{2\pi} j^l \tau^L / \frac{1}{2\pi} j^{l-1}\tau^L.$$

As before with Theorem 9 and Theorem 16 the expansions of Theorem 3 and Theorem 23 are not uniform in the point on the diagonal. We next give uniform estimates on the Bergman kernel useful in Section 4.4. Below we set $C_{r_1} \coloneqq \inf_{|R^V|=1} \Pi^{g^V, R^V, J^V}(0,0)$ for each $0 \neq R^V \in S^{r_1-2}V^* \otimes \Lambda^2 V^*$, $r_1 \geq 2$. Furthermore, the Bergman kernel $\Pi^{g_y^{TY}, j_y^0 R^L, J_y^{TY}}(0,0)$ of the model operator (A.8) is extended (continuously) by zero from Y_2 to Y.

Lemma 25. *The Bergman kernel satisfies*

(4.35)
$$\left[\inf_{y\in Y_r} \Pi^{g_y^{TY}, j_y^{r-2} R^L, J_y^{TY}}(0,0)\right][1+o(1)]\,k^{2/r} \leq \Pi_k(y,y) \leq \left[\sup_{y\in Y} \Pi^{g_y^{TY}, j_y^0 R^L, J_y^{TY}}(0,0)\right][1+o(1)]\,k,$$

with the $o(1)$ terms being uniform in $y \in Y$.

Proof. Note that theorem Theorem 3 already shows

$$\Pi_k(y,y) \geq C_{r_y}\left(\left|j^{r_y-2}R^L\right| k\right)^{2/r_y} - c_y \tag{4.36}$$

$\forall y \in Y$, with $c_y = c\left(\left|j^{r_y-2}R^L(y)\right|^{-1}\right) = O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)$ being a (y-dependent) constant given in terms of the norm of the first non-vanishing jet. The norm of this jet affects the choice of ϱ needed for (4.13); which in turn affects the C^∞-norms of the coefficients of (4.25) via (3.23). We first show that this estimate extends to a small ($\left|j^{r_y-2}R^L(y)\right|$- dependent) size neighborhood of y. To this end, for any $\varepsilon > 0$ there exists a uniform constant c_ε depending only on ε and $\left\|R^L\right\|_{C^r}$ such that

$$\left|j^{r_y-2}R^L(\mathsf{y})\right| \geq (1-\varepsilon)\left|j^{r_y-2}R^L(y)\right|, \tag{4.37}$$

$\forall \mathsf{y} \in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y)$.

We begin by rewriting the model Kodaira Laplacian $\tilde{\Box}_k$ (4.15) near y in terms of geodesic coordinates centered at y. In the region

$$\mathsf{y} \in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y) \cap \left\{C_0\left(\left|j^0R^L(\mathsf{y})\right| k\right) \geq k^{2/r_y}\Pi^{g_y^{TY}, j_y^{r_y-2}R^L, J_y^{TY}}(0,0)\right\}$$

a rescaling of $\tilde{\Box}_k$ by $\delta_{k^{-1/2}}$, now centered at y, shows

$$\begin{aligned}\Pi_k(\mathsf{y},\mathsf{y}) &= k\Pi^{g_{\mathsf{y}}^{TY}, j_{\mathsf{y}}^0 R^L, J_{\mathsf{y}}^{TY}}(0,0) + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)\\ &= k\left|j^0R^L(\mathsf{y})\right|\Pi^{g_{\mathsf{y}}^{TY}, \frac{j_{\mathsf{y}}^0 R^L}{\left|j^0R^L(\mathsf{y})\right|}, J_{\mathsf{y}}^{TY}}(0,0) + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)\\ &\geq k^{2/r_y}\Pi^{g_y^{TY}, j_y^{r_y-2}R^L, J_y^{TY}}(0,0) + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)\end{aligned} \tag{4.38}$$

as in (4.36). Now, in the region

$$\mathsf{y} \in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y) \cap \Big\{ C_1\left(\left|j^1R^L(\mathsf{y})/j^0R^L(\mathsf{y})\right|k\right)^{2/3} \geq k^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0) \geq C_0\left(\left|j^0R^L(\mathsf{y})\right|k\right)\Big\}$$

a rescaling of $\tilde{\square}_k$ by $\delta_{k^{-1/3}}$ centered at y similarly shows

$$\begin{aligned}\Pi_k(\mathsf{y},\mathsf{y}) &= k^{2/3}\left[1+O\left(k^{2/r-2/3}\right)\right]\Pi^{g_{\mathsf{y}}^{TY},j_{\mathsf{y}}^1R^L/j_{\mathsf{y}}^0R^L,J_{\mathsf{y}}^{TY}}(0,0) \\ &\quad + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1) \\ &= k^{2/3}\left[1+O\left(k^{2/r-2/3}\right)\right]\left|j_{\mathsf{y}}^1R^L/j_{\mathsf{y}}^0R^L\right|^{2/3}\Pi^{g_{\mathsf{y}}^{TY},\frac{j_{\mathsf{y}}^1R^L/j_{\mathsf{y}}^0R^L}{\left|j_{\mathsf{y}}^1R^L/j_{\mathsf{y}}^0R^L\right|},J_{\mathsf{y}}^{TY}}(0,0) \\ &\quad + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1) \qquad (4.39) \\ &\geq (1-\varepsilon)\,k^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0) + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1) \qquad (4.40)\end{aligned}$$

Next, in the region

$$\mathsf{y} \in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y) \cap \Big\{ C_2\left(\left|j^2R^L(\mathsf{y})/j^1R^L(\mathsf{y})\right|k\right)^{1/2} \geq k^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0) \geq \max\left[C_0\left(\left|j^0R^L(\mathsf{y})\right|k\right), C_1\left(\left|j^1R^L(\mathsf{y})/j^0R^L(\mathsf{y})\right|k\right)^{2/3}\right]\Big\}$$

a rescaling of $\tilde{\square}_k$ by $\delta_{k^{-1/4}}$ centered at y shows

$$\begin{aligned}\Pi_k(\mathsf{y},\mathsf{y}) &= k^{1/2}\left[1+O\left(k^{2/r-1/2}\right)\right]\Pi^{g_{\mathsf{y}}^{TY},j_{\mathsf{y}}^2R^L/j_{\mathsf{y}}^1R^L,J_{\mathsf{y}}^{TY}}(0,0) + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1) \\ &= k^{1/2}\left[1+O\left(k^{2/r-1/2}\right)\right]\left|j_{\mathsf{y}}^2R^L/j_{\mathsf{y}}^1R^L\right|^{1/2}\Pi^{g_{\mathsf{y}}^{TY},\frac{j_{\mathsf{y}}^2R^L/j_{\mathsf{y}}^1R^L}{\left|j_{\mathsf{y}}^2R^L/j_{\mathsf{y}}^1R^L\right|},J_{\mathsf{y}}^{TY}}(0,0) + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1) \\ &\geq (1-\varepsilon)\,k^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0) + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1) \qquad (4.41)\end{aligned}$$

Continuing in this fashion, we are finally left with the region

$$\mathsf{y} \in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y) \cap \Big\{ k^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0) \geq \max\left[C_0\left(\left|j^0R^L(\mathsf{y})\right|k\right),\ldots,C_{r_y-3}\left(\left|j^{r_y-3}R^L(\mathsf{y})/j^{r_y-4}R^L(\mathsf{y})\right|k\right)^{2/(r_y-1)}\right]\Big\}.$$

In this region we have

$$\left|j^{r_y-2}R^L(\mathsf{y})/j^{r_y-3}R^L(\mathsf{y})\right| \geq (1-\varepsilon)\left|j^{r_y-2}R^L(y)\right| + O\left(k^{2/r_y-2/(r_y-1)}\right)$$

following (4.37) with the remainder being uniform. A rescaling by $\delta_{k^{-1/r_y}}$ then giving a similar estimate in this region, we have finally arrived at

$$\Pi_k(\mathsf{y},\mathsf{y}) \geq (1-\varepsilon)\,k^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0) + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)$$

$\forall \mathsf{y} \in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y)$.

Finally a compactness argument finds a finite set of points $\{y_j\}_{j=1}^N$ such that the corresponding $B_{c_\varepsilon\left|j^{r_{y_j}-2}R^L\right|}(y_j)$'s cover Y. This gives a uniform constant $c_{1,\varepsilon} > 0$ such that

$$\Pi_k(y,y) \geq (1-\varepsilon)\left[\inf_{y\in Y_r}\Pi^{g_y^{TY},j_y^{r-2}R^L,J_y^{TY}}(0,0)\right]k^{2/r} - c_{1,\varepsilon}$$

$\forall y \in Y$, $\varepsilon > 0$ proving the lower bound (4.35). The argument for the upper bound is similar. $\square$

We now prove a second lemma giving a uniform estimate on the derivatives of the Bergman kernel. Again below, the model Bergman kernel $\Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}(0,0)$ and its relevant ratio

$$\frac{\left|\left[j^l \Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}\right](0,0)\right|}{\Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}(0,0)}$$

are extended (continuously) by zero from $\left\{y | j_y^1 R^L / j_y^0 R^L \neq 0\right\}$ to Y.

Lemma 26. *The l-th jet of the Bergman kernel satisfies*

$$\left|j^l\left[\Pi_k(y,y)\right]\right| \leq k^{l/3}\left[1+o(1)\right]\left[\sup_{y\in Y} \frac{\left|\left[j^l \Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}\right](0,0)\right|}{\Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}(0,0)}\right]\Pi_k(y,y)$$

with the $o(1)$ term being uniform in $y \in Y$.

