Some results around sub-Riemannian spectral asymptotics

Nikhil Savale

Universität zu Köln

Sep. 8, 2020

Sub-Riemannian Geometry and Interactions

Sub-Riemannian (sR) geometry

Sub-Riemannian (sR) manifold: $(X,\, E \subset TX,\, g^E)$ inside the tangent space.
Subbundle E is assumed to be *bracket-generating*.

(Chow-Rashevsky '37)
E bracket-generating $\implies$ X connected by *horizontal* paths
$\gamma \in H^1([0,1]\,;X)$ s.t.$\dot{\gamma}(t) \in E_{\gamma(t)}$ a.e.
X metric space with $d^E = \inf_{\gamma \text{ horizontal}} \int_0^1 dt\, |\dot{\gamma}(t)|$.

Examples of bracket-generating distributions (Contact, even/quasi - contact, Martinet, Engel distributions..)

Sub-Riemannian (sR) geometry

Two peculiar phenomena:

1. Hausdorff dimension:
Canonical flag: $\underbrace{E_0}_{=\{0\}} \subset \underbrace{E_1}_{=E} \subset \ldots \subset \underbrace{E_j}_{\substack{\text{span of} \\ j\text{th brackets}}} \subset \ldots \subset \underbrace{E_{r(x)}}_{=TX}, \quad r(x) = \text{step}$

$$Q(x) = \lim_{\varepsilon \to 0} \frac{\ln \mathrm{vol} B_\varepsilon(x)}{\ln \varepsilon} \underset{\text{thm}}{\overset{\text{Ball-Box}}{=}} \sum_{j=1}^{r(x)} j \left[\dim E_j(x) - \dim E_{j-1}(x)\right] > n$$

2. Abnormal geodesics: Do not satisfy variational equations.
Pontryagin max. principle: abnormal γ is critical for end-point map (i.e. singular)
Calculating $d\mathrm{end}(\gamma)$: γ is projection of characteristic $\lambda \in H^1\left([0,1]; E^\perp\right)$,
$i_{\dot\lambda(t)}\left(\omega|_{E^\perp}\right) = 0$

Textbooks: Montgomery ('02), Bellaiche-Risler ('96), Rifford ('14), Jean ('14), Le Donne ('17), Agrachev-Barilari-Boscain ('19)..

sR Laplacian

Let $(X, E \subset TX, g^E)$ sR manifold.

$$\text{sR Laplacian :} \quad \Delta_{g^E,\mu} := \left(\nabla^{g^E}\right)^*_\mu \circ \nabla^{g^E}$$

sR gradient: $g^E\left(\nabla^{g^E} f, U\right) = U(f)$, $\forall U \in C^\infty(E)$.
μ =auxiliary density

Characteristic variety: $\Sigma = \left\{\sigma\left(\Delta_{g^E,\mu}\right) = 0\right\} = E^\perp \subset T^*X$.
(Hormander '67) E bracket generating $\implies$ $\Delta_{g^E,\mu}$ is hypoelliptic

Discrete spectrum: $\Delta_{g^E,\mu}\varphi_j = \lambda_j\varphi_j$, $\varphi_j \in L^2(X)$ on a compact manifold.
Spectral asymptotics questions: Weyl law, trace formula, propagation, ergodicity ...
(mostly open)

sR heat trace

Theorem (Ben Arous 1989, Léandre 1992...)

There exist $a_j(x) \in C^\infty(X)$, $j = 0, 1, \ldots,$

$$e^{-t\Delta_{g^E,\mu}}(x,x) \sim t^{-Q(x)/2}\left[\sum_{j=0}^{\infty} a_j(x)\, t^j\right].$$

The expansion is in general not uniform in $x \in X$. Does not yield trace asymptotics.

Theorem (Métivier '76)

If E is equiregular (i.e. dimensions in canonical flag constant)

$$N(\lambda) \sim \frac{\lambda^{Q/2}}{\Gamma(Q/2+1)} \int_X a_0.$$

Fefferman-Phong '81 (rough general estimates),

Colin de Verdière-Hillairet-Trélat (singular Weyl law, ongoing)
Martinet eg.: $N(\lambda) \sim c\lambda^2 \ln \lambda$.

Spectrum and dynamics

(X^3, E^2) 3D contact.

Theorem (Melrose '84)

$$N(\lambda) \sim \lambda^2 \left(\int \mu_{Popp} \right) + O\left(\lambda^{3/2}\right).$$

$$sing\ spt\left(tr\ e^{it\sqrt{\Delta_{g^E,\mu}}}\right) \subset \{0\} \cup \{lengths\ of\ (normal)\ geodesics\}$$

Spectrum and dynamics

(X^3, E^2) 3D contact.

