Introduction to AGT duality

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The aim of this master thesis is to introduce the AGT conjecture and sketch proofs in some special cases. The AGT conjecture relates two a priori different topics of physics to each other. These topics are two dimensional conformal field theory and $\mathcal{N} = 2$ supersymmetric Yang-Mills theory in four dimensions. This relation is somehow expected from the M-theory picture where both theories are limits of a M5 brane compactification. In the following, the two topics needed to understand the conjecture will be presented.

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1 Introduction

$\mathcal{N} = 2$ supersymmetric gauge theories are known as exactly solvable non-trivial quantum field theories which serve as a toy model to understand asymptotically free theories and gauge dualities. This theories are at the border of solvable and non-trivial quantum field theories and serve as a laboratory of gauge/string dualities. In 2009 Alday, Gaiotto and Tachikawa (AGT) established an extraordinary duality conjecture [1]. AGT supposed a dictionary between certain $\mathcal{N} = 2$ SCFT in four dimensions and between two dimensional Liouville theory [2]. The original work related the Nekrasov partion function [3] that encodes the non-perturbative effects of $\mathcal{N} = 2$ theories for SU(2) to the conformal blocks of a certain CFT - Liouville theory. This theory arises naturally, quantizing the bosonic string in non-critical dimensions. The Liouville theory can be quantized as a conformal field theory with continuous spectrum. In four dimensional gauge theories with extended Supersymmetry it is possible to compute the partition function of the theory as a path integral over the space of fields.This approach is applicable for every theory with a Lagrangian description. Gaiotto [4] argued that the boundary of moduli space of complex structures of Riemann surfaces is the same as the weakly coupled limit of the moduli space of gauge couplings where a Lagrangian description exists. Thus there is a map from a Riemann surface to a superconformal theory with two supersymmetries, which is a hint for the AGT duality. The dictionary of AGT was also generalized to higher rank groups. The main question is: Where does this duality come from? This thesis presents the intermediate steps to understand the duality conjecture. Firstly, common subjects like $\mathcal{N} = 2$ field theory in general and the solution via the Seiberg Witten approach will be discussed. In chapter 3 Liouville theory and to computations of correlator in the theory will be introduced. Next, in chapter 4 we will explain the direct integration method of Nekrasov to compute the instanton partition function via localization and see that the Nekrasov partition function reproduce the Seiberg-Witten prepotential in a certain limit. In Chapter 5 Gaiottos classification [4] of certain SCFTs in four dimensions will be explained using methods anticipated long time ago by Witten [5]. This will give a first hint to the AGT conjecture because of the deep relation between Riemann surfaces and SCFT in 4 dimensions. In chapter 6 the AGT conjecture is presented and it is shown that at $g = 0$ the conjecture is true by experimentally testing it. It turns out that even for asymptotically free theories the conjecture is true if one modifies the notion of conformal blocks to the irregular blocks. In chapter 7 we briefly want to present the idea of Wyllard to generalize the AGT duality conjecture to higher rank groups. It can be concluded that the AGT conjecture can naturally be understood in the context of topological strings and matrixmodels. This fact gives a possibility to prove the conjecture and this approach will be explained in chapter 8. The ninth chapter deals with speculations related to calculations that have been done during the thesis.

2 $\mathcal{N} = 2$ supersymmetric gauge theories and the Seiberg-Witten Solution

2.1 Fieldcontent of supersymmetric theories

2.1.1 Fieldcontent of $\mathcal{N} = 1$ SUSY

Firstly we want to review how to construct the field content of $\mathcal{N} = 1$ supersymmetric theories, before starting to construct higher SUSY theories. This is done by analysing the irreducible representations[1] of the SUSY algebra, which is the only possible extension of the Poincare algebra without violating the unitarity of the S-matrix. Therefore it is introduced a complex spinor denoted by Q_α and its conjugate $Q^\dagger_{\dot\alpha}$. These spinors are the generators of the new transformations that relate bosons and fermions which are called SUSY transformations. It is clear that the SUSY generator has to be a spinor from the assumption that a boson is transformed to a fermion. In the following the Poincare part is skipped to restrict the equation to the extension of the algebra:

$$\begin{aligned} \left\{Q_\alpha, Q^\dagger_{\dot\alpha}\right\} &= 2\sigma^\mu_{\alpha\dot\alpha}P_\mu \\ \left\{Q_\alpha, Q_\beta\right\} &= 0 \\ \left\{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\right\} &= 0 \end{aligned} \tag{1}$$

The algebra has a $U(1)$-symmetry, called R-symmetry. Obviously the vacuum is only supersymmetric if and only if the potential energy vanishes, otherwise SUSY is spontaneously broken. The algebra is the starting point to construct the field content. Here it is important to distinguish between massive and massless representations.

Massive particles are labeled by mass, spin and one component of spin s_3.Therefore, choosing $P^\mu = (M, 0, 0, 0)$ reduces the algebra to a Clifford algebra

$$\begin{aligned} \left\{Q_\alpha, Q^\dagger_{\dot\alpha}\right\} &= 2M\delta_{\alpha\dot\alpha} \\ \left\{Q_\alpha, Q_\beta\right\} &= 0 \\ \left\{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\right\} &= 0 \end{aligned} \tag{2}$$

Then define the "Clifford vacuum" of spin s to be the state $|\Omega_s\rangle$ that is annihilated by the SUSY generators.

$$Q_1|\Omega_s\rangle = Q_2|\Omega_s\rangle = 0 \tag{3}$$

So we can use Q_α as a lowering operator and $Q^\dagger_{\dot\alpha}$ as a raising operator. From this highest weight state we can now construct the entire representation of the $\mathcal{N} = 1$ algebra by acting with the conjugate spinor on the vacuum. The field content is:

$$|\Omega_s\rangle, Q^\dagger_1|\Omega_s\rangle, Q^\dagger_2|\Omega_s\rangle, Q^\dagger_1 Q^\dagger_2|\Omega_s\rangle \tag{4}$$

Choosing different values of the spin, we can construct different supermultiplets. The case where we do not want to create spin higher than 1, restricts the "Clifford vacua" to

[1] This is done by the Wigner method because the Poincare group is non-compact

spin 0 and $\frac{1}{2}$. For $s = 0$ we get a so-called chiral multiplet with the following field content:

state	s_3
$\vert\Omega_0\rangle$	0
$Q_1^\dagger\vert\Omega_0\rangle, Q_2^\dagger\vert\Omega_0\rangle$	$\frac{1}{2}$
$Q_1^\dagger Q_2^\dagger\vert\Omega_0\rangle$	0

This is a Majorana spinor and a complex scalar which is a scalar particle and its supersymmetric partner. Now assuming that the vacuum has spin $\frac{1}{2}$ the supermultiplet consists of these different fields:

state	s_3
$\vert\Omega_{\frac{1}{2}}\rangle$	$\pm\frac{1}{2}$
$Q_1^\dagger\vert\Omega_{\frac{1}{2}}\rangle, Q_2^\dagger\vert\Omega_{\frac{1}{2}}\rangle$	0,1,0, -1
$Q_1^\dagger Q_2^\dagger\vert\Omega_{\frac{1}{2}}\rangle$	$\pm\frac{1}{2}$

That are two Majorana fermions, a massive vector particle and a real scalar. A similar construction can be done for massless particles. We boost our reference frame to the following one: $P_\mu = (E, 0, 0, E)$ and analyse the SUSY algebra. The algebra reduces to:

$$\begin{aligned}\left\{Q_1, Q_1^\dagger\right\} &= 4E \\ \left\{Q_2, Q_2^\dagger\right\} &= 0 \\ \left\{Q_\alpha, Q_\alpha\right\} &= 0 \\ \left\{Q_{\dot\alpha}^\dagger, Q_{\dot\beta}^\dagger\right\} &= 0\end{aligned} \tag{5}$$

This is a Clifford algebra with one raising operator. For massless particles helicity is a good quantum number and we choose a Clifford vacuum of fixed helicity. It is defined as the state that is annihilated by one of the SUSY generators.

$$Q_1|\Omega_\lambda\rangle = 0 \tag{6}$$

Concerning the algebra and the anticommutation relation of Q_2, $Q_2^\dagger$ produces states of norm zero. From this analysis we now see that a massless supermultiplet has the following states:

state	helicity
$\vert\Omega_\lambda\rangle$	λ
$Q_1^\dagger\vert\Omega_\lambda\rangle$	$\lambda + \frac{1}{2}$

If we want to construct a CPT-invariant theory we have to add states of inverse helicity to avoid CPT violating terms. So there a two more states:

state	helicity
$\vert\Omega_{-\lambda-\frac{1}{2}}\rangle$	$-\lambda - \frac{1}{2}$
$Q_1^\dagger\vert\Omega_{-\lambda-\frac{1}{2}}\rangle$	$-\lambda$

Again we can choose different helicities and see the field content of a supermultiplet. Adding the CPT-conjugated states a supersymmetric CPT invariant theory with massless and massive particles can be constructed. For helicity 0 a Weyl fermion and a complex scalar is determined for the massless chiralmultiplet while a Weyl fermion and a massless vectorparticle[2] is calculated for helicity 1/2 called the vector multiplet
Here some problems are arising. Where are the supersymmetric particles? The analysis suggests that the particles and their superpartner have the same mass. Therefore it is clear that SUSY must be spontaneously broken at some energy scale higher than the observed scales.

2.1.2 Extended SUSY

One of our assumptions was, that there is one SUSY generator, but we can now easily generalize this to a set of generators Q_I^a with a $= 1, ..., \mathcal{N}$ that fulfills the extended SUSY algebra[3]:

$$\begin{aligned} \left\{Q^a_\alpha, Q^\dagger_{\dot{\alpha}b}\right\} &= 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^a_b \\ \left\{Q^a_\alpha, Q^b_\beta\right\} &= 0 \\ \left\{Q^\dagger_{a\dot{\alpha}}, Q^\dagger_{b\dot{\beta}}\right\} &= 0 \end{aligned} \tag{7}$$

The R-symmetry is now $U(\mathcal{N})_R$. We construct in the extended SUSY representations in the same manner as the representations for one supercharge starting with the massless representations $P_\mu = (E, 0, 0, E)$. By this assumption the algebra reduces to the following:

$$\begin{aligned} \left\{Q^a_1, Q^\dagger_{\dot{1}b}\right\} &= 4E\delta^a_b \\ \left\{Q^a_2, Q^\dagger_{\dot{2}b}\right\} &= 0 \end{aligned} \tag{8}$$

We see again, as in the $\mathcal{N} = 1$ case, that Q_{2b} produces states of norm zero for $\forall b = 1, ..., \mathcal{N}$.Analogous to the simpler case a Clifford vacuum of fixed helicity is defined as a state that is annihilated by a SUSY generator. We create new states by acting with the conjugate spinor on this Clifford vacua. The extended SUSY algebra will be degenerated in these different states. In general a massless supermultiplet of extended SUSY has the following field content with certain degeneracies:

state	helicity	degeneracy
$\vert\Omega_\lambda\rangle$	λ	1
$Q^\dagger_{1a}\vert\Omega_\lambda\rangle$	$\lambda + \frac{1}{2}$	$\mathcal{N}$
$Q^\dagger_{1a}Q^\dagger_{1b}\vert\Omega_\lambda\rangle$	$\lambda + 1$	$\mathcal{N}(\mathcal{N}-1)/2$
.	.	.
.	.	.
$Q^\dagger_{11}Q^\dagger_{12}Q^\dagger_{1\mathcal{N}}\vert\Omega_\lambda$	$\lambda + \mathcal{N}/2$	1

[2]Gauge particle
[3]There are also $\mathcal{N}$ conjugated complex spinors

In this thesis we will not consider general supersymmetry. Instead we will focus on $\mathcal{N}=2$ SUSY. The field content of $\mathcal{N}=2$ SUSY consists of a massless vectormultiplet: [4]

state	helicity	degeneracy
$\vert\Omega_{-1}\rangle$	-1	1
$Q_1^\dagger\vert\Omega_{-1}\rangle$	$-\frac{1}{2}$	2
$Q_{11}^\dagger Q_{12}^\dagger\vert\Omega_{-1}\rangle$	0	1

Nevertheless we have also to add the CPT-conjugate states. Finally we end up with the following field content: two Majorana spinors, one complex scalar and one vectorparticle. Another clue is that the $\mathcal{N}=2$ vectormultiplet can be built out of a chiral multiplet and a vector multiplet in the $\mathcal{N}=1$ language. We want to move further and study another supermultiplet built from another Clifford vacuum. For $\lambda=1/2$ we get the hypermultiplet. The hypermultiplet consists of two Majorana spinors and two complex scalars. This is CPT invariant for real representations of a gauge group. From this hypermultiplet we can already deduce that something can not be correct. Take some gauge group G. From the two Majorana spinors we can construct mass terms for the fermions that are gauge invariant. Theories with this property are called vector-like theories. In our standard model description of particle physics particles become massive via the higgs mechanism, because the gauge symmetry does not allow to write massterms. Corresponding theories are called chiral-like theories. So somehow $\mathcal{N}=2$ does not seem to be something realized in our physical world, but it is a huge laboratory to learn about string and gauge dualities and even at this stage new interesting phenomena will appear.

2.1.3 Central charge

The algebra for the extended supersymmetry is not the most general possibility for the extension of the Poincare algebra. If we define the SUSY charges as the integral over spacetime of the zero component of the supercurrent (like the definition of charge implies) and calculate the anticommutator we normally neglect boundary terms. Nevertheless in general there can be some source on the boundary of spacetime that contributes to the anticommutator and so we have to modify our SUSY algebra. We will directly restrict to $\mathcal{N}=2$. The new algebra reads:

$$\begin{aligned}\left\{Q^a_\alpha, Q^\dagger_{\dot{\alpha}b}\right\} &= 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu\delta^a_b \\ \left\{Q^a_\alpha, Q^b_\beta\right\} &= 2\sqrt{2}\epsilon_{\alpha\beta}\epsilon^{ab}Z \\ \left\{Q^\dagger_{a\dot{\alpha}}, Q^\dagger_{b\dot{\beta}}\right\} &= 2\sqrt{2}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{ab}Z\end{aligned} \tag{9}$$

Here it is $a,b=1,2$ and the number of components of the SUSY charges depends on the spacetime dimensions. The R-symmetry will be broken by this central charge Z. We introduce a new operator built out of the SUSY charges, to gain more information

[4] we just want particles with helicity smaller or equal than 1

about the representations of the SUSY algebra with central charge:

$$\tilde{Q}_\alpha = \frac{1}{2}\left[Q^1_\alpha + \epsilon_{\alpha\beta}\left(Q^1_\beta\right)^\dagger\right] \quad (10)$$

$$\tilde{\tilde{Q}}_\alpha = \frac{1}{2}\left[Q^1_\alpha - \epsilon_{\alpha\beta}\left(Q^1_\beta\right)^\dagger\right] \quad (11)$$

Taking this new definitions into account it is possible to calculate the new anticommutation relations. Using the following identity for the epsilon matrices: $\epsilon_{\alpha\beta}\epsilon^\dagger_{\beta\gamma} = \delta_{\alpha\gamma}$ you get for the restframe by using equation (9)

$$\{\tilde{Q}_\alpha, \tilde{Q}_\beta{}^\dagger\} = \delta_{\alpha\beta}\left(M + \sqrt{2}Z\right) \quad (12)$$

$$\{\tilde{\tilde{Q}}_\alpha,\ \tilde{\tilde{Q}}^\dagger_\beta\} = \delta_{\alpha\beta}\left(M - \sqrt{2}Z\right) \quad (13)$$

$$(14)$$

We now do a simple calculation with this new algebra. We introduce a state $|M, Z\rangle$ that is characterized by the mass and the central charge. We assume that the norm of the state is normalized to unity and consider that we are in a unitary theory, where the norm of states induces a semidefinite innerproduct.

$$\begin{aligned} 0 &\leq \langle M, Z|\tilde{\tilde{Q}}_\alpha \tilde{\tilde{Q}}^\dagger_\alpha|M, Z\rangle + \langle M, Z|\tilde{\tilde{Q}}^\dagger_\alpha \tilde{\tilde{Q}}_\alpha|M, Z\rangle \\ &= \langle M, Z|\{\tilde{\tilde{Q}}_\alpha \tilde{\tilde{Q}}^\dagger_\alpha\}|M, Z\rangle \\ &= M - \sqrt{2}Z \end{aligned}$$

From this simple calculation it can be deduced a dependence of the central charge and the mass of supermultiplet

$$\Rightarrow M - \sqrt{2}Z \geq 0 \quad (15)$$

A direct consequence is that if we tune the mass to zero, that the central charge will also tend to zero. A BPS-state is defined to be a state that fulfills the following equality[5]:

$$M - |Z| = 0 \quad (16)$$

Taking this condition into account it becomes clear that $\tilde{\tilde{Q}}_\alpha$ produces zero norm states or in other words, the BPS-states are annihilated by half of the SUSY generators. So in fact it is a massive multiplet that looks like a $\mathcal{N} = 2$ massless multiplet or $\mathcal{N} = 1$ massive supermultiplet.The number of states in this multiplet is extremely reduced. This is the reason why we call the supermultiplets that fulfill the BPS-bound "short multiplets" and the other "long multiplets". For example a short chiral multiplet of $\mathcal{N} = 2$ has 4 states, but a long chiral multiplet has 16 states! Quantum corrections can not produce the missing degrees of freedom in the multiplets, so the BPS-bound is protected from quantum corrections, even from non-perturbative corrections.

[5]By rescaling (10) (11) we can bring the BPS bound to the form (16)

2.2 Some facts about the $\mathcal{N}=2$ Lagrangian

2.2.1 $\mathcal{N}=2$ Lagrangian from $\mathcal{N}=1$ Lagrangian

We have studied the field content of extended supersymmetric theories and now we have to construct Lagrangians for this theory that respects the symmetries of the algebra and of some gauge group following [8]. We extend our fourdimensional spacetime to a superspace by introducing a non-commutative coordinate, strictly speaking a spinor θ^α and its conjugate $\bar{\theta}_{\dot{\alpha}}$ The index structure is lowered and raised with the ϵ-tensor. The transformations generated by the SUSY algebra are the following:

$$\begin{aligned} x^\mu \rightarrow x'^\mu &= x^\mu + i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta} \\ \theta \rightarrow \theta' &= \theta + \xi \\ \bar{\theta} \rightarrow \bar{\theta}' &= \bar{\theta} + \bar{\xi} \end{aligned} \tag{17}$$

Here ξ is a transformation parameter of infinitesimal SUSY transformation that is implemented by the operator $\xi^\alpha Q_\alpha + \bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}$ and by comparing the transformations we can deduce a local representation of the SUSY charges as a differential operator acting on the superspace:

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \tag{18}$$

There is also a similar expression for the conjugate supercharge. We also introduce a super covariant derivative that commutes with the SUSY charges.

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\theta^{\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \tag{19}$$

The main idea is to introduce superfields. These are fields expanded in the grassmann coordinates. This expansions are exact due to anticommutation of the superspace extension. In general the superfield can be written as:

$$F(x,\theta,\bar{\theta}) = f(x) + \theta\phi(x) + \bar{\theta}\bar{\xi}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) \tag{20}$$
$$+ \theta\theta\bar{\theta}\bar{\lambda}(x) + \theta\bar{\theta}\bar{\theta}\psi(x) + \theta\theta\bar{\theta}\bar{\theta}d(x) \tag{21}$$

This formalism allows to expand the supersymmetric variations in simple calculations because we define: $\delta F = \left[\xi Q + \bar{\xi}\bar{Q}\right] F$. We can read off the different variations of the different superfield components. Another very important observation is that the highest component transforms into a total derivative. So the first guess how to construct susy invariant theories is as the highest component of some special superfield, since we want to construct the field content of the representations constructed in the first subchapter. Therefore we have to find the correct constraints on the superfield. We need two different constraints to obtain the two different supermultiplets. The first one is as follows:

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{22}$$

This is called a chiral superfield but may also be described as a left chiral superfield. The direct conclusion is that it can be expanded[6]:

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \tag{23}$$

Here $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$. The different components consist of a scalar and a fermion and some auxiliary field which is also needed for an off shell formalism because otherwise the fermionic and the bosonic degrees do not match. We can also introduce a antichiral superfield: $D_\alpha\Phi^\dagger = 0$ that now depends on $(y^\dagger, \bar{\theta})$ and is similarly expanded as the chiral superfield. In general any arbitrary function of chiral or anti chiral superfields is a chiral or antichiral superfield. The grassmann formalism allows to expand arbitrary functions exactly. So for Φ chiral superfield we get by Taylor expansion and the anticommutativity of the grassmann numbers[7]:

$$W(\Phi) = W(\phi + \sqrt{2}\theta\psi + \theta\theta F) \tag{24}$$

$$= W(\phi) + \frac{\partial W}{\partial\phi}2\sqrt{2}\theta\psi + \theta\theta\left(\frac{\partial W}{\partial\phi}F - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^2}\psi^2\right) \tag{25}$$

Or for many fields:

$$W(\Phi_i) = W(\phi_i + \sqrt{2}\theta\psi_i + \theta\theta F_i) \tag{26}$$

$$= W(\phi_i) + \frac{\partial W}{\partial\phi_i}2\sqrt{2}\theta\psi_i + \theta\theta\left(\frac{\partial W}{\partial\phi_i}F - \frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\phi_j}\psi_i\psi_j\right) \tag{27}$$

This function is called superpotential. We now turn to the other supermultiplet and thereby introduce another constrain. This new superfield should be real, i.e. $V = V^\dagger$ and can be written in the component fields as:

$$V(x,\theta,\bar{\theta}) = C + i\theta\chi - i\bar{\theta}\bar{\chi} + \frac{i}{2}\theta^2\left(M + iN\right) - \frac{i}{2}\bar{\theta}^2(M - iN) - \theta\sigma^\mu\bar{\theta}A_\mu +$$
$$i\theta^2\bar{\theta}(\bar{\lambda} + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi) - i\bar{\theta}^2\theta(\lambda + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}) + \frac{1}{2}\theta^2\bar{\theta}^2(D - \frac{1}{2}\Box C) \tag{28}$$

Nevertheless fortunately not every component is physical. We can get rid of many components by an Abelian gauge transformation: $V \to V + \Lambda + \Lambda^\dagger$ with Λ chiral superfield and $\Lambda^\dagger$ anti chiral superfield. This means: $C = M = N = \chi = 0$ .This gauge is called Wess Zumino gauge. In this gauge SUSY is not manifest, but we have a gauge symmetry of the Abelian gauge field. In fact, including the gauge transformation, the subsequent superfield is gained:

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D \tag{29}$$

[6]This is one possibility. One could also introduce two different subalgebra. In that case the representation of SUSY charges becomes easier, but since the chiral fields are in different representations we must do some manipulations to multiply them

[7]We get terms proportional to θ^3 and θ^4 but they vanish identically

The vector superfield consists of a vector boson and its superpartner the gaugino λ. Further there is an auxiliary field to have an off shell formalism. From this superfield we can define an Abelian field strength that is a chiral superfield and can be computed in the Wess Zumino gauge:

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V \ , \bar{W}_{\dot{\alpha}} = -\frac{1}{4}D^2 \bar{D}_{\dot{\alpha}} V$$

By expanding this definition in the Wess Zumino gauge it is noticable that the usual Abelian field strength is a component field of the SUSY Abelian field strength.
Now we should extend our definitions to the non-Abelian case. The vector superfield is in the adjoint representation of the gauge group. The gauge transformations are now:

$$e^{-2V} \rightarrow e^{-i\Lambda^\dagger} e^{-2V} e^{i\Lambda}, \text{with}, \Lambda = \Lambda_A T^A \tag{30}$$

These T^A are the gauge group generators. From this we can define the non-Abelian field strength

$$W_\alpha = \frac{1}{8}\bar{D}^2 e^{2V} D_\alpha e^{-2V} \tag{31}$$

that transforms in the following way

$$W_\alpha \rightarrow W'_\alpha = e^{-i\Lambda} W_\alpha e^{i\Lambda}$$

We can expand this in components and we get the following form:

$$W_\alpha = T^a \left(-i\lambda^a_\alpha + \theta_\alpha D^a - \frac{i}{2}(\sigma^\mu \bar{\sigma}^\nu \theta)_\alpha F^a_{\mu\nu} + \theta^2 \sigma^\nu D_\mu \bar{\lambda}^a \right) \tag{32}$$

Here we have:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \ \ , D_\mu \bar{\lambda}^a = \partial_\mu \bar{\lambda}^a + f^{abc} A^b_\mu \bar{\lambda}^c$$

We have constructed SUSY non-Abelian field strength and certain superfields with the components we were searching for. Now we are in the position to write down SUSY Lagrangians as the highest component of special combinations of superfields. We multiply a chiral superfield with an anti chiral superfield, take the highest component, drop sum derivatives and sum over all possible fields: This is a free field Lagrangian:

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \Phi^\dagger_i \Phi_i = \partial_\mu \phi^\dagger_i \partial^\mu A_i + F^\dagger_i F_i - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi_i \tag{33}$$

Apparently this is a free field Lagrangian for a massless scalar and a massless fermion. The auxiliary field can be eliminated on shell because the equation of motion is just algebraic. There could also be a more general form of this free field Lagrangian with a nontrivial metric on the space of fields. Therefore it is possible to write a general function called Kaehler potential which induces locally a metric. This comes from the notion of Kaehler manifolds where a closed 2-form exists which restricts the metric locally to be the second mixed derivative of some scalar function called Kaehler potential.