Proof. The proof follows a similar argument as the previous lemma. Given $\varepsilon > 0$ we find a uniform c_ε such that (4.37) holds for each $y \in Y$ and $\mathsf{y} \in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y)$. Then rewrite the model Kodaira Laplacian $\tilde{\square}_k$ (4.15) near y in terms of geodesic coordinates centered at y. In the region

$$\mathsf{y} \in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y) \cap \left\{C_0\left(\left|j^0 R^L(\mathsf{y})\right| k\right) \geq k^{2/r_y}\Pi^{g_y^{TY}, j_y^{r_y-2}R^L, J_y^{TY}}(0,0)\right\}$$

a rescaling of $\tilde{\square}_k$ by $\delta_{k^{-1/2}}$, now centered at y, shows

$$\partial^\alpha \Pi_k(\mathsf{y},\mathsf{y}) = \frac{k}{2\pi}\left(\partial^\alpha \tau^L(\mathsf{y})\right) + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)$$

following 24 as $r_{\mathsf{y}} = 2$. Diving the above by (4.38) gives

$$\begin{aligned}
\frac{\left|\partial^\alpha \Pi_k(\mathsf{y},\mathsf{y})\right|}{\Pi_k(\mathsf{y},\mathsf{y})} &\leq \frac{\left|\partial^\alpha \tau^L(\mathsf{y})\right|}{\tau^L(\mathsf{y})} + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}\left(k^{-1}\right)\\
&\leq k^{|\alpha|/3}\left[\sup_{y\in Y} \frac{\left|\left[j^{|\alpha|} \Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}\right](0,0)\right|}{\Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}(0,0)}\right]\Pi_k(\mathsf{y},\mathsf{y})\\
&\quad + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}\left(k^{-1}\right)
\end{aligned}$$

Next, in the region

$$\begin{aligned}
\mathsf{y} \in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y) \cap \Big\{ & C_1\left(\left|j^1 R^L(\mathsf{y}) / j^0 R^L(\mathsf{y})\right| k\right)^{2/3}\\
& \geq k^{2/r_y}\Pi^{g_y^{TY}, j_y^{r_y-2}R^L, J_y^{TY}}(0,0) \geq C_0\left(\left|j^0 R^L(\mathsf{y})\right| k\right)\Big\}
\end{aligned}$$

a rescaling of $\tilde{\square}_k$ by $\delta_{k^{-1/3}}$ centered at y similarly shows

$$\begin{aligned}
\partial^\alpha \Pi_k(\mathsf{y},\mathsf{y}) = {} & k^{(2+|\alpha|)/3}\left[1+O\left(k^{2/r-2/3}\right)\right]\left[\partial^\alpha \Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_{\mathsf{y}}^{TY}}\right](0,0)\\
& + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}\left(k^{(1+|\alpha|)/3}\right)
\end{aligned}$$

as in Theorem 23. Dividing this by (4.40) gives

$$\begin{aligned}\frac{\left|\partial^{\alpha}\Pi_{k}\left(\mathsf{y},\mathsf{y}\right)\right|}{\Pi_{k}\left(\mathsf{y},\mathsf{y}\right)} &\leq k^{|\alpha|/3}\left(1+\varepsilon\right)\frac{\left|\left[\partial^{\alpha}\Pi^{g_{\mathsf{y}}^{TY},j_{\mathsf{y}}^{1}R^{L}/j_{\mathsf{y}}^{0}R^{L},J_{\mathsf{y}}^{TY}}\right]\left(0,0\right)\right|}{\left[\Pi^{g_{\mathsf{y}}^{TY},j_{\mathsf{y}}^{1}R^{L}/j_{\mathsf{y}}^{0}R^{L},J_{\mathsf{y}}^{TY}}\right]\left(0,0\right)}\\ &\quad+O_{\left|j^{r_{y}-2}R^{L}(y)\right|^{-1}}\left(k^{(|\alpha|-1)/3}\right)\\ &\leq k^{|\alpha|/3}\left(1+\varepsilon\right)\left[\sup_{y\in Y}\frac{\left|\left[j^{|\alpha|}\Pi^{g_{y}^{TY},j_{y}^{1}R^{L}/j_{y}^{0}R^{L},J_{y}^{TY}}\right]\left(0,0\right)\right|}{\Pi^{g_{y}^{TY},j_{y}^{1}R^{L}/j_{y}^{0}R^{L},J_{y}^{TY}}\left(0,0\right)}\right]\\ &\quad+O_{\left|j^{r_{y}-2}R^{L}(y)\right|^{-1}}\left(k^{(|\alpha|-1)/3}\right).\end{aligned}$$

Continuing in this fashion as before eventually gives

$$\begin{aligned}\frac{\left|\partial^{\alpha}\Pi_{k}\left(\mathsf{y},\mathsf{y}\right)\right|}{\Pi_{k}\left(\mathsf{y},\mathsf{y}\right)} &\leq k^{|\alpha|/3}\left(1+\varepsilon\right)\left[\sup_{y\in Y}\frac{\left|\left[j^{|\alpha|}\Pi^{g_{y}^{TY},j_{y}^{1}R^{L}/j_{y}^{0}R^{L},J_{y}^{TY}}\right]\left(0,0\right)\right|}{\Pi^{g_{y}^{TY},j_{y}^{1}R^{L}/j_{y}^{0}R^{L},J_{y}^{TY}}\left(0,0\right)}\right]\\ &\quad+O_{\left|j^{r_{y}-2}R^{L}(y)\right|^{-1}}\left(k^{(|\alpha|-1)/3}\right)\end{aligned}$$

$\forall y\in Y$, $\mathsf{y}\in B_{c_{\varepsilon}\left|j^{r_{y}-2}R^{L}\right|}(y)$, $\forall\alpha\in\mathbb{N}_{0}^{2}$. By compactness one again finds a uniform $c_{1,\varepsilon}$ such that

$$\frac{\left|\partial^{\alpha}\Pi_{k}\left(y,y\right)\right|}{\Pi_{k}\left(y,y\right)}\leq k^{|\alpha|/3}\left(1+\varepsilon\right)\left[\sup_{y\in Y}\frac{\left|\left[j^{|\alpha|}\Pi^{g_{y}^{TY},j_{y}^{1}R^{L}/j_{y}^{0}R^{L},J_{y}^{TY}}\right]\left(0,0\right)\right|}{\Pi^{g_{y}^{TY},j_{y}^{1}R^{L}/j_{y}^{0}R^{L},J_{y}^{TY}}\left(0,0\right)}\right]+c_{1,\varepsilon}$$

$\forall y\in Y$, proving the lemma. □

Example 27. (Branched coverings) We end this section by giving an example where semi-positive bundles arise and where the first term of the Bergman kernel expansion (1.7) can be made explicit. Here $f:Y\to Y_0$ is a branched covering of a Riemann surface Y_0 with branch points $\{y_1,\ldots,y_M\}\subset Y$. The Hermitian holomorphic line bundle on Y is pulled back $\left(L,h^{L}\right)=\left(f^{*}L_{0},f^{*}h^{L_{0}}\right)$ from one on Y_0. If $\left(L_{0},h^{L_{0}}\right)$ is assumed positive, then $\left(L,h^{L}\right)$ is semi-positive with curvature vanishing at the branch points. In particular, near a branch point $y\in Y$ of local degree $\frac{r}{2}$ one may find holomorphic geodesic coordinate such that the curvature is given by $R^{L}=\frac{r^{2}}{4}\left(z\bar{z}\right)^{r/2-1}R_{f(y)}^{L_{0}}+O\left(y^{r-1}\right)$. The leading term of (1.7) is given by the model Bergman kernel $\Pi^{\boxdot_{0}}\left(0,0\right)$ of the operator $\boxdot_{0}=bb^{\dagger}$, $b^{\dagger}=2\partial_{\bar{z}}+a$, $a=\frac{r}{4}z\left(z\bar{z}\right)^{r/2-1}R_{f(y)}^{L_{0}}$. An orthonormal basis for $\ker\left(\boxdot_{0}\right)$ is then given by

$$\begin{aligned}s_{\alpha}&:=\left(\frac{1}{2\pi}\frac{r}{\Gamma\left(\frac{2(\alpha+1)}{r}\right)}\left[R_{f(y)}^{L_{0}}\right]^{\frac{2(\alpha+1)}{r}}\right)^{1/2}z^{\alpha}e^{-\Phi},\quad\text{with}\\ \Phi&:=\frac{1}{4}\left(z\bar{z}\right)^{r/2}R_{f(y)}^{L_{0}}.\end{aligned}$$

This gives the first term of the expansion

$$c_{0}\left(y\right)=\Pi^{\boxdot_{0}}\left(0,0\right)=\frac{1}{2\pi}\frac{r}{\Gamma\left(\frac{2}{r}\right)}\left[R_{f(y)}^{L_{0}}\right]^{\frac{2}{r}}$$

at the vanishing/branch point y in this example.

4.2. **Induced Fubini-Study metrics.** A theorem of Tian [60], with improvements in [17, 61] (see also [44, S 5.1.2, S 5.1.4]), asserts that the induced Fubini-Study metrics by Kodaira embeddings given by kth tensor powers of a positive line bundle converge to the curvature of the bundle as k goes to infinity. In this Section we will give a generalization for semi-positive line bundles on compact Riemann surfaces.