Theorem (Melrose '84)

$$N(\lambda) \sim \lambda^2 \left(\int \mu_{Popp} \right) + O\left(\lambda^{3/2}\right).$$

$$sing\ spt\left(tr\ e^{it\sqrt{\Delta_{g^E,\mu}}} \right) \subset \{0\} \cup \{lengths\ of\ (normal)\ geodesics\}$$

Theorem (Colin de Verdière-Hillairet-Trélat '18)

If Reeb flow is ergodic. Then one has quantum ergodicity (QE).

Semiclassical analog:

trace of magnetic Dirac operator S. (CMP '14, APDE '17, GAFA '18) QE '20.

Spectrum and dynamics

$(X^4, E^3 = \ker a)$ 4D quasi-contact.
Abnormals (characteristics) are integral curves of $L^E = \ker(a \wedge da) \subset E$.
T^E =shortest closed period of L^E, Z =unit generator of L^E.

Theorem (S. '19)

$$\textit{sing spt}\left(tr\ e^{it\sqrt{\Delta_{g^E,\mu}}}\right) \subset \begin{array}{c}\{0\} \cup \{\textit{lengths of (normal) closed geodesics}\}\\ \cup\left(-\infty, -T^E_{abnormal}\right] \cup \left[T^E_{abnormal}, \infty\right)\end{array}$$

$$N(\lambda) \sim \lambda^{5/2}\left(\int \mu_{Popp}\right) + O\left(\lambda^2\right).$$

Union of L^E closed curves is of measure zero :

$$N(\lambda) \sim \lambda^{5/2}\left(\int \mu_{Popp}\right) + o\left(\lambda^2\right).$$

Spectrum and dynamics

$(X^4, E^3 = \ker a)$ 4D quasi-contact.
Abnormals (characteristics) are integral curves of $L^E = \ker(a \wedge da) \subset E$.
T^E =shortest closed period of L^E, Z =unit generator of L^E.

Theorem (S. '19)

$$\textit{sing spt}\left(tr\ e^{it\sqrt{\Delta_{g^E,\mu}}}\right) \subset \begin{array}{c}\{0\} \cup \{\textit{lengths of (normal) closed geodesics}\} \\ \cup \left(-\infty, -T^E_{abnormal}\right] \cup \left[T^E_{abnormal}, \infty\right)\end{array}$$

$$N(\lambda) \sim \lambda^{5/2}\left(\int \mu_{Popp}\right) + O\left(\lambda^2\right).$$

Union of L^E closed curves is of measure zero :

$$N(\lambda) \sim \lambda^{5/2}\left(\int \mu_{Popp}\right) + o\left(\lambda^2\right).$$

Theorem (S. '19)

If L^E ergodic & $L_Z \mu_{Popp} = 0$. Then one has quantum ergodicity (QE).

Circle bundles

Natural sR-structure: $\left(\underbrace{X^n}_{S^1 L}, \underbrace{E^{n-1}}_{HX \text{ horizontal}}, \underbrace{g^E}_{\pi^* g^{TY}}\right)$ with
$\left(L, h^L, \nabla^L\right) \to \left(Y^{n-1}, g^{TY}\right)$ is a Hermitian line bundle with connection.
Equivalently consider sR structure invariant by a free and transversal S^1 action.

Proposition: $\underbrace{r(x)}_{\text{step of } E} - 2 = \text{ord}\left(R^L_{\pi(x)}\right)$

E bracket generating $\iff R^L$ vanishes to finite order (i.e. $r = \max_{y \in Y} r_y < \infty$)

Decomposition by order: $Y = \cup_{j=2}^{r} Y_j$; $Y_j = \{y | r_y = j\}$

Bochner Laplacian

Fourier modes: $C^{\infty}(X) = \oplus_{k=-\infty}^{\infty} C^{\infty}\left(Y, L^k\right)$

$$\underbrace{\Delta_{g^E,\mu_X}}_{\text{sR Laplacian}} = \oplus_{k=-\infty}^{\infty} \underbrace{\left(\nabla^{L^k}\right)^* \nabla^{L^k}}_{=\Delta_k \text{ Bochner}}$$

sR heat kernel expansion analogously gives $e^{-\frac{t}{k^{2/r}}\Delta_k}(y,y)$, $k \to \infty$.

Theorem (Marinescu-S. '18)

The first eigenfunction/eigenvalue (ψ_0^k, λ_0^k) of the Bochner Laplacian Δ_k satisfy

$$\lambda_0^k \sim C k^{2/r}$$
$$\left|\psi_0^k(y)\right| = O\left(k^{-\infty}\right), \quad y \notin Y_r.$$

Generalizes:
R. Montgomery '95 (Martinet case $\dim Y = 2$, $r = 3$),
Subsequently several cases: see Helffer-Kordyukov '14, Raymond '17 (still quite restrictive)