The interaction terms can be built out of the superpotential, which has to be holomorphic or antiholomorphic. So the most general Lagrangian is

$$\mathcal{L} = \int d^4\theta K(\Phi^\dagger, \Phi) + \int d^2\theta W(\Phi) + \int d^2\bar{\theta}\bar{W}(\Phi^\dagger) \tag{34}$$

The other chiral superfield combination is just the highest component of the contraction of the Abelian field strength and in the non-Abelian case we have to take the trace to sum over the lie algebra index of the gauge group. So we have

$$\mathrm{Tr}\int d^2\theta W^\alpha W_\alpha = -2i\lambda^a\sigma^\mu D_\mu\bar{\lambda}^a + D^a D^a - \frac{1}{2}F^{a\mu\nu}F^a_{\mu\nu} + \frac{i}{4}\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \tag{35}$$

Due to this we know now how to construct the Super Yang Mills Lagrangian with theta term coming from the topological charge, This theta term is a possible explanation for CP-violation in non-supersymmetric QCD. Upon introduction of the constant chiral field $\tau = \theta/2\pi + 4\pi i/g^2$, it is easy to see how to introduce the theta term.

$$\mathcal{L} = \frac{1}{8\pi}\mathrm{Im}\left(\tau\mathrm{Tr}\int d^2\theta W^\alpha W_\alpha\right) \tag{36}$$

The last question is how to built a gauge invariant kinetic term. This is done by extending the Lagrangian built out of the chiral and anti chiral superfields to a locally gauge invariant term. This is the Kaehler potential. In the renormalisable case it is simply $\Phi^\dagger e^{-2V}\Phi$, but it may have a much more complicated form when renormalisability is not a criterion. In the end we can list the entire Yang-Mills Lagrangian with $\mathcal{N} = 1$ supersymmetry.

$$\mathcal{L} = \frac{1}{8\pi}\mathrm{Im}\left(\tau\mathrm{Tr}\int d^2\theta W^\alpha W_\alpha\right) + \int d^4\theta\Phi^\dagger e^{-2V}\Phi + \int d^2\theta W(\Phi) + \int d^2\bar{\theta}\bar{W}(\Phi^\dagger) \tag{37}$$

Now it is possible to expand the Lagrangian into the different field in the supermultiplets. Every term is $\mathcal{N} = 1$ Susy invariant so the whole Lagrangian is invariant under SUSY, but the normalization between the scalar part and the Yang-Mills part is not fixed by SUSY. We set the scalar normalization part to one and thus obtain the general super Yang-Mills action action in components:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \frac{\theta}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu} - \frac{i}{g^2}\lambda^a\sigma^\mu D_\mu\bar{\lambda}^a + \frac{1}{2g^2}D^a D^a \\ &+ (\partial_\mu\phi - iA^a_\mu T^a\phi)^\dagger(\partial^\mu\phi - iA^{a\mu}T^a\phi) - i\bar{\psi}\bar{\sigma}^\mu(\partial_\mu\psi - iA^a_\mu T^a\psi) \\ &- D^a\phi^\dagger T^a\phi - i\sqrt{2}\phi^\dagger T^a\lambda^a\psi + i\sqrt{2}\bar{\psi}T^a\phi\bar{\lambda}^a + F_i^\dagger F_i \\ &+ \frac{\partial W}{\phi_i}F_i + \frac{\partial\bar{W}}{\phi_i^\dagger}F_i^\dagger - \frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\phi_j}\psi_i\psi_j - \frac{1}{2}\frac{\partial^2\bar{W}}{\partial\phi_i^\dagger\phi_j^\dagger}\bar{\psi}_i\bar{\psi}_j\end{aligned} \tag{38}$$

The next question is what could be the $\mathcal{N} = 2$ super Yang Mills action. One interesting point has already been discussed in the representation theory of the extended SUSY algebra. The on-shell $\mathcal{N} = 2$ vector multiplet can be build out of a chiral and a vector multiplet in $\mathcal{N} = 1$ Susy. The fields needed are already in (38), however the question is which constraints come from extended supersymmetry? With the fields now being in the same multiplet these have to be in the same representation of the gauge group. So there are as many fields as the rank of the gauge group. Moreover the superpotential vanishes in $\mathcal{N} = 2$ SUSY because only one fermionic component couples to the superpotential while the other does not. However these two fermionic components that appear now in the $\mathcal{N} = 2$ vector multiplet are on an equal footing and this restricts the super potential to zero.This is the reason why both fermionic kinetic terms should have the same normalization, which is achieved by rescaling the chiral superfield $\Phi \to \Phi/g$. The new Lagrangian has $\mathcal{N} = 2$ supersymmetry.

$$\mathcal{L}_{\mathcal{N}=2} = \frac{1}{8\pi}\mathrm{ImTr}\left[\tau\left(\int d^2\theta W^\alpha W_\alpha + 2\int d^2\theta d^2\bar{\theta}\Phi^\dagger e^{-2V}\Phi\right)\right] \tag{39}$$

with the following scalar potential after eliminating the D-term and F-term:

$$V = -\frac{1}{2g^2}\mathrm{Tr}\left([\phi^\dagger, \phi]^2\right) \tag{40}$$

2.2.2 The $\mathcal{N} = 2$ superspace formalism

Another approach to construct Lagrangians for $\mathcal{N} = 2$ supersymmetric gauge theories is by making some constraints on $\mathcal{N} = 2$ Superfields that propagate in the extended superspace. In this extendend superspace there are two grassman coordinates instead of one. The $\mathcal{N} = 2$ superfield can be written as a function of of the following variables: $F = F(x, \theta, \bar{\theta}, \tilde{\theta}, \bar{\tilde{\theta}})$ and formulating some constraints reduces the field content to a $\mathcal{N} = 2$ vector supermultiplet. Therefore we introduce two super covariant derivatives with respect to the different grassmann coordinates and make the following constraints:

$$\bar{D}_{\dot{\alpha}}\Psi = \bar{\tilde{D}}_{\dot{\alpha}}\Psi = 0 \tag{41}$$

We expand the function w.r.t. one variable:

$$\Psi = \Psi^{(1)}(\tilde{y}, \theta) + \sqrt{2}\tilde{\theta}^\alpha\Psi^{(2)}_\alpha(\tilde{y}, \theta) + \tilde{\theta}^\alpha\tilde{\theta}_\alpha\Psi^{(3)}(\tilde{y}, \theta) \tag{42}$$

Here is $\tilde{y} = x^\mu + i\theta\sigma^\mu\bar{\theta} + i\tilde{\theta}\sigma^\mu\bar{\tilde{\theta}}$. As a result of the $\mathcal{N} = 1$ analysis and the constraints on the superfields $\Psi^{(1)}$ has to be the $\mathcal{N} = 1$ chiral superfield and $\Psi^{(2)}$ has to be the $\mathcal{N} = 1$ field strength. The last term is:

$$\Psi^{(3)} = -\frac{1}{2}\int d^2\bar{\theta}\left[\Phi(\tilde{y} - i\theta\sigma\bar{\theta}, \theta, \bar{\theta})^\dagger\right]\exp\left[-2gV(\tilde{y} - i\theta\sigma\bar{\theta}, \theta, \bar{\theta})\right] \tag{43}$$

This is just the highest component evaluated at fixed $\tilde{y}$. Now it is possible to state that the action in $\mathcal{N} = 2$ language is

$$S = \mathrm{Im}\left[\frac{\tau}{32\pi}\int d^4x d^2\theta d^2\tilde{\theta}\Psi^2\right] \tag{44}$$

If we integrate out the second grassman coordinate we will regain the same action as constructed from $\mathcal{N}=1$ language by setting the superpotential to zero and rescaling the chiral superfield. The quadratic dependence of the Lagrangian is due to renormalizability of the action, but if we study effective actions this is no longer a criterion. In this case the Lagrangian is controlled by a function called prepotential $\mathcal{F}$ which is holomorphic because it is integrated over half of the superspace. In conclusion the effective action can have a more complicated form than this renormalisable Yang Mills action.
Here it is important to mention that there are two different effective actions - the 1PI action and the Wilsonian action. It is necessary to briefly review the difference. The 1PI effective action is just the Legendre transformation of the vacuum energy as a function of an external source. It is called 1PI effective action because it can be shown that this effective action is the generating functional of the one-particle irreducible (1PI) correlation functions. It is possible to reconstruct the information of a quantum field system after spontaneous breakdown of some symmetry. Another means of the effective action is the Wilsonian action which is achieved by integrating out all massive modes down to some scale and all momenta above this scale. For a theory witout interacting massless particles the two effective actions are the same but in case of gauge theory, where massless gauge particles exists they differ. The 1PI effective action does not depend holomorphically on the cut off scale while the Wilsonian action does. In general we write:

$$S=\mathrm{Im}\left[\frac{\tau}{4\pi}\int d^4x d^2\theta d^2\tilde{\theta}\mathcal{F}\right] \tag{45}$$

where $\mathcal{F}$ can be any holomorphic function.n this case the central charge will have the following form:

$$Z=an_e+a_Dn_m \tag{46}$$

This formula can be explained via duality. a is the vev of the scalar in the vectormultiplet. We will see that there is a dual description of the effective theory. In this dual description electrically charged objects are replaced by magnetic monopoles. In the dual description n_e and n_m[8] are interchanged and also a and a_D where a_D is the vev of the dual higgs field characterizing a monopole. The BPS bound of a monopole can be calculated explicitly to be a_Dn_m. We will see that the duality group is $SL(2,\mathbb{Z})$ and (a_D,a) transform as a doublet. This explains (46).

2.2.3 Coupling gauge theory to matter

This chapter will briefly review how to include matter into the gauge theory. Matter comes in a different representation than the gaugefield so it can not be in the vector multiplet. So matter has to appear in hypermultiplets. We will directly formulate everything in $\mathcal{N}=1$ language, because then a hypermultiplet contains a chiral field Q and an anti chiral superfield $\tilde{Q}^\dagger$ transforming under complex conjugated representations of the gauge group. We can directly write down the Lagrangian for N_f hypermultiplets

[8] n_e and n_m are the charge multiplicities of electric and magnetic interaction

interacting with an $\mathcal{N}=2$ vector multiplet:

$$\mathcal{L} = \mathcal{L}_{pureSYM} + \int d^4\theta (Q_i^\dagger e^{-2V} Q_i + \tilde{Q}_i e^{2V} \tilde{Q}_i^\dagger) + \int d^2\theta (\sqrt{2}\tilde{Q}_i \Phi Q_i + m_i \tilde{Q}_i Q_i) \qquad (47)$$

Here we have a modification of the Kaehler potential by the appearance of more chiral multiplets and a superpotential. If all m_i are equal we will have a flavor symmetry. From this new terms we can also calculate the modification of the central charge. There will be an non-homogenous term that appears as:

$$Z = a n_e + a_D n_m + \sum_i \frac{1}{\sqrt{2}} m_i S_i \qquad (48)$$

Hence S_i depend only on the field content of the chiral fields that come into play by adding hypermultiplets to the pure gauge theory. The S_i are the flavor charges of the the unbroken flavor symmetry or for different bare masses the flavor charge of the broken flavor symmetry which is $U(1)^{N_f}$.

2.3 Seiberg-Witten Solution for pure $\mathcal{N}=2$ SU(2) gauge theory

The following subsection will analyze the low energy effective action starting with the classical action with no matter and gauge group $G = SU(2)$ [6]. From this it is possible to generalize to matter theories. First we have to study the moduli space of inequivalent vacua to get a physical theory. Then we compute the low energy effective theory and see that there are inconsistencies where electric-magnetic duality will help us to overcome these inconsistencies. We will see that there are monodromies on the quantum moduli space of vacua. These come from singularities where the Wilsonian description breaks down. Analyzing these monodromies it is possible to deduce that solving the theory is equivalent to find a certain section of a principal G-bundle where G is the group generated by the monodromies.Due to this analyzis we see that there is an elliptic curve that encodes the spectrum and the masses of the BPS states and the gauge couplings called the Seiberg-Witten curve.

2.3.1 The moduli space of vacua

We know that Susy is unbroken if the scalar potential vanishes, so we have to study the possible solutions of the following equation

$$V = -\frac{1}{2g^2}[\phi, \phi^\dagger] = 0 \rightarrow [\phi, \phi^\dagger] = 0 \qquad (49)$$

We see that the solutions of this equation take values in the Cartan subalgebra of the gauge group. The gauge group is broken to G/H. We can parameterize the solutions of this equation by $\phi = \frac{1}{2} a \sigma_3$ where a is some complex number and the vev of the higgs field. There are still Weyl reflections changing $a \rightarrow -a$, so the gauge invariant quantity parameterizing the moduli space of vacua is $\frac{1}{2}a^2$ or $\mathrm{tr}\phi^2$, which is the same

semiclassically, but it will not be the same in quantum theory.
For the quantum theory we will define the following quantities:

$$u = \left\langle \mathrm{tr}\phi^2 \right\rangle \ , \langle \phi \rangle = \frac{1}{2} a\sigma_3 \tag{50}$$

u is a complex number parameterizing the gauge inequivalent vacua. So u is a coordinate on the moduli space of the $\mathcal{N} = 2$ pure super Yang Mills theory called $\mathcal{M}$. $\mathcal{M}$ is one dimensional complex Kaehler manifold and it is essentially a plane that has certain singularities. These singularities and the behavior near this points will allow to determine the effective action. For non vanishing $\langle \phi \rangle$ it is known that the SU(2) gauge symmetry is broken down to U(1) by the higgs mechanism. So we have to study an U(1) effective theory. We will briefly discuss the relations for higher rank gauge groups later.

2.3.2 R-symmetry breaking

In the extended SUSY algebra we have a $U(\mathcal{N})_{\mathcal{R}}$ symmetry. To briefly repeat how R-symmetry acts on the multiplets of $\mathcal{N} = 2$ theory it is valuable to organize the field content as follows:

$$\begin{array}{ccc} & A_\mu & \\ \lambda & & \psi \\ & \phi & \end{array}$$

This is the vector multiplet we introduced already. Here we have a $U(2)_{\mathcal{R}} = SU(2)_{\mathcal{R}} \times U(1)_{\mathcal{R}}$ symmetry. The $SU(2)$ acts on the rows by rotating the fermions in the table. The gauge field and the scalar are invariant. For the hypermultiplet[9] the transformation acts in the same way as the scalars and leaves the fermions invariant.

$$\begin{array}{ccc} & \psi_q & \\ \mathrm{q} & & \tilde{\psi}_q^\dagger \\ & \tilde{q}^\dagger & \end{array}$$

In the $\mathcal{N} = 1$ decomposition this gives rise to a coupling of the hypermultiplet to the vectormultiplet.

$$\mathcal{W} = \sqrt{2}\tilde{Q}\Phi Q \tag{51}$$

The surviving $\mathcal{R}$-symmetries in the $\mathcal{N} = 1$ language are broken by a chiral anomaly. This can be seen by constructing a Dirac spinor out of the Weyl fermions in every multiplet which is invariant under $\mathcal{R}$-symmetry and that looks like a chiral transformation. This invariance is broken in the quantum theory and so we have a chiral anomaly in the theory that is the breakdown of the $\mathcal{R}$-symmetry. As we will explain, $U(1)_{\mathcal{R}}$ is broken to $\mathbb{Z}_{4N_c-2N_f}$. There is also a redundant symmetry that has to be projected out and the discrete group is broken further by the higgs vacuum. In the case of $SU(2)$ gauge group we have the following breakdown in the end:

$$SU(2)_{\mathcal{R}} \times U(1)_{\mathcal{R}} \rightarrow (SU(2)_{\mathcal{R}} \times \mathbb{Z}_4)/\mathbb{Z}_2 \tag{52}$$

[9]In the $\mathcal{N} = 1$ language one can write for a $\mathcal{N} = 2$ hypermultiplet a chiral and a anti chiral field Q and $\tilde{Q}$

2.3.3 The low energy effective action

So lets start to discuss the low energy effective action that controls the infrared behavior of the theory, where A is the chiral superfield and $\mathcal{F}$ is the prepotential. The Lagrangian (45) in the $\mathcal{N} = 1$ language is:

$$\mathcal{L}_{eff} = \frac{1}{4\pi}\mathrm{Im}\left[\int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A}\bar{A} + \int d^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial A^2} W^\alpha W_\alpha\right] \tag{53}$$

Here we directly see that we have a Kaehler potential inducing a metric on the space of fields. We will denote the scalar component of the chiral superfield with a.

$$K = \mathrm{Im}\left(\frac{\partial \mathcal{F}(A)}{\partial A}\bar{A}\right) \rightarrow ds^2 = \mathrm{Im}\frac{\partial^2 \mathcal{F}}{\partial a^2} da d\bar{a} \tag{54}$$

This equation (53) shows that the gauge coupling for the field strength term is the same as the metric. By constructing a metric on the moduli space of vacua, we also gain the gauge couplings of the theory. The aim is to construct the prepotential $\mathcal{F}$ beginning with the 1-loop corrections. Therefore it is necessary to explain that there are no higher order corrections due to renormalization theorems which state that the one loop expansion is exact in perturbation theory. However there will be instantons that will contribute to the prepotential. The 1-loop prepotential can be deduced in two different ways. One way is to integrate the β-function with certain integrations limits. Another approach is to use the anomaly that breaks R-symmetry. As already stated before this is a chiral anomaly and so we know how the current has to look like. In this special case for $SU(N_c)$ we know that:

$$\partial_\mu J^\mu = -\frac{N_c}{8\pi^2} F_{\mu\nu} F^{\mu\nu} \tag{55}$$

So we know how the Lagrangian changes under an anomalous R-transformation. This is a one-loop effect. (For N_f fermions in the fundamental representation)

$$\delta \mathcal{L}_{eff} = -\frac{\alpha(N_c - N_f/2)}{8\pi^2} F_{\mu\nu} F^{\mu\nu} \tag{56}$$

From this we see that the $U(1)_\mathcal{R}$ is broken down to $Z_{4N_c - 2N_f}$. This can easily be seen by noting that $(32\pi^2)^{-1}\int F\tilde{F}$ is integer valued. It is obvious that the variation of the Lagrangian and the topological charge differ from a multiplicity of four so we get this R-symmetry breaking. Regarding the term it is clear that we can see this as a θ-angle shifting and therefore we set θ to zero.
We can deduce the 1-loop prepotential by performing an R-symmetry transformation with the relevant terms. Then we assume that the Lagrangian changes as in (56). From this we get:

$$\mathcal{F}''(e^{2i\alpha}A) = \mathcal{F}'' - \frac{2\alpha N_c}{\pi} \tag{57}$$

We expand this equation for infinitesimal α and integrate the expression resulting in:

$$\mathcal{F}_{1-loop} = \frac{i}{2\pi} A^2 \ln \frac{A^2}{\Lambda^2} \tag{58}$$

Λ is a dynamically generated scale that comes from the integration of the prepotential and is the same as the dynamically generated scale in quantum chromodynamics. In perturbation theory this prepotential is exact. Next, the reason for the absence of higher order terms will be discussed. This argument is due to Seiberg [9].
We know that the effective action is gauge invariant under $U(1)$. This confines the perturbative prepotential to the following form:

$$\mathcal{F}_{eff} = (1/8g^2)\Phi^2[1 + A_2\log(\Phi^2/\Lambda^2)] \tag{59}$$

The Lagrangian itself is not $U(1)$ gauge invariant because varying the Lagrangian under a $U(1)$ gauge transfomration results in topological terms which change the action, but not the equation of motion. We get:

$$\delta\mathcal{L}_{eff} = -\alpha(A_2/g^2)F_{\mu\nu}F^{\mu\nu} \tag{60}$$

So we have for $N_c = 2$ and by comparing (56) with the variation of the effctive Lagrangian that $A_2 = g^2/4\pi^2$. Thus it can be seen that the log term in the prepotential is a 1-loop effect and the beta function have the following form:

$$\beta(g) = -(1/4\pi^2)g^3 + \text{instantons} \tag{61}$$

This is exactly the 1-loop beta function and so it is shown that the higher order corrections do not modify the prepotential. Now we turn to the non-perturbative effects: the instantons. This subject will be discussed further during this thesis, here we want to argue how the prepotential changes in the presence of instantons. Firstly a k-instanton contribution should be suppressed by the k-instanton factor. Considering this and including the 1-loop β function we get:

$$\mathcal{F} \propto e^{-8\pi^2 k/g^2} = \left(\frac{\Lambda}{a}\right)^{4k} \tag{62}$$

By restoring the $U(1)_{\mathcal{R}}$ symmetry we can assign charge of 2 to Λ and since the prepotential now has an R-charge of 4 we know that the instanton part also has to be proportional to the square of the chiral superfield. In the end we have:

$$\mathcal{F} = \mathcal{F}_{1-loop} + \mathcal{F}_{inst} = \frac{i}{2\pi}A^2\ln\frac{A^2}{\Lambda^2} + \sum\mathcal{F}_k\left(\frac{\Lambda}{A}\right)^{4k}A^2 \tag{63}$$

2.3.4 Duality

At this point we will see that the effective action description is not the right one for the whole moduli space of vacua. In a physical theory the metric on a space should be positive definite. Nevertheless it holds that $\text{Im}\tau(a) > 0$. Since the prepotential is a holomorphic function we directly see that $\text{Im}\tau$ is a harmonic function, because it is the second derivative of the prepotential. It can not take a minimum on the (compactified) u-plane. If it can not take a minimum, it is not bound from below and so to be positiv is violated by the metric. Have we done something wrong? In order to circumwent this problem duality can be used. Since we approach in the moduli space to some singularity where the metric goes to zero we have to choose another coordinate system where the metric is positive. So we will have different local descriptions that are glued together to get an everywhere consistend picture. The different descriptions are related via duality transformation. We will show how this can work out. We introduce first two dual fields:

$$A_D = \frac{\partial \mathcal{F}}{\partial A}, \frac{\partial \mathcal{F}_D}{\partial A_D} = -A \tag{64}$$

From this definition we directly see that the gauge coupling is related to the dual field via $\tau = \partial A_D / \partial A$. Now we need to reformulate the effective Lagrangian (53) in the $\mathcal{N} = 1$ language. We can rewrite the first term easily with dual fields:

$$\begin{aligned}
\text{Im}\left[\int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A}\bar{A}\right] &= \text{Im}\left[\int d^4\theta A_D \bar{A}\right] \\
&= \text{Im}\left[\int d^4 A_D(-\bar{\mathcal{F}}'_D)\right] \\
&= \text{Im}\left[\int d^4 \bar{A}_D \mathcal{F}'_D\right]
\end{aligned}$$

The possibility to rewrite the term means also that the action (53) is invariant under duality transformation, i.e. under (64). Now we have to rewrite the second term of the effective action resulting in a dual Lagrangian by implementation of a constraint. The Bianchi identity for superspace is $\text{Im}(D_\alpha W^\alpha) = 0$ with D_α super covariant derivative. We will in fact just use the Lagrangian method to introduce constraints in the Path integral.Usually the path integral is integrated over the vectormultiplet (here called) V, but you can equivalently also integrate over the fieldstrength and some Lagrangian multiplier, which is a superfield. The Path integral is then equal to:

$$\begin{aligned}
&\int DV \exp\left[\frac{i}{4\pi}\text{Im}\int d^4x d^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial A^2} W^\alpha W_\alpha\right] \\
&= \int DW DV_D \exp\left[\frac{i}{4\pi}\text{Im}\int d^4x \left(\int d^2\theta \frac{\partial^2 \mathcal{F}}{\partial A^2} W^\alpha W_\alpha + \frac{1}{2}\int d^2\theta d^2\bar{\theta} V_D D_\alpha W^\alpha\right)\right]
\end{aligned} \tag{65}$$

The second term can be rewritten as:

$$\text{Im}\int d^4x d^4\theta V_D D_\alpha W^\alpha = \text{Re}\int\int d^4x d^4\theta i D_\alpha V_D W^\alpha = -\text{Im}\int d^4x d^2\theta W_{\alpha D} W^\alpha \tag{66}$$

Here we made use of the fact that the integral over $d^2\bar{\theta}$ acts as quadratic super covariant derivative on the integrand and that the derivative commutes with the dual superfield. So we could introduce the dual field strength $W_D = \bar{D}^2 D_\alpha V_D$. Performing the gaussian integral over the field strength W_α in the path integral yields the dual Lagrangian:

$$\frac{1}{8\pi}\text{Im}\int d^2\theta \frac{-1}{\tau} W_D^2 \tag{67}$$

What happened? We have seen that we can write down an equivalent Lagrangian which exchanges a gauge field coupled to electric charges by an dual gauge field that couples to magnetic charges if we also exchange the gauge coupling by an S-duality transformation:

$$\tau \rightarrow \tau_D = -\frac{1}{\tau} \tag{68}$$

The action is also invariant under T-transformation so we so we recognized that the action is invariant under $SL(2, Z)$ called the duality group. Returning to the metric on the moduli space it can be seen that the metric can be written as:

$$ds^2 = \mathrm{Im}\tau da d\bar{a} \tag{69}$$

We have seen that we can not have the same coordinate on the whole moduli space and have defined[10] $a_D = \partial\mathcal{F}/\partial a$. Now the metric can be written

$$ds^2 = \mathrm{Im} da_D d\bar{a} \tag{70}$$

This formula is symmetric within the field and its dual and and it is also invariant under the duality group. In fact the problem can be formulated in a more mathematical language: Let $\mathcal{M}$ be the moduli space of vacua parameterized by the u-plane so it is a one dimensional special Kaehler manifold. Then we introduce some space $\mathcal{P} \simeq \mathbb{C}^2$ where (a_D, a) are the coordinates. We choose a symplectic form $\omega = \mathrm{Im} a_D \wedge a$ on $\mathcal{P}$. In fact we have a map $f : u \rightarrow (a_D(u), a(u))$ so $f : \mathcal{M} \rightarrow \mathcal{P}$. This is a section of an $SL(2, \mathbb{Z})$-bundle over the moduli space of vacua and we know how to introduce a a metric. We simply pullback the metric from the space $\mathcal{P}$ to the moduli space $\mathcal{M}$. In fact solving the low energy effective theory is equivalent to find (a_D, a).

2.3.5 Monodromies on the moduli space of vacua

Now it is possible to study the singularities on the moduli space. The aim is to find out how the coordinates change if we move along the moduli space. If we take a contour on the u-plane and go around this contour, the coordinates (a_D, a) change in case of a singularity encircled by this contour. This is called a non-trivial monodromy. We begin in the limit of large u. The theory is asymptotic free so in this limit it is possible to use the perturbative expression for the prepotential $\mathcal{F}_{1-loop} = iA^2 \ln(A^2/\Lambda^2)/2\pi$ and compute for large a:

$$a_D = \frac{\partial \mathcal{F}}{\partial a} \approx \frac{2ia}{\pi} \ln(a/\Lambda) + ia/\pi \tag{71}$$

In this region we can approximate the coordinate by the following equation $u = 1/2a^2$. For encircling the point at ∞ in a clockwise sense we get $\ln(u) \rightarrow \ln(u) + 2\pi i$. Out of the semiclassical approximation of the moduli space coordinate it can be deduced that $\ln(a) \rightarrow \ln(a) + \pi i$. From this we get:

$$a_D \rightarrow -a_D + 2a \tag{72}$$

$$a \rightarrow -a \tag{73}$$

[10]We have defined the dual field condition for the chiral superfield therefore it holds for the scalar component as well

This monodromy may also also be written in a more sophisticated form:

$$\left(\begin{array}{c} a_D(u) \\ a(u) \end{array} \right) \rightarrow M_\infty \left(\begin{array}{c} a_D(u) \\ a(u) \end{array} \right) \quad , M_\infty = \left(\begin{array}{cc} -1 & 2 \\ 0 & -1 \end{array} \right). \tag{74}$$

Through the non-trivial monodromy we know that there has to be at least some other non-trivial monodromy in the moduli space of vacua because the monodromies generate a group and so otherwise the this non-trival monodromy would be the identify element which is a contradiction. Due to R-symmetry observations it is known that we have the following symmetry on the modulispace $u \rightarrow -u$. The fix points of this group action are at infinity and at zero. We have already seen that ∞ is a singularity. If there is just one singularity more then it has to be at 0.This is self-contradictory because it indicates that the monodromy is the same at zero and infinity.This indicates that a^2 is global coordinate on $\mathcal{M}$. However this violates the former analysis so we have conclude that we need three singularities or more. Due to the R-symmetry we would have $\infty, u_0, -u_0$. We assume that we have 3 singularities. Probably one would expect a singularity at the origin. In the classical picture where $u = 1/2a^2$ this is correct because there is a gauge enhancement at the origin and all the fields that have gotten mass through the higgs mechanism get massless there. In consequence the Wilsonian description breaks down there. Asymptotic freedom makes it possible to analyze everything classically within the large u limit but if we tune the values of u towards the origin we will encounter monodromies and singularities, because we enter the strong coupled region, where $u \neq 1/2a^2$. Nevertheless why do we get these singularities? A first guess would be that gauge bosons become massless, but in [6] it was explained why this can not be correct. Massless gauge bosons would imply that we have an asymptotically and conformally invariant theory within the infrared limit. This would imply $u \neq \langle \mathrm{tr}\phi^2 \rangle$ unless $\mathrm{tr}\phi^2$ has dimension zero which would lead to the conclusion that $\mathrm{tr}\phi^2$ is the unity operator which makes no sense. The singularities do not result in gauge bosons becoming massless, so why do we have this singularities in the moduli space? There are no other elementary particles since we are dealing with pure gauge theory, but we have noticed that we need massive particles with spin $\leq 1/2$ that become massless. That should be collective excitations like magnetic monopoles or dyons. If a magnetic monopole becomes massless we know from the BPS-bound that the dual coordinate tends to zero. The singularity is denoted bby u_0 so there holds: $a_D(u_0) = 0$. Monopoles are described by $\mathcal{N} = 2$ hypermultiplets. These supermultiplets couple to the dual fields A_D and W_D like the electron supermultiplet would couple to fields A and W. We know that we have to deal with SQED because of the breakdown of the non-abelian gauge symmetry to $U(1)$-gauge theory. Therefore we can analyze the β function of the theory. This leads to the following magnetic coupling of the monopole near the singularity:

$$\tau_D \approx -\frac{i}{\pi} \mathrm{ln} a_D \tag{75}$$

Since from our definitions we have $\tau_D = -\frac{\partial a}{\partial a_D}$, we can integrate the expression to obtain

$$a \approx a_0 + \frac{i}{\pi} a_D \mathrm{ln} a_D + O(a_D) \tag{76}$$

For small neighborhood of the singularity we can also write

$$a_D \approx c_0(u - u_0) \tag{77}$$

In fact this means:

$$a_D \approx c_0(u - u_0) \tag{78}$$

$$a \approx a_0 + \frac{i}{\pi} c_0(u - u_0)\ln(u - u_0) \tag{79}$$

By encircling and going around we now get: $\ln(u - u_0) \to \ln(u - u_0) + 2\pi i$ and so the monodromy around the singularity can be computed directly:

$$M_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \tag{80}$$

A special condition will help us to deduce the monodromy of the third singularity. The possible monodromy is restricted because of the R-symmetry acting on the moduli space. If we take the monodromies counterclockwise then, we have to have that multipling the two monodromies around the finite u yields the same result as the monodromy around infinity $M_\infty = M_{u_0} M_{-u_0}$ because the homology cycles[11] build up a group. From this we can compute:

$$M_{-u_0} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \tag{81}$$

The question that directly arises is which kind of particle gets massless at the singularity. The charge of every particle has to be invariant under monodromy. This is due to the fact that a monopole is invariant under its monodromy and via duality has to account for every particle. The simplest answer to this monodromy invariance is that a $(1, -1)$ dyon gets massless at singularity. From now on assume that $u_0 = 1$. Now we will solve the model via elliptic curves