Let us review first Tian's theorem. Let (Y, J, g^{TY}) be a compact Hermitian manifold, (L, h^L), (F, h^F) be holomorphic Hermitian line bundles such that (L, h^L) is positive. We endow $H^0(Y; F \otimes L^k)$ with the L^2 product induced by g^{TY}, h^L and h^F. This induces a Fubini-Study metric ω_{FS} on the projective space $\mathbb{P}\left[H^0\left(Y; F \otimes L^k\right)^*\right]$ and a Fubini-Study metric h_{FS} on $\mathcal{O}(1) \to \mathbb{P}\left[H^0\left(Y; F \otimes L^k\right)^*\right]$ (see [44, S 5.1]). Since (L, h^L) is positive the Kodaira embedding theorem shows that the Kodaira maps $\Phi_k : Y \to \mathbb{P}\left[H^0\left(Y; F \otimes L^k\right)^*\right]$ (see (4.48)) are embeddings for $k \gg 0$. Moreover, the Kodaira map induces a canonical isomorphism $\Theta_k : F \otimes L^k \to \Phi_k^*\mathcal{O}(1)$ and we have (see e.g. [44, (5.1.15)])

$$(\Theta_k^* h_{FS})(y) = \Pi_k(y, y)^{-1} h^{F\otimes L^k}(y), \;\; y \in Y. \tag{4.42}$$

This implies immediately (see e.g. [44, (5.1.50)])

$$\frac{1}{k}\Phi_k^*\omega_{FS} - \frac{i}{2\pi}R^L = \frac{i}{2\pi k}R^F - \frac{i}{2\pi k}\bar{\partial}\partial \ln \Pi_k\left(y, y\right). \tag{4.43}$$

Applying now the Bergman kernel expansion in the positive case one obtains Tian's theorem, which asserts that we have

$$\frac{1}{k}\Phi_k^*\omega_{FS} - \frac{i}{2\pi}R^L = O\left(k^{-1}\right), \;\; k \to \infty, \;\text{ in any } C^\ell\text{-topology}. \tag{4.44}$$

Let us also consider the convergence of the induced Fubini-Study metric $\Theta_k^* h_{FS}$ to the initial metric h^L. For this purpose we fix a metric h_0^L on L with positive curvature. We can then express $h^L = e^{-\varphi}h_0^L$, $\Theta_k^* h_{FS} = e^{-\varphi_k}(h_0^L)^k \otimes h^F$, where $\varphi, \varphi_k \in C^\infty(Y)$ are the global potentials of the metrics h and $\Theta_k^* h_{FS}$ with respect to h_0^L and $(h_0^L)^k \otimes h^F$. Note that

$$R^{(L,h^L)} = R^{(L,h_0^L)} + \partial\overline{\partial}\varphi, \;\; R^{(L^k,\Theta_k^* h_{FS})} = kR^{(L,h_0^L)} + R^{(F,h^F)} + \partial\overline{\partial}\varphi_k,$$

and $\frac{i}{2\pi}R^{(L,\Theta_k^* h_{FS})} = \Phi_k^*\omega_{FS}$. Then (4.42) can be written as

$$\frac{1}{k}\varphi_k(y) - \varphi(y) = \frac{1}{k}\ln \Pi_k(y, y), \;\; y \in Y. \tag{4.45}$$

We obtain by (1.7) that

$$\left|\frac{1}{k}\varphi_k - \varphi\right|_{C^0(Y)} = O\big(k^{-1}\ln k\big), \;\; k \to \infty, \tag{4.46}$$

that is, the normalized potentials of the Fubini-Study metric converge uniformly on Y to the potential of the initial metric h^L with speed $k^{-1}\ln k$. Moreover,

$$\left|\frac{1}{k}\partial\varphi_k - \partial\varphi\right|_{C^0(Y)} = O\big(k^{-1}\big), \;\; \left|\frac{1}{k}\partial\overline{\partial}\varphi_k - \partial\overline{\partial}\varphi\right|_{C^0(Y)} = O\big(k^{-1}\big), \;\; k \to \infty, \tag{4.47}$$

and we get the same bound $O\big(k^{-1}\big)$ for higher derivatives, obtaining again (4.44). Note that if g^{TY} is the metric associated to $\omega = \frac{i}{2\pi}R^L$, then we have a bound $O(k^{-2})$ in (4.44) and (4.47).

We return now to our situation and consider that Y is a compact Riemann surface and (L, h^L), (F, h^F) be holomorphic Hermitian line bundles on Y such that (L, h^L) is semi-positive and its curvature vanishes at finite order. An immediate consequence of Lemma 25 is that the base locus

$$\mathrm{Bl}\left(F \otimes L^k\right) := \left\{y \in Y | s\left(y\right) = 0, \; s \in H^0\left(Y; F \otimes L^k\right)\right\} = \emptyset$$

is empty for $k \gg 0$. This shows that the subspace

$$\Phi_{k,y} := \left\{ s \in H^0\left(Y; F \otimes L^k\right) | s\left(y\right) = 0 \right\} \subset H^0\left(Y; F \otimes L^k\right),$$

is a hyperplane for each $y \in Y$. One may identify the Grassmanian $\mathbb{G}\left(d_k - 1; H^0\left(Y; F \otimes L^k\right)\right)$, $d_k := \dim H^0\left(Y; F \otimes L^k\right)$, with the projective space $\mathbb{P}\left[H^0\left(Y; F \otimes L^k\right)^*\right]$ by sending a non-zero dual element in $H^0\left(Y; F \otimes L^k\right)^*$ to its kernel. This now gives a well-defined Kodaira map

$$\begin{aligned} &\Phi_k : Y \to \mathbb{P}\left[H^0\left(Y; F \otimes L^k\right)^*\right], \\ &\Phi_k\left(y\right) := \left\{ s \in H^0\left(Y; F \otimes L^k\right) | s\left(y\right) = 0 \right\}. \end{aligned} \tag{4.48}$$

It is well known that the map is holomorphic.

Theorem 28. *Let Y be a compact Riemann surface and (L, h^L), (F, h^F) be holomorphic Hermitian line bundles on Y such that (L, h^L) is semi-positive and its curvature vanishes at most at finite order. Then the normalized potentials of the Fubini-Study metric converge uniformly on Y to the potential of the initial metric h^L with speed $k^{-1}\ln k$ as in* (4.46)*. Moreover,*

$$\left|\frac{1}{k}\partial\varphi_k - \partial\varphi\right|_{C^0(Y)}, \ \left|\frac{1}{k}\overline{\partial}\varphi_k - \overline{\partial}\varphi\right|_{C^0(Y)} = O\big(k^{-2/3}\big), \ \ k \to \infty, \tag{4.49}$$

and

$$\left|\frac{1}{k}\partial\overline{\partial}\varphi_k - \partial\overline{\partial}\varphi\right|_{C^0(Y)} = O\big(k^{-1/3}\big), \ \ k \to \infty, \tag{4.50}$$

especially

$$\frac{1}{k}\Phi_k^*\omega_{FS} - \frac{i}{2\pi}R^L = O\left(k^{-1/3}\right), \ \ k \to \infty, \tag{4.51}$$

uniformly on Y. On compact sets of Y_2 the estimates (4.44) *and* (4.47) *hold.*

Proof. The proof follows from (4.43), (4.45) and the uniform estimate of Lemma 26 on the derivatives of the Bergman kernel. □

As we noted before, the bundle L satisfying the hypotheses of Theorem 28 is ample, so for $k \gg 0$ the Kodaira map is an embedding and the induced Fubini-Study forms $\frac{1}{k}\Phi_k^*\omega_{FS}$ are indeed metrics on Y. Due to the possible degeneration of the curvature R^L the rate of convergence in (4.51) is slower than in the positive case (4.44).

One can easily prove a generalization of Theorem 28 for vector bundles (F, h^F) of arbitrary rank (see [44, S 5.1.4] for the case of a positive bundle (L, h^L)). We have then Kodaira maps $\Phi_k : Y \to \mathbb{G}\left(\mathrm{rk}\left(F\right); H^0\left(Y; F \otimes L^k\right)^*\right)$ into the Grassmanian of $\mathrm{rk}\left(F\right)$-dimensional linear spaces of $H^0\left(Y; F \otimes L^k\right)^*$ and we introduce the Fubini-Study metric on the Grassmannian as the curvature of the determinant bundle of the dual of the tautological bundle (cf. [44, (5.1.6)]). Then by following the proof of [44, Theorem 5.1.17] and using Lemma 26 we obtain

$$\frac{1}{k}\Phi_k^*\omega_{FS} - \mathrm{rk}\left(F\right)\frac{i}{2\pi}R^L = O\left(k^{-1/3}\right), \ \ k \to \infty, \tag{4.52}$$

uniformly on Y.

4.3. **Toeplitz operators.** A generalization of the projector (4.9) and Bergman kernel (4.10) is given by the notion of a Toeplitz operator. The Toeplitz operator $T_{f,k}$ operator corresponding to a section $f \in C^{\infty}(Y; \mathrm{End}(F))$ is defined via

$$\begin{aligned} &T_{f,k} : C^{\infty}\left(Y; F \otimes L^k\right) \to C^{\infty}\left(Y; F \otimes L^k\right) \\ &T_{f,k} := \Pi_k f \Pi_k, \end{aligned} \tag{4.53}$$

where f denotes the operator of pointwise composition by f. Each Toeplitz operator above further maps $H^0\left(Y; F \otimes L^k\right)$ to itself.