Bergman kernel

Assume (Y, h^{TY}), (L, h^L) holomorphic Hermitian

Kodaira Laplacian: $\Box_k : \Omega^{0,*}(Y; L^k) \to \Omega^{0,*}(Y; L^{\otimes k})$.
Bergman projector: $\Pi_k : L^2(Y; L^k) \to \ker(\Box_k) = H^0(Y; L^{\otimes k})$

Assume R^L semipositive (i.e. $R^L(w, \bar{w}) \geq 0$, $w \in T^{1,0}Y$)
Lichnerowicz + Mckean-Singer: $\text{Spec}(\Box_k) \subset \{0\} \cup [c_1 k^{2/r} - c_2, \infty)$ in 2D
Local index theory of Bismut-Lebeau '91, Dai-Liu-Ma '06, Ma-Marinescu '07 gives

Theorem (Marinescu-S. '18)

For $dimY = 2$ & R^L semi-positive of finite order

$$\Pi_k(y) \sim k^{2/r_y} \left[\sum_{j=0}^{N} c_j(y) k^{-j/r_y} \right]$$

as $k \to \infty$, where $r_y - 2 = ord(R^L_y)$.

Tian '91, Catlin '97, Zelditch '99 (positive case);
R. Berman '09, Hsiao-Marinescu '14 (on positive part in some cases).

Szegő kernel

More generally, CR manifold $(X^3, T^{1,0}X \subset TX \otimes \mathbb{C})$
-non-degeneracy: $T^{1,0}X \cap \overline{T^{1,0}X} = \emptyset$
-codim: $\dim_{\mathbb{C}} T^{1,0}X = 1$
-integrability: $\left[T^{1,0}X, T^{1,0}X\right] \subset T^{1,0}X$ (in higher dim.)

Levi distribution: $HX = \mathrm{Re}\ \left[T^{1,0}X \oplus T^{0,1}X\right]$
Levi form: $\mathscr{L} \in (HX^*)^{\otimes 2} \otimes (T_xX/H_xX)$
$\mathscr{L}(U,V) := [[U,V]] \in T_xX/H_xX$

X is weakly/ strongly pseudoconvex $\iff \mathscr{L}(.,J.)$ is positive definite/semi-definite
X is finite type $\iff HX$ is bracket generating

Szegő kernel

Tangential CR operator: $\bar{\partial}_b : C^\infty(X) \to C^\infty\left(T^{0,1*}X\right)$,

$$\bar{\partial}_b = \pi^{T^{0,1*}X} \circ d$$

Szego kernel: $\Pi_{X,\mu} : L^2(X,\mu) \to \ker\left(\bar{\partial}_b\right)$

Theorem (Hsiao-S. '20)

$\left(X^3, T^{1,0}X\right)$ *weakly pseudoconvex, finite type. Assume* $\bar{\partial}_b$ *closed range. Then*

$$\Pi_{X,\mu}\left(x,x'\right) = \int_0^\infty dt\, e^{itx_3} a\left(x_1,x_2;t\right) + C^\infty(X)$$

$$a(x;t) \sim t^{\frac{2}{r}}\left[\sum_{j=0}^{\infty} t^{-\frac{1}{r}} a_j\left(t^{\frac{1}{r}}x\right)\right] \in S^{\frac{2}{r}}_{\frac{1}{r},cl}\left(\mathbb{R}_{x_1,x_2}\times\mathbb{R}_t\right)$$

in some local coord near $x' \in X$ *(fixed) with* $r = r\left(x'\right)$ *type/step of* x'.

Boutet de Monvel-Sjostrand '75 (strongly pseudoconvex),
Christ '89, McNeal '89, Nagel-Rosay-Stein-Wainger '89 (pointwise bounds)
Open in higher dimensions

Szegő kernel

Classical case, $X^3 = \partial D$, $D \subset \mathbb{C}^2$(smoothly bounded domain).
Boundary defining function: $D = \{\rho \leq 0\}$, $d\rho|_{\partial D} \neq 0$.

Bergman kernel: $\Pi_D : L^2(D) \to \ker(\bar{\partial})$

Theorem (Hsiao-S. '20)

$D \subset \mathbb{C}^2$ weakly pseudoconvex, finite type. Then

$$\Pi_D(z,z) \sim \sum_{j=0}^{\infty} \frac{1}{(-\rho)^{2+\frac{2}{r}-\frac{1}{r}j}} a_j + \sum_{j=0}^{\infty} b_j (-\rho)^j \log(-\rho)$$

as $z \to x'$ (fixed) with $r = r(x')$ type/step of x'.

Fefferman '74 (strongly pseudoconvex),
D'Angelo '78 (ellipsoids), Boas-Straube-Yu '95 (leading term in $\mathbb{C}^2$), Kamimoto '04 (toric domains).

Higher dim is open
Herbort '83: eg. $D \subset \mathbb{C}^3$ shows $c_{x'} \leq \frac{\Pi_D(z,z)}{(-\rho)^3 \log(-\rho)} \leq C_{x'}$ pointwise .

Thank you.