2.3.6 The Seiberg Witten curve

Seiberg and Witten recognized that the solution to low energy theory of pure $\mathcal{N} = 2$ super Yang-Mills can be beautifully encoded in an elliptic curve of a certain genus. We summarize our conlusions uptil mow: we are studying the quantum moduli space $\mathcal{M}$ of vacua. This is the u-plane with singularities at 1,-1 and at ∞ where we have put $u_0 = 1$. We have a $\mathbb{Z}_2$-symmetry acting on the moduli space. Previously it was shown that duality indicates a flat $SL(2, \mathbb{Z})$-bundle over the moduli space of vacua with three monodromies computed in the last subsection where (a_D, a) is a section of this bundle. We know the behavior of the section near special points. For $u = \infty$

$$a = \sqrt{2u} \tag{82}$$

$$a_D = i\frac{\sqrt{2u}}{\pi}\ln(u) \tag{83}$$

[11] Remember the definiton of a monodromy

Near $u = 1$

$$a_D = c_0(u-1) \tag{84}$$

$$a = a_0 + \frac{i}{\pi} a_D \ln(a_D) \tag{85}$$

Near $u = -1$

$$a = c_0(u-1) + a_D \tag{86}$$

$$a_D = a_0 + \frac{i}{\pi}(a - a_D)\ln(a - a_D) \tag{87}$$

Here a_0 and c_0 are constants. We have a metric on the u-plane

$$ds^2 = \mathrm{Im}(\tau) \cdot |da|^2 \tag{88}$$

with

$$\tau = \frac{\partial a_D}{\partial a} = \frac{\partial a_D / \partial u}{\partial a / \partial u} \tag{89}$$

The solution of the model, which consists of knowing (a, a_D) not only in a local patch, can be achieved in two different ways. One way is via differential equations where one studies the Schroedinger equation with periodic potential Through introduction of the correct potential to the problem it is possible to solve the Schroedinger equation. From this you can compute the coordinates (a_D, a) explicitly. The other approach is via elliptic curves and was noted in [6]. The monodromy matrices generate a certain subgroup $\Gamma(2) \subset SL(2,\mathbb{Z})$. This is the set of all matrices that are congruent modulo two to the identity. The moduli space of vacua $\mathcal{M}$ is nothing then the upper halfplane moded out by the monodromy group. This fact is due to the definition of coordinate u and the fact that the monodromies does not change the physical theory. We directly see that three cusps of the quotient correspond to the three singularities. The other known fact is that the space $H/\Gamma(2)$ is the moduli space of elliptic curves of genus 1 and parameterizes the family of elliptic curves E_u via the following equation:

$$y^2 = (x+1)(x-1)(x-u) \tag{90}$$

On this curve there is also a $\mathbb{Z}_2$-symmetry for u by introduction of corresponding symmetry transformations to the other variables of the curve. The curve (90) is double cover of the x-plane[12] branched over -1,1,u and ∞. We want to fix a branch cut between 1 and -1 and another between u and ∞. The a-cycle corresponds to a path around one of the branch cuts and a b-cycle is intersecting both cuts. This can be seen by transforming a torus into the x-plane with certain branch cuts. Owing to this it is clear that in case two branch points coincide, a cycle of the curve goes to zero and the curve becomes singular. This happens at the singularities of the u-plane or at the zeros of the discriminant of the curve:

$$\Delta(u) = 0 = u^4 - 2u^2 + 1 \tag{91}$$

[12]This is due to the quadratic term in y

The family of curves gets singular at the singularities of the u-plane $u = \pm 1$. To identify the pair (a_D, a) we have to choose two cycles γ_1 and γ_2 on the Riemann surface that want should fulfill:

$$\gamma_1 \cdot \gamma_2 = 1 \tag{92}$$

They form a basis of the the first homology group. The homology group $V_u = H^1(E_u, C)$ are the fibers of the constructed bundle over the moduli space. Locally we have cycles and by gluing this together we get the $SL(2,\mathbb{Z})$-bundle. The elements of V_u can be paired with elements of the cohomology group via:

$$\gamma \to \oint_\gamma \lambda \tag{93}$$

So γ is a meromorphic (1,0)-form on the family of curves, but with a vanishing residue. This condition protects the pairing form corrections if γ is deformed across poles. γ is an element of the cohomology group so also defined modulo exact forms. In fact we think that Poincare duality can be used to identify λ as an element of the V_u. This can easily be seen by noting that the pairing induces a isomorphism between the k-th comohomology and the (n-k)-homology. However on a surface we have $n = 2$ so we see that the first homology group is isomorphic to the first cohomology group. For the 1-forms we now choose a basis:

$$\lambda_1 = \frac{dx}{y} \ , \lambda_2 = \frac{xdx}{y} \tag{94}$$

Define a new parameter:

$$b_i = \oint_{\gamma_i} \lambda_1 \tag{95}$$

We can then write for the complex structure of the torus:

$$\tau_u = b_1/b_2 \tag{96}$$

Here we have $\text{Im}(\tau_u) > 0$. Now we build an arbitrary section out of this basis

$$\lambda = f(u)\lambda_1 + \tilde{f}(u)\lambda_2 \tag{97}$$

We define:

$$\begin{aligned} a_D &= \oint_{\gamma_1} \lambda \\ a &= \oint_{\gamma_2} \lambda \end{aligned} \tag{98}$$

A different choice of the cycles will lead to a $SL(2,\mathbb{Z})$-transformed (a_D, a) pair. In regard to (98) it can be deduced that the method was correct. The main point is to show that the metric on the moduli space is positive. So we calculate the derivative of the section:

$$
\begin{aligned}
\frac{\partial a_D}{\partial u} &= \oint_{\gamma_1} \frac{\partial \lambda}{\partial u} \\
\frac{\partial a}{\partial u} &= \oint_{\gamma_2} \frac{\partial \lambda}{\partial u}
\end{aligned} \tag{99}
$$

We assume:

$$
\frac{\partial \lambda}{\partial u} = f(u)\lambda_1 = f(u)\frac{dx}{y} \tag{100}
$$

From this assumption we can directly conclude that τ is now positive on the entire moduli space of vacua.

$$
\begin{aligned}
\frac{\partial a_D}{\partial u} &= f(u)b_1 \\
\frac{\partial a}{\partial u} &= f(u)b_2
\end{aligned} \tag{101}
$$

Here we used the definition of (95). The last step is:

$$
\tau = \frac{\partial a_D}{\partial a} = \frac{\partial a_D/\partial u}{\partial a/\partial u} = \frac{b_1}{b_2} = \tau_u \tag{102}
$$

So we have shown that if $\tau_u > 0$ then τ is positive on the whole moduli space. In [6] it is argued that the implication holds also in the other direction and so the derivative of λ w.r.t. u is indeed independent of λ_2. The arbitrary function $f(u)$ is fixed by the asymptotic behavior near the singularities: $f(u) = -\sqrt{2}/4\pi$. From the asymptotic behavior and (100) we can directly calculate λ

$$
\lambda = \frac{\sqrt{2}}{2\pi}\frac{ydx}{x^2-1} \tag{103}
$$

Now we need an explicit basis of 1-cycles on E_u to evaluate the desired expressions. We have already discussed that a circle around 1 and -1 lifts to an a-cycle on the Riemann surface and a circle that intersects both branch cuts (e.g. encircling 1 and u) lifts to a b-cycle.So we define the γ_s as the encircling a-cycle and b-cycle. The result is:

$$
\begin{aligned}
a_D &= \oint_{\gamma_1} \lambda = \frac{\sqrt{2}}{\pi}\int_{-1}^{1} \frac{dx\sqrt{x-u}}{\sqrt{x^2-1}} \\
a &= \oint_{\gamma_2} \lambda = \frac{\sqrt{2}}{\pi}\int_{1}^{u} \frac{dx\sqrt{x-u}}{\sqrt{x^2-1}}
\end{aligned} \tag{104}
$$

From this expression one can check that this gives the desired behavior near the singularities, which means that the right monodromies are generated by a coordinate transformation around the singularity. The integrals (104) can be evaluated in terms of hypergeometric functions, which are defined as follows:

$$
F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 dx x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-zx)^{-\alpha} \tag{105}
$$

At the end we have:

$$a_D = \frac{i}{2}(u-1)F(1/2, 1/2, 2; (1-u)/2)$$
$$a = \sqrt{2(1+u)}F(-1/2, 1/2, 1; 2/(1+u) \tag{106}$$

Now we turn our interest to gauge theory coupled to matter and especially to the conformal case.

2.4 Seiberg-Witten Solution for $\mathcal{N}=2$ SU(2) gauge theory coupled to Matter: the superconformal case

In the following we will study conformal SQCD with $SU(2)$ gauge group on the coulomb branch. For nonconformal cases we want to refer to [8] or to the Appendix. First of all we want to make some general remarks. Firstly it is convenient to rescale some formulas, because our matter could have half-integer charges. So in (46) we multiply the electric charge by 2 and divide a by 2. That will not change the central charge formula. Still the subsequent equation can be formed (in the large u limit):

$$2a_D = \partial\mathcal{F}/\partial a \ , a \approx \frac{1}{2}\sqrt{2u} \ , a_D \approx i\frac{4}{\pi}a\ln u \tag{107}$$

The effective coupling is changed, too: $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$. The most important fact is that in contrast to the physics the curve might change. In fact we will try to solve the theory again via elliptic curves and to try to find the Seiberg-Witten differential. So the problem we are facing is to find an elliptic curve similar to (90) that describes the pure gauge theory coupled to 4 flavor with arbitrary bare masses. We start to analyze the superpotential that now appears in SQCD[13] with N_f fundamental hypermultiplets (47):

$$W = \sum_{i=1}^{N_f}(\sqrt{2}\tilde{Q}_i\Phi Q_i + m_i\tilde{Q}_iQ_i) \tag{108}$$

Here Φ is the chiral superfield. The global symmetry group for zero masses is a product of a flavor symmetry and R-symmetry, which will be broken in quantum theory. When the gauge group is $SU(2)$ we have the flavor symmetry enlarging from $SU(N_f)$ to $O(2N_f)$ which can be seen by analyzing the superpotential[14] we have an additional $\mathbb{Z}_2$-symmetry that acts as:

$$\rho : Q_1 \leftrightarrow \tilde{Q}_1 \tag{109}$$

This symmetry leads the other "squarks" invariant. This symmetry will be broken in the superconformal case. The Coulomb branch is defined as the directions in which the gauge group is broken to $U(1)$. There is also another branch for $N_f > 1$ where the gauge

[13] SUSY Yang-Mills coupled to matter

[14] This is due to the fact that the fundamental representation of $SU(2)$ is pseudoreal. This will be explained in more detail in the chapter about $\mathcal{N}=2$-dualities

group is completely broken called Higgs branch. However, this thesis is confined to the first one. This is due to the fact that there are no quantum corrections on the Higgs branch so the classical moduli space of vacua does not change in case we quantize the theory.
The one-loop beta function is proportional to $4-N_f$, thus the interesting physics appear for $N_f < 4$ and $N_f = 4$. We will restrict to the second case, in case we have to do assumptions about the number of flavors. There is a $\mathbb{Z}_{4-N_f}$ symmetry acting on the u-plane and in the conformal case this symmetry is absent. We will now analyze the quantum moduli space of vacua on the Coulomb branch starting again by defining: $u =< \mathrm{Tr}\phi^2 >$. For $N_f = 4$ the $U(1)_{\mathcal{R}}$ symmetry is anomaly free and we can assign to u a charge of 4. The parity transformation is anomalous but can be restored by assigning odd parity to $e^{-\frac{8\pi^2}{g^2}}$.For $N_f \neq 0$ it is expected that terms with odd instanton number vanish from the anomalous parity symmetry and that the instantons can not contribute to the even metric, because the odd instanton number is odd under parity. So we have even contributions from the instanton sectors:

$$a = \frac{1}{2}\sqrt{2u}\left(1+\sum_{n=1}^{\infty} a_n(N_f)\left(\frac{\Lambda_{N_f}^2}{u}\right)^{n(4-N_f)}\right) \tag{110}$$

$$a_D = i\frac{(4-N_f)}{2\pi}a(u)\log\frac{u}{\Lambda_{N_f}^2} + \sqrt{u}\sum_{n=1}^{\infty} a_{Dn}(N_f)\left(\frac{\Lambda_{N_f}^2}{u}\right)^{n(4-N_f)} \tag{111}$$

2.4.1 The singularities in the superconformal case

The aim of this section is to study the singularities on moduli space of vacua for $\mathcal{N} = 2$ gauge theory with four flavors. First of all it is known that the 1-loop beta function is vanishing. So we know from non-renormalization theorems that the higher loop beta functions vanish to all orders in perturbation theory. So there could be non-perturbative effects that lead to the conclusion that the exact beta function does not vanish. There are some reason to assume that the contributions to the beta function vanishe also non-perturbativly. One reason is that the metric on the moduli space is not positive definite and we know no other non-perturbative effects that could modify this in the right way. So the exact quantum theory is scale invariant and is characterized by the marginal coupling $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$. The exact answer to the coordinates is

$$a = \frac{1}{2}\sqrt{2u} \tag{112}$$

$$a_D = \tau a \tag{113}$$

Through introduction of masses of matter the scale invariance and $U(1)_{\mathcal{R}}$-symmetry is broken. If we send some mass to infinity, we should obtain the moduli space of the theories with less flavors because we decouple flavors. These theories are discussed in [8]. If we take masses m_i and label $i = n+1, ..., 4$ and let this masses go to infinity, the number of flavors is reduced and the low energy theory should have n flavors with the corresponding scale parameter that differ for each matter configuration. In each singularity there can be more hypermultiplets that become massless. If we denote every singularity and its order by a number, the overall sum this number has to be 6. This comes from the fact that we describe the physics via curves. So if we denote the weight of the i^{th} singularity by k_i then we have $\sum k_i = 6$. These values are not arbitrary and we now quote some examples from [7] of certain mass configuration with certain singularity structure. That help us to extrapolate to arbitrary masses. First of all we bring the masses of the flavor into a row vector.

Ex. 1:
$m_i = (m, 0, 0, 0)$ has global symmetry $SU(4) \times U(1)$. This has three singularities and $k_i = (4, 1, 1)$, where the massless particles in the first singularity transform in the fundamental of SU(4)

Ex. 2:
$m_i = (m, m, m, m)$ has global symmetry $SU(4) \times U(1)$ This has three singularities and $k_i = (1, 1, 4)$, where the massless particles in the last singularity transform in the fundamental of SU(4)

Ex. 3:
$m_i = (m, m, 0, 0)$ has global symmetry $SU(2) \times SU(2) \times SU(2) \times U(1)$ This has three singularities and $k_i = (2, 2, 2)$, where the massless particles in the every singularity transform in the fundamental of SU(2)

Ex. 4:
$m_i = (m + \mu, m - \mu, 0, 0)(\mu \neq m)$ has global symmetry $SU(2) \times SU(2) \times U(1) \times U(1)$ This has four singularities and $k_i = (1, 1, 2, 2)$, where 2 massless particles transform as doublet under $SU(2)$.

Ex. 5:
$m_i = (m, m, \mu, \mu)(\mu \neq m)$ has global symmetry $SU(2) \times SU(2) \times U(1) \times U(1)$ This has four singularities and $k_i = (1, 1, 2, 2)$, where 2 massless particles transform as doublet under $SU(2)$.

Ex. 6:
$m_i = (m, m, m, m)(\mu \neq m)$ has global symmetry $SU(3) \times U(1) \times U(1)$ This has four singularities and $k_i = (1, 1, 1, 3)$, where 3 massless particles transform in the fundamental under $SU(3)$.

2.4.2 The curve for massless $N_f = 4$

Now we are in position to deduce the right family of curves. The coupling constant is dimensionless in this case so the curve $y^2 = F(x, u, m_i, \tau)$ will depend on this coupling. Strictly speaking this means that the coefficients of the curve will be functions of the marginal coupling constant instead of being a function of the renormalization scale. We want to find a curve so that the differential form

$$\omega = \frac{\sqrt{2}}{8\pi}\frac{dx}{y} \tag{114}$$

has the right periods like in (113). To find the right genus one curve one first notices that a_D is just a multiple of a. Now introduce the curve as the complex plane moded out by a lattice that is generated by π and $\tau\pi$. This is a torus. Let $w_0 = dz$. The periods of this form are the generators of the lattice. We introduce the Weierstrass $\wp$ function that obeys:

$$\wp(z) = \wp(z+1) = \wp(z+\tau) = \wp(-z) \tag{115}$$

This has a double pole at the origin. This $\wp$ obeys a certain differential equation:

$$\wp'(z) = 4\wp^3(z) - g_2(\tau)\wp(z) - g_3(\tau) \tag{116}$$

Here $g_2 = 60\pi^{-4}G_4\, g_3 = 140\pi^{-6}G_6$ with G_4 , G_6 the Eisenstein series.
Now we define $x_0 = \wp(z)$ and $y_0 = \wp'(z)$ and directly see from (116):

$$y_0 = 4x_0^3 - g_2(\tau)x_0 - g_3(\tau)\ ,\ \omega_0 = \frac{dx_0}{y_0} \tag{117}$$

Setting $x = x_0 u$ and also $y = \frac{1}{2}y_0 u^{3/2}$ and inserting we get:

$$\omega = \sqrt{2/u}/4\pi\omega_0 \tag{118}$$

and the curve becomes:

$$y^2 = x^3 - \frac{1}{4}g_2(\tau)xu^2 - \frac{1}{4}g_3(\tau)u^3 \tag{119}$$

This leads to the curve with the right periods. It is important to know more about the structure of the curve. For this reason the curve has to be factorized first (119):

$$y^2 = (x - e_1(\tau)u)(x - e_2(\tau)u)(x - e_3(\tau)u) \tag{120}$$

Here the e_i are the roots of $4x^3 - g_2 x - g_3$ with $\sum e_i = 0$. The roots can be expanded in terms of *modular theta functions* and look like:

$$e_1 = \frac{2}{3} + 16q + 16q^2 + ... \tag{121}$$
$$e_2 = -\frac{1}{3} - 8q^{1/2} - 8q - 32q^{3/2} - 8q^2 16q^2 + ... \tag{122}$$
$$e_3 = -\frac{1}{3} + 8q^{1/2} - 8q + 32q^{3/2} - 8q^2 16q^2 + ... \tag{123}$$
$$\tag{124}$$

Here q is the usual instanton parameter. It can be shown that the e_i are in one-to-one correspondence to the even spinstructures on the torus [7].If S-duality should permute the three 8-dimensional representations of $SO(8)$, which is universally covered by the corresponding spin group ,the same way it acts on the even spinstructures we can deduce that the this S-duality corresponds to permute the different e_i in the curve.

2.4.3 The mass deformed curve

Now we want to deduce the curve for arbitrary masses of the hypermultiplets: To do so the above-mentioned examples will be used. This will restrict in an extent resulting in a symmetry from which it is possible to deduce the exact form of the curve . First of all we start with the following mass vector: $m_i = (m, m, 0, 0)$ resulting in three singularities with two particles in each transforming as a doublet under $SU(2)$. From this it can be shown that the monodromy is conjugate to T^2, where T is one generator of the $SL(2,\mathbb{Z})$. From this one can deduce that the curve has to look like:

$$y^2 = \prod_i (x - e_i\tilde{u} - e_i^2 f) \tag{125}$$

Here $\tilde{u}$ is a constant shift of the coordinate on the moduli space of vacua and f is proportional to m^2. f has to be constant to avoid τ dependency for the residues of the Seiberg-Witten differential. In the weak coupling limit one can fix the constant to $f = m^2$. So we now have the curve for the special case and we try now to extrapolate to the general case with arbitrary bare masses. Therefore we introduce $SO(8)$ mass invariants.
First of all the quadratic term

$$R = \frac{1}{2}\sum_i m_i^2 \tag{126}$$

Then we introduce quartic invariants which are linearly independent

$$\begin{aligned} T_1 &= \frac{1}{12}\sum_{i>j} m_i^2 m_j^2 - \frac{1}{24}\sum_i m_i^4 \\ T_2 &= -\frac{1}{2}\prod_i m_i - \frac{1}{24}\sum_{i>j} m_i^2 m_j^2 + \frac{1}{48}\sum_i m_i^4 \\ T_3 &= \frac{1}{2}\prod_i m_i - \frac{1}{24}\sum_{i>j} m_i^2 m_j^2 + \frac{1}{48}\sum_i m_i^4 \end{aligned} \tag{127}$$

(128)

and the fourth quartic invariant R^2. We are now able to combine the invariants to invariants of order six. That are: R^3 , RT_i and the following one:

$$N = \frac{3}{16}\sum_{i>j>k} m_i^2 m_j^2 m_k^2 - \frac{1}{96}\sum_{i\neq j} m_i^2 m_j^4 + \frac{1}{96}\sum_i m_i^6 \tag{129}$$

Now we will impose certain conditions to generalize the curve (125) to arbitrary bare masses:
1) The curve is a polynomial in the masses, because in the limit of vanishing masses the curve should be smooth.
2) By assigning $U(1)_{\mathcal{R}}$ charges the powers of m are constraint. The charges are for $(\tilde{u}, x, y, m_i)(4, 4, 6, 2)$.

3) In the special case of two identical masses and two zero bare masses, we recover (125). We can directly restrict the form of the curve from these conditions using the invariants and (125):

$$y^2 = \prod_i W_i + x \sum_i T_i f_i + \tilde{u} \sum_i T_i g_i + R \sum_i T_i hi + pN \tag{130}$$

Here the $W_i = x - e_i \tilde{u} - e_i^2 R$. For the special case $m_i = (m, m, 0, 0)$ we have $T_i = N = 0$ and the curve is directly reduced to (125). In (130) we have to fix some constants to obtain the correct curve, which will be explained in this paragraph. First of all we know that we have three singularities at $\tilde{u}_i = e_i m^2$. Consequently we know that the discriminant have double zeros at these values. As it was done in example 4, we can perturb this configuration of masses through addition and substraction of the same amount of masses. From this, four singularities will result and, as it was evident in the example, one singularity has to split to two while the other singularities stay the same. This can also be seen from the transformation properties. The two "old" singularities transform in the fundamental representation of $SU(2)$ while the other transform under $U(1)$. Now we assume $\tilde{u}_1$ splits and $\tilde{u}_2, \tilde{u}_3$ are double zeros, which still can move on the moduli space of vacua. Through analyzis of the discriminant and the assumption that the second and third zeros are double zeros we can deduce equations that restricting the coefficients in the curve (130). In order to have enough equations to be able to solve these for the coefficients, this can be done in the same way with $m_i = (m, m, \mu, \mu)$ and $m_i = (m, m, \mu, -\mu)$.
Finally the following Seiberg-Witten curve is calculated for the $\mathcal{N} = 2$ gauge theory with $N_f = 4$:

$$y^2 = \prod_i W_i + A(W_1 T_1 (e_2 - e_3) + W_2 T_2 (e_3 - e_1) + W_3 T_3 (e_1 - e_2)) - A^2 N \tag{131}$$

Here we have:

$$A = (e_1 - e_2)(e_2 - e_3)(e_3 - e_1) \tag{132}$$

2.4.4 S-duality?

This curve has a full $SL(2, \mathbb{Z})$ invariance in case it is combined with a $Spin(8)$ triality that permutes the linearly independent quartic invariants of $SO(8)$. More precise this means that if we act on the gauge coupling with a $SL(2, \mathbb{Z})$ transformation the curve is not longer invariant under this transformation. Nevertheless if we map the mass parameters of the four hypermultiplets to linear combinations, we can restore the symmetry of the curve:

$$\begin{aligned}
m_1 &\to \frac{1}{2}\left(m_1+m_2+m_3-m_4\right)\\
m_2 &\to \frac{1}{2}\left(m_1+m_2-m_3+m_4\right)\\
m_3 &\to \frac{1}{2}\left(m_1-m_2+m_3-m_4\right)\\
m_4 &\to \frac{1}{2}\left(-m_1+m_2+m_3-m_4\right)
\end{aligned} \tag{133}$$

These transformations must be performed in order to have a symmetry under the map $\tau \to -\frac{1}{\tau}$

$$\begin{aligned}
m_1 &\to m_1\\
m_2 &\to m_2\\
m_3 &\to m_3\\
m_4 &\to -m_4
\end{aligned} \tag{134}$$

These transformation must be performed in order to have a symmetry under the map $\tau \to \tau + 1$. The conclusion is that the transformation on the mass parameteres correspond to the permutation of the three different 8-dimensional representations of the flavor group, which is called triality. These representations are called $8_v, 8_s, 8_c$ and correspond to the spinor representation, the conjugated spinor representation and the vector representation of the flavor symmetry group $SO(8)$. So the invariance group of the curve is a semi direct product of $SL(2,\mathbb{Z})$ with S_3 and this will be used a lot to explore a general class of $\mathcal{N}=2$ SCFTs in Chapter 5. Please note that talking about S-duality means invariance under $SL(2,\mathbb{Z})$ combined with triality.

2.5 Generalization to higher rank gauge groups

In this chapter the extension to higher rank gauge groups will be discussed [10]. The Lagrangian of the classical theory for arbitrary gauge group of rank r broken down by the higgs effect to $U(1)^r$ is:

$$\mathcal{L}_{eff} = \frac{1}{4\pi}\mathrm{Im}\left[\int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A_i}\bar{A}_i + \int d^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial A_i \partial A_j} W^{\alpha i} W^j_\alpha\right] \tag{135}$$

We see that the Kaehler potential is given by $K = \mathrm{Im}(\bar{A}_i \partial\mathcal{F}(A)/\partial A_i)$. Thus the metric on the moduli space is:

$$ds^2 = \mathrm{Im}\frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} da^i d\bar{a}^j \tag{136}$$

Here a_i is the scalar component of the $i-th$ chiral superfield. Again we see that the metric is the gauge coupling. Equivalently to the rank one case, the prepotential can be written as:

$$\mathcal{F} = \frac{2q}{\pi i}\sum_{i=1}^{r} a_i^2 + \frac{i}{4\pi}[\sum_{\alpha}(\alpha\cdot a)^2 \ln\frac{(\alpha\cdot a)^2}{\Lambda^2} - \sum_{i}\sum_{j=1}^{N_f}(\lambda_i\cdot a + m_j)^2 \ln\frac{(\lambda_i\cdot a + m_j)^2}{\Lambda^2}] + \sum_{m=1}^{\infty}\frac{\Lambda^{2mq}}{2m\pi i}\mathcal{F}^{(m)}(a) \tag{137}$$

q depends on the gauge group and ensures asymptotic freedom. λ_i also depends on the gauge group and is an orthonormal basis up to a sign in the case of classical gauge groups. α is the root vector of the corresponding gauge group and the $\mathcal{F}$ are the instanton expansion coefficients that we want to deduce. Analogous to (130), the curve can be given as:

$$y^2 = A^2(x) + B(x) \tag{138}$$

Here for $SU(r+1)$:

$$A(x) = \prod_{i=1}^{r}(x - a_k)\ , B(x) = \Lambda^{2q}\prod_{j=1}^{N_f}(x + m_j) \tag{139}$$

$q = N_c - N_f/2$. The a_k parameterizes the coordinates on the moduli space of vacua. The coordinates, in the case of $SU(r+1)$, are given as the Weyl invariant Chern classes. For other classical gauge groups we can form certain combinations on the a_k to parameterize the moduli space of vacua. For the groups $SO(2r+1), Sp(2r), SO(2r)$ we have

$$A(x) = x^a\prod_{k=1}^{r}(x^2 - a_k^2)\ , B(x) = \Lambda^{2q}x^b\prod_{j=1}^{N_f}(x - m_j) \tag{140}$$

The meromorphic 1-form on the curve is:

$$d\lambda = \frac{x}{y}\left(A - \frac{AB'}{2B}\right)dx \tag{141}$$

For $SO(2r+1)$ we have $q = 2r - 1 - N_f$ and $a = 0, b = 2$; for Sp(2r) we have we have $q = 2r + 2 - N_f$ and $a = 2, b = 0$ and for $SO(2r)$ we have $q = 2r - 2 - N_f$ and $a = 0, b = 4$. These results are in compliance with the literature and were evaluated for classical gauge group groups (e.g. for $SU(3)$ this is done in [11]) according to [6]. In case we have a curve and the differential we can evaluate the periods again to deduce the higgs fields and their dual to obtain the exact prepotential.

2.6 Seiberg-Witten curves from M-theory

Now we are interested in the construction of Seiberg-Witten curves like (130), (125), (90) from string theory. In fact there are two approaches to this. The first approach makes us of geometric engineering. Here we compactify the ten dimensions on a local Calabi Yau manifold with certain singularities. A more detailed description (of this approach) can be found in [12], [13]. On the other hand the Seiberg-Witten curve can be constructed as in [**?**]. Here it was noticed that $\mathcal{N} = 2$ field theories can be constructed as effective theories on certain brane configurations in type IIA string theory. This construction can be lifted to M-theory.

2.6.1 Field theory from D6/NS5/D4-branes

Firstly we introduce intersecting brane configurations in Type IIA string theory on $\mathbb{R}^{10}$.