We now prove the expansion for the kernel of a Toeplitz operator generalizing Theorem 3. For positive line bundles the analogous result was proved in [18, Theorem 2] for compact Kähler manifolds and $F = \mathbb{C}$ and in [44, Lemma 7.2.4 and (7.4.6)], [46, Lemma 4.6], in the symplectic case.

Theorem 29. *Let Y be a compact Riemann surface, $(L, h^L) \to Y$ a semi-positive line bundle whose curvature R^L vanishes to finite order at any point. Let $(F, h^F) \to Y$ be a Hermitian holomorphic vector bundle. Then the kernel of the Toeplitz operator* (4.53) *has an on diagonal asymptotic expansion*

$$T_{f,k}(y, y) = k^{2/r_y}\left[\sum_{j=0}^{N} c_j(f, y)\, k^{-2j/r_y}\right] + O\left(k^{-2N/r_y}\right), \quad \forall N \in \mathbb{N}$$

where the coefficients $c_j(f, \cdot)$ are sections of End(F) with leading term

$$c_0(f, y) = \Pi^{g_y^{TY}, R_y^{TY}, J_y^{TY}}(0, 0)\, f(y).$$

Proof. Firstly from the definition (4.53) and the localization/rescaling properties (4.20), (4.27) one has

$$\begin{aligned} T_{f,k}(y, y) &= \int_Y dy'\, \Pi_k(y, y')\, f(y')\, \Pi_k(y', y) \\ &= \int_{B_\varepsilon(y)} dy'\, \tilde{\Pi}_k(0, y')\, f(y')\, \tilde{\Pi}_k(y', 0) + O\left(k^{-\infty}\right) \\ &= \int_{B_\varepsilon(y)} dy'\, k^{4/r_y} \Pi^{\boxdot}\left(0, y' k^{1/r_y}\right) f(y')\, \Pi^{\boxdot}\left(y' k^{1/r_y}, 0\right) + O\left(k^{-\infty}\right) \\ &= \int_{k^{1/r_y} B_\varepsilon(y)} dy'\, k^{2/r_y} \Pi^{\boxdot}(0, y')\, f\left(y' k^{-1/r_y}\right) \Pi^{\boxdot}(y', 0) + O\left(k^{-\infty}\right). \end{aligned} \tag{4.54}$$

Next as in Section A, $\varphi(\boxdot)(., 0) \in \mathcal{S}(V)$ for $\varphi \in \mathcal{S}(\mathbb{R})$ in the Schwartz class. Thus plugging (4.31) and a Taylor expansion

$$f\left(y' k^{-1/r_y}\right) = \sum_{|\alpha| \le N+1} \frac{1}{\alpha!} (y')^{\alpha} k^{-\alpha/r_y} f^{(\alpha)}(0) + O\left(k^{-(N+1)/r_y}\right)$$

into (4.54) above gives the result with the leading term again coming from (4.26). Finally and as in the proof of Theorem 3, there are no odd powers of k^{-j/r_y} as the corresponding coefficients are given by odd integrals (the integrands change sign by $(-1)^j$ under $\delta_{-1} x := -x$) which are zero. □

We now show that the Toeplitz operators (4.53) can be composed up to highest order generalizing the results of [14] in the Kähler case and $F = \mathbb{C}$ and [44, Theorems 7.4.1–2], [46, Theorems 1.1 and 4.19] in the symplectic case.

Theorem 30. *Given $f, g \in C^\infty(Y; End(F))$, the Toeplitz operators* (4.53) *satisfy*

$$\lim_{k\to\infty} \|T_{f,k}\| = \|f\|_\infty := \sup_{\substack{y\in Y \\ u\in F_y\setminus 0}} \frac{|f(y)u|_{h^F}}{|u|_{h^F}}, \tag{4.55}$$

$$T_{f,k}T_{g,k} = T_{fg,k} + O_{L^2\to L^2}\left(k^{-1/r}\right). \tag{4.56}$$

Proof. The first part of (4.55) is similar to the positive case. Firstly, $\|T_{f,k}\| \leq \|f\|_\infty$ is clear from the definition (4.53). For the lower bound, let us consider $y \in Y_2$ where the curvature is non-vanishing and $u \in F_y$, $|u|_{h^F} = 1$. It follows from the proof of [44, Theorem 7.4.2] (see also [4, Proposition 5.2, (5.40), Remark 5.7]) that

$$|f(y)(u)|_{h^F} + O_{y,u}\left(k^{-1/2}\right) \leq \|T_{f,k}\|. \tag{4.57}$$

If $\|f\|_\infty = |f(y_0)(u_0)|_{h^F}$ is attained at a point $y_0 \in Y_2$, it follows immediately from (4.57) that

$$\|f\|_\infty + O\left(k^{-1/2}\right) \leq \|T_{f,k}\|,$$

so one obtains the lower bound. Next let $\|f\|_\infty = |f(y_0)(u_0)|_{h^F}$ be attained at $y_0 \in Y \setminus Y_2$, a vanishing point of the curvature. As $Y \setminus Y_2 \subset Y$ is open and dense one may find for any $\varepsilon > 0$ a point $y_\varepsilon \in Y \setminus Y_2$ and $u_\varepsilon \in F_{y_\varepsilon}$, $|u_\varepsilon|_{h^F} = 1$, with $\|f\|_\infty - \varepsilon \leq |f(y_\varepsilon)(u_\varepsilon)|_{h^F}$. Combined with (4.57) this gives

$$\begin{aligned} \|f\|_\infty - \varepsilon + O_\varepsilon\left(k^{-1/2}\right) &\leq \|T_{f,k}\|, \quad \text{and} \\ \|f\|_\infty - \varepsilon &\leq \liminf_{k\to\infty} \|T_{f,k}\|. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this implies $\|f\|_\infty \leq \liminf_{k\to\infty} \|T_{f,k}\|$ proving the lower bound.

Next, to prove the composition expansion (4.56) it suffices to prove a uniform kernel estimate

$$\|[T_{f,k}T_{g,k} - T_{fg,k}](.,y)\|_{L^2} = O\left(k^{-1/r}\right), \quad \forall y \in Y.$$

To this end we again compute in geodesic chart centered at y

$$\begin{aligned} T_{f,k}T_{g,k}(.,0) &= \int_{Y\times Y} dy_1 dy_2\, \Pi_k(.,y_1) f(y_1) \Pi_k(y_1,y_2) g(y_2) \Pi_k(y_2,0) \\ &= O_{L^2}\left(k^{-\infty}\right) + \int_{B_\varepsilon(y_2)} dy_1 \int_{B_\varepsilon(y)} dy_2\, \tilde{\Pi}_k(.,y_1) f(y_1) \tilde{\Pi}_k(y_1,y_2) g(y_2) \tilde{\Pi}_k(y_2,0) \\ &= O_{L^2}\left(k^{-\infty}\right) + \int_{B_\varepsilon(y_2)} dy_1 \int_{B_\varepsilon(y)} dy_2 k^{6/r_y} \left\{\Pi^{\boxdot}\left(k^{1/r_y}.,k^{1/r_y}y_1\right)\right. \\ &\qquad \left. f(y_1) \Pi^{\boxdot}\left(k^{1/r_y}y_1, k^{1/r_y}y_2\right) g(y_2) \Pi^{\boxdot}\left(k^{1/r_y}y_2, 0\right)\right\} \\ &= O_{L^2}\left(k^{-\infty}\right) + \int_{k^{1/r_y}B_\varepsilon(y_2)} dy_1 \int_{k^{1/r_y}B_\varepsilon(y)} dy_2 k^{2/r_y} \left\{\Pi^{\boxdot}(.,y_1)\right. \\ &\qquad \left. f\left(y_1 k^{-1/r_y}\right) \Pi^{\boxdot}(y_1,y_2) g\left(y_2 k^{-1/r_y}\right) \Pi^{\boxdot}(y_2,0)\right\} \\ &= O_{L^2}\left(k^{-1/r_y}\right) + \int_{k^{1/r_y}B_\varepsilon(y_2)} dy_1 \int_{k^{1/r_y}B_\varepsilon(y)} dy_2 k^{2/r_y} \left\{\Pi^{\boxdot}(.,y_1)\right. \\ &\qquad \left. \Pi^{\boxdot}(y_1,y_2) fg\left(y_2 k^{-1/r_y}\right) \Pi^{\boxdot}(y_2,0)\right\} \\ &= O_{L^2}\left(k^{-1/r_y}\right) + \int_{B_\varepsilon(y_2)} dy_1 \int_{B_\varepsilon(y)} dy_2\, \tilde{\Pi}_k(.,y_1) \tilde{\Pi}_k(y_1,y_2) fg(y_2) \tilde{\Pi}_k(y_2,0) \\ &= O_{L^2}\left(k^{-1/r_y}\right) + T_{fg,k} \end{aligned}$$

with all remainders being uniform in $y \in Y$. Above we have again used the localization/rescaling properties (4.20), (4.27) as well as the first order Taylor expansion $f\left(y_1 k^{-1/r_y}\right) = f\left(y_2 k^{-1/r_y}\right) + O_{\|f\|_{C^1}}\left(k^{-1/r_y}\right)$. □