We will have :

NS5	1	2	3	4	5	-	-	-	-
D4	1	2	3	-	-	6	7	8	9

So the fivebranes are located at $x^7 = x^8 = x^9 = 0$ and have a fixed x^6 value in the classical theory. We want to assume that there are $n + 1$ NS5 branes and k_α D4 branes suspended between the $(\alpha - 1)$th brane and the αth NS5 brane. The resulting low

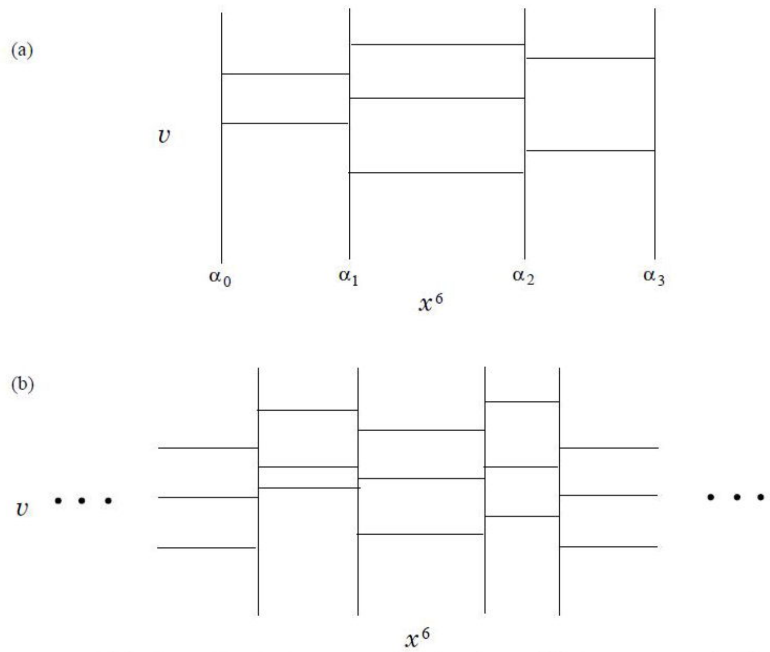

Figure 1: There are several fivebranes with parallel fourbranes in between

energy theory on the D4-world volume is an $\mathcal{N} = 2$ gauge theory with $\prod SU(k_\alpha)$ and bifundamental hypermultiplets $(k_1, \bar{k}_2) \oplus \cdots \oplus (k_{n-1}, k_n)$. The important fact is that we have special unitary groups. This is due to the fact that in case the kinetic energy of the five brane is studied, terms will results which are only convergent if the difference of the scalar parts of the $U(1)$ vector multiplet in the $U(k_\alpha)$ and in the $U(k_{\alpha+1})$ is fixed. This is the case for every fivebrane so a $U(1)$ vectormultiplet is missing in the spectrum. Therefore we have special unitary gauge groups. If the four branes are parallel, the gauge symmetry is broken by the higgs effect to gauge symmetry on the Coulomb branch. For a four dimensional observer the effective gauge coupling grows logarthmically. That can be seen as pulling the NS5 brane along the x^6-direction and the displacement of two branes is growing logarithmically. By adding some D4-branes on the left and the right we can built hypermultiplets in the fundamental of the first and last gauge group. Now we can also include D6 branes in the following brane configuration:

NS5	1	2	3	4	5	-	-	-	-
D4	1	2	3	-	-	6	7	8	9
D6	1	2	3	-	-	-	7	8	9

If there are d_α D6 branes between the $(\alpha-1)$th brane and the αth NS5 brane, we will have additional d_α hypermultiplets in the fundamental representation of the gauge group $SU(k_\alpha)$.

Let us quickly review how to construct the familiar supersymmetric theories. Pure $\mathcal{N}=2$ gauge theory with gauge group $SU(N)$ is constructed by two NS5-branes with N D4-branes between them. SQCD can be constructed by ddtion of a number of D4-branes which are semi-infinite[15] on both sides of the two NS5-branes or by inclusion of D6 branes between the NS5 branes.

2.6.2 Lift to M-theory

Now we have seen that we can construct supersymmetric gauge theories from Brane configurations. Still, how can they be analyzed in general? The basic observation is that the D4 branes and NS5 branes come from the be same fundamental object in M-theory. The M5 brane. By compactifying the 11th dimension on a circle one gets that a D4 brane is an M5 brane that wraps the extra dimension and the NS5 brane is an M5 brane located at some point on the circle. The whole brane configuration comes from one curved M5-brane. To preserve $\mathcal{N}=2$ supersymmetry the world volume of the M5 brane should have a complex structure on the space transverse to the observed four dimensional space so should be of the form $\mathbb{R}^4 \times \Sigma$ where Σ is a Riemannian surface embedded into $\mathbb{R}^3 \times S^1$.

The D6 branes are lifted to Kaluza Klein monopoles of M-theory. This is due to the observation that D6 branes are magnetically charged dual objects to D0 branes which are Kaluza Klein excitations. So D6 branes can be seen as magnetically charged under U(1) and are called Kaluza Klein monopoles or Taub-NUT spaces. These are 4-dimensional hyperkaehler manifolds. After the lifting a single M5-brane to M-theory we will have that the world volume will be $\mathbb{R}^4 \times \Sigma$ where Σ is now embedded into a Taub-NUT space. So we need to understand the geometry of the M5-brane to compute various quantities.

[15] Massive modes from the strings stretching between this branes and the four branes in between of the NS5 branes lead to massive multiplets

Now let T be a self dual field strength coming from the M5-brane[16]. We decompose the fieldstrength

$$T = (1 + *)F \wedge \omega \tag{142}$$

Here ω is a harmonic one-form on the Riemannian surface and F is the gauge field strength. If we have a genus g surface we will get a $U(1)^g$ gauge theory on the coulomb branch. The gauge couplings are obtained from the dimensional reduction along the Riemannian surface:

$$\int_{\mathbb{R}^4 \times \Sigma} |T|^2 \tag{143}$$

Hence we see that we get the gauge coupling by evalution of the period matrix of the surface. This means that the surface is the Seiberg-Witten curve and the gauge coupling depends only on the complex structure.

2.6.3 Solutions to $\mathcal{N} = 2$ theories

The aim of this section is to write down Seiberg-Witten curves for the several brane configurations starting with pure gauge. It was already noticed before that this corresponds to two NS5-branes an k D4-branes. We write $v = x^4 + ix^5$ and $t = \exp(-s) = \exp(-(x^6 + ix^{10})/R$. Here x^{10} is the extra dimension form M-theory and R the size of the extra dimension. So we are searching for an equation $F(v,t) = 0$ that defines the Seiberg Witten curve. First we observe that for fixed v the roots of the defining equation are the position of the NS5-branes. So the number of NS5-branes is equal to the degree of the function in t. For pure gauge F has to be quadratic in t. Accordingly we see that the degree in v is equal to the number of fourbranes between the fivebranes. So we have:

$$F(v,t) = A(v)t^2 + B(v)t + C(v) = 0 \tag{144}$$

Here A, B, C are polynomials in v of degrees k. There are no fourbranes on the left or right of the two NS5-branes so we can argue that in the large and zero limit of t there should be no solution to $F = 0$ and therefore we can set $A = C = 1$. So the curve is

$$t^2 + B(v)t + 1 = 0 \tag{145}$$

Shifting t we can bring this to

$$\tilde{t} = \frac{B(v)}{4} - 1 \tag{146}$$

with $\tilde{t} = t + B/2$. The last step is to bring B to the form

$$B(v) = v^k + u_2 v^{k-1} + \ldots + u_k \tag{147}$$

(146) is the solution of pure $\mathcal{N} = 2$ gauge theory. Now we directly want to incorporate flavors. So we need zeros of the polynomials we have chosen to be constant. In this

[16] This fact comes from the representation theory of the low-energy theory of M5 branes which is $(2,0)$-theory in 6 dimension and its representation

case we take $A = 1$, $C(v) = f\prod_{j=1}^{N_f}(v - m_j)$ where the m_j are the bare masses of the hypermultiplets and f is a complex constant. Inserting the new expression for C will lead to the Seiberg-Witten curve. The above analysis leads to the following general expression:

$$F(t,v) = t^{n+1} + f_1(v)t^n + ... + 1 \equiv \prod_{\alpha=0}^{n}(t - t_\alpha(v)) \tag{148}$$

We have to deduce the structure of this f_i. This is done by noting that the solution $t_\alpha v$ to $F(v,t) = 0$ are polynomials in v and so for large v:

$$t_\alpha(v) \approx h_\alpha v^{a_\alpha} \tag{149}$$

Here $a_0 \geq a_1 \geq ...a_n$ and h_α constants. The behavior can be expressed by the 1-loop beta function:

$$-i\tau_\alpha \approx s_\alpha - s_{\alpha-1} \tag{150}$$

Notice that $t = \exp(-s)$. One can also argue:

$$a_\alpha - a_{\alpha-1} = -b_\alpha = -2k_\alpha + 2k_{\alpha+1} + k_{\alpha-1} \tag{151}$$

We know that the last term of the polynomial is independent of v. So if we multiply out an expression like (148) then there is term $\prod_\alpha t_\alpha = 1$, thus $\sum a_\alpha = 0$. From this and (150) we now can express the powers of the polynomials from the number of fourbranes:

$$a_\alpha = k_{\alpha+1} - k_\alpha \tag{152}$$

Consequently:

$$F(t,v) = t^{n+1} + p_{k_1}(v)t^n + ... + p_{k_n}(v)t + 1 \tag{153}$$

Here p_{k_α} are polynomials of the degree corresponding to the number of fourbranes in between of two fivebranes. The polynomials can be expanded and the coefficients can be seen as the gauge coupling, the bare mass of the hypermultiplets or order parameters on the coulomb branch in the αth gauge group factor.
To include matter in an arbitrary fundamental representation of the gauge group we need to put D6-branes between the NS5-branes. We have seen that this brane configuration is lifted to an M5-brane that is embedded into a Taub-NUT space. Let us assume we have d D6-branes. The Taub-NUT space is written:

$$yz = P(v) = \prod_{a=1}^{d}(v - e_\alpha) \tag{154}$$

Here the e_α are the positions of the D6-branes in the v-plane and for t going to zero we have a D6-brane on the right and for very large t we have a D6 brane on the left. This is seen by the definition of t and the fact that the e_α are the positions of the D6 branes. We assume that there are two NS5-branes, so the curve quadractiv in y:

$$F(y,v) = A(v)y^2 + B(v)y + C(v) = 0 \tag{155}$$

By avoiding semi-finite four branes on the left and right one can normalize the polynomial such that $A = 1$. Now multiply by z^2 and use (154):

$$y^2 + B(v)y + C(v) = 0 \Leftrightarrow C(v)z^2 + B(v)P(v)z + P(v)^2 = 0 \tag{156}$$

This restricts the solution to have the following property:

$$C(v)|B(v)P(v), C(v)|P(v)^2 \tag{157}$$

From this consideration it can be shown that the right Seiberg-Witten curve appears for $\mathcal{N} = 2$ QCD with fundamental matter. So in fact we have learned that we can construct effective four dimensional field theories with $\mathcal{N} = 2$ supersymmetry by compactification of M5 branes that are a product of space time and a genus one Riemann surface. This is important because later on this approach will be extended by construction of arbitrary superconformal field theories from compactifications on more general Riemann surfaces of genus g and with n punctures. The constructions were also generalized to classical gauge groups in the same spirit as in [?] in [14] by introduction of orientifold planes in the typ IIA setup.

3 Liouville field theory

In the following chapter we want to introduce Liouville field theory. Liouville theory is a two-dimensional non-rational field theory which can be quantized as a conformal field theory. In 1981 Polyakow studied the quantization of the bosonic string via the path integral formalism [15]. He noted that the Weyl anomaly can be measured by the Liouville action and that in the critical dimension of 26 the bosonic string is anomaly free. However, to solve non-critical bosonic string theories we have to quantize Liouville field theory or for the superstring we have to quantize super Liouville theory if the case we do not want to work in ten spacetime dimensions. Beside this, Liouville theory was introduced a long time ago when Liouville studied the uniformization theorem of Riemann surfaces which gives a relation to Teichnueller theory. The path integral of bosonic string theory is:

$$\begin{aligned} Z_{bos} = & \int [Dg_{ab}] \exp(-\lambda \int \sqrt{g} d^2\xi) \\ & \times \int DX(\xi) \left[\exp\left(\frac{-1}{2} \int \sqrt{g} g^{mn} \partial_m X^\mu \partial_n X_\mu d^2\xi \right) \right] \\ & \times \phi[X(\xi)] \end{aligned} \tag{158}$$

for any functional ϕ.The Weyl moduli is given by: $g_{ab} = e^\phi \delta_{ab}$. By gauge fixing and introduction of ghosts we can bring this to the subsequent form:

$$Z_{bos} = \int D\phi(\xi) \exp\left(-\frac{26 - D}{48\pi} \int [\frac{1}{2}(\partial_\mu \phi)^2 + \mu^2 e^\phi] \right) \tag{159}$$

From this equation it can be seen that the Liouville action principally measures the Weyl anomaly and we can call the field ϕ the Weyl moduli. In a more general setup[17] the classical Liouville action is defined to be

$$S_{Liou} = \frac{1}{8\pi} \int d^2z \sqrt{g} (g^{mn} \partial_m \phi \partial_n \phi + QR(z)\phi(z) + \mu e^{2b\phi}) \tag{160}$$

Firstly we want to remember how the 3-point function was conjectured and which important steps have to be taken to evaluate this correlator. Then we want to introduce the notion of conformal blocks and go to higher point functions. In the semiclassical approximation we can write $V_\alpha(z) = e^{2\alpha\phi(z)}$ which will be our Vertex operators up to quantum corrections[18] or in [16]. We define the n-point function to be:

$$\left\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \right\rangle = \int \mathcal{D}\phi e^{-S_{Liou}} \prod_{i=1}^{n} V_{\alpha_i}(z_i) \tag{161}$$

Liouville theory is conformal if we assume following relation between the central charge and the number Q which is called the charge at infinity which comes from the free field

[17] On an arbitrary Riemann surface

[18] Many more details can be found in [2]

representation of Liouville theory.

$$c = 1 + 6Q^2, \;\; Q = b^{-1} + b \tag{162}$$

The conformal dimensions are related to the momenta of the vertex in the following way:

$$\Delta_\alpha = \alpha(Q - \alpha) \tag{163}$$

To be complete we state the OPE of the Vertex operators:

$$\begin{aligned} T(w)T(z) =& \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2} T(w) \frac{1}{(z-w)} \partial_w T(w) \\ T(w)V_\alpha(z,\bar{z}) =& \frac{\Delta_\alpha}{(z-w)^2} V_\alpha(w) + \frac{1}{(z-w)} \partial_w V_\alpha(w) \end{aligned} \tag{164}$$

Beside the primary fields which are the vertex operator, there are also families of decandants that comesfrom the acting of the Virasoro modes on the primary fields:

$$L_K V_\alpha(z,\bar{z}) := L_{-K_1} \cdot L_{-K_2} \cdots L_{-K_n} V_\alpha(z,\bar{z}) \tag{165}$$

Here K is a partition. $K = k_1 \geq k_2 ... \geq k_n$. For completion of the Liouville setup we recall how the Virsoro modes $T(z) = \sum_{k=-\infty}^{\infty} \frac{L_n}{z^{n-2}}$ act on the the primary fields. For $n > 0$ the modes annihilate the the fields whereas for $n < 0$ it acts as a creation operator which is clear from (165). For the zero mode we have:

$$L_0 V_\alpha = \Delta_\alpha V_\alpha \tag{166}$$

For the descandants we get:

$$L_0 V_{\tilde{\alpha}} = \Delta_{\tilde{\alpha}} V_{\tilde{\alpha}} = \left(\Delta_\alpha + \sum_{i=1}^{n} k_i \right) V_{\tilde{\alpha}} \tag{167}$$

3.1 The 3 point function

Now we want to start to investigate the correlators in the Liouville theory. As the Liouville theory is conformal the position dependence of the 3-point function is completely fixed:

$$\left\langle \prod_{i=1}^{3} V_{\alpha_i}(z_i) \right\rangle = \frac{C(\alpha_1, \alpha_2, \alpha_3)}{|z_{12}|^{2\Delta_{12}} |z_{13}|^{2\Delta_{13}} |z_{23}|^{\Delta_{23}}} \tag{168}$$

Here we have $z_{ij} = |z_i - z_j|$ and $\Delta_{ij} = \Delta_k - \Delta_i - \Delta_j$ for unequal indices. As long as there are no technical problems arising we want to calculate the general n-point function. So lets expand the correlator in a power series in the cosmological constant μ.

$$\left\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \right\rangle = \sum_{N=0}^{\infty} V^{(N)}_{\alpha_1, \cdots, \alpha_n}(z_1, \cdots, z_n) \tag{169}$$

It is imporant to stress that this correlator we are investigating will be on genus 0 Riemann surface so the action can be rewritten in the following form:

$$\mathcal{L} = \frac{1}{4}(\partial_m \phi)^2 + \mu e^{2b\phi} \tag{170}$$

Now boundary conditions have to be fixed to be able to study Liouville on the sphere and we have to fix what happends at infinity. The following condition can be seen as a source of curvature because we have chosen the metric to be flat.

$$\phi(z, \bar{z}) = -Q\ln|z|^2 + O(1), \; |z| \to \infty \tag{171}$$

Here we have:

$$V^{(N)}_{\alpha_1,\dots,\alpha_n} = \frac{(-\mu)^N}{N!} \int = \langle V_{\alpha_1}(z_1) \cdots V_{\alpha_n}(z_n) V_b(u_1) \cdots V_b(u_N) \rangle_{free} \, d^2u_1 \cdots d^2u_N \tag{172}$$

The bracket in the integration is defined as the path integral over the free field and so we have

$$\begin{aligned} \left\langle \prod_{\alpha_i} V(z_i) \right\rangle_{free} &= \int \prod_{i=1}^{N} e^{2\alpha_i \phi(z_i)} \exp\left(-\frac{1}{4\pi} \int (\partial_a \phi)^2 d^2 z \right) D\phi \\ &= \prod_{i>j} |z_i - z_j|^{-4\alpha_i \alpha_j} \end{aligned} \tag{173}$$

But it is known that the derivation only makes sense if the follwing condition is fulfilled:

$$\sum_{i=1}^{n} \alpha_i = Q - Nb \tag{174}$$

This is due to the fact that the n-th term does not match the boundary condition (171) when the relation (174) is not correct. One can say that the n-point function has a pole every time the equation (174) is fulfilled for every N. So we have the following equation:

$$\text{res}_{\sum \alpha_i = Q - Nb} \left\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \right\rangle = V^{(N)}_{\alpha_1,\dots,\alpha_n}(z_1, \dots, z_n)|_{\sum \alpha_i = Q - Nb} \tag{175}$$

It is of great value that in case $n = 3$, this equation leads to a possible solution for the 3-point function. The key is that we have the following equation:

$$V^{(N)}_{\alpha_1,\dots,\alpha_3}(z_1, \dots, z_3)|_{\sum \alpha_i = Q - Nb} = \frac{I_N(\alpha_1, \alpha_2, \alpha_3)}{|z_{12}|^{2\Delta_{12}} |z_{13}|^{2\Delta_{13}} |z_{23}|^{\Delta_{23}}} \tag{176}$$

where the coeffiecents have been calculated in the mid 80s by Fateev and Dotensko to be:

$$I_N(\alpha_1, \alpha_2, \alpha_3) = \left(\frac{-\pi\mu}{\gamma(-b^2)} \right)^N \frac{\prod_{j=1}^{N} \gamma(-jb^2)}{\prod_{k=0}^{N-1} [\gamma(2\alpha_1 b + kb^2)\gamma(2\alpha_2 b + kb^2)\gamma(2\alpha_3 b + kb^2)]} \tag{177}$$

Here we have:

$$\gamma(x) = \Gamma(x)/\Gamma(1-x) \tag{178}$$

Comparing the position dependence of the 3-point function with the derivation done by Fateev and Dotensko we can conclude that the condition (174) for $n = 3$ is equivalent to the following statement:

$$\text{res}_{\sum \alpha_i = Q - Nb} C(\alpha_1, \alpha_2, \alpha_3) = I_N(\alpha_1, \alpha_2, \alpha_3) \tag{179}$$

In fact, now it is possible to solve for C (168). However, we know that it is only defined in this derivation for integer screening charge $Q - \alpha/b$. We will now make a choice for the function and this function will pass the condition (176). The analytic continuation of this function is one of the main subjects of [17] that was published recently.
Now we want to introduce a special function $\Upsilon(x, b)$ that depending on a variable and on the parameter b . We define the function for $0 < \text{Re}x < Q$ as

$$\log\Upsilon(x, b) = \int_0^\infty \frac{dt}{t}\left[\left(\frac{Q}{2} - x\right)^2 e^{-t} - \frac{\sinh^2(\frac{Q}{2} - x)\frac{t}{2}}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right] \tag{180}$$

Lets fix b. The following equations can be deduced:

$$\Upsilon(x) = \Upsilon(Q - x), \ \Upsilon\left(\frac{Q}{2}\right) = 1$$
$$\Upsilon(x + b) = \gamma(bx)b^{1-2bx}\Upsilon(x), \ \Upsilon(x + 1/b) = \gamma(x/b)b^{-(1-2b^{-1}x)}\Upsilon(x) \tag{181}$$
$$\tag{182}$$

Obviously this function stays the same after the transformation $b \to b^{-1}$. The last definition is:

$$\Upsilon_0 = \frac{d\Upsilon(x)}{dx}|_{x=0} \tag{183}$$

With this we can define a function which will satisfy the condition (179) so this is conjectured to be the exact 3-point function [18] [19] and is called the DOZZ function named after the scientist that evaluated the 3-point function along the lines stated here:

$$C(\alpha_1, \alpha_2, \alpha_3) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{(Q-\sum\alpha_i)/b} \times \frac{\Upsilon_0\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1 + \alpha_2 + \alpha_3 - Q)\Upsilon(\alpha_1 + \alpha_2 - \alpha_3)\Upsilon(\alpha_2 + \alpha_3 - \alpha_1)\Upsilon(\alpha_3 + \alpha_1 - \alpha_2)} \tag{184}$$

This function has more poles as as we were searching for but this can be resolved by introduction of a potential that scales with the inverse of b which is no problem because we have seen that Liouville theory is self dual.

3.2 Conformal blocks

After anticipating the form of the 3-point function we can now go further and look for other correlators. The next natural step is to ask for the 4-point function. It is a central object in the AGT conjecture and we want to go through the analysis of this function. First of all we can fix some dependence on the coordinates on the Riemann surface, where we will restrict to a sphere. On the sphere we will have positions $z_i, i = 1, ..4$. Regarding the conformal invariance we can bring the fourpoint function to the following form:

$$\begin{aligned}\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)V_{\alpha_4}(z_4)\rangle =&|z_{34}|^{2(\Delta_2+\Delta_1-\Delta_4-\Delta_3)}|z_{14}|^{2(\Delta_3+\Delta_2-\Delta_4-\Delta_1)}\times \\ &|z_{23}|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)}|z_{24}|^{-4\Delta_2}G_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(z,\bar{z})\end{aligned} \tag{185}$$

Here z is the cross ratio of the four coordinates. Thus there will be a dependence on the coordinates of at least one vertex operator. The four point function can be decomposed into pairs using OPEs. Depending on which pair you will have different channels like in the Feynmann graph expansion. We begin by introducing a Bra/Ket notation

$$\langle\alpha_1|V_{\alpha_2}(1)V_{\alpha_3}(q)|\alpha_4\rangle := \langle V_{Q-\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(q)V_{\alpha_4}(\infty)\rangle \tag{186}$$

Consider two partitions K, K'. We define a primary field $e^{2\alpha\psi}$ in this notation as $|\alpha>$ and the family of descendants as $|\psi_K(\alpha)>= L_{-K_1}\cdots L_{-K_r}|\alpha>$. Further we define the matrix $K =<\psi_K|\psi_{K'}>$. As a next step we have a completeness relation with respect to the primary fields.

$$1 = \int d\alpha \sum_{K,K'} |\psi_K(\alpha)> (K^{-1})_{K,K'} < \psi_{K'}(\alpha)| \tag{187}$$

We insert this relation into the four-point function

$$\langle\alpha_1|V_{\alpha_2}(1)V_{\alpha_3}(q)|\alpha_4\rangle = \int d\alpha \sum_{K,K'} \langle\alpha_1|V_{\alpha_2}(1)|\psi_K(\alpha)\rangle\, (K^{-1})_{K,K'}\, \langle\psi_{K'}(\alpha)|V_{\alpha_3}(q)|\alpha_4\rangle \tag{188}$$

We can now define the normalized conformal blocks of the theory as:

$$\mathcal{B}(\alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,q) = \frac{\sum_{K,K'} \langle\alpha_1|V_{\alpha_2}(1)|\psi_K(\alpha)\rangle\, (K^{-1})_{K,K'}\, \langle\psi_{K'}(\alpha)|V_{\alpha_3}(q)|\alpha_4\rangle}{\langle\alpha_1|V_{\alpha_2}(1)|\alpha\rangle\, \langle\alpha|V_{\alpha_3}(q)|\alpha_4\rangle} \tag{189}$$

With this, one can write the four-point function as:

$$\langle\alpha_1|V_{\alpha_2}(1)V_{\alpha_3}(q)|\alpha_4\rangle = \int d\alpha\, \langle\alpha_1|V_{\alpha_2}(1)|\alpha\rangle\, \langle\alpha|V_{\alpha_3}(q)|\alpha_4\rangle\, \mathcal{B}(\alpha,\alpha_i,q) \tag{190}$$

The four point function can be deduced from the 3-point function by gluing them together and integrating over the primary fields weighted by the conformal blocks. The conformal block is simply the summation over all the descendants famliles in a certain intermediate channel which depends on how you choose the different pairs in the correlator. It turns out that the conformal blocks will have an interpretation in supersymmetric gauge theories which can be anticipated by following fact: If we have a sphere with four punctures each corresponding to a vertex operator, we will have four external conformal dimensions and one internal dimensions coming from the integration over the whole spectrum. Furthermore we will have the cross ratio of the four puncture posititions. If we turn to the easiest example of $\mathcal{N} = 2$ gauge theory with vanishing β-function, we will see that this theory can be represtend in the exact same way. There, the parameters are the masses of the hypermulitplets, the gauge coupling constant. Even the central charge will have an interpretation in Ω-deformed theories, which will be the topic of the next chapter. However before we turn to this topic we want to present another way of rewriting the four point function. Start with the scalar product of two states, which is the two-point function at zero and infinty. Then remember that, as we are in a CFT, we have an OPE. So take the four-point function and insert two OPEs. E.g. for the first two operators and for the second two operators. You end up with a function that depends on a two point function because the OPE reduces the dependence from two operators to one. By using the conformal symmetry to change the coordinates and inserting for the two-point function a well known identity. The you read off the conformal Block for a sphere with four insertions:

$$\mathcal{B}(\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, q) = \sum_{|Y|=|Y'|} q^{|Y|} \lambda^{\Delta}_{\Delta_1 \Delta_2}(Y) K^{-1}_{YY'} \lambda^{\Delta}_{\Delta_3 \Delta_4}(Y') \tag{191}$$

Where the λ are the three point functions with two primary and one decendant corresponding to partition Y. In formula:

$$\lambda_{\Delta \Delta_1 \Delta_2}(Y) = \prod_{k=0}^{l(Y)} \left(\Delta + k_l \Delta_1 - \Delta_2 + \sum_{j<k} k_j \right) \tag{192}$$

Before turning to the Nekrasov partition function we should mentioned that we have only dealed with the complex variables without taking care of the conjugated dependency. Normally the four point function does not only depend on the cross ratio but also on the complex conjugated cross ratio. This should be reflected in the expansion of the four point function. The evaluation is almost the same and the four point function now factorizes into the holomorphic conformal blocks and the anti-holomorphic conformal blocks.

$$\langle \alpha_1 | V_{\alpha_2}(1) V_{\alpha_3}(q, \bar{q}) | \alpha_4 \rangle = \int d\alpha \, \langle \alpha_1^* | V_{\alpha_2}(1) | \alpha \rangle \, \langle \alpha^* | V_{\alpha_3}(q) | \alpha_4 \rangle \, \mathcal{B}(\alpha, \alpha_i, q) \bar{\mathcal{B}}(\alpha, \alpha_i, \bar{q}) \tag{193}$$

4 Nekrasovs Partition Function

This chapter continues with our studies around $\mathcal{N} = 2$ gauge theories focussing on another approach more direct to gauge theories [3]. In the Seiberg-Witten approach we deduced an elliptic curve enconding the low energy pysics of $\mathcal{N} = 2$ theories from monodromies on the moduli of space of vacua. Now we directly want to calculate the instanton contributions to the prepotential via direct integration and localization. Therefore we have to deal with many problems: On which field configurations does the path integral localize and how can the integration be resolved?
We summarize the strategy to obtain the Partition function of $\mathcal{N} = 2$ gauge theories [3, 20–26]:
1) We rewrite the Lagrangian in a slightly modified way because we want to be more general then in the $SU(2)$ case.
2) We twist the theory. This means we gauge the R-symmetry and redefine the Lorentz group. This will lead to a topological theory where the states are certain cohomology classes.
3) We show that the twisted Lagrangian comes from the topological action by the typical gauge fixing procedure.
4) The topological twisted theory (in mathematics Donaldson theory) will be protected by addition of certain terms. This fact will lead to the conclusion that the partition function localizes to the instanton contributions and, in more geneneral, case to the solution of the Seiberg-Witten monopole equation, which is the instanton equation modified by the matter fields and the Dirac equation in the instanton background.
5)Via localization of the path integral to instanton solutions we have to deal with the integration of the Moduli space of framed instantons and its construction
6)We reduce the integral to a finite integral of the equivariant Euler class over the moduli space.
7)By turning on more symmetries the integration can be resolved.
8)We evaluate the instanton partition function.

4.1 Twisting $\mathcal{N} = 2$ SYM

4.1.1 Twisting pure gauge theory

Firstly we write the action (39) in a slightly modified version, because we deal with gauge groups of higher rank so the dual coxeter number is different from 1. We also rewrite the action in terms of covariant derivatives and commutators and suppress lie algebra indices. The new $\mathcal{N} = 2$ Lagrangian is now:

$$\begin{aligned}\mathcal{L}_{\mathcal{N}=2} = &\frac{\theta}{32\pi h^V}\mathrm{Tr}F_{\mu\nu}\star F^{\mu\nu}+\\ &+\frac{1}{2g^2h^V}\mathrm{Tr}\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+(\nabla_\mu\phi)^\dagger\nabla_\mu\phi-\frac{1}{2}[\phi,\phi^\dagger]^2\\ &+i\psi^A\sigma^\mu\nabla_\mu\bar{\psi}_A-\frac{i}{\sqrt{2}}\psi_A[\phi^\dagger,\psi^A]+\frac{i}{\sqrt{2}}\bar{\psi}_A[\phi^\dagger,\bar{\psi}^A]\}\end{aligned} \tag{194}$$

Here the $\psi_A,\ A = 1,2$ represents the two different gluinos of the vectormultiplet. First of all we will do a redefinition of the Lorentzgroup to obtain a topological field theory. We construct a scalar operator:

$$\bar{Q} = \epsilon^{A\dot{\alpha}}\bar{Q}_{A,\dot{\alpha}} \tag{195}$$

This means that we have mixed spinors and spacetime indices that come from the redefinition of the Lorentz group by choosing the right chiral part to be the diagonal embedding of the old right chiral part with the rotation group of the R-symmetry: $SU(2)_{R'} = \text{diag}SU(2)_R \times SU(2)_I$. We have to redefine all fields in this twisting procedure, write down the new Lagrangian and calculate how this new operator acts on the new field content of the twisted version. We will make a suprsing observation noticed by Witten [22]: the twisted theory can be written as an exact "form". The twist does not change the theory if it is put on a manifold with trivial holonomy. So we can now write for the fields:

$$\psi_{A\alpha} = \frac{1}{2}\sigma^{\mu}_{A\alpha}\psi_\mu, \bar{\psi}_{A\alpha} = \frac{1}{2}\epsilon^{A\alpha}\bar{\psi} + \frac{1}{2}\bar{\sigma}^{A\dot{\alpha}}_{\mu\nu}\bar{\psi}^{\mu\nu} \tag{196}$$

The action changes in all terms with spinorial fields. To rewrite this action in the twisted form we need some equations for the spinor contraction of the Pauli matrices tensor and have to integrate by parts. After a short calculation, noting that the hodge star dual can be written in local coordinates as the Levi-Civita tensor, we arrive at:

$$\begin{aligned}\mathcal{L}_{twi\mathcal{N}=2} = & \frac{\theta}{32\pi h^V}\text{Tr}F_{\mu\nu} \star F^{\mu\nu} + \frac{1}{2g^2h^V}\text{Tr}\left\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\nabla_\mu\phi)^\dagger\nabla_\mu\phi - \frac{1}{2}[\phi,\phi^\dagger]^2\right\} \\ & + \frac{1}{2g^2h^V}\text{Tr}\left\{\frac{i}{2}\psi^\mu\nabla_\mu\bar{\psi} - \frac{i}{2}(\nabla_\mu\psi_\nu - \nabla_\nu\psi_\mu)^-\bar{\psi}^{\mu\nu} + \frac{i}{2\sqrt{2}}\psi_\mu[\phi^\dagger,\psi^\mu]\right\} \\ & + \frac{1}{2g^2h^V}\text{Tr}\left\{\frac{i}{2\sqrt{2}}\bar{\psi}[\phi,\bar{\psi}] - \frac{i}{2\sqrt{2}}\bar{\psi}_{\mu\nu}[\phi,\bar{\psi}^{\mu\nu}]\right\}\end{aligned} \tag{197}$$

Here it is indicated that $D^-_{\mu\nu} = \frac{1}{2}(D_{\mu\nu} - i \star D_{\mu\nu})$. We can now define the complete set of twisted supercharges and evaluate the variation on the fields after transforming every spinor [19] in the same way we transformed the gluino fields:

$$Q_\mu = \bar{\sigma}^{A\alpha}Q_{A\alpha}, Q_{\mu\nu} = \bar{\sigma}^{A\dot{\alpha}}_{\mu\nu}Q_{A\dot{\alpha}}. \tag{198}$$

Witten observed that the action can be written in the following form:

$$S_{twi\mathcal{N}=2} = S_{SYM} + S_{top} = \bar{Q}\tilde{S}_{SYM} + S_{top} \tag{199}$$

It can be shown that the topological term is even closed under action of the scalar symmetry. For completeness we write down the variations of the fields under the scalar symmetry:

$$\bar{Q}\phi = 0,\ \bar{Q}\phi^\dagger = \sqrt{2}\bar{\psi},\ \bar{Q}\psi_\mu = 2i\sqrt{2}\nabla_\mu\phi \tag{200}$$

$$\bar{Q}\bar{\psi} = 2i[\phi,\phi^\dagger],\ \bar{Q}\psi_{\mu\nu} = -2(F_{\mu\nu})^-,\ \bar{Q}A_\mu = -i\psi_\mu \tag{201}$$

[19]Here it is indicated the SUSY transformation parameter that is a spinor

We can now easily extract S_{SYM} in terms of the scalar symmetry

$$S_{SYM} = \text{Im}\left[\bar{Q}\left\{\frac{\tau}{16\pi h^V}\int \mathrm{d}^4x Tr((F_{\mu\nu})^- \bar{\psi}^{\mu\nu} - i\sqrt{2}\psi^\mu \nabla_\mu \phi^\dagger + i\bar{\psi}[\phi, \phi^\dagger]\right\}\right] \tag{202}$$

Here one needs the equation of motion for the twisted tensorial spinor field. The scalar SUSY generator squares to zero up to a gauge transformation: $\bar{Q}^2 = G(\phi)$.

4.1.2 Lagrangian from gauge fixing

We will now show that the twisted topological Lagrangian can be understood by gauge fixing the ordinary topological action [24]:

$$S_{top} = \frac{\theta}{32\pi^2 h^V}\int d^4x \text{Tr}\left\{F_{\mu\nu} * F^{\mu\nu}\right\} \tag{203}$$

We have to introduce ghosts that come from the surving R-symmetry under the twist. The first observation is that the topological action is invariant under certain tranformations of the gauge field that are larger than the Yang-Mills symmetry:

$$\delta A_\mu = -\nabla_\mu \alpha + \alpha_\mu \tag{204}$$

It is clear that the function α has to be a Lie algebra valued function like α_μ because the gauge field is a Lie algebra valued 1-form. The only restriction is that $A_\mu + \alpha_\mu$ are in the same orbit of the gauge group. The bigger symmetry comes from the family of 1-forms that makes the symmetry bigger than the usual Yang-Mills symmetry. We introduce ghosts for each symmetry. Obviously there are two symmetries, but there is also a third hidden symmetry that comes from the fact that α_μ and $\alpha_\mu - \nabla_\mu \beta$ produce the same trafo for the gauge field strength. So we introduce ghosts ψ_μ, c, H with ghost numbers $1, 1, 2$. Now we fix the gauge.

$$\begin{aligned} \nabla^\mu A_\mu &= 0 \\ (F_{\mu\nu})^- &= 0 \\ \nabla^\mu \psi_\mu &= 0 \end{aligned}$$

We further have to introduce antighosts and of course Lagrangian multipliers to implemend the constraints. We start with the multipliers that are b, $H^{\mu\nu}$[20] that are bosons and a fermion η. The ghost numbers are $(0, 0, 1)$. The antighosts are $\bar{c}, \chi_{\mu\nu}, \lambda$ with ghost number $(-1, -1, -2)$. The next step is to introduce a BRST charge that corresponds to the symmetries. This can be extracted by reproduction of the transformation law for the gauge field corresponding to the ghost and by the requierment that the BRST

[20]This tensor is antiselfdual

charge is nilpotent. We get:

$$\begin{aligned}
\bar{Q}A_\mu =& -\nabla_\mu c - i\psi_\mu \\
\bar{Q}c =& -\frac{i}{2}\{c,c\} - \phi \\
\bar{Q}\psi_\mu =& -i\nabla_\mu\phi - i\{c,\psi_\mu\} \\
\bar{Q}\phi =& -i[c,\phi] \\
\bar{Q}\bar{c} =& b \\
\bar{Q}\chi_{\mu\nu} =& H_{\mu\nu} - i\{c,\chi_{\mu\nu}\} \\
\bar{Q}\lambda =& \eta - i[c,\lambda] \\
\bar{Q}b =& 0 \\
\bar{Q}H_{\mu\nu} =& -i[\phi,\chi_{\mu\nu}] - i[c,H_{\mu\nu}] \\
\bar{Q}\eta =& -i[\phi,\lambda] - i\{c,\eta\}
\end{aligned} \tag{205}$$

(206)

Indeed a short calculation shows that the BRST charge is nilpotent. We arrive at the following expression which is equivalent to the twisted theory:

$$S_{gf} = S_{top} + \bar{Q}V_{SYM} \tag{207}$$

Here the function V_{SYM} is given as:

$$V_{SYM} = \frac{1}{h^V g^2}\int d^4x \mathrm{Tr}\left\{\frac{1}{2}\chi^{\mu\nu}\left(F^-_{\mu\nu} + \frac{1}{4}H_{\mu\nu}\right) + \frac{i}{8}\lambda\nabla_\mu\psi^\mu + \bar{c}(\nabla_\mu A^\mu + b) - \frac{1}{128}\eta[H,\lambda]\right\} \tag{208}$$

If certain fields are identified with the topological fields at the end, the direct conclusion is that the twisted action (197) that makes the theory topological is indeed the same as the gauge fixed version of the standard topological action (203)

$$\begin{aligned}
H = -2\sqrt{2}\phi, \lambda = -2\sqrt{2}\phi^\dagger \\
\chi_{\mu\nu} = \bar{\psi}_{\mu\nu}, \eta = -4\bar{\psi}
\end{aligned}$$

4.1.3 Twisting matter fields

We have twisted the Lagrangian of $\mathcal{N}=2$ pure Super Yang Mills and now we have to deal with theories coupled to matter. Thus we have to look how the hypermultiplets change and hope that we can write the the action again as a $\bar{Q}$-exact form. The hypermultiplet consists of two complex scalars and two complex scalars that can be written as half-hypermultiplets in the conjugated representations of the gauge group in the $\mathcal{N}=1$ language. We take the lowest component as $q^A \to q^{\dot{\alpha}}$ as the twist is performed[21]. Then we go through all the terms in the Lagrangian for matter multiplets coupled to a vector multiplet. As an example see (47).

[21]The chiral multiplet is given by $Q = q + \sqrt{2}\theta\chi + \theta\theta X$

Let us begin with the massless case. We use the equation of motion[22] for the auxiliary field in the $\mathcal{N}=1$ chiral superfields which build the matter of the $\mathcal{N}=2$ theory as the hypermultiplets consist of two conjugated chiral superfields in the $\mathcal{N}=1$ language. We arrive at the following expression:

$$\begin{aligned} S_{mat} = \frac{1}{h^V g_0^2} \int d^4x \mathrm{Tr} \nabla_\mu q_A^\dagger \nabla^\mu q^A + i\chi^\alpha \sigma^\mu_{\alpha\dot\alpha} \nabla_\mu \bar\chi^{\dot\alpha} + i\tilde\chi^\alpha \sigma^\mu_{\alpha\dot\alpha} \nabla_\mu \bar{\tilde\chi}^{\dot\alpha} \\ + \tilde\chi^\alpha \phi \chi_\alpha - \bar\chi_{\dot\alpha} \phi^\dagger \bar{\tilde\chi}^{\dot\alpha} + \sqrt{2} q_A^\dagger \psi^{A,\alpha} \chi_\alpha - \sqrt{2} \bar\chi_{\dot\alpha} \psi^{\dot\alpha}_A q^A + \sqrt{2} q_A^\dagger \bar\psi^A_{\dot\alpha} \bar{\tilde\chi}^{\dot\alpha} - \sqrt{2} \tilde\chi^\alpha q^A \psi_{A,\alpha} \\ + .q_A^\dagger \left(\phi\phi^\dagger + \phi^\dagger\phi\right) q^A - \frac{1}{2}\left(q^{\dagger A} T^{\rho a} q^B + q^{\dagger B} T^{\rho a} q^A\right) q_A^\dagger T_a^\rho q_B \end{aligned} \tag{209}$$

We have to introduce a new pair of auxiliary fields as the SUSY-trafos are not closed off-shell: h_α and $\tilde h_\alpha$. This is due to the fact that we have integrated out[23] the auxiliary fields in the chiral superfields. We introduce new fields - to avoid some numerical issues - $\bar\mu^{\dot\alpha}$, $\mu_{\dot\alpha}$, ν_α and $\bar\nu^\alpha$ as follows:

$$\begin{aligned} \sqrt{2}\bar{\tilde\chi}^{\dot\alpha} = \mu^{\dot\alpha}, \chi_\alpha = \sqrt{2}\nu_\alpha, \\ \sqrt{2}\bar\chi_{\dot\alpha} = \bar\mu_{\dot\alpha}, \tilde\chi^\alpha = \sqrt{2}\bar\nu^\alpha \end{aligned}$$

Closed off-shell (up to a gauge transformations) BRST operator $\bar Q$ is given by the following relations:

$$\begin{aligned} \bar Q q^{\dot\alpha} = \mu^{\dot\alpha},\ \bar Q \mu^{\dot\alpha} = \phi q^{\dot\alpha} \\ \bar Q q^\dagger_{\dot\alpha} = \bar\mu_{\dot\alpha},\ \bar Q \bar\mu_{\dot\alpha} = -q^\dagger_{\dot\alpha}\phi \\ \bar Q \bar\nu^\alpha = \bar h^\alpha,\ \bar Q \bar h^\alpha = -\bar\nu^\alpha \phi \\ \bar Q \nu_\alpha = h_\alpha,\ \bar Q h_\alpha = \phi \nu_\alpha. \end{aligned} \tag{210}$$

Using these formulae one can check that the matter action can be rewritten as a $\bar Q$-exact expression: $S_{mat} = \bar Q V_{mat}$ where

$$\begin{aligned} V_{mat} = \frac{1}{h^V g_0^2} \int d^4x \mathrm{Tr} - \frac{i}{2} \chi_{\mu\nu} q^\dagger_{\dot\alpha} \bar\sigma^{\mu\nu,\dot\alpha}_{\dot\beta} q^{\dot\beta} - \frac{1}{4}\left(\bar\mu_{\dot\alpha}\lambda q^{\dot\alpha} - q^\dagger_{\dot\alpha}\lambda\mu^{\dot\alpha}\right) \\ + .2\bar\nu^\alpha \left(\sigma^\mu_{\alpha\dot\alpha}\nabla_\mu q^{\dot\alpha} - h_\alpha\right) - 2\left(\nabla_\mu q^\dagger_{\dot\alpha} \bar\sigma^{\mu,\dot\alpha\alpha} - \bar h^\alpha\right)\nu_\alpha. \end{aligned} \tag{211}$$

Now consider the general case, where the mass is not zero. After integrating out all the auxiliary fields we obtain the following terms in the action:

$$S_{mass} = \frac{1}{h^V g_0^2} \int d^4 \mathrm{Tr} \left\{ -m^2 q_A^\dagger q^A + \sqrt{2} m q_A^\dagger H q^A + \sqrt{2} m q_A^\dagger H^\dagger q^A - m \bar{\tilde\chi}^{\dot\alpha} \bar\chi_{\dot\alpha} - m\tilde\chi^\alpha \chi_\alpha \right\}. \tag{212}$$

[22] which is algebraic and thus it does not change the action

[23] We mean using the e.o.m

The mass term deforms the supersymmetry transformation further. The proper version of the off-shell BRST transformation is now given by

$$\begin{aligned} \bar{Q}q^{\dot{\alpha}} &= \mu^{\dot{\alpha}},\ \bar{Q}\mu^{\dot{\alpha}} = \phi q^{\dot{\alpha}} + mq^{\dot{\alpha}} \\ \bar{Q}q^{\dagger}_{\dot{\alpha}} &= \bar{\mu}_{\dot{\alpha}}, \bar{Q}\bar{\mu}_{\dot{\alpha}} = -q^{\dagger}_{\dot{\alpha}}\phi - mq^{\dagger}_{\dot{\alpha}} \\ \bar{Q}\bar{\nu}^{\alpha} &= \bar{h}^{\alpha}, \bar{Q}\bar{h}^{\alpha} = -\bar{\nu}^{\alpha}\phi - m\bar{\nu}_{a} \\ \bar{Q}\nu_{\alpha} &= h_{\alpha}, \bar{Q}h_{\alpha} = \phi\nu_{\alpha} + m\nu_{\alpha}. \end{aligned} \tag{213}$$

The deformation leads to a new characteristic of the BRST operator. Before we had

$$\bar{Q}^2 = G(\phi) \tag{214}$$

where $G(\phi)$ is the gauge transformation with the parameter ϕ. The new BRST operator satisfies the new relation:

$$\bar{Q}^2 = G(\phi) + F(m). \tag{215}$$

Here $F(m)$ is an operator which does not act on the gauge multiplet,as it seen from (213) but multiplies matter fields by $\pm m$ which is the the linear term of the following transformation:

$$Q \to Q' = e^{m}Q,\ \tilde{Q} \to \tilde{Q}' = e^{-m}\tilde{Q} \tag{216}$$

Thus it is the infinitesimal form of this operator and can be identified with the flavor group action if one redfine $m \to im$. This is due to the fact that the flavor symmetry gets broken by a mass deformation to $U(1)$. To require the full mass terms we have to add to to the action a BRST exact term $\bar{Q}V_{mass}$ where

$$V_{mass} = \frac{1}{h^V g_0^2}\int d^4x \mathrm{Tr}\left\{-\frac{1}{4}m\left(q^{\dagger}_{\dot{\alpha}}\mu^{\dot{\alpha}} + \bar{\mu}_{\dot{\alpha}}q^{\dot{\alpha}}\right)\right\} \tag{217}$$

The fact that the full action is BRST invariant follows from the fact that every term is invariant with respect to the transformation of the nilpotent operator

$$S = S_{top} + \bar{Q}\left(V_{SYM} + V_{mat} + V_{mass}\right) \tag{218}$$

4.1.4 Localizing the solutions to instantons

Now we can localize the solutions to instanton configurations in the pure gauge scenario. This is done by deformation the Lagrangian by a Q-exact term to the Lagrangian that depends on a parameter λ after integrating out some auxiliary fields that were ghosts. So we lead to an equivalent Lagrangian of the following form:

$$S = S_{\mathsf{T}} + \int d^4x \mathrm{Tr}\left(-t^2\left(F_{\mu\nu}\right)^{-}\left(F^{\mu\nu}\right)^{-} + O(t^1) + O(t^0)\right) \tag{219}$$

Since the twisted theory does not depend on this parameter because it is topological, we can change λ as we want. By sending $\lambda \to \infty$ we see that the only term that contributes to the Lagrangian is the selfdual part. These are the selfdual connections called

Instantons. These configurations are non-perturbative extrema of the supersymmetric Yang-Mills action. In fact we just have to integrate over the moduli space of instantons, to obtain the partition function because our system localizes to these configurations and as we want to count gauge inequivalent instantons. Here it is important that we dealing talking about framed instantons. This framing corresponds to the higgs expectation value or to trivial gauge transformation at infinity. The basic information is that our Path integral reduced from integrating over the space of fields to a finite dimensional manifold which is the moduli space of instantons.

4.2 Some facts about the moduli space of Instantons

Here, the construction of the moduli space of Instantons will be introduced in brief and some known mathematical facts will be quoted about. We have seen that the partition function reduces to an integral over the moduli space of instantons. This is the space of all self dual connections of certain vector bundle modulo gauge transformations. These transformations are the transition functions of the associated principal G-bundle. The moduli space is constructed by some data called the ADHM(Atiyah-Drinfeld-Hitchin-Manin)-data.

4.2.1 ADHM construction

First of all we introduce a complex vector bundle E of rank N over R^4 where the fiber is an N-dimensional vector space framed at infinity. This means that the fiber at infinity is isomorphic to the N-dimensional complex vector space which correspond to a fixed higgs vev at infinity for the scalar field. On this bundle we introduce a connection which will be the instanton. Instantons are characterized by

$$F_A = *F_A \tag{220}$$

where $*$ is the hodge star operator. For $U(N)$ constructions we want to summarize the construction. We introduce two hermitian complex vector spaces V (k-dimensional) and W (N-dimensional) and certain linear mappings between them. In the literature this mappings are called B_1, B_2, I, J

$$B_1 \in \mathrm{Hom}(V,V),\ B_2 \in \mathrm{Hom}(V,V),\ I \in \mathrm{Hom}(W,V), J \in \mathrm{Hom}(V,W) \tag{221}$$

The vector space W is isomorphic to the fiber of E. This is clear because the vector bundle has rank N, which is the same as the dimension of W. Due to the framing it can be seen as the fiber at infinity and there is a natural action of $U(N)$ because of the freedom to change the basis in a complex hermitean vector space.[24] There is also the freedom of framing in V, so we have a natural action of $U(k)$ on V. In fact we have an action of $U(N) \times U(k)$ on the vector spaces and so on the ADHM data. We

[24] This can be seen as the rigid gauge transformation at infinity

introduce the momentum maps $\mu_{\mathbb{R}}, \mathrm{Re}\mu_{\mathbb{C}}, \mathrm{Im}\mu_{\mathbb{C}}$ associated to $U(k)$ action so it is a map $\mu_i = (\mu_{\mathbb{R}}, \mathrm{Re}\mu_{\mathbb{C}}, \mathrm{Im}\mu_{\mathbb{C}}) : \mathcal{X} \to \mathfrak{u}(k)^* \otimes \mathbb{R}^3$ with

$$\mathcal{X} = \mathrm{Hom}(V,V) \oplus \mathrm{Hom}(V,V) \oplus \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W) \tag{222}$$

The ADHM equations are explicitly given by:

$$\begin{aligned} \mu_{\mathbb{R}} &= \left[B_1, B_1^\dagger\right] + \left[B_2, B_2^\dagger\right] + II^\dagger - J^\dagger J \\ \mu_{\mathbb{C}} &= [B_1, B_2] + IJ \end{aligned} \tag{223}$$

In [23] the moduli space of instantons was indentified with the following Hyperkaehler quotient

$$\mathcal{M}_{k,N} = \mathcal{X}///U(k) = \{\mu_{\mathbb{R}} = \mu_{\mathbb{C}} = 0\}/U(k) = \mu^{-1}(0)/U(k) \tag{224}$$

In the following we want to explain why this is correct. Let $X = (B_1, B_2, I, J) \in \mathcal{X}$ be a solution to the ADHM-equations (223). We want to construct a self dual connection over $\mathbb{R}^4$. Let us parametrize an operator by the two complex variables $(z_1, z_2) = x \in \mathbb{R}^4$ by introduction of a complex structure on the total space $\mathbb{R}^4$. We introduce the bundels of dotted and undotted spinors, the positive and negativ chirality spinor bundles. The spinors are sections of this bundle. We define an operator:

$$\triangledown^\dagger(x) = \begin{pmatrix} I & -(B2 - z_2) & B_1 - z_1 \\ J^\dagger & (B_1 - z_1)^\dagger & (B_2 - z_2)^\dagger \end{pmatrix} : W \oplus S^- \otimes V \to S^+ \otimes V \tag{225}$$

Then we compute $\triangledown^\dagger(x) \triangledown (x)$ which acts on the right hand side $S^+ \otimes V$. By simple matrix multiplication the following equation results:

$$\triangledown^\dagger(x) \triangledown (x) = \mathrm{id}_{S^+} \otimes \boxdot \tag{226}$$

The structure of $\boxdot$ brings us to the conlusion that $\triangledown^\dagger(x)$ is surjective and hence we can construct a vector bundle as the kernel of the map with a self dual connection coming from the hermitean metric on $W \oplus V \oplus V$. This will be a $U(N)$-connection where the curvature will be selfdual and the instanton number will be k. It turns out that the moduli space of $U(N)$ k-instantons has the dimension:

$$\mathrm{dim}\mathcal{M}_{k,N} = 4kN \tag{227}$$

4.3 Equivariant cohomology and localization theorems

Here we briefly want to summarize technical facts that allow the evaluation of integrals over complicated manifolds as fixpoint contributions of some torus group action [27]. Therefore we have to introduce the notion of equivariant cohomology and quote some deep mathematical results of localization which may be used often in topological theories like $\mathcal{N} = 2$ twisted SYM.

4.3.1 Equivariant cohomology

Let us begin with a smooth group action of a compact Lie group G on some oriented, compact n-dimensional manifold[25] M. This is a map

$$G \times M \to M, (g, x) \to g \cdot x, x \in M, g \in G \tag{228}$$

If the action is free, we define the equivariant cohomology as:

$$H_G^*(M) := H^*(M/G) \tag{229}$$

Two homotopy equivalent spaces have the same cohomology[26]. From this observation we can construct the equivariant cohomology even if the group action is not free[27]. One can show that for every space there is a homotopy equivalent space on which the group G acts freely. So if the action on M is not free, define $\tilde{M} = M \times E$ where E is a contractible space define the equivariant cohomology as:

$$H_G^*(M) := H^*((M \times E)/G) \tag{230}$$

One can show that indeed the cohomology as it was just defined is independent of the choice of contractible space.
Now we have a Lie algebra $\mathfrak{g}$ that acts on the Lie group via the adjoint action. So we define a G-equivariant form to be a C^∞ map $\alpha : \mathfrak{g} \to \Omega^*(M)$ that fulfills the following relation

$$\alpha(X) = g^{-1} \cdot \alpha(Ad(g)X), \forall X \in \mathfrak{g}, g \in G \tag{231}$$

$\Omega^*(M)$ is the algebra of complex valued k-forms on M. We will need this to introduce the twisted deRham complex which will be the key step to the localization theorems. The wedge product respects the equivariance so we have an algebra of equivariant forms which we denote with $\Omega_G^*(M)$. The group action induces an action on the space of functions:

$$(g \cdot f)(x) := f(g^{-1} \cdot x) \tag{232}$$

From this we can define the Lie derivative by taking the derivative of (232) which is a vector field:

$$(L_X f)(g) := \frac{d}{dt} f(\exp(-tX) \cdot x)|_{t=0} \tag{233}$$

As a next step we want to define the interior product which is a contraction of a differential form with a vector field. For $X \in \mathfrak{g}$ we define the $\iota_X = \iota(L_X)$ and this is defined as $\iota_X(\omega)(X_1, ..., X_{p-1}) = \omega(X, X_1, ..., X_{p-1})$ so it maps a k-form to a (k-1)-form and thereby inducing a map from the algebra of forms to itself. Now we can define the twisted deRham differential, which is an operator d_{tdR} from the algebra of G-equivariant

[25]We will need the condition to have a well-defined integral
[26]We mean Cech or singular cohomology
[27]Non-free group action can destroy the smooth manifold structure. E.g. the orbifold case

forms to itself. More precise this means:
Let $\alpha \in \Omega_G(M)$, then $d_{tdR}\alpha : \mathfrak{g} \to \Omega^*(M)$ with

$$d_{tdR}(\alpha)(X) = d\alpha(X) - \iota_X(\alpha(X)) \tag{234}$$

Here the d is the ordinary deRham differential and one can show that this twisted deRham differential maps into the algebra of G-equivariant forms and is nilpotent with degree two like in ordinary deRham theory, in formula: $d_{tdR}^2 = 0$. This allows us to define the twisted deRham complex as in the usual deRham cohomology theory as follows: We take a subalgebra of $\Omega_G^*(M)$ consisting of all polynomial maps $\alpha : \mathfrak{g} \to \Omega^*(M)$ which is the space $(\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M))^G$ where the G indicates that the polynomials are G-invariant. Let us define the twisted deRham complex: It is $((\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M))^G, d_{tdR})$.
One important fact is a theorem proved by Cartan. He noticed the twisted deRham complex is the equivariant cohomology of the manifold M. So if we want to compute cohomology classes of some topological field theory which is acted by e.g. a torus action we can compute the cohomology groups in the twisted deRham complex. For the forms that represent the cohomology classes of this complex we will evaluate localization theorems that make the evaluation of complicated integrals simpler.