Remark 31. Similar to the previous remark 24, we can recover the usual algebra properties of Toeplitz operators when f, g are compactly supported on the set Y_2 where the curvature R^L is positive. In particular we define a generalized Toeplitz operator to be a sequence of operators $T_k : L^2(Y, F \otimes L^k) \longrightarrow L^2(Y, F \otimes L^k)$, $k \in \mathbb{N}$, such that there exist $K \Subset Y_2$, $h_j \in C_c^\infty(K; \mathrm{End}(F))$, $C_j > 0$, $j = 0, 1, 2, \ldots$ satisfying

$$\left\| T_k - \sum_{j=0}^{N} k^{-j} T_{h_j,k} \right\| \leqslant C_N\, k^{-N-1}, \quad \forall N \in \mathbb{N}. \tag{4.58}$$

Then this class is closed under composition and one may define a formal star product on $C_c^\infty(Y_2)[[h]]$ via

$$f *_h g = \sum_{j=0}^{\infty} C_j(f, g)\, h^j \in C_c^\infty(Y_2)[[h]] \quad \text{where}$$

$$T_{f,k} \circ T_{g,k} \sim \sum_{j=0}^{\infty} T_{C_j(f,g)} k^{-j},$$

(cf. [14, 18, 46]). Furthermore

$$T_{f,k} \circ T_{g,k} = T_{fg,k} + O_{L^2 \to L^2}\left(k^{-1}\right)$$
$$[T_{f,k}\,, T_{g,k}] = \frac{i}{k} T_{\{f,g\},k} + O_{L^2 \to L^2}(k^{-2})$$

$\forall f, g \in C_c^\infty(Y_2; \mathrm{End}(F))$, with $\{\cdot,\cdot\}$ being the Poisson bracket on the Kähler manifold (Y_2, iR^L).

Finally we address the asymptotics of the spectral measure of the Toeplitz operator (4.53), called Szegő-type limit formulas [15, 27]. The spectral measure of $T_{f,k}$ is defined via

$$u_{f,k}(s) := \sum_{\lambda \in \mathrm{Spec}\left(T_{f,k}\right)} \delta(s - \lambda) \in \mathcal{S}'(\mathbb{R}_s). \tag{4.59}$$

We now have the following asymptotic formula.

Theorem 32. *The spectral measure* (4.59) *satisfies*

$$u_{f,k} \sim \frac{k}{2\pi} f_* R^L \tag{4.60}$$

in the distributional sense as $k \to \infty$.

Proof. Since $\mathrm{Spec}(T_{f,k}) \subset \left[-\|f\|_\infty, \|f\|_\infty \right]$ by (4.55), the equation (4.60) is equivalent to

$$\mathrm{tr}\, \varphi(T_{f,k}) = \sum_{\lambda \in \mathrm{Spec}\left(T_{f,k}\right)} \varphi(\lambda) \sim \frac{k}{2\pi} \int_Y [\varphi \circ f]\, R^L,$$

for all $\varphi \in C_c^\infty\left(-\|f\|_\infty - 1, \|f\|_\infty + 1\right)$. We first prove that the trace of a Toeplitz operator (4.53) satisfies the asymptotics

$$\mathrm{tr}\, T_{f,k} \sim \frac{k}{2\pi} \int_Y f R^L. \tag{4.61}$$

To this end first note that the expansion of Theorem 29 is uniform on compact subsets $K \subset Y_2$ while $|T_{f,k}(y,y)| = O(k)$ uniformly in $y \in Y$ as in Lemma 25. Further as with Proposition 12 $Y_{\geq 3}$ is a closed subset of a hypersurface and has measure zero. Then with $K_j \subset Y_2$, $j = 1, 2, \ldots$, being a sequence of compact subsets satisfying we have $K_j \subset K_{j+1}$, $\cap_{j=1}^{\infty} K_j = Y_{\geq 3}$. One may then breakup the trace integral

$$\begin{aligned} \frac{1}{k}\mathrm{tr} T_{f,k} &= \frac{1}{k}\int_{K_j} \mathrm{tr}\ T_{f,k}(y,y) + \frac{1}{k}\int_{Y\setminus K_j} \mathrm{tr}\ T_{f,k}(y,y) \\ &= \frac{1}{2\pi}\int_{K_j} fR^L + O_j\left(\frac{1}{k}\right) + O\left(\mu\left(Y \setminus K_j\right)\right) \end{aligned}$$

from which (4.61) follows on knowing $\frac{1}{2\pi}\int_{K_j} fR^L \to \frac{1}{2\pi}\int_Y fR^L$, $\mu(Y \setminus K_j) \to 0$ as $j \to \infty$.

Following this one has

$$\mathrm{tr}\ T_{f,k}^l = \mathrm{tr}\ T_{f^l,k} + O_f\left(k^{1-1/r}\right)$$

$\forall l \in \mathbb{N}$ from (4.8), (4.56). A polynomial approximation of the compactly supported function $\varphi \in C_c^{\infty}\left(-\|f\|_{\infty} - 1, \|f\|_{\infty} + 1\right)$ then gives

$$\begin{aligned} \mathrm{tr}\ \varphi\left(T_{f,k}\right) &= \mathrm{tr}\ T_{\varphi\circ f,k} + o(k) \\ &= \frac{k}{2\pi}\int_Y [\varphi \circ f]\, R^L + o(k) \end{aligned}$$

by (4.61) as required. □

The analogous result for projective manifolds endowed with the restriction of the hyperplane bundle was originally proved in [15, Theorem 13.13], [27] and for arbitrary positive line bundles in [9], see also [42]. In [36, Theorem 1.6] the asymptotics (4.61) are proved for a semi-classical spectral function of the Kodaira Laplacian on an arbitrary manifold.

4.4. **Random sections.** In this section we generalize the results of [57] to the semi-positive case considered here. Let us consider Hermitian holomorphic line bundles (L, h^L) and (F, h^F) on a compact Riemann surface Y. To state the result first note that the natural metric on $H^0\left(Y; F \otimes L^k\right)$ arising from g^{TY}, $h^F$and h^L gives rise to a probability density μ_k on the sphere

$$SH^0\left(Y; F \otimes L^k\right) := \left\{s \in H^0\left(Y; F \otimes L^k\right) \mid \|s\| = 1\right\},$$

of finite dimension $\chi\left(Y; F \otimes L^k\right) - 1$ (4.8). We now define the product probability space $(\Omega, \mu) := \left(\Pi_{k=1}^{\infty} SH^0\left(Y; F \otimes L^k\right), \Pi_{k=1}^{\infty}\mu_k\right)$. To a random sequence of sections $s = (s_k)_{k\in\mathbb{N}} \in \Omega$ given by this probability density, we then associate the random sequence of zero divisors $Z_{s_k} = \{s_k = 0\}$ and view it as a random sequence of currents of integration in $\Omega_{0,0}(Y)$. We now have the following.

Theorem 33. *Let (L, h^L) and (F, h^F) be Hermitian holomorphic line bundles on a compact Riemann surface Y and assume that (L, h^L) is semi-positive line bundle and its curvature R^L vanishes to finite order at any point. Then for μ-almost all $s = (s_k)_{k\in\mathbb{N}} \in \Omega$, the sequence of currents*

$$\frac{1}{k} Z_{s_k} \rightharpoonup \frac{i}{2\pi} R^L$$

converges weakly to the semi-positive curvature form.

Proof. The proof follows [44] Thm 5.3.3 with some modifications which we point out below. With Φ_k denoting the Kodaira map (4.48), we first have

$$\mathbb{E}\left[Z_{s_k}\right] = \Phi_k^*\left(\omega_{FS}\right) \tag{4.62}$$

as in [44] Thm 5.3.1. For a given $\varphi \in \Omega_{0,0}(Y)$, one has

$$\left\langle \frac{1}{k} Z_{s_k} - \frac{i}{2\pi} R^L, \varphi \right\rangle = \left\langle \frac{1}{k} Z_{s_k} - \frac{1}{k} \Phi_k^* (\omega_{FS}), \varphi \right\rangle + O\left(k^{-1/3} \|\varphi\|_{C^0}\right)$$

following (4.51) and it thus suffices to show $Y^\varphi(s_k) \to 0$, μ-almost surely with

$$Y^\varphi(s_k) := \left\langle \frac{1}{k} Z_{s_k} - \frac{1}{k} \Phi_k^* (\omega_{FS}), \varphi \right\rangle$$

being the given random variable. But (4.62) gives

$$\begin{aligned} \mathbb{E}\left[|Y^\varphi(s_k)|^2\right] &= \frac{1}{k^2} \mathbb{E}\left[\langle Z_{s_k}, \varphi\rangle^2\right] - \frac{1}{k^2} \mathbb{E}\left[\langle \Phi_k^* (\omega_{FS}), \varphi\rangle^2\right] \\ &= O\left(k^{-2}\right) \end{aligned}$$

as in [44] Thm 5.3.3. Thus $\int_\Omega d\mu \left[\sum_{k=1}^\infty |Y^\varphi(s_k)|^2\right] < \infty$ proving the theorem. □

The above result may be alternatively obtained using L^2 estimates for the $\bar{\partial}$-equation of a modified positive metric as in [22, S 4].