4.3.2 Localization

Now we can state some strong results with the help of the equivariant cohomology which can not be deduced for ordinary cohomology.
Theorem (Berline-Vergne)
Let G be a compact group with Lie algebra $\mathfrak{g}$ acting on some Manifold M and let $\alpha : \mathfrak{g} \to \Omega^*$ be a C^∞ map such that it is closed under the twisted deRham differential. Let $X \in \mathfrak{g}$ such that L_X has isolated zeros. Then

$$\int_M \alpha(X) = (-2\pi)^{dimM/2} \sum_{p \in M_0} \frac{\alpha(X)_{[0]}(p)}{det^{1/2}(L(X,p))} \tag{235}$$

Here M_0 indicates the set of zeros of the vector field and on the l.h.s. we evaluate the zero degree part of α at a point $p \in M$. $L(X,p)$ is the transformation on the fiber of the tangent bundle induced by the Lie action and can be evaluated explicitly.
From this we come to our main tool in the computation of the partition function.
Theorem (Duistermaat-Heckman)
Let (M, ω) be a symplectic manifold with momentum map $\mu : M \to \mathfrak{g}^*$ where we want to restrict to $G = \mathbb{T}$ and so $\mathfrak{g}^* = Lie(\mathbb{T})^*$ or at least to maximal torus of the Lie group. Let x_f be the $\mathbb{T}$-stable points, so fixed under the group action of the torus. Then we get:

$$\int_M \frac{\omega^{\dim M/2}}{\dim M/2!} e^{-\langle \mu, \xi \rangle} = \sum_{x_f} \frac{e^{-\langle \mu(x_f), \xi \rangle}}{\prod_\alpha (\langle \omega_\alpha(x_f), \xi \rangle)} \tag{236}$$

So for simplicity we write $\langle \omega_\alpha(x_f), \xi \rangle = \omega_\alpha$, which are called the *weights*. In fact, if we have an equivariant closed form with respect to some torus and a manifold that have fixed points under the torus action we just have to evaluate the fixed points of this action and the weights to calculate the integral. This will lead to the evaluation of the partition function. There is one more aspect which is very imporant to mention. As we are dealing with supersymmetric theories we have to deform the localiztion formula to supermanifolds. This can easily be done by taking into account that the weights of different statistics will contribute inversly to the formula.

4.4 The partition function

Now we have learned some ways to calculate certain integrals and we already know that we will have to deal with integrals over the moduli space of instantons. In fact, we need to know how to integrate over the zero locus of some function which in our case will be the momentum map from which the ADHM constrution works and of factors which will be the the quotient over the gauge group. We will quote the results for this integration which are prestended in a field theoretical way in [25]. First of all define the following vacuum expectation value for a $\bar{Q}$-closed gauge invariant operator:

$$\langle \mathcal{O} \rangle = \int \mathcal{D}X \mathcal{O} e^{S_{top} + \bar{Q}V_{SYM}} \tag{237}$$

This definition corresponds to the case of pure gauge theory as we have put the matter and mass potential to zero. The integration measure is the measure in the space of all fields that appear in the exponent. We already know that the pure SYM localizes to instantons by allowed deformations of the action so that the a priori infinite dimensional integral (237) will be reduced to something computable and finite dimensional in the end. In [25] the following two formulae were deduced. First of all for a manifold $M \subset X$ with $\iota : M \to X$ the inclusion map and x^μ local coordinates on a patch with differential $\mathrm{d}x^\mu = \psi^\mu$ and a supersymmetric multiplet (H, χ) in the fiber of a bundle over the ambient space X with s a section:

$$\int_X \alpha = \int \mathcal{D}x \mathcal{D}\psi \mathcal{D}H \mathcal{D}\chi \iota^* \alpha e^{i\bar{Q}\chi\left(s(x) - \frac{1}{2t}H\right)} \tag{238}$$

The next formula already uses the notion of equivariant cohomology and especially the fact that the cohomology of a quotient is isomorphic to the equivariant cohomology of the manifold.

$$\int_{s^{-1}(0)/G} \tilde{\alpha} = \oint_X \int \mathcal{D}\eta \mathcal{D}\lambda \mathcal{D}H \mathcal{D}\chi e^{i\bar{Q}(\chi s + \psi_\mu V^\mu(\lambda))} \iota^* \alpha(\phi, x, \psi) \tag{239}$$

To project on the factor one has to introduce a projection multiplet (η, λ) It is essential that the two forms correspond to the same cohomology class and the contour integration is called the *equivariant integration* and is defined as follows:

$$\oint_X \alpha = \frac{1}{\mathrm{Vol}(G)} \int_{\mathfrak{g}} \prod_{a=1}^{\dim G} \frac{\mathrm{d}\phi}{2\pi i} \int_M \alpha(\phi) \tag{240}$$

where $\mathfrak{g}$ is the Lie algebra and the integration measure is the Haar measure on the identity of the group which acts on the manifold equivariantly. Using formula (238) and using the fact that the function does not depend on the parameter t we can use the equation of motion show that (238) changes to the following integral:

$$\int_M \alpha = \int_X \iota^* \alpha \mathrm{Eu}_g(E) \tag{241}$$

where we defined the equivariant Euler class of the bundle $E \to X$ as the following expression:

$$\mathrm{Eu}_g(E) = \int \mathcal{D}\chi e^{\frac{1}{4}\chi R_{\mu\nu}\psi^\mu\psi^\nu\chi} \tag{242}$$

At this point we can give an interesting formula for the vev defined in (237). First of all we know that the instanton part of the partition function collapses into different sectors of different instanton numbers $q = e^{2\pi i\tau}$:

$$\langle \mathcal{O} \rangle_a = \sum_{k=0}^{\infty} e^{2\pi i k\tau} \int_{\mathcal{M}_k} \tilde{\mathcal{O}}_k \tag{243}$$

Analogous to this pure gauge situation one can study the theory with matter. So we have to deal with solutions to the Seiberg-Witten monopole equations. Again we can deform the action that the gauge fields localize to the self dual part, but in additon we have to solve Weyl equations in an instanton background. If we solve these equations, we see that the matter action gives additional contributions to the vev of the operator defined in (243). The action gets contributions as defined in (242), for the solutions to the Weyl equations. If we define the bundle of solutions to the Weyl equations over the moduli space of framed instantons as $\mathcal{D}_k \to \mathcal{M}_k$, we can now write the vev of the operator as follows [20]:

$$\langle \mathcal{O} \rangle_a = \sum_{k=0}^{\infty} e^{2\pi i k\tau} \int_{\mathcal{M}_k} \tilde{\mathcal{O}}_k \mathrm{Eu}_g(\mathcal{D}_k) \tag{244}$$

And in fact we know the partition functions of the pure gauge theory and also the matter theory:

$$Z_{pure} = \langle 1 \rangle_a = \sum_{k=0}^{\infty} e^{2\pi i k\tau} \int_{\mathcal{M}_k} 1 \tag{245}$$

$$Z = \langle 1 \rangle_a = \sum_{k=0}^{\infty} e^{2\pi i k\tau} \int_{\mathcal{M}_k} \mathrm{Eu}_g(\mathcal{D}_k)$$

4.5 Six dimensional origin and Ω-backround

The integration over the moduli space of instantons has some inconsistences that can be resolved in a beautiful way. Often it is convenient to introduce supersymmetric gauge theories in four dimension as a dimensional reduction of a higher dimensional theory on some torus. This can also be done in some nontrivial way using the symmetries of the four dimensional space time through introduction of a background with a nontrivial metric. To be more precise, we mean that we can come to a four dimensional Lagrangian from a higher dimensional construction. The theory in four dimension depends on the choice of compactification.

4.5.1 trival metric

Let us take $d = 6$ with $\mathcal{N} = 1$. Then we compactify the two extra dimensions on a Torus T^2 with a flat metric. We assume that the fields do not have a dependency on the two compactified dimensions because they are small. In 6 dimension we have a sympletic Majorana spinor ψ_A. The $\mathcal{N} = 1$ Lagrangian is given by the follwing action

$$S_{d=6} = \frac{1}{g^2 h^V} \int d^4x \mathrm{Tr} \left\{ -\frac{1}{4} F_{IJ} F^{IJ} + \frac{i}{2} \bar{\Psi}_A \Gamma^I \nabla_I \Psi^A \right\} \tag{246}$$

It is already a 4-dimensional integral because the extra dimensions are integrated out. We have $F_{\mu 4} = \nabla_\mu A_4$ and $F_{\mu 5} = \nabla_\mu A_5$ because the derivatives with respect to compact direction vanish. We define:

$$\phi = \frac{A_4 + A_5}{\sqrt{2}} \tag{247}$$

So we get $F_{45} = [\phi, \phi^\dagger]$. From the gauge term we see the following expression in the Lagrangian::

$$-\frac{1}{4} F_{IJ} F^{IJ} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \nabla_\mu \phi \nabla^\mu \phi^\dagger - \frac{1}{2} [\phi, \phi^\dagger]^2 \tag{248}$$

We already see the structure of the $\mathcal{N} = 2$ Lagrangian. Through the additional reduction of the Majorana spinor with the sympletic constraint the missing terms appear and we have shown that the Lagrangian can be construted in this way. Now we come to the question what happens if we compactify on a torus with nontrival metric. It is clear that we have to do some extra work because you could break all the supersymmetries in this procedure.

4.5.2 Ω-background

Now we will explore what happens if the torus T^2 acts on the four dimensional space by Lorentz tranformation. This is also a symmetry of the Yang-Mills theory. So let T^2 act on $\mathbb{R}^{1,3}$ or, after a Wick rotation, on $\mathbb{R}^4$ on the the euclidean space:

$$V_4 = \Omega^\mu_{4\nu} x^\nu, V_5 = \Omega^\mu_{5\nu} x^\nu \tag{249}$$

These are the vector fields that act on space time. The matrices are Lorentzmatrices. Now we define a nontrivial six dimensional metric on the product of space time with the torus and reduce the theory from six dimensions to four. This will lead to the conclusion that the Ω-background can only be choosen to have discrete matrices [3] because otherwise we would break all the supersymmetries because we could not be sure that we have at least one covariant constant spinor. The metric is

$$\begin{aligned} ds^2 &= g_{\mu\nu}\left(dx^\mu + V_a^\mu dx^a\right)\left(dx^\nu + V_b^\nu dx^b\right) - (dx^4)^2 - (dx^5)^2 \\ &= G_{IJ} dx^I dx^J \end{aligned}$$

So now we get:

$$\begin{aligned} G_{\mu\nu} = g_{\mu\nu},\ G^{\mu\nu} &= g^{\mu\nu} - V_a^\mu V_a^\nu, \\ G_{a\mu} = V_{a,\mu},\ G^{a\mu} &= V_a^\mu, \\ G_{ab} = -\delta_{ab} + V_a^\mu V_{b,\mu},\ G^{ab} &= -\delta^{ab} \end{aligned} \tag{250}$$

Now we will use a six dimensional Vielbein to calculate the action in the Ω-deformed background. First of all, for the metric on the six dimensional space we can write:

$$ds_6^2 = g_{\mu\nu} e_I^{(\mu)} e_J^{(\nu)} dx^I \mathrm{d}x^J - e_I^{(a)} e_J^{(a)} dx^I dx^J. \tag{251}$$

We can simply read off the components of the Vielbein and define the new Field strength tensor in the curved space time with Vielbein indices:

$$-\frac{1}{4}\sqrt{-G} F_{IJ} F_{KL} G^{IK} G^{JL} = -\frac{1}{4} F_{(I)(J)} F^{(I)(J)}. \tag{252}$$

From this consideration we can conclude that the field strength tensor has the following components in the background reducing on a two-torus:

$$\begin{aligned} F_{(\mu)(\nu)} &= F_{\mu\nu} \\ F_{(a)(\mu)} &= F_{a\mu} - V_a^\rho F_{\rho\mu} \\ F_{(a)(b)} &= V_a^\mu V_b^\nu F_{\mu\nu} - F_{a\nu} V_b^\nu - V_a^\mu F_{\mu b} + F_{ab}. \end{aligned} \tag{253}$$

Now we define

$$\begin{aligned} V^\mu &= \frac{1}{\sqrt{2}}\left(V_4^\mu + iV_5^\mu\right) & \bar{V}^\mu &= \frac{1}{\sqrt{2}}\left(V_4^\mu - iV_5^\mu\right) \\ \Omega_\nu^\mu &= \frac{1}{\sqrt{2}}\left(\Omega_\nu^\mu + i\Omega_{5\nu}^\mu\right) & \bar{\Omega}_\nu^\mu &= \frac{1}{\sqrt{2}}\left(\Omega_{4\nu}^\mu - i\Omega_{5\nu}^\mu\right). \end{aligned} \tag{254}$$

Now we expand the term (252) to see how the action changes in this Ω-background.

$$-\frac{1}{4}\sqrt{-G} F_{IJ} F_{KL} G^{IK} G^{JL} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \left(\nabla_\mu \phi + V^\rho F_{\rho\mu}\right)\left(\nabla^\mu \phi^\dagger + \bar{V}^\rho F_\rho^\mu\right) \tag{255}$$

$$-\frac{1}{2}\left\{[\phi,\phi^\dagger] - i\bar{V}^\mu V^\nu F_{\mu\nu} - i\left(V^\mu \nabla_\mu \phi^\dagger - \bar{V}^\mu \nabla_\mu \phi\right)\right\}^2. \tag{256}$$

Note what happends in this expression. It has the same structure as the standard Lagrangian and if the Ω-matrices commute the last bracket can be written as the square of commutator $[\Phi, \Phi^\dagger]$ built out of the adjoint valued higgs plus a shift:

$$\Phi = \phi - iV^\mu \nabla_\mu, \qquad \Phi^\dagger = \phi^\dagger - i\bar{V}^\mu \nabla_\mu \tag{257}$$

We also have to be aware of the fact that the connection on the bundle changes in the Ω-background so we have to shift the Higgs field further by the spinoperator of the Lorentzgroup that acts on the spinors nontrival. Let us repeat the calculation for the fermionic part of the sixdimensional action

$$\frac{i}{2}\bar{\Psi}_A\Gamma^I e^I_{(J)}\nabla_J\Psi^A \tag{258}$$

Here ∇ is the Ω-deformed connection. If the Ω-matrices commute again, we can manipulate this term further by inserting the Vielbein and using the relation between spin connection and the Vielbein. We come to the following expression:

$$\frac{i}{2}\bar{\Psi}_A\Gamma^I e^J_{(I)}\nabla_J\Psi^A = i\psi_A\sigma^\mu\nabla_\mu\bar{\psi}^A - \frac{i}{\sqrt{2}}\psi_A[\Phi^\dagger,\psi^A] + \frac{i}{\sqrt{2}}\bar{\psi}^A[\Phi,\bar{\psi}_A] \tag{259}$$

$$- \frac{1}{2\sqrt{2}}\Omega_{\mu\nu}\bar{\psi}^A\bar{\sigma}^{\mu\nu}\bar{\psi}_A - \frac{1}{2\sqrt{2}}\bar{\Omega}_{\mu\nu}\psi_A\frac{1}{2}\sigma^{\mu\nu}\psi^A. \tag{260}$$

If you compare the action (39) with the one deduced from dimensional reduction on the non trivial background, you find at they are the same in case you shift $\phi \to \Phi$ and accept that the coupling constant gets superspace dependence:

$$\tau \to \tau(x,\theta) = \tau - \bar{\Omega}^+_{\mu\nu}\theta^\mu\theta^\nu \tag{261}$$

The great idea was to not only use the scalar part of the twisted algebra but the fermionic operator to construct a twisted deRham operator. The first idea is that one could simply try to repeat the calculations as in (202) with another operator of the twisted algebra and to see how one can obtain a exact action w.r.t. a new operator. The modified SYM action will be Q_Ω-exact in case we shift the coupling constant further by the following term:

$$\tau \to \tau(x,\theta) = \tau - \frac{1}{\sqrt{2}}\left[\left(\bar{\Omega}^+_{\mu\nu}\theta^\mu\theta^\nu - \frac{1}{2\sqrt{2}}\bar{\Omega}_{\mu\nu}\Omega^\mu_\rho x^\rho x^\nu\right)\right] \tag{262}$$

This coupling constant is annihilated by the following supercharge:

$$\bar{Q}_\Omega = \bar{Q} + \frac{1}{2\sqrt{2}}\Omega^\mu{}_\nu x^\nu Q_\mu. \tag{263}$$

4.6 Obtaining the prepotential and the partition function

4.6.1 The prepotential

This last observation in fact shows how we can compute the prepotential. The partition function now reads:

$$Z(a;\epsilon_1,\epsilon_2) = \langle 1\rangle_a = \int \mathcal{D}X \mathrm{e}^{-S_{\text{micro}}(X)} = \int_{|k|<\Lambda} \mathcal{D}\tilde{X}\mathrm{e}^{-S_{\text{eff}}(\tilde{X})}$$

$$= \exp\left\{\frac{1}{4\pi}\Im\mathfrak{m}\frac{1}{2\pi i}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\mathcal{F}(-2\sqrt{2}a,\Lambda(x,\theta))\right\} = \exp\frac{1}{\varepsilon_1\varepsilon_2}\mathcal{F}(a,\Lambda,\varepsilon_1,\varepsilon_2) \tag{264}$$

In the first line we integrated out the massive degrees of freedom and localized the integral to the zero modes, which are parametrized by the vev. The function also does depend on the cutoff scale which becomes now superspace dependent because we have seen that the coupling is superace dependent and they are related. As we perform the integral this will lead to a regularized integral in the Ω-deformed $\mathbb{R}^4_{\epsilon_1\epsilon_2}$ which gives the denominator and cancels the π's. By using that the prepotential is homogenous of degree two we see that the other factors cancel and this will finally lead to the above-mentioned expression. This brings us directly to the evaluation of the prepotential because if we turn off the Ω-background, we should really obtain the quantities in the flat space which is the regular prepotential. We conclude the remarkable formula:

$$\lim_{\epsilon_1,\epsilon_2\to 0} \ln Z(a,\epsilon_1,\epsilon_2,\Lambda,\tau) = \mathcal{F}(a,\tau,\Lambda) \tag{265}$$

From our discussions in the previous subsection we obtain the following formula

$$Z_k(a,m,\epsilon_1,\epsilon_2) = \int_{\mathcal{M}_k} \mathrm{Eu}_g(\mathcal{D}_k) \tag{266}$$

Here g denotes the torus action of the symmetry groups which is a product of the flavor group, the gauge group, the rigid gauge transformations and the Ω-deformation. We should really comment on the perturbative part of the partition function. It is a remarkable property that this part arises as we fix the gauge. We will not comment on this in the following. The final solution is:

$$Z(a,m,\Lambda,\epsilon_1,\epsilon_2) = Z^{pert}(a,m,\Lambda,\epsilon_1,\epsilon_2) \times \sum_{k=0}^{\infty} q^k \int_{\mathcal{M}_k} \mathrm{Eu}_g(\mathcal{D}_k) = \exp\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}(a,m,\Lambda,\epsilon_1,\epsilon_2) \tag{267}$$

4.6.2 Evaluating the partition function explicitly

Now we are in the position to obtain the partition function by using the Duistermaat-Heckmann formula. Therefore we have to remember how the moduli space of framed instantons is constructed and deduce the action of the different tori which is used to equivariantly localize the contributions to the path integral. We have introduced certain multiplets which we want to integrate out for doing the integration of the equivariant Euler class over the moduli space of framed instantons. However we want to restrict to the pure gauge partition function. So the Euler class is trivial in this case and we have "just" to deal with the "volume" of the moduli space. See (245). Remember that the moduli space of framed instantons is the subset spanned by the solution of ADHM-equations moded out by the "dual" gauge group. This dual means that we have fixed the field configuration at infinity which corresponds to the framing of the instantons but we now have the possibility to transform our field configuration at infinity in case it has *fixed* framing. This is the framed instanton moduli space $\mathcal{M}_{k,N}$ where N comes from the gauge group $U(N)$ or $SU(N)$ and k is the instanton number. In [25] it was shown how to deduce formula (245). If you want to integrate over a zero locus you have to introduce a multiplet living in the fiber of the vector bundle. This multiplet gets supersymmetrized. For the ADHM equation we know that the moduli space is the intersection of two different zero loci $\mu_{\mathbb{C}} = 0,\ \mu_{\mathbb{R}} = 0$. So we have to introduce two additional supermultiplets to the action $(\chi_{\mathbb{C}}, H_{\mathbb{C}}), (\chi_{\mathbb{R}}, H_{\mathbb{R}})$. The algebra now reads: The transformation properties of the matrices $B_1,\ B_2,\ I,\ J$ and the ADHM equation with respect to $\mathbb{T}_L$, which is the Lorentzsymmetry are

$$B_1 \to e^{i\epsilon_1} B_1, \qquad B_2 \to e^{i\epsilon_2} B_2, \tag{268}$$

$$I \to e^{-i\epsilon_+} I, \qquad J \to e^{-i\epsilon_+} J, \tag{269}$$

$$\mu_{\mathbb{R}} \to \mu_{\mathbb{R}}, \qquad \mu_{\mathbb{C}} \to e^{i\epsilon} \mu_{\mathbb{C}}, \tag{270}$$

where $\epsilon = \epsilon_1 + \epsilon_2$ and $\epsilon_+ = \frac{1}{2}(\epsilon_1 + \epsilon_2)$. Taking into account and the Lorentz deformation of the BRST operator we can write

$$\bar{Q} B_{1,2} = \psi_{1,2}, \qquad \bar{Q}\psi_{1,2} = [\phi, B_{1,2}] + i\epsilon_{1,2} B_{1,2}, \tag{271}$$

$$\bar{Q} I = \psi_I, \qquad \bar{Q}\psi_I = \phi I - Ia - i\epsilon_+ I, \tag{272}$$

$$\bar{Q} J = \psi_J, \qquad \bar{Q}\psi_J = -J\phi + aJ - i\epsilon_+ J, \tag{273}$$

$$\bar{Q}\chi_{\mathbb{R}} = H_{\mathbb{R}}, \qquad \bar{Q} H_{\mathbb{R}} = [\phi, \chi_{\mathbb{R}}], \tag{274}$$

$$\bar{Q}\chi_{\mathbb{C}} = H_{\mathbb{C}}, \qquad \bar{Q} H_{\mathbb{C}} = [\phi, \chi_{\mathbb{C}}] + i\epsilon\chi_{\mathbb{C}}, \tag{275}$$

$$\bar{Q}\eta = \lambda, \qquad \bar{Q}\lambda = [\phi, \lambda]. \tag{276}$$

If we integrate over all the operators in the theory and remembering the previous results, we can immediatly write by puting together equations (238) and (239):

$$Z_k(a;\epsilon) = \int \frac{\mathcal{D}\phi}{\mathrm{Vol}(G_D)} \mathcal{D}\eta \mathcal{D}\lambda \mathcal{D}H \mathcal{D}\chi \mathcal{D}B_1 \mathcal{D}B_2 \mathcal{D}I \mathcal{D}J \mathcal{D}\psi e^{i\bar{Q}(\chi\cdot\mu + t\chi\cdot H + \psi\cdot V(\lambda))} \tag{277}$$

where

$$\chi \cdot \mu = \mathrm{Tr}\left\{ \chi_{\mathbb{R}}\mu_{\mathbb{R}} + \frac{1}{2}\left(\chi_{\mathbb{C}}^\dagger \mu_{\mathbb{C}} + \chi_{\mathbb{C}}\mu_{\mathbb{C}}^\dagger \right) \right\}, \tag{278}$$

$$\chi \cdot H = \mathrm{Tr}\left\{ \chi_{\mathbb{R}} H_{\mathbb{R}} + \frac{1}{2}\left(\chi_{\mathbb{C}}^\dagger H_{\mathbb{C}} + \chi_{\mathbb{C}} H_{\mathbb{C}}^\dagger \right) \right\}, \tag{279}$$

(note that torus action on $\chi_{\mathbb{R}}$ and $\chi_{\mathbb{R}}$ is chosen in a way that $\chi \cdot \mu$ is invariant) and $V(\lambda)$ is the dual group flow vector field:

$$\psi \cdot V(l) = \mathrm{Tr}\left\{ \sum_{i=1}^{2} \psi_i[\lambda, B_i^\dagger] + \sum_{i=1}^{2} \bar{\psi}_i[\lambda, B_i] + \psi_I \lambda I - I^\dagger \lambda \bar{\psi}_I - J\lambda\bar{\psi}_J + \psi_J \lambda J^\dagger \right\}. \tag{280}$$

This integral can be evaluated through the deduction of the equivariant action on the fields, especially the stable points[28] and the corresponding weights to compute the finite integral via the Duistermaat-Heckmann localization formula. First of all we introduce

[28] These are the fixed points under the torus action

the torus resulting from the different symmetries of the theory. We have the dual group transforming the field keeping the framing fixed, the Lorentz symmetry and the flavor symmetry and the transformations at infinity. In pure gauge we restrict to $\mathbb{T} = \mathbb{T}_G \times \mathbb{T}_{dual} \times \mathbb{T}_L$. So we have to see how the different fields get transformed under this action of the torus:

$$B_i \to e^{\phi} B_i e^{-\phi} e^{i\epsilon_i}, \tag{281}$$

$$I \to e^{\phi} I e^{-a} e^{-i\epsilon_+}, \tag{282}$$

$$J \to e^{a} J e^{-\phi} e^{-i\epsilon_+}, \tag{283}$$

$$\chi_{\mathbb{C}} \to e^{\phi} \chi_{\mathbb{C}} e^{-\phi} e^{-i\epsilon}. \tag{284}$$

The other fields have been integrated out. The stable point is the origin and if you infinitesimally write the transformations,that the weights for the different fields are given as [3]

$$\phi_i - \phi_j + \epsilon_t, \text{ for } B_{t,ij}, \tag{285}$$

$$\phi_i - a_l - \epsilon_+, \text{ for } I_{il}, \tag{286}$$

$$a_l - \phi_i - \epsilon_+, \text{ for } J_{li}, \tag{287}$$

$$\phi_i - \phi_j - \epsilon, \text{ for } \chi_{\mathbb{C},ij}. \tag{288}$$

Performing this the same way for $\chi_{\mathbb{R}}$ and using (236) we end up with the following expression:

$$\begin{aligned} Z_k = \oint \prod_{i=1}^{k} \frac{d\phi_i}{2\pi i} \frac{1}{k!} \frac{\epsilon^k}{\epsilon_1^k \epsilon_2^k} \frac{\prod_{i<j\leq k}\left((\phi_i-\phi_j)^2\right)\left((\phi_i-\phi_j)^2-\epsilon^2\right)}{\prod_{i<j\leq k}\left((\phi_i-\phi_j)^2-\epsilon_1^2\right)\left((\phi_i-\phi_j)^2-\epsilon_2^2\right)} \\ \times \prod_{i=1}^{k} \frac{1}{\prod_{i=1}^{N}(\phi_i+\epsilon_+-a_i)\prod_{i=1}^{N}(\phi_i-\epsilon_+-a_i)} \end{aligned} \tag{289}$$

This is the contribution of the k-th instanton sector to the partition function computed in [3]. There was also given the result with fundamental matter hypermultiplets. Here we have to consider the equivariant Euler class of the bundle of solutions to the Weyl equation. A nice observation in [3] was that the integral $\oint$ can be interpreted as a contour integral. Thus the poles will contribute to the partition function. These poles can be classified by introduction of Young tableaux. We end up with the partition function of $SU(2)$ with N_f flavors where the theory is asymptotically free or conformal as a sum over Young tableaux [3, 20]. From the partition function we can obtain the prepotential by turning off the Ω-background. So one has a nontrival check of formula (289) through comparison of the result to the solution of Seiberg-Witten and others.

5 $\mathcal{N} = 2$ Dualities

In 2009 Gaiotto introduced a possibility to study a large class of superconformal theories in four dimensions by using duality arguments and quiver diagrams [28]. We will mostly restrict to gauge group $SU(2)$ because then we can use powerful statements learned from the Seiberg-Witten analysis about S-duality. We want to argue that the moduli space of exactly marginal[29] deformations coincides with with moduli space of complex structures of Riemann surfaces of genus g and n punctures.

Let us sketch the argumentation. We will explore the boundary of the moduli space of

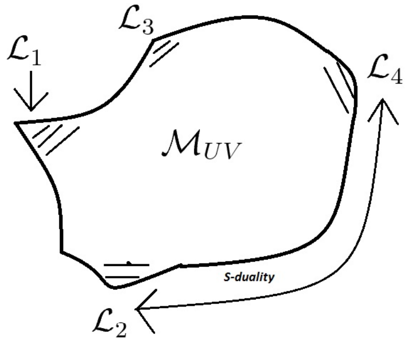

Figure 2: The moduli space of marginal couplings and the Lagrangian description at corner or boundaries

exactly marginal gauge couplings of quivers up to three gauge groups. On the boundary theories are weakly coupled and a Lagrangian description exists that will be given by a quiver. We will use S-duality of one gauge group to analyze dual theories and see that (once we specify the gauge group and the flavors) that the theories are the same modulo S-duality and this information can be encdoded in two numbers (g, n) characterzing a quiver. Instead of trying to investigate the moduli space of gauge couplings we will introduce a moduli space that matches the moduli space of exactly marginal gauge couplings on the boundary. This is the complex structure moduli space of certain Riemann surfaces. Matching our quivers from which we have constructed a Lagrangian to certain Riemann surfaces and using the same rules for the quivers as for the Riemann surfaces we can explore the interior of the moduli space. We have to mention that we are dealing with the ultraviolet theory here so we can really write down all possible Lagrangians.

5.1 Quivers for $SU(2)$-gauge groups

The notion of quiver describes a systematic way of writing down a gauge theory with a gauge group built as a product of certain other groups and the corresponding flavor symmetry of the matter content. If the gauge group is always the same e.g. $G = SU(2)$

[29] marginal means that the beta function vanishes and implies that the gauge coupling does not depend on the scale

and the flavor symmetry for every state is the same, we can represent these quivers as a skeleton where internal lines correspond a gauge group and external lines the flavor symmetry. The possibilities of connecting these lines are restricted by the assumption of vanishing β-function as we are describing conformal theories.

5.1.1 Possible Lagrangians

The philosophy of exploring the UV-parameter space can be compared to the one of Seiberg-Witten. In Seiberg-Witten theory we study the moduli space of vacua. In the large vev limit we know that a semiclassical approximation can be done carefully and the theory can be analyzed exactly. Then, by using the powerfull restrictions of $\mathcal{N}=2$ theories in the low energy effective theory, we try to explore the interior of the moduli space of vacua following the Lagrangian into the strongly coupled region. We will pick up singluarities and monodromies as we try to interpolate the function into this region. In some sense this way can be also used for the moduli space of exactly marginal gauge couplings. As the theory has marginal couplings we can tune them. So the first question is concerning this weakly coupled region analogous to the large vev limit in Seiberg-Witten theory. We know that in $\mathcal{N}=2$ theories there are two different supermultiplets from which we can build Lagrangians. The vectormultiplet comes in the adjoint representation of the gauge group as we are dealing with gauge theories and the hypermultiplets which are the matter that consist of two chiral $\mathcal{N}=1$ multiplets $Q,\bar{Q}$ that comes in the representation $R,\bar{R}$. For $\mathcal{N}=2$ the information $(G,\oplus_I n_I R_I \oplus \bar{R}_I)$ fixes the UV-Lagrangian completely, where n_I is the multiplicity of different representations[30]. The flavor symmetry is $\prod_I U(n_I)$ which can be seen by writing down the superpotential in the $\mathcal{N}=1$ language with vanishing mass deformations. If the matter representation of the gauge group is real or pseudoreal, one can define a "doublet" $(Q,\bar{Q})$ out of the chiral fields and the new superpotential has an enhanced symmetry by rewriting the corresponding term in the superpotential. For R real the flavor symmetry gets enhanced to $\prod_I Sp(2n_I)$ and for R pseudoreal to $\prod_I SO(2n_I)$ which will be important, at least in the case of $SU(2)$ quivers.
The next question is what are these exactly marginals gauge couplings? From the 1-loop beta function we know the following relation for $SU(N_c)$ gauge group and N_f flavors.

$$\beta \propto -(N_f - 2N_c) \tag{290}$$

This equation hold if the vector multiplet is in the adjoint representation of the gauge group and the flavors in the fundamental representation. The exactly marginal couplings are the ones where the matter contributes so much that the beta function vanishes. As can be seen from (290) it should be $N_f = 2N_c$. Another example could be that $G = SU(2)$ with $\mathcal{N}=2^*$. This theory corresponds to a vector multiplet in the adjoint and a matter hypermultiplets in the adjoint with non-vanishing mass deformation. It can be easily seen that this is the mass deformed version of $\mathcal{N}=4$ SYM because with vanishing mass deformation the vector multiplet and the hypermultiplet are in the same

[30]E.g. $SU(2)$ $N_f = 4$ has $n_1 = 4$ for R_1 fundamental representation and $n_i = 0$ for $i > 1$

adjoint representation and make up the field content of the maximal supersymmetric theory in four dimensions.
Now we look further for an example of a superconformal theory with marginal couplings. As we have seen from the beta function the theory with gauge group $SU(N)$ with $2N$ flavors is a conformal theory. However one can also have products of groups. One example could be $SU(N)_1 \times SU(N)_2$ with $2N$ fundamentals and one bifundamental, because the first gauge coupling is coupled to exactly $2N$ flavors. The flavorsymmetry group is $U(N)^2U(1)$ because every fundamental of $SU(N)$ has a flavor symmetry of $U(N)$, but the bifundamental only has a $U(1)$ flavor symmetry. We will now restrict to the case $G = SU(2)$. Starting from two gauge groups we will have $SU(2)_1 \times SU(2)_2$ with four fundamental and one bifundamental so every gauge parameter is coupled to four fundamental flavors. We will denote the flavor symmetry group by an index of small latin capital and the gauge groups by numbers. The bifundamental representation is $(2)_1 \otimes (2)_2$

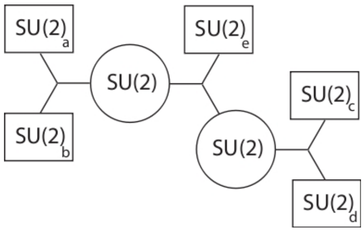

Figure 3: The flavor groups are all the same

where the fundamental representation of $SU(2)$ is pseudoreal. The tensorproduct of two pseudoreal representations is a real representation and so the flavor symmetry of the bifundamental representation is real. However it is already known that flavor symmetry enhances from $U(1)$ to $Sp(2) = SU(2)$. The flavor symmetry of the four fundamentals $SO(8)$ can be decomposed as four $SU(2)$, see (293). Thus the flavor symmetry group is $SU(2)^5$ and the gauge group $SU(2)^2$. One might ask if there is a systematic way to obtain the possible flavor symmetries and gauge groups by assuming conformal symmetry for more complicated quiver theories. The answer is yes. We can go further and add more and more gauge groups and the corresponding matter to keep the couplings exactly marginal. One can even construct loops by this graph technics and we denote the number of loops by g. One can simplify these quivers further by painting skeletons. Here the internal lines correspond to a gauge group $SU(2)$ and the external lines correspond to a flavor symmetry group $SU(2)$ Then we count the number of gauge groups and the number of flavor groups. As the bifundamental always gets enhanced flavor symmetry all the symmetry groups will be $SU(2)$. Doing some combinatorial work we end up for g loops and n external lines which always corresponds to a $SU(2)$ flavor symmetry with the following theory which is superconformal as the mass and the higgs are not turned on:

$$\begin{aligned} G &= SU(2)^{3g-3+n} \\ G_f &= SU(2)^n \end{aligned} \tag{291}$$

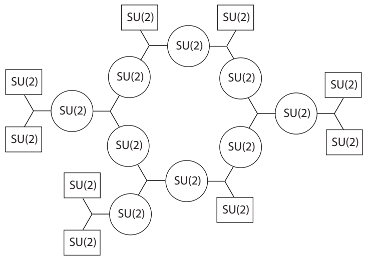

Figure 4: A quiver with a loop

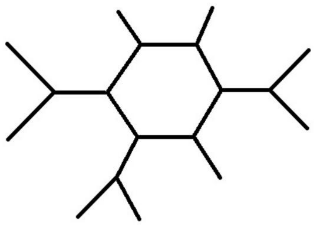

Figure 5: A simplified quiver

So for every g, n the theory has an unique gauge group and an unique flavor symmetry group. As we will see, the difference comes from the coupling of the matter to the gauge theory. The main question is what happen if we project out the different ways the matter couples to the gauge fields. Are the theories then equal after the projection? Obviously they will be same. Formally we have constructed a map from the space of graphs to the space of $\mathcal{N} = 2$ superconformal theories. It turns out that the building blocks of the boundary are S-duality of $SU(2)$ with $N_f = 4$ and trifundamentals[31] which will turn out to be the theory of four free hypermultiplets.

5.1.2 Building blocks of quiver diagrams in the case of SU(2)

We know that the Seiberg Witten curve of $SU(2)$ $N_f = 4$ is invariant under $SL(2, Z)$ in case we permute the three 8-dimensional representations $8_v, 8_c, 8_s$ of the flavor symmetry group which gets enhanced to $SO(8)$ in this special case. Now we can try to follow the action of S-duality on some subgroup. For example we can split into two Lorentz groups:

$$SO(8) \supset SO(4) \times SO(4) = SU(2)_a \times SU(2)_b \times SU(2)_c \times SU(2)_d \qquad (292)$$

We know how the different representations of the flavor symmetry group get decomposed under this supgroup. Thus we can show the following fact: S-duality permutes the different labels of the SU(2) subgroups. The argumentation is as follows: Under the subgroup of the flavorsymmetry group the different representations get decomposed,

[31]We will define this quantity later on

where we have chosen the vector representation to this specific one:

$$8_v = 2_a \otimes 2_b \oplus 2_c \otimes 2_d$$
$$8_s = 2_a \otimes 2_c \oplus 2_d \otimes 2_b$$
$$8_c = 2_a \otimes 2_d \oplus 2_b \otimes 2_c \quad (293)$$

Now we know that the $8_v, 8_s, 8_c$ are permuted by the outer automorphisms of the duality group which is S_3. So under S-duality the labels are permuted as in the figure indicated. What happens if the mass parameter are turned on? As we can associate a

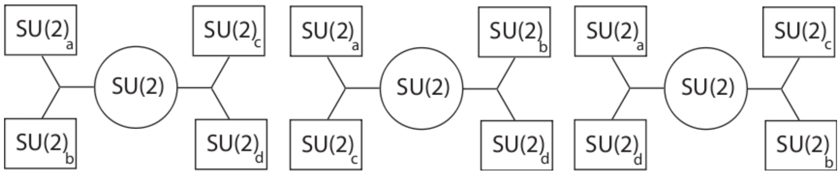

Figure 6: The SU(2) flavor groups are permuted

mass parameter to every $SU(2)$ subgroup S-duality will permute the mass deformations the same way as it permutes the labels of the different $SU(2)$ flavor subgroups.
Are there more elementary building blocks from which we can built out this SU(2) theory with four flavors? The answer is yes. Think about a vertex labeled by three indices, each label come from a flavor group. Now take two of this vertices and gauge two flavor indices by rotating the indices that are identified. The resulting theory will be SU(2)

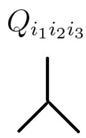

Figure 7: A vertex with 3 flavor indicies

with 4 flavors and there will be no anomaly because from the perspective of the gauge group the other vertices are just flavors .Now we have building blocks and dualities that will be important to explore the boundary of the moduli space of gauge couplings. The next step is to deduce what happens if we take two gauge groups with two marginal couplings. We have already seen that this corresponds to four fundamentals and one bifundamental that have $SU(2)^5$ flavor symmetry.

5.1.3 Triality at work

We have seen how S-duality acts on the flavor subgroups of the corresponding hypermultiplets for a theory of one gauge group. We now can use this at a gauge coupling to explore a different frame in which the theory appears. This can be done for every

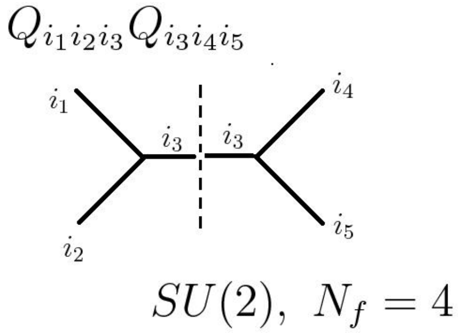

Figure 8: We glue 2 vertices by gauging one flavor index

gauge coupling in a quiver diagram by turning off a gauge coupling almost decoupling the gauge group and by simultaneously assuming that S-duality is still valid on the other gauge couplings. As a first example we analyze what happends for two gauge groups.
Firstly we tune the coupling on the first node to be arbitrary weak and we assume that we can do an S-duality transformation on the second node. We deduce a whole set of S-dual theories by permuting the different labels of the flavor symmetry group. Then we repeat the same procedure on the second node. What are the main conclusions? First of all we see that S-duality transformations do not commute, so that the moduli space of marginal couplings is not just the product of two moduli spaces of theories with one gauge group. This can easily be seen by just reversing the choice of node and observing that this does not bring us to the same result. This seems like a trivial statement but it tells us that the space of exactly marginal couplings has a nontrivial structure. The other conclusion is that the pictures closes. So by tuning the coupling on and off and using the S-duality arguments we will end up in the same theory. This means that there are only finitly many S-dual theories.
Next we go a step further and analayze the theory with three gauge groups. The same arguments as for two gauge groups lead to the conclusion that we are dealing with four fundamentals and two bifundamentals corresponding to gauge group $SU(2)^3$ and $SU(2)^6$ flavor symmetry or equivalently $g = 0,\ n = 6$. Again we tune off the different gauge

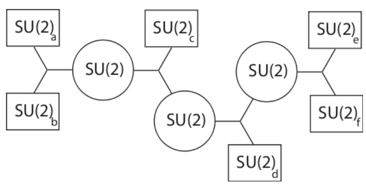

Figure 9: The SU(2) flavor groups are permuted

couplings which are all exactly marginal and perform S-duality transformations. In the case of two gauge groups the structure of the quiver does not change but here we will find an S-dual frame that has three internal lines coming together. This can be seen by taking the quiver of this theory and tuning the gauge coupling arbitrary weak at the first node and the third. Then we can do an S-duality transformation on the second node and deduce a new class of S-dual theories that does not look like a typical quiver: This

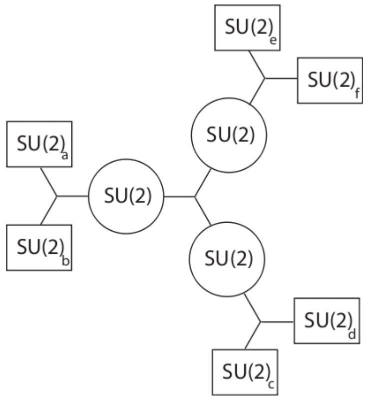

Figure 10: A new example of a Sicilian quiver

quivers are called "generalized" or "sicilian" quivers, depending on which authors one wants to follow. In the middle of the three gauge groups there are states transforming as $(2)_1 \otimes (2)_2 \otimes (2)_3$. These are called trifundamentals. If we decouple the six flavors we are left with the building block where every link is given a flavor index. This analysis can be further extended to many more gauge groups but the main issues were already adressed in the case of zero, one, two and three gauge groups.

5.1.4 Riemann surfaces from trifundamentals

We briefly state what are the results until now. It is known that the moduli space of exactly marginal gauge couplings or, to be more precise, of exactly marginal deformations has boundaries on which a Lagrangian description of superconformal theories exists. S-duality relates all the theories with the same number of loop and external legs in the graphs to each other and so, after projecting out the S-duality grou,p the theories are the same if the two numbers (g, n) are fixed, at least on the boundary. Another observation was that the different theories can be glued together from the graphs with just three legs. Another fact which we have not mentioned yet is the number of gauge groups that is determined by the number of loops and external legs. The theory of arbitrary (g, n) has $3g - 3 + n$ exactly marginal gauge couplings τ_i and can be built out of $2g - n + 2$ graphs with three legs. In fact we can introduce the same amount of coordinates and move around in the space. Nevertheless we know that this description only works locally on the boundaries of the space. One way out is to search for a space that fullfils all the requirements to be the moduli space of marginal gauge couplings. Let us introduce the moduli space of Riemann surfaces $C_{g,n}$ of genus g and n punctures or marked points. For $g > 1$ the dimension of this space is:

$$\dim \mathcal{M}_{g,n} = 3g - 3 + n \tag{294}$$

Actually, the dimensions match. So a first educated guess could be to map the graphs of the three external legs to a sphere with three punctures which can be fixed by conformal symmetry to be at $0, 1, \infty$. Then remember that we have constructed $SU(2)$ with four flavors from two graphs. So now one can ask how to glue the spheres around the punctures. If one associates flavor symmetry to the punctures and introduces local coordinates at one sphere around infinity and on the other around the zero point, you can glue the spheres together in the same way you "glue" together the graphs by gauging the flavor symmetry of each graph. For example, take a sphere with coordinate $z = \frac{1}{w}$ around infinity and on the other sphere a local coordinate z around 0. Define the following gluing parameter:

$$\frac{z}{w} = q = e^{2\pi i \tau_{UV}} \tag{295}$$

You end up with a sphere with four punctures at $0, 1, \infty, q$. Analogously, this represents

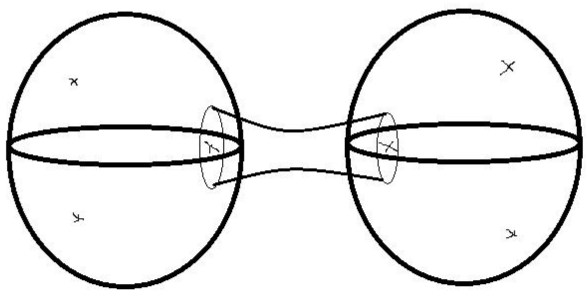

Figure 11: The 3-punctured spheres are glued like the vertices

$SU(2),\ N_f = 4$ theory, where q is directly related to the gauge coupling of the system. So what is S-duality in this picture? We know that S-duality just permutes the different labels of the flavor symmetry. If we assume that every puncture has a certain flavor symmetry index, we conclude a dramatic observation. The boundary of the moduli space of complex structures is the region where the Riemann surface degenerates. So the S-duality frames are nothing more than the different ways a Riemann surface like the sphere with four punctures can degenerate into the two spheres with three punctures glued together by a long tube which gives the strength of the gauge coupling. This even bring us to the next conclusion. Remember that the Teichmueller space has the following relation:

$$\mathcal{M}^{Teich}_{C_{g,n}}/MCG(C_{g,n}) = \mathcal{M}_{g,n} \tag{296}$$

Where MCG is the mapping class group which are the diffeomorphisms of the Riemann surface which preserve orientation and where the trival homotopy cycles are projected out. We we conclude that MCG is the S-duality group and the Teichmueller space is the full UV-parameter space. Taking a Riemann surface $C_{g,n}$ we define the class of theories which are all the same up to S-duality as $\mathcal{T}_{g,n}$ as the theory coming from the quiver corresponding to the Riemann surface $C_{g,n}$. Gaiotto conjectures that the moduli space of gauge couplings of the theory $\mathcal{T}_{g,n}$ is the same as the moduli space of complex structure of the corresponding Riemann surface $C_{g,n}$. This conclusion directly induces a map from the space of Riemann surfaces to the space of different $\mathcal{N} = 2$ conformal theories:

$$F : \mathcal{M}_{g,n} \to \mathcal{T}_{g,n},\ C_{g,n} \to \tau_i,\ i = 1, ..., 3g - 3 + n \tag{297}$$

The τ_is correspond to the gauge coupling because the gauge group is $SU(2)^{3g-3+n}$. This map strongly suggests that there is a relation between $\mathcal{N} = 2$ conformal theories in four dimensions and between Riemann surfaces or a theory defined on a Riemann surface, depending on how much information can be extracted from the Riemann surface. In the end this connection leads to the AGT conjecture.
One could argue even more sophisticated. Study the category of all Riemann surfaces and and a category which is not well-defined but known to physicist of supersymmetric theories of $\mathcal{N} = 2$. Then the map should be something like a functor from the category of the Riemann surfaces to the category of theories denoted $\cup_{g,n=0}^{\infty} \mathcal{T}_{g,n}$ and behaves well under degenerations of the Riemann surface. This means that the morphisms in the category of Riemann surfaces are the gluing of two different Riemann surfaces and the $\mathcal{N} = 2$ theories constructed from this glued Riemann surface have to have the right properties. In fact, we already know a large number of ill-defined functors. The hyperkaehler functor which maps an $\mathcal{N} = 2$ theory to its higgs branch or a special kaehler functor which maps the theory to its Coulomb branch or the partition function functor. Composing this with the functor from Riemann surfaces to the different $\mathcal{N} = 2$ theories $\mathcal{T}_{g,n}$ defines a well-defined functor from the category of Riemann surfaces to the category of hyperkaehler, special kaehler manifolds and to the complex numbers [29].

5.1.5 Seiberg-Witten curves

Starting with the observations of Witten about brane configurations and related Seiberg Witten curves[32] we can begin with a SW-curve of a system of n $SU(2)$ groups (corresponding to the theory $\mathcal{T}_{n+3,0}$) with the right fundamental matter at the end and rewrite the curve in some way. The polynomial is a function in two variables depending on the position of the 5-branes and 4-branes $(v, t) \in \mathbb{C} \times (\mathbb{C}^* - (t_0, \cdots, t_n))$ [33]. The degree in v is two because we are dealing with $SU(2)$ gauge group and of order $n + 1$ in t. Lets restrict to $n = 1$ with four flavors and turn off the mass deformations. We can write the curve as:

$$(t - 1)(t - t_1)\, v^2 = ut \tag{298}$$

[32] See 2.6

[33] The t_i are the positions of the D4 branes in the brane configuration

In the Seiberg-Witten theory the gauge coupling is identified with the complex structure parameter of a torus

$$y^2 = t(t-1)(t-t_1) \tag{299}$$

By doing a coordinate transformation we can bring the curve to the following form $v = tx$:

$$t(t-1)(t-t_1)x^2 = u \tag{300}$$

The S-duality invariant moduli space is the same as the complex structure moduli space of a sphere with four punctures at $0, 1, t_1, \infty$. Through a further transformation, puting this four punctures on the same footing $t_1, ..., t_4$, one deduces the following form of the Seiberg-Witten curve:

$$x^2 = \frac{u}{\Delta_2(z)} = \phi_2(z) \tag{301}$$

t_1 is the cross ratio of the four roots of this function Δ_2 and z is a coordinate on the Riemann surface which is a sphere with four punctures. The Seiberg-Witten differential is $\lambda = xdz$ and so we conclude that the Seiberg-Witten curve lives in the cotangent bundle of the Riemann surface and the canonical 1-form that exists on such Riemann surfaces is just the Seiberg-Witten differential. The quadractic differential $\phi_2(z)dz^2$ has poles at the punctures of the Riemann surface. In [?] the mass deformed curve was also computed from the brane configuration. Also in the mass deformed case we can, in fact, bring the SW-curve to the following form:

$$x^2 = \phi_2(z) \tag{302}$$

The $\phi_2(z)$ is a little bit more involved and the mass parameter are now the coefficents of the double poles in the quadratic differential $\phi_2(z)dz^2$. We conjecture a Seiberg-Witten curve for the $\mathcal{T}_{g,n}$. The curve lives in the cotangent bundle of a Riemann surface, called the "Gaiotto" curve. For every brane configuration it will be of the form

$$x^2 = \phi_2(z) \tag{303}$$

The canoncial one-form will be the Seiberg-Witten differential and the quadratic differential has simple poles at the punctures. Mass deformations are residues of double poles with quadratic divergences. The nice oberservation is that if a Riemann surface degenerates, the quadratic differential actually develops a double pole. This means, as the curve degnerates, a gauge coupling has to become something like a flavor symmetry information with corresponding mass deformation.

5.2 $\mathcal{N} = 2$-dualities for higher rank gauge groups

To generalize the above-mentioned construction we have to find building blocks for the higher rank gauge groups. Here we have to deal with some problems. In the case of rank two we will see that the construction for higher rank gauge groups are much more difficult because there are blocks of strong interacting theories that can not be reduced by S-duality to weak interacting theories. If we explore the moduli space of exactly gauge couplings for $SU(3)$ gauge group coupled to six hypermultiplets in the fundamental representation of the gauge group, we will see that there is a point on the moduli space where the system gets infinitely strongly coupled. At this point the system is the same as a SU(2) theory coupled to some strange flavor symmetry. This duality is called Argyres-Seiberg duality [30], which we want to explain in the following lines.

5.2.1 Argyres-Seiberg duality

Let us begin with this conformal theory of $SU(3)$ $N_f = 6$. We can study the moduli space of gauge couplings. For $SU(2)$ we have seen that there exists an S-duality frame that is the semidirect product of $SL(2, Z)$ and $Spin(8)$ and we have used this fact as a building block of general quiver or Sicilian quiver. In the case of rank two it does not work out that simple. First of all we want to give a dual theory in the infinitely strongly coupled fixed point. The S-duality group of this theory for six massless fundamental hypermultiplets is Γ_2^0, which is a subgroup of $SL(2, Z)$. So the fundamental domain for this subgroup is not bound away from infinite strongly coupled points. Argyres and Seiberg proposed that there is a dual frame in this point which is the following duality:

$$SU(3)w,\ 6 \cdot (3 \oplus \bar{3}) = SU(2)w,\ (2 \cdot 2 \oplus SCFT_{E_6}) \tag{304}$$

That means that the conformal theory with one gauge group $SU(3)$ is the same as the a $SU(2)$ gauge theory coupled to one fundamental and a strongly interacting non-Lagrangian theory with flavor symmetry E_6 where a subgroup is gauged. So we study the Seiberg Witten curve that encodes the low energy effective action given in [31]:

$$y^2 = \left[\left(1 - \sqrt{f}\right) x^3 - ux - v\right] \left[\left(1 + \sqrt{f}\right) x^3 - ux - v\right] \tag{305}$$

Here u, v are the order parameter that parameterize the two-dimensional moduli space of vacua and are of dimension two and three. The corresponding one forms $\omega_u = \frac{x\mathrm{d}x}{y}, \omega_v = \frac{\mathrm{d}x}{y}$ with the corresponding cycles determining the BPS spectrum. The coupling is parameterized in a way that $f \to 1$ corresponds to the infinitely strongly coupled point. From the curve it is directly clear that in this limit the leading order vanishes at one side of the factorization so that it is a genus one curve in contrast to the case $f \neq 1$ where it is a genus 2 curve. The one form corresponding to the dimension two operator develops a pair of poles at $x = \infty$. This can be shown by introducing a local patch on the sphere or by computing the integral around infinity picking up contributions as residua. Now think about a 1-dimensional Coulomb branch with a dimension two operator u and a mass deformation m. This has a rank 1 global flavor symmetry which is broken by the mass parameter. The central charge is

$$Z = n_{electric} a(u) + n_{magnetic} a_D(u) + nm \tag{306}$$

The third contribution comes from the broken flavor symmetry which we want to gauge weakly. Then the quark number n becomes a new electric charge and the mass deformation is the vev of some vectormultiplet. From Seiberg-Witten theory we know the following equations hold:

$$\frac{\partial Z}{\partial u} = \oint_\gamma \omega_u, \frac{\partial Z}{\partial m} = \oint_\gamma \omega_m \tag{307}$$

The γ are cycles on a genus two curve and the ω are one-forms. If we turn off the gauging, the genus two curve degenerates to a genus one curve because a cycle degenerates and by computing $\partial Z/\partial m = n$ we see that the one form ω_m has to develop a pair of poles. This setup is exactly the same as by considering the limit where f goes to 1 in the Seiberg-Witten curve and so we can give an explanation. If f goes to zero we can see v as a coordinate on the moduli space of vacua of a rank 1 SCFT and u appears as a mass deformation. For f almost 1 we will have that u is a vev of a vectormultiplet of a global symmetry of the same SCFT, exactly like in the example discussed above. From this discussion we should now have the possibility to rederive the Seiberg-Witten curve of some rank 1 theory. There we have to put $f = 1, u = 0$, because this limit restores the SCFT symmetry that is broken by the mass deformation which is u. With this assumptions we can identity the SCFT as E_6-theory by reproducing the Seiberg Witten curve of this theory. The other limit is the conformal point of the E_6-theory which means that we have to put $v = 0$. This reproduces the Seiberg-Witten curve $SU(2)$-theory.

5.2.2 Argyres-Seiberg duality is not enough

We want to see that the same strategy we use for $G = SU(2)$ is not that powerful in the case of rank two. Consider e.g. a quiver of three $SU(3)$ groups[34] turn off the all gauge couplings and then turn on the middle gauge coupling. By Argyres-Seiberg we find an E_6-theory with a weakly coupled $SU(2)$ gauge group. We will get $SU(3)$ gauge theories coupled to the E_6-theory. To obtain the S-dual frames we would have to guess the strong coupling limit of such theories which is not known. So we can not deduce the set of all S-dual frames by just using Argyres-Seiberg as the building block instead of S-duality in the rank one case. We just can give the theory by hand and then show that the weakly coupled limits produce all S-dual frames.

5.2.3 Short comments on the higher rank generalizations

In fact we can go on and study quivers with higher rank gauge groups. One problem that arises directly is the problem of flavor symmetry or the classification of punctures. Every puncture is characterized by a flavor symmetry information and the divergences of higher differentials that controll the Seiberg-Witten curve by certain Young tableaux. The Seiberg-Witten curves can be deduced from the multibrane scenario and will have

[34] To be conformal we will have 2 bifundamentals with flavor symmetry $U(1)$ and the we pick a flavor subgroup $SU(3)^2$ similar to the rank one case. The big difference is that the fundamental representation is not pseudoreal in the rank two case and so we have two different types of flavor symmetry. Here $SU(3)^2U(1)^1$

the same canonical form as in the rank one case:

$$x^N = \sum_{i=2}^{N} \phi_i(z) x^{N-i} \tag{308}$$

This curve lives in the cotangent bundle of the Gaiotto curve and its canonical 1-form is the Seiberg-Witten differential.