Example 34. (Random polynomials) The last theorem has an interesting specialization to random polynomials. To this end, let $Y = \mathbb{CP}^1 = \mathbb{C}_w^2 \setminus \{0\} / \mathbb{C}^*$ with homogeneous coordinates $[w_0 : w_1]$. A semi-positive curvature form for each even $r \geq 2$, is given by

$$\begin{aligned} \omega_r :=& \frac{i}{2\pi} \partial\bar{\partial} \ln\left(|w_0|^r + |w_1|^r\right) \\ =& \frac{i}{2\pi} \frac{r^2}{4} \frac{|w_0|^{r-2} |w_1|^{r-2}}{\left(|w_0|^r + |w_1|^r\right)^2} dw_0 \wedge dw_1, \end{aligned} \tag{4.63}$$

which has two vanishing points at the north/south poles of order $r-2$. This is the curvature form on the hyperplane line bundle $L = \mathcal{O}(1)$ for the metric with potential $\varphi = \ln\left(|w_0|^r + |w_1|^r\right)$. An orthogonal basis for $H^0\left(X, L^k\right)$ is given by $s_\alpha := z^\alpha$, $0 \leq \alpha \leq k$, in terms of the affine coordinate $z = w_0/w_1$ on the chart $\{w_1 \neq 0\}$ and a $\mathbb{C}^*$ invariant trivialization of L. The normalization is now given by

$$\begin{aligned} \|s_\alpha\|^2 &= \frac{1}{2\pi} \frac{r^2}{4} \int_{\mathbb{C}} \frac{|z|^{2\alpha + r - 2}}{\left(1 + |z|^r\right)^{k+2}} \\ &= \frac{1}{\frac{2}{r}(k+1) \binom{k}{\frac{2}{r}\alpha}} \end{aligned}$$

with the binomial coefficient $\binom{k}{\frac{2}{r}\alpha} = \frac{\Gamma(k+1)}{\Gamma\left(\frac{2}{r}\alpha + 1\right)\Gamma\left(k - \frac{2}{r}\alpha\right)}$ given in terms of the Gamma function. We have now arrived at the following.

Corollary 35. *For each even $r \geq 2$, let*

$$p_k(z) = \sum_{\alpha=0}^{k} c_\alpha \sqrt{\binom{k}{\frac{2}{r}\alpha}} z^\alpha$$

be a random polynomial of degree k with the coefficients c_α being standard i.i.d. Gaussian variables. The distribution of its roots converges in probability

$$\frac{1}{k} Z_{p_k} \rightharpoonup \frac{1}{2\pi} \frac{r^2}{4} \frac{|z|^{r-2}}{\left(1 + |z|^r\right)^2}.$$

The above theorem interpolates between the case of $SU(2)$/elliptic polynomials ($r = 2$) [13] and the case of Kac polynomials ($r = \infty$) [29, 38, 56]. For recent results on the distribution of zeroes of more general classes of random polynomials we refer to [5, 12, 37].

4.5. **Holomorphic torsion.** In this section we give an asymptotic result for the holomorphic torsion of the semi-positive line bundle L generalizing that of [11] (see also [44, S 5.5]). First recall that the holomorphic torsion of L is defined in terms of the zeta function

$$\zeta_k(s) := \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \mathrm{tr}\left[e^{-t\Box_k^1}\right], \quad \mathrm{Re}(s) > 1. \tag{4.64}$$

The above converges absolutely and defines a holomorphic function of $s \in \mathbb{C}$ in this region. It possesses a meromorphic extension to $\mathbb{C}$ with no pole at zero and the holomorphic torsion is defined to be $\mathcal{T}_k := \exp\left\{-\frac{1}{2}\zeta_k'(0)\right\}$.

Next, with τ^L, $\omega\left(R^L\right)$ as in (4.4) and $t > 0$, set

$$R_t(y) := \begin{cases} \frac{1}{2\pi}\tau^L\left(1 - e^{-t\tau^L}\right)^{-1} e^{-t\omega\left(R^L\right)}; & \tau^L(y) > 0 \\ \frac{1}{2\pi}\frac{1}{t}; & \tau^L(y) = 0. \end{cases} \tag{4.65}$$

Note that the above defines a smooth endomorphism $R_t(y) \in C^\infty\left(Y; \mathrm{End}\left(\Lambda^{0,*}\right)\right)$. Further, let $A_j \in C^\infty\left(Y; \mathrm{End}\left(\Lambda^{0,*}\right)\right)$ be such that

$$\rho_t^N := R_t(y) - \sum_{j=-1}^{N} A_j(y)\, t^j = O\left(t^{N+1}\right). \tag{4.66}$$

We now prove the following uniform small time asymptotic expansion for the heat kernel.

Proposition 36. *There exist $A_{k,j} \in C^\infty\left(Y; End\left(\Lambda^{0,*}\right)\right)$, $j = -1, 0, 1, \ldots$, satisfying $A_{k,j} - A_j = O\left(k^{-1}\right)$, such that for each $t > 0$*

$$\left| k^{-1} e^{-\frac{t}{2k}D_k^2}(y,y) - \sum_{j=-1}^{N} A_{k,j}(y)\, t^j - \rho_t^N \right| = O\left(t^{N+1}k^{-1}\right) \tag{4.67}$$

uniformly in $y \in Y$, $k \in \mathbb{N}$.

Proof. We again work in the geodesic coordinates and local orthonormal frames centered at $y \in Y$ introduced in Section 4.1. With $\tilde{D}_k$ as in (4.14), similar localization estimates as in Section 4.1 (cf. also Lemma 1.6.5 in [44]) give

$$e^{-\frac{t}{k}D_k^2}(y,y) - e^{-\frac{t}{k}\tilde{D}_k^2}(0,0) = O\left(e^{-\frac{\varepsilon^2}{32t}k}\right)$$

uniformly in $t > 0$, $y \in Y$ and k. It then suffices to consider the small time asymptotics of $e^{-\frac{t}{k}\tilde{D}_k^2}(0,0)$. We again introduce the rescaling $\delta_{k^{-1/2}}y = k^{-1/2}y$, under which

$$\left(\delta_{k^{-1/2}}\right)_* \tilde{D}_k^2 = k\Big(\underbrace{\boxdot_0 + k^{-1}E_1}_{=:\boxdot}\Big),$$

with $\boxdot_0 = bb^\dagger + \tau_y \bar{w} i_{\bar{w}}$; $b := -2\partial_z + \frac{1}{2}\tau_y \bar{z}$ and where $E_0 = a_{pq}(y;k)\,\partial_{y_p}\partial_{y_q} + b_p(y;k)\,\partial_{y_p} + c(y;k)$ is a second order operator whose coefficient functions are uniformly (in k) C^∞ bounded (cf.

[44] Thms. 4.1.7 and 4.1.25). Again $e^{-\frac{t}{k}\tilde{D}_k^2}(0,0) = ke^{-t\boxdot}(0,0)$ and following a standard small time heat kernel expansion of an elliptic operator [26, 55] one has

$$e^{-t\boxdot}(y_1, y_2) = \frac{1}{4\pi t} e^{-\frac{1}{4t}d^2(y_1,y_2)} \left[\sum_{j=-1}^{N} A_{k,j} t^j\right] + \rho_{k,t}^N \tag{4.68}$$

with $d(y_1, y_2)$ denoting the distance function for the metric $\tilde{g}^{TY}$ on $\mathbb{R}^2$. Moreover

$$A_{k,j}(y_1, y_2) = A_j(y_1, y_2) + O\left(k^{-1}\right) \tag{4.69}$$

where A_j denotes an analogous term in the small time expansion of $e^{-t\boxdot_0}$ satisfying $A_j(0,0) = A_j(4.66)$ ([44, (1.6.68)]). Finally, the remainders in (4.66), (4.68) being given by

$$\rho_t^N = -\int_0^t ds\, e^{-(t-s)\boxdot_0} s^N (\boxdot_0 A_N), \quad \rho_{k,t}^N = -\int_0^t ds\, e^{-(t-s)\boxdot} s^N (\boxdot A_{k,N}),$$

the proposition now follows from (4.69) along with $e^{-t\boxdot} = e^{-t\boxdot_0} + O(k^{-1})$ uniformly in $t > 0$. $\square$

We now prove the the asymptotic result for holomorphic torsion. Below we denote by $x \ln x$ the continuous extension of this function from $\mathbb{R}_{>0}$ to $\mathbb{R}_{\geq 0}$ (i.e. taking the value zero at the origin).

Theorem 37. *The holomorphic torsion satisfies the asymptotics*

$$\ln \mathcal{T}_k := -\frac{1}{2}\zeta_k'(0) = -k \ln k \left[\frac{\tau^L}{8\pi}\right] - k\left[\frac{\tau^L}{8\pi} \ln\left(\frac{\tau^L}{2\pi}\right)\right] + o(k)$$

as $k \to \infty$.

Proof. First define the rescaled zeta function $\tilde{\zeta}_k(s) := \frac{k^{-1}}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \mathrm{tr}\left[e^{-\frac{t}{k}\boxdot_k^1}\right] = k^{-1}k^s \zeta_k(s)$ satisfying

$$\zeta_k'(0) = k\tilde{\zeta}_k'(0) - (k \ln k)\,\tilde{\zeta}_k(0). \tag{4.70}$$

With $a_{k,j} := \int_Y \mathrm{tr}[A_{k,j}]\, dy$, $j = -1, 0, \ldots$, and the analytic continuation of the zeta function being given in terms of the heat trace, one has

$$\tilde{\zeta}_k(0) = a_{k,0} \to \int_Y dy\, \mathrm{tr}[A_0], \tag{4.71}$$

$$\begin{aligned} \tilde{\zeta}_k'(0) = & \underbrace{\int_0^T dt\, t^{-1}\left\{k^{-1}\mathrm{tr}\left[e^{-\frac{t}{k}\boxdot_k^1}\right] - a_{k,-1}t^{-1} - a_{k,0}\right\}}_{=\int_0^T dt\, t^{-1}\rho_t^0 + O\left(\frac{T}{k}\right)} \\ & + \int_T^\infty dt\, t^{-1}k^{-1}\mathrm{tr}\left[e^{-\frac{t}{k}\boxdot_k^1}\right] \\ & - a_{k,-1}T^{-1} + \Gamma'(1)\, a_{k,0} \end{aligned} \tag{4.72}$$

following (4.67).