5.3 4d $\mathcal{N} = 2$ SU(2)-quivers from compactifications of $6d(2,0) - A_1$ theories and higher rank realizations

We have seen that there is a whole class of SCFTs which are the same up to S-duality uniquely determined by the topology of a Riemann surface we want to call the "Gaiotto curve". We will give some evidence that this class can be constructed by a twisted compactification of low energy theory of k coincident M5-branes wrapping the Gaiotto curve. We will relate the Gaiotto curve to the Seiberg-Witten curve and explain how we can define this class of theories for higher rank. This is strongly related to Witten's construction in Typ IIA string theory and the lift to M-theory. In this way we see that the AGT relation arises naturally as two different compactification limits compared after the reduction.
The low energy theory of k coincident M5 branes is given by a six-dimensional super conformal theory of type $\mathcal{N} = (2,0)$ with Lie algebra A_{K-1}. From Nahms classification of super Lie algebras we know that there is a $(2,0)$ theory $\mathcal{T}[\mathfrak{g}]$ with simply laced Lie algebras $\mathfrak{g}$ in six dimensions corresponding to different M5 branes setup. The field content of a theory with one M5 brane decoupled from gravity is a tensormultiplet containing a two-form which has a corresponding self dual 3-form. 5 scalars transforming in the vector representation of the R-symmetry group $SO(5)$ and four fermions making the theory supersymmetric which fulfill a reality condition. What do we mean by twisted compactification? We begin with a six-manifold of the following form:

$$M_6 = \mathbb{R}^4 \times C_{g,n} \tag{309}$$

We reduce the sixdimensional theory along the Gaiotto curve $C_{g,n}$ to a four dimensional theory or, to be more precise, we compactify one M5 brane on a curve $C_{g,n}$. However doing this would break all the supersymmetries. To avoid this we have to do a twisted compactification. We break the R-symmetry group:

$$SO(5)_R \to SO(3)_R \times SO(2)_R \tag{310}$$

and the spacetime symmetry group:

$$SO(1,5) \to SO(1,3) \times SO(2) \tag{311}$$

The spinors now transform under the decomposition as a product of positive and negative chirality spinors, where the reality condition in six dimensions removes half of the components, which is good because we just want to have half of the supercharge algebra. By identification of the surving R-symmetry group with the structure group of the Riemann surface $C_{g,n}$, which is exactly a $SO(2)$-bundle, we can get rid of the terms that would avoid having covariant constant spinors. In fact, turning on the curvature of Riemann surface would bring us to a term of the following form:

$$D_\mu \epsilon = (\partial_\mu + \Gamma)\,\epsilon \tag{312}$$

Obviously supersymmetry could be broken. Therefore now we set $SO(2)_R = SO(2)$ and so we can have:

$$D_\mu \epsilon = (\partial_\mu + \Gamma \pm A_\mu)\,\epsilon \tag{313}$$

Here A_μ is a R-symmetry connection. In fact, we can bring our spinors to be covariantly constant. By elimination of the curvature of the R-symmetry with the structure group of the Riemann surface (or vice versa) we are left with the following symmetry group under which the spinors now transform:

$$SO(1,3) \times SO(3)_R = SO(1,3) \times SU(2)_R \cong \mathcal{N} = 2 \tag{314}$$

This equation indicates that we have actually constructed an $\mathcal{N} = 2$-symmetry group by the twisted compactification. But what happend to the scalars of the tensormultiplet. In the six-dimensional theory the five scalars ϕ_i, $i = 1, ...5$ transformed as a vector of the R-symmetry group $SO(5)$. Now ϕ_1, ϕ_2 couple to the structure group of the Riemann surface by the twisted compactification. We define a 1-form of the following form $\phi(z, \bar{z}) = \phi_1 + i\phi_2 dz$ where z is a coordinate on the Riemann surface. Now we can think of the supersymmetry conditions. A short calculation shows that supersymmetry is preserved if and only if the field $\phi(z, \bar{z})$ is a holomorphic function so $\bar{\partial}\phi(z, \bar{z}) = 0$. So this function can be nontrivial. If we take k M5-branes wrapped on a curve, we will have a collection of one-forms

$$\phi^{(1)}(z), \phi^{(2)}(z), ..., \phi^{(k)}(z), \tag{315}$$

Now we want to define an equation where all the information is concentrated on. So we take an arbitrary one-form λ and build the following equation

$$\left(\lambda - \phi^{(1)}(z)\right)\left(\lambda - \phi^{(2)}(z)\right)\cdots\left(\lambda - \phi^{(k)}(z)\right) = 0 = \lambda^k - u_1(z)\lambda^{k-1} + \cdots \lambda^k - u_k(z) \tag{316}$$

Obviously these $u_k(z)$'s are k-differentials which can be written as $u_k(z) = a(z)dz^k$. We end up with the following conclusion: If we take k coincident M5 branes or, to be more precise, the low energy limit of this theory which should be six-dimensional $\mathcal{N} = (2,0)$ of type A_{k-1} and wrap[35] them around an arbitrary Riemann surface of genus g with n punctures the resulting theory has $cn = 2$ supersymmetry in four dimensions and is controlled by the following equation:

$$0 = \lambda^N + u_1(z)\lambda^{N-1} + \cdots u_N(z) \tag{317}$$

[35]We mean by this wrapping the twisted compactification of the directions of the Riemann surface

Rewriting this equation further by defining

$$u_k(z) = u_k \left(\frac{dz}{z}\right)^k, u_N = (\Lambda^N z + u_N + \Lambda^N/z) \left(\frac{dz}{z}\right)^N, \ \lambda = xdz/z \tag{318}$$

we see that the equation reproduces Seiberg-Witten solutions studied in [32]. For each $z \in C$ we will have N solutions to the equation. So the Seiberg-Witten curve is the N-sheeted branched cover of the Gaiotto curve. We have seen the same relation in (301) or in more detail for the $SU(2)$ case. It can be shown that certain paths corresponds to certain states and that it can be shown that this description will lead to $\mathcal{N} = 2$ vector multiplets and also hypermultiplets. For example, take the sphere with three punctures wrapped by two coincident M5 branes. The defining equation is:

$$x^2 = \phi_2(z) \tag{319}$$

Every puncture is labeled by a mass parameter. This will give four paths and two zeros of the differential which is given as

$$\begin{aligned} \phi_2(z) &= \mathrm{m}_a^2 dz/z, \ z = 0 \\ \phi_2(z) &= \mathrm{m}_b^2 dz/z - 1, \ z = 1 \\ \phi_2(z) &= \mathrm{m}_c^2 d(1/z)/(1/z), \ z = \infty \end{aligned} \tag{320}$$

From four paths of this type one can deduce that this corresponds to hypermultiplets with certain mass deformations given as linear combination of the mass parameters of the punctures. So the conclusion is that wrapping two M5 branes on a sphere with three punctures we have constructed four dimensional $\mathcal{N} = 2$ theory of four *free* hypermultiplets what was expected from the four dimensional point of view. One can actually argue that connecting two copies of this theory and gauging together one global symmetry will lead to the right paths and zeros of the differential to have $SU(2)$ gauge theory with four flavors. If you tune the coupling you will have the same geometry again but the mass parameters will be permuted. This is nothing than S-duality!
Again, one can ask how the different local descriptions are glued together. We are dealing with exactly marginals deformations and as we tune on the vev of the higgs we are moving in the moduli space of vacua that is fibered over the moduli space of gauge couplings. So the Lagrangian changes in the following form:

$$L \to L' = L + \delta\tau_i \int d^4x d\theta^2 \left\langle \mathrm{Tr}\phi_i^2 \right\rangle \tag{321}$$

If we define $u_i = \mathrm{Tr}\phi_i^2$, we can see $u_i \delta\tau_i$ as a 1-form and the Coulomb branch which is fibered over $\mathcal{M}_{UV}$ gets as a fiber bundle glued together like the cotangent bundle of the moduli space $T^*\mathcal{M}_{g,n}$ and we can go further and study this space of complex structure deformations and its corresponding 1-forms. It turns out that the Seiberg-Witten curve lives in the cotangent bundle of the Gaiotto curve and can be given as N-sheeted branched covering of the Gaiotto curve with the following equation:

$$x^N = \sum_{i=1}^{N} \phi_{i+1}(z) x^{N-i} \tag{322}$$

This was expected from the analysis done before. On the cotangent bundle of the Riemann surface with local coordinates (x, z) there is a canoncial one form written as:

$$\lambda = xdz \tag{323}$$

So we have come to the same conclusion by three different approaches. First of all we rewrote the Seiberg-Witten curve with a non-bijectiv transformation obtaining the Gaiotto curve. This transformation lead to the possibility to have a branched covering. The second way was to construct a curve with the information from the twisted compactification of the six-dimensional theory of type A_{k-1} with $\mathcal{N} = (2, 0)$ SUSY. The last approach is, but not explicitly performed here, to derive the relation of the moduli space of gauge couplings with the Coulomb branch. The Coulomb branch is fibered over the moduli space. The different local descriptions are glued together like the cotangent bundle of the moduli space which was identified with the moduli space of complex structures of a Riemann surface. From this it follows that the Seiberg-Witten curve lives in the cotangent bundle.

6 The AGT conjecture

Now we come to the central statements of this master thesis: The AGT conjecture which relates conformal field theory and supersymmetric gauge theories. We will explain two different cases. The first one, which is the one presented in [1], studies a relation between SCFT in 4 dimensions with $\mathcal{N} = 2$ supersymmetry and Liouville theory on a Riemann surface. The other case is a similar relation between asymptotically free theories and CFT first noted in [4]. We will formulate the conjecture and do some tests of the AGT duality at lower genus of the Riemann surface. We will see that there is some evidence that a dictionary exists.

6.1 General Statement for conformal theories

We have a map from Riemann surfaces to SCFT in 4 dimensions with $\mathcal{N} = 2$ supersymmetry. Therefore we may ask if there is a correspondence between certain objects in the two theories. One natural object in the SCFT is the Nekrasov partition function. Start with $U(2)$ gauge group factor out a $U(1)$ part and set $a_1 = -a_2 = a$. So we conjecture that for $G = SU(2), N_f = 4$ the instanton part of the partition function is the conformal block of the CFT on the Riemann surfaces which causes the four-dimensional $\mathcal{N} = 2$ theory:

Conjecture AGT I

$$\boxed{Z^{U(1)} \times Z^{nek}_{inst}(a, \mu_\alpha, \epsilon_1, \epsilon_2) = B(\Delta_1, \Delta_2, \Delta_3, \Delta_4; \Delta_0, c)} \tag{324}$$

Remembering (191) and defining

$$\mathcal{B}^{\alpha}_{Y,Y'} = \lambda^{\Delta}_{\Delta_1\Delta_2}(Y) K^{-1}_{YY'} \lambda^{\Delta}_{\Delta_3\Delta_4}(Y') \tag{325}$$

we can also state (324) as

$$\sum_{|Y|=|Y'|} q^{|Y|} \mathcal{B}^{\alpha}_{Y,Y'}(\alpha_1, \alpha_2; \alpha_3, \alpha_4) = (1-q)^{-\nu} \sum_{Y,Y'} q^{|Y|+|Y'|} Z^{SU(2)}_{Y,Y'} = \sum_{Y,Y'} q^{|Y|+|Y'|} Z^{U(2)}_{Y,Y'} \tag{326}$$

Here we have $(1-q)^{-\nu} = Z^{U(1)}$. We need to identify the parameters:

$$\mu_1 = \alpha_1 - \alpha_2 + \frac{\epsilon}{2},\ \mu_2 = \alpha_1 + \alpha_2 - \frac{\epsilon}{2},\ \mu_3 = \alpha_3 - \alpha_4 + \frac{\epsilon}{2},\ \mu_4 = \alpha_3 + \alpha_4 - \frac{\epsilon}{2}$$
$$a = \alpha_{(0)} + \epsilon/2,\ \Delta_k = \frac{\alpha_k(\epsilon - \alpha_k)}{\epsilon_1\epsilon_2},\ c = 1 + \frac{6(\epsilon_1 + \epsilon_2)^2}{\epsilon_1\epsilon_2} \tag{327}$$

If we want to compute the correlator of four primary fields in a CFT, we have seen that we have to combine the conformal block with the 3-point function. If there is a relation between the conformal block and the instanton part of the Nekrasov partition function, could there be an object in the four-dimensional theory we have to identify to compute the correlator in the Liouville theory? The AGT conjecture states that the perturbative part of the Nekrasov function can be identified with the DOZZ-formula and the classical part of the partition function as simple function in q. So the four-point correlator in Liouville theory living on the Gaiotto curve is the integral over the Coulomb branch (as the coulomb branch parameter is identified with the internal conformal dimension of the intermediated channel in the expansion of the correlator) of the squared complete Nekrasov function:

Conjecture AGT II

$$\langle V_{\alpha_0}(\infty)V_{\alpha_1}(1)V_{\alpha_2)}(q)V_{\alpha_3}(0)\rangle = \int daa^2|Z_{nek}(a,\mu_i,q)|^2 \tag{328}$$

The last conjecture relates the field that comes from the compactification of the six-dimensional theory and lives on the Gaiotto curve to the energy-momentum tensor of the CFT. CFTs have powerfull Ward identities and by inserting the Energy momentum tensor we will have expansions that depend on the OPEs of the other operators. From this AGT concluded:

Conjecture AGT III

$$\phi_2(z)dz^2 = -\langle T(z)\rangle \tag{329}$$

ϕ_2 has double poles of fixed coefficents. The space of quadratic differential with fixed residua actually has the same dimension as the Coulom branch for the theories of class $\mathcal{T}_{g,n}$. Indeed, we can see this $\phi_2(z)$ as the field coming from the Gaiotto curve as we wrap the M5 branes around the curve. The conjecture can be checked experimentally. If you calculate the residua of the quadratic differential the fixed coefficents of the quadratic differential in the limit of vanishing Ω-background should be the same as the mass parameters for certain cycles and for the other cycles the Coulomb branch parameter. This statement was checked in [1].

6.1.1 Experiments at genus 0

After formulating the AGT conjecture we will see that (324) is experimentally true. What we exactly mean is that we will see that the two expressions (324) will match in the first orders in the expansion. The Gaiotto curve is a sphere with four punctures located at $0, 1, \infty$ and q. On the on the hand we will expand the instanton partition function in the instanton counting parameter and on the other side in the position on the sphere which is not fixed by conformal symmetry. This reflects the fact that in the construction of the $\mathcal{T}_{g,n}$-theories the gluing parameter was identified with the gauge coupling which can be tuned. So let us start with the expansion in the lowest order of the conformal block (189). The first nontrivial term will be:

$$\mathcal{B}^{(1)}(\Delta_1,\Delta_2,\Delta_3,\Delta_4) = \frac{\langle\alpha_1|V_{\alpha_2}L_{-1}|\alpha\rangle\left(\langle\alpha|L_1L_{-1}|\alpha\rangle\right)^{-1}\langle\alpha|L_1V_{\alpha_3}(q)|\alpha_4\rangle}{\langle\alpha_1|V_{\alpha_2}(1)|\alpha\rangle\langle\alpha|V_{\alpha_3}(q)|\alpha_4\rangle} \tag{330}$$

Evaluating the different operations explictly leads to the following linear term for arbitrary external dimension and to the following term in the conformal block expansion

$$\mathcal{B}^{(1)}(\Delta_1,\Delta_2,\Delta_3,\Delta_4) = -\frac{2a^4+2a^2\Big(\epsilon(-\alpha_1+\alpha_2-\alpha_3+\alpha_4)+\alpha_1^2-\alpha_2^2+\alpha_3^2-\alpha_4^2-\epsilon^2/2\Big)}{\epsilon_1\epsilon_2(4a^2-\epsilon^2)} - \\ -\frac{2\Big(\alpha_1(\epsilon-\alpha_1)-\alpha_2(\epsilon-\alpha_2)+\epsilon^2/4\Big)\Big(\alpha_3(\epsilon-\alpha_3)-\alpha_4(\epsilon-\alpha_4)+\epsilon^2/4\Big)}{\epsilon_1\epsilon_2(4a^2-\epsilon^2)} \tag{331}$$

Now expand the partition function to linear order so for $k=|Y|+|Y'|=1$:

$$Z_{[1][0]} = \frac{1}{\epsilon_1\epsilon_2}\frac{\prod_{r=1}^4(a+\mu_r)}{2a(2a+\epsilon)} \tag{332}$$

And we also need:

$$Z_{[0][1]} = \frac{-1}{\epsilon_1\epsilon_2}\frac{\prod_{r=1}^4(a-\mu_r)}{2a(2a-\epsilon)} \tag{333}$$

Now we combine the contributions from the easiest Young tableux and we also have to incorporate the $Z^{U(1)}$-factor by also expansion in the instanton counting parameter. In the Nekrasov partition function this gives a contribution of ν in the linear order. At least we have to calculate

$$\mathcal{B}^{(1)}(\Delta_1,\Delta_2,\Delta_3,\Delta_4) - \big(Z_{[0][1]}+Z_{[1][0]}+\nu\big) = ? \tag{334}$$

We use (327) and see that for $\nu = 2\alpha_1\alpha_2/\epsilon_1\epsilon_2$ both sides really match! The quadratic expansion in the instanton counting parameter produces horrible expressions which can be analyzed the same way with mathematica. It is easier to show the powerfull results of mathematica in the case of asymptotically free theories and we will come back to this. Now we compute the four point function in Liouville theory. We have seen that the four point function can be written as

$$\langle V_{\alpha_0}(\infty)V_{\alpha_1}(1)V_{\alpha_2}(q)V_{\alpha_4}(0)\rangle \\ = \int d\alpha C(\alpha_0^*,\alpha_1,\alpha)C(\alpha^*,\alpha_3,\alpha_4)|q^{\Delta_\alpha-\Delta_{\alpha_3}+\Delta_{\alpha_4}}\tilde{\mathcal{B}}(\alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,q)|^2 \tag{335}$$

where the $\tilde{\mathcal{B}}$ mean that we have already factored out a certain dependence (191) on the cross ratio out of the conformal block. It is already clear that this conformal block have the full instanton information (at least this is the conjecture). By plugin in the DOZZ-formula we can deduce that the four point function can be written in the following form:

$$= F(\alpha_0^*)F(m_0)F(m_1)F(\alpha_1)\left|q^{Q^2/4-\Delta_{\alpha_3}-\Delta_{\alpha_4}}\right|^2\int a^2da\,|Z_{\alpha_0}{}^{m_0}{}_{\alpha}{}^{m_1}{}_{\alpha_1}(q)|^2 \tag{336}$$

up to a constant which only depends on b. Here

$$F(\alpha) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{-\alpha/b}\Upsilon(2\alpha) \tag{337}$$

This Z have the following form [1]:

$$Z_{\alpha_0}{}^{m_0}{}_{\alpha}{}^{m_1}{}_{\alpha_1}(q) = \\ q^{-a^2}\frac{\prod\Gamma_2(\hat{m}_0 \pm \tilde{m}_0 \pm a + Q/2)\prod\Gamma_2(\hat{m}_1 \pm \tilde{m}_1 \pm a + Q/2)}{\Gamma_2(2a+b)\Gamma_2(2a+1/b)}\tilde{\mathcal{B}}(\alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,q) \tag{338}$$

The first factor is nothing than the contribution of the classical prepotential! Also the rest of the functions are the contributions to the 1-loop of four fundamental hypermultiplets! So we conclude that the Liouville four-point correlator is up to some constants the same as the full square of the Nekrasov partition function integrated over the Coulom branch with the natural measure which means that we have to encounter the Vandermonde determinant which is a^2.
The AGT conjecture can also be formulated in a more abstract way. It states that the conformal block of a CFT defined on a Riemann surface $C_{g,n}$ is the same as the instanton part of the Nekrasov partition function of a theory in four dimensions coming from two M5 branes wrapping the Riemann surface $C_{g,n}$. In [1] it was also computed up to some order that the 1-point function on a torus is the same as the Nekrasov partition function integrated over the Coulom branch. This should hold at least for mulitple punctures on a sphere and on a torus by just integrating over the multidimensional Coulomb branch.The complete dictionary can be found in table 1.

6.2 General Statement for asymptotically free theories

After introduction of the AGT conjecture the question appeared if it possible to establish such a duality also between asymptotically free theories and CFT. For the gauge group $SU(2)$ we can have $N_f = 0, 1, 2, 3$. On the gauge theory side there is still the possibility to compute the Nekrasov partition function. So we have to find a related object on the CFT side. Therefore Gaiotto [4] introduced a special coherent state whose norm should reproduce the Nekrasov partition function called Gaiotto state. The motivation of this idea comes from the M5-brane construction. Another possibility is to go to the gauge theory side and to decouple fundamental by sending their mass to infinity keeping the renormalization scale finite. Then use the dictionairy and do the corresponding steps on the CFT side. The object which we will obtain we will defined as "irregular conformal block“ and we will see that this will indeed match the Nekrasov function. To define this Gaiotto states we have to deal with extensions to the program in [4] because the compactifications obtained are in order to construct superconformal theories and want to deduce asymptotically free theories from M5 brane compactifications. The punctures that control the theory are no longer regular so are of higher degree as two and the construction can be find in [?] and [33]. We will restrict to punctures that are not of higher degree than four. We will do the calculation for the pure gauge case but the

$\mathcal{N}=2$ **theory of class** $\mathcal{T}_{g,n}$	**Liouville theory on the Gaiotto curve** $C_{g.n}$
Deformation parameters ϵ_1, ϵ_2	Liouville parameters $\epsilon_1 : \epsilon_2 = b : 1/b$ $c = 1 + 6Q^2$, $Q = b + 1/b$
four free hypermultiplets	a three-punctured sphere
Mass parameter m associated to an $SU(2)$ flavor	Insertion of a Liouville exponential $e^{2m\phi}$
one $SU(2)$ gauge group with UV coupling τ	a thin tube (or channel) with gluing parameter $q = \exp(2\pi i\tau)$
Vacuum expectation value a of an $SU(2)$ gauge group	Primary $e^{2\alpha\phi}$ for the intermediated channel, $\alpha = Q/2 + a$
Instanton part of Z	Conformal blocks
One-loop part of Z	Product of DOZZ factors
Integral of $\lvert Z^2_{\text{full}}\rvert$	Liouville correlator

Table 1: Dictionary between the Liouville theory and Nekrasovs partition function Z.

matter theories can be worked out in the complete same fashion. The main statement is:

$$\langle A, B\rangle = Z_{nek}^{N_f=0,1,2,3} \tag{339}$$

Here A, B are certain coherent states in the Verma module. The definiton of these states is defined from the M5-brane perspective and we will now present the pure gauge example.

6.2.1 Experiments for $N_f = 0$

In the following chapter we restrict to the case of pure gauge theory. The cases $N_f = 1, 2, 3$ work the same way. First let us quote that the Gaiotto curve is a sphere with two punctures such that the quadratic differential on the sphere has a degree three pole at the puncture that can be put to zero and infinity by conformal symmetry. The curve is given as [33]:

$$x^2 = \phi_2(z) = \frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z} \tag{340}$$

The parameters are the Coulomb branch parameter and the scale of the theory that differ for every matter content. Now we define the state $|\Delta, \Lambda^2 >$ living in the Verma module of highest weight of conformal dimension $\Delta = \frac{Q^2}{4} - a^2$. The curve restricts the choice of the Gaiotto state to:

$$L_1|\Delta,\Lambda^2\rangle = \Lambda^2|\Delta,\Lambda^2\rangle \qquad\qquad L_2|\Delta,\Lambda^2\rangle = 0 \tag{341}$$

because of the relation of the energy momentum tensor and the quadratic differential living on the Gaiotto curve. By using the Virasoro algebra one deduces that for all $n > 2$ the modes of the energy-momentum tensor act as annihilation operator. We expand the Gaiotto state in the scale parameter:

$$|\Delta,\Lambda^2\rangle = v_0 + \Lambda^2 v_1 + \Lambda^4 v_2 + \cdots \tag{342}$$

Here v_0 is the highest weight vector $|\Delta\rangle$ and v_n is a level n descendant such that $L_1 v_n = \Lambda^2 v_{n-1}$ and $L_2 v_n = 0$. Using the Virasoro algebra and by studying the requierments from the Gaiotto curve one can deduce the functions v_n up to every power recursivly. We state the coefficents up to second order:

$$\begin{aligned} v_0 &= |\Delta\rangle \\ v_1 &= \frac{1}{2\Delta} L_{-1}|\Delta\rangle \\ v_2 &= \frac{1}{4\Delta\,(2c\Delta + c + 16\Delta^2 - 10\Delta)}((c+8\Delta)L_{-1}^2 - 12\Delta L_{-2})|\Delta\rangle \end{aligned} \tag{343}$$

Our proposal is to identify the Norm of the Gaiotto state with the Nekrasov partition function. So we calculate the Norm of this Gaiotto state. We assumed that the highest weight state is normalized to unity. For the norm we get:

$$\begin{aligned} \langle\Delta,\Lambda^2|\Delta,\Lambda^2\rangle =& \sum \Lambda^{4n}|v_n|^2 \\ =& 1 + \frac{\Lambda^4}{2\Delta} + \frac{\Lambda^8(c+8\Lambda)}{4\Delta(2c\Delta + c + 16\Delta^2 - 10\Delta)} + \cdots \end{aligned} \tag{344}$$

Now we compare the Nekrasov function in the pure gauge theory for $U(2)$. We calculate the Nekrasov function up to quadratic order and set the casimirs as $a_1 = -a_2$[36] and get the following equation

$$\begin{aligned} Z_{nek}^{N_f=0} =& \sum_0^\infty q^{|n|} Z_n^{N_f} = 1 + \frac{2q}{\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2-2a)(\epsilon_1+\epsilon_2+2a)} \\ &+ \frac{q^2(8(\epsilon_1+\epsilon_2)+\epsilon_1\epsilon_2-8a)}{\epsilon_1^2\epsilon_2^2((\epsilon_1+\epsilon_2)^2-4a^2)((2\epsilon_1+\epsilon_2)^2-4a^2)((\epsilon_1+2\epsilon_2)^2-4a^2)} + \cdots \end{aligned} \tag{345}$$

Here something remarkable has happened. If we identify the instanton counting parameter as the fourth power of the scale parameter in the irregular conformal block $q = \Lambda^4$, the two quantities match up to second order in case we use the standard AGT dictionairy. Let us be explicit in the linear term. First of all we have $\epsilon_1\epsilon_2 = 1$ and $\epsilon_1 = b$.

[36]This is the $SU(2)$ case

Furthermore we have $Q = (\epsilon_1 + \epsilon_2)^2$ and so we get $4\Delta = ((\epsilon_1 + \epsilon_2)^2 - 4a^2)$. So the linear term gets:

$$\frac{2q}{\epsilon_1\epsilon_2(\epsilon_1 + \epsilon_2 - 2a)(\epsilon_1 + \epsilon_2 + 2a)} = \frac{2\Lambda^4}{1(\epsilon_1 + \epsilon_2)^2 - 4a^2} = \frac{2\Lambda^4}{4\Delta} = \frac{\Lambda^4}{2\Delta} \tag{346}$$

We see that the linear term matches and the same calculation can be done for every order e.g. order 3:

$$\frac{\left(16c\Delta + 2c^2\Delta - 52\Delta^2 + 22c\Delta^2 + 48\Delta^3\right)\Lambda^{12}}{96c\Delta^2 + 48c^2\Delta^2 - 960\Delta^3 - 624c\Delta^3 + 144c^2\Delta^3 + 4896\Delta^4 - 240c\Delta^4 + 96c^2\Delta^4 - 6816\Delta^5 + 1056c\Delta^5 + 2304\Delta^6} \tag{347}$$

Now we calculate the Nekrasov function up to third order:

$$\begin{aligned} q^3(& \frac{1}{4a\epsilon_1^3(2a+\epsilon_1)(-2a+2\epsilon_1)\epsilon_2^2(-2a+\epsilon_2)(-\epsilon_1+\epsilon_2)(2a-\epsilon_1+\epsilon_2)(-2a+\epsilon_1+\epsilon_2)} \\ & - \frac{1}{4a\epsilon_1^3(-2a+\epsilon_1)(2a+2\epsilon_1)\epsilon_2^2(2a+\epsilon_2)(-\epsilon_1+\epsilon_2)(-2a-\epsilon_1+\epsilon_2)(2a+\epsilon_1+\epsilon_2)} \\ & + \frac{1}{12a(2a-2\epsilon_1)(2a-\epsilon_1)\epsilon_1^3\epsilon_2(-2\epsilon_1+\epsilon_2)(-\epsilon_1+\epsilon_2)(-2a+\epsilon_1+\epsilon_2)(-2a+2\epsilon_1+\epsilon_2)(-2a+3\epsilon_1+\epsilon_2)} \\ & - \frac{