Choosing $T = k^{1-2/r}$, gives

$$\begin{aligned} t^{-1}k^{-1}\mathrm{tr}\left[e^{-\frac{t}{k}\boxdot_k^1}\right] &\leq e^{-\frac{(t-1)}{k}\left[c_1 k^{2/r} - c_2\right]} t^{-1}k^{-1}\mathrm{tr}\left[e^{-\frac{1}{k}\boxdot_k^1}\right] \\ &\leq Ct^{-1}k^{-1}e^{-\frac{(t-1)}{k}\left[c_1 k^{2/r} - c_2\right]}, \quad t \geq T, \end{aligned}$$

on account of (4.7), (4.67). The last expression having a uniformly in k bounded integral on $[T,\infty)$, dominated convergence gives

$$\begin{aligned}\tilde{\zeta}_k'(0) &\longrightarrow \int_Y dy\,\alpha(y)\,,\qquad \text{where}\\ \alpha(y) &:= \int_0^T dt\,t^{-1}\left\{\operatorname{tr}\left[R_t(y)\right]-\operatorname{tr}\left[A_{-1}\right]t^{-1}-\operatorname{tr}\left[A_0\right]\right\}\\ &\quad+\int_T^\infty dt\,t^{-1}\operatorname{tr}\left[R_t(y)\right]\\ &\quad-\operatorname{tr}\left[A_{-1}\right]t^{-1}+\Gamma'(1)\operatorname{tr}\left[A_0\right].\end{aligned}\tag{4.73}$$

Finally, using (4.65) one has

$$\begin{aligned}\operatorname{tr}\left[A_0\right] &= -\frac{\tau^L}{4\pi}\\ \alpha(y) &= \frac{\tau^L}{4\pi}\ln\left(\frac{\tau^L}{2\pi}\right)\end{aligned}\tag{4.74}$$

with again the extension of the function $x\ln x$ to the origin being given by continuity to be zero as before. The proposition now follows from putting together (4.70), (4.71), (4.72), (4.73) and (4.74). □

Appendix A. Model operators

Here we define certain model Bochner/Kodaira Laplacians and Dirac operators acting on a vector space V. First the Bochner Laplacian is intrinsically associated to a triple $\left(V,g^V,R^V\right)$ with metric g^V and tensor $0\neq R^V\in S^{r-2}V^*\otimes\Lambda^2V^*$, $r\geq 2$. We say that tensor R^V is nondegenerate if

$$S^{r-s-2}V^*\otimes\Lambda^2V^*\ni i_v^s\left(R^V\right)=0,\ \forall s\leq r-2 \implies T_yY\ni v=0.\tag{A.1}$$

Above i^s denotes the s-fold contraction of the symmetric part of R^V.

For $v_1\in V$, $v_2\in T_{v_1}V=V$, contraction of the antisymmetric part (denoted by ι) of R^V gives $\iota_{v_2}R^V\in S^{r-2}V^*\otimes V^*$. The contraction may then be evaluated $\left(\iota_{v_2}R^V\right)(v_1)$ at $v_1\in V$, i.e. viewed as a homogeneous degree $r-1$ polynomial function on V. The tensor R^V now determines a one form $a^{R^V}\in\Omega^1(V)$ via

$$a_{v_1}^{R^V}(v_2):=\int_0^1 d\rho\left(\iota_{v_2}R^V\right)(\rho v_1)=\frac{1}{r}\left(\iota_{v_2}R^V\right)(v_1)\,,\tag{A.2}$$

which we may view as a unitary connection $\nabla^{R^V}=d+ia^{R^V}$ on a trivial Hermitian vector bundle E of arbitrary rank over V. The curvature of this connection is clearly R^V now viewed as a homogeneous degree $r-2$ polynomial function on V valued in Λ^2V^*. This now gives the model Bochner Laplacian

$$\Delta_{g^V,R^V}:=\left(\nabla^{R^V}\right)^*\nabla^{R^V}:C^\infty(V;E)\to C^\infty(V;E)\,.\tag{A.3}$$

An orthonormal basis $\{e_1,e_2,\ldots,e_n\}$, determines components $R_{pq,\alpha}:=R^V\left(e^{\odot\alpha};e_p,e_q\right)\neq 0$, $\alpha\in\mathbb{N}_0^{n-1}$, $|\alpha|=r-2$, as well as linear coordinates $(y_1,\ldots,y_n)$ on V. The connection form in these coordinates is given by $a_p^{R^V}=\frac{i}{r}y^qy^\alpha R_{pq,\alpha}$. While the model Laplacian (A.3) is given

$$\Delta_{g^V,R^V}=-\sum_{q=1}^n\left(\partial_{y_p}+\frac{i}{r}y^qy^\alpha R_{pq,\alpha}\right)^2.\tag{A.4}$$

As in (3.5), the above may now be related to the (nilpotent) sR Laplacian on the the product $S^1_\theta \times V$ given by

$$\hat{\Delta}_{g^V,R^V} := -\sum_{q=1}^{n}\left(\partial_{y_p} + \frac{i}{r} y^q y^\alpha R_{pq,\alpha}\partial_\theta\right)^2, \tag{A.5}$$

and corresponding to the sR structure $\left(S^1_\theta \times V, \ker\left(d\theta + a^{R^V}\right), \pi^* g^V, d\theta \mathrm{vol} g^V\right)$ where the sR metric corresponds to g^V under the natural projection $\pi : S^1_\theta \times V \to V$. Note that the above differs from the usual nilpotent approximation of the sR Laplacian since it acts on the product with S^1. As (3.6), the heat kernels of (A.3), (A.5) are now related

$$e^{-t\Delta_{g^V,R^V}}(y, y') = \int e^{-i\theta} e^{-t\hat{\Delta}_{g^V,R^V}}(y, 0; y', \theta)\, d\theta. \tag{A.6}$$

Next, assume that the vector space V of even dimension and additionally equipped with an orthogonal endomorphism $J^V \in O(V)$; $\left(J^V\right)^2 = -1$. This gives rise to a (linear) integrable almost complex structure on V , a decomposition $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$ into $\pm i$ eigenspaces of J and a Clifford multiplication endomorphism $c : V \to \mathrm{End}\left(\Lambda^* V^{0,1}\right)$. We further assume that R^V is a $(1, 1)$ form with respect to J (i.e. $S^k V^* \ni R^V(w_1, w_2) = 0$, $\forall w_1, w_2 \in V^{1,0}$). The $(0, 1)$ part of the connection form (A.2) then gives a holomorphic structure on the trivial Hermitian line bundle $\mathbb{C}$ with holomorphic derivative $\bar{\partial}_{\mathbb{C}} = \bar{\partial} + \left(a^V\right)^{0,1}$. One may now define the Kodaira Dirac and Laplace operators, intrinsically associated to the tuple $\left(V, g^V, R^V, J^V\right)$, via

$$D_{g^V,R^V,J^V} := \sqrt{2}\left(\bar{\partial}_{\mathbb{C}} + \bar{\partial}^*_{\mathbb{C}}\right) \tag{A.7}$$

$$\square_{g^V,R^V,J^V} := \frac{1}{2}\left(D_{g^V,R^V,J^V}\right)^2 \tag{A.8}$$

acting on $C^\infty\left(V; \Lambda^* V^{0,1}\right)$. The above (A.3), (A.8) are related by the Lichnerowicz formula

$$\square_{g^V,R^V,J^V} = \Delta_{g^V,R^V} + c\left(R^V\right) \tag{A.9}$$

where $c\left(R^V\right) = \sum_{p<q} R^{i_1\ldots i_{r-2}}_{pq} y_{i_1\ldots} y_{i_{r-2}} c(e_p)\, c(e_q)$. We may choose complex orthonormal basis $\{w_j\}_{j=1}^m$ of $V^{1,0}$ that diagonalizes the tensor R^V: $R^V(w_i, \bar{w}_j) = \delta_{ij} R_{j\bar{j}}$; $R_{i\bar{j}} \in S^{r-2}V^*$. This gives complex coordinates on V in which (A.8) may be written as

$$\begin{aligned} \square_{g^V,R^V,J^V} &= \sum_{q=1}^{\mathrm{dim}V/2} b_j b_j^\dagger + 2\left(\partial_{z_j} a_j + \partial_{\bar{z}_j}\bar{a}_j\right)\bar{w}_j i_{\bar{w}_j} \\ b_j &= -2\partial_{z_j} + \bar{a}_j \\ b_j^\dagger &= 2\partial_{\bar{z}_j} + a_j \\ a_j &= \frac{1}{r} R_{j\bar{j}} z_j \end{aligned} \tag{A.10}$$

with each $R_{j\bar{j}}(z)$, $1 \le j \le \mathrm{dim}V/2$, being a real homogeneous function of order $r-2$.

Being symmetric with respect to the standard Euclidean density and semi-bounded below, both Δ_{g^V,R^V} and $\square^V$ are essentially self-adjoint on L^2. The domains of their unique self-adjoint extensions are

$$\begin{aligned} \mathrm{Dom}\left(\Delta_{g^V,R^V}\right) &= \left\{\psi \in L^2 | \Delta_{g^V,R^V}\psi \in L^2\right\}, \\ \mathrm{Dom}\left(\square_{g^V,R^V,J^V}\right) &= \left\{\psi \in L^2 | \square_{g^V,R^V,J^V}\psi \in L^2\right\}, \end{aligned}$$

respectively. We shall need the following information regarding their spectrum.

Proposition 38. *For some $c>0$, one has Spec $\left(\Delta_{g^V,R^V}\right)\subset[c,\infty)$. For R^V satisfying the non-degeneracy condition* (A.1) *one has EssSpec $\left(\Delta_{g^V,R^V}\right)=\emptyset$. Finally, for dim$V=2$ with $R^V(w,\bar{w})\geq0,\,\forall w\in V^{1,0}$ semi-positive one has Spec $\left(\Box_{g^V,R^V,J^V}\right)\subset\{0\}\cup[c,\infty)$.*

Proof. The proof is similar to those of Proposition 15 and Corollary 21. Introduce the deformed Laplacian $\Delta_k:=\Delta_{g^V,kR^V}$ obtained by rescaling the tensor R^V. From (A.4) $\Delta_k=k^{2/r}\mathscr{R}\Delta_{g^V,R^V}\mathscr{R}^{-1}$ are conjugate under the rescaling $\mathscr{R}:C^\infty(V;E)\to C^\infty(V;E)$, $(\mathscr{R}u)(x):=u\left(yk^{1/r}\right)$ implying

$$\begin{aligned}\operatorname{Spec}(\Delta_k)&=k^{2/r}\operatorname{Spec}\left(\Delta_{g^V,R^V}\right)\\ \operatorname{EssSpec}(\Delta_k)&=k^{2/r}\operatorname{EssSpec}\left(\Delta_{g^V,R^V}\right)\end{aligned}\tag{A.11}$$

By an argument similar to Proposition 15, one has $\operatorname{Spec}(\Delta_k)\subset\left[c_1k^{2/r}-c_2,\infty\right)$ for some $c_1,c_2>0$ for $R^V\neq0$. From here $\operatorname{Spec}\left(\Delta_{g^V,R^V}\right)\subset[c,\infty)$ follows. Next, under the non-degeneracy condition, the order of vanishing of the curvature homogeneous curvature R^V (of the homogeneous connection a^{R^V} (A.2)) is seen to be maximal at the origin: $\operatorname{ord}_y\left(R^V\right)<r-2$ for $y\neq0$. Following a similar sub-elliptic estimate (2.15) on $V\times S^1_\theta$ as in Proposition 15, we have

$$k^{2/(r-1)}\|u\|^2\leq C\left[\langle\Delta_ku,u\rangle+\|u\|^2_{L^2}\right],\quad\forall u\in C_c^\infty\left(V\setminus B_1(0)\right),$$

holds on the complement of the unit ball centered at the origin. Combining the above with Persson's characterization of the essential spectrum (cf. [53, 1] Ch. 3)

$$\operatorname{EssSpec}(\Delta_k)=\sup_R\inf_{\substack{\|u\|=1\\ u\in C_c^\infty(V\setminus B_R(0))}}\langle\Delta_ku,u\rangle,$$

we have $\operatorname{EssSpec}(\Delta_k)\subset\left[c_1k^{2/(r-1)}-c_2,\infty\right)$. From here and using (A.11), $\operatorname{EssSpec}\left(\Delta_{g^V,R^V}\right)=\emptyset$ follows.

The proof of the final part is similar following $k^{2/r}\operatorname{Spec}\left(\Box_{g^V,R^V,J^V}\right)=\operatorname{Spec}\left(\Box_{g^V,kR^V,J^V}\right)=\operatorname{Spec}(\Box_k)\subset\{0\}\cup\left[c_1k^{2/r}-c_2,\infty\right)$, $\Box_k:=\Box_{g^V,kR^V,J^V}$, by an argument similar to Corollary 21. $\square$

Next, the heat $e^{-t\Delta_{g^V,R^V}}$, $e^{-t\Box_{g^V,R^V,J^V}}$ and wave $e^{it\sqrt{\Delta_{g^V,R^V}}}$, $e^{it\sqrt{\Box_{g^V,R^V,J^V}}}$ operators being well-defined by functional calculus, a finite propagation type argument as in Lemma 8 gives $\varphi\left(\Delta_{g^V,R^V}\right)(.,0)\in\mathcal{S}(V)$, $\varphi\left(\Box_{g^V,R^V,J^V}\right)(.,0)\in\mathcal{S}(V)$ are in the Schwartz class for $\varphi\in\mathcal{S}(\mathbb{R})$. Further, when $\operatorname{EssSpec}\left(\Delta_{g^V,R^V}\right)=\emptyset$ any eigenfunction of Δ_{g^V,R^V} also lies in $\mathcal{S}(V)$. Finally, on choosing φ supported close to the origin, the Schwartz kernel $\Pi^{g^V,R^V,J^V}(.,0)\in\mathcal{S}(V)$ of the projector Π^{g^V,R^V,J^V} onto the kernel of $\Box_{g^V,R^V,J^V}$ is also of Schwartz class.

We now state another proposition regarding the heat kernel of Δ_{g^V,R^V}. Below we denote $\lambda_0\left(\Delta_{g^V,R^V}\right):=\inf\operatorname{Spec}\left(\Delta_{g^V,R^V}\right)$.

Proposition 39. *For each $\varepsilon>0$ there exist $t,R>0$ such that the heat kernel*

$$\frac{\int_{B_R(0)}dx\left[\Delta_{g^V,R^V}e^{-t\Delta_{g^V,R^V}}\right](x,x)}{\int_{B_R(0)}dx\,e^{-t\Delta_{g^V,R^V}}(x,x)}\leq\lambda_0\left(\Delta_{g^V,R^V}\right)+\varepsilon$$

Proof. Setting $P:=\Delta_{g^V,R^V}-\lambda_0\left(\Delta_{g^V,R^V}\right)$ it suffices to show

$$\frac{\int_{B_R(0)}dx\left[Pe^{-tP}\right](x,x)}{\int_{B_R(0)}dx\,e^{-tP}(x,x)}\leq\varepsilon$$

for some $t, R > 0$. With $\Pi^P_{[0,x]}$ denoting the spectral projector onto $[0, x]$, we split the numerator

$$\int_{B_R(0)} dx \left[Pe^{-tP}\right](x,x) = \int_{B_R(0)} dx \left[\Pi^P_{[0,4\varepsilon]} Pe^{-tP}\right](x,x) + \int_{B_R(0)} dx \left[\left(1 - \Pi^P_{[0,4\varepsilon]}\right) Pe^{-tP}\right](x,x).$$

From $P \geq 0$, $\Pi^P_{[0,4\varepsilon]} Pe^{-tP} \leq 4\varepsilon e^{-tP}$ and $\left(1 - \Pi^P_{[0,4\varepsilon]}\right) Pe^{-tP} \leq ce^{-3\varepsilon t}$, $\forall t \geq 1$, we may bound

$$\frac{\int_{B_R(0)} dx \left[Pe^{-tP}\right](x,x)}{\int_{B_R(0)} dx\, e^{-tP}(x,x)} \leq 4\varepsilon + \frac{ce^{-3\varepsilon t} R^{n-1}}{\int_{B_R(0)} dx\, e^{-tP}(x,x)} \tag{A.12}$$

$\forall R, t \geq 1$. Next, as $0 \in \mathrm{Spec}(P)$ there exists $\|\psi_\varepsilon\|_{L^2} = 1$, $\|P\psi_\varepsilon\|_{L^2} \leq \varepsilon$. It now follows that $\left\|\psi_\varepsilon - \Pi^P_{[0,2\varepsilon]}\psi_\varepsilon\right\| \leq \frac{1}{2}$ and hence

$$\begin{aligned} \frac{1}{2} = -\frac{1}{4} + \int_{B_{R_\varepsilon}(0)} dx \left|\psi_\varepsilon(x)\right|^2 &\leq \int_{B_{R_\varepsilon}(0)} dx \left|\int dy \Pi^P_{[0,2\varepsilon]}(x,y)\,\psi_\varepsilon(y)\right|^2 \\ &\leq \int_{B_{R_\varepsilon}(0)} dx \left(\int dy \Pi^P_{[0,2\varepsilon]}(x,y)\,\Pi^P_{[0,2\varepsilon]}(y,x)\right) = \int_{B_{R_\varepsilon}(0)} dx \Pi^P_{[0,2\varepsilon]}(x,x), \end{aligned}$$

for some $R_\varepsilon > 0$, using $\left(\Pi^P_{[0,2\varepsilon]}\right)^2 = \Pi^P_{[0,2\varepsilon]}$ and Cauchy-Schwartz. This gives

$$\int_{B_{R_\varepsilon}(0)} dx\, e^{-tP}(x,x) \geq \frac{e^{-2\varepsilon t}}{2}, \quad t > 1.$$

Plugging this last inequality into (A.12) gives

$$\frac{\int_{B_{R_\varepsilon}(0)} dx \left[Pe^{-tP}\right](x,x)}{\int_{B_{R_\varepsilon}(0)} dx\, e^{-tP}(x,x)} \leq 4\varepsilon + ce^{-\varepsilon t} R_\varepsilon^{n-1}$$

from which the theorem follows on choosing t large. □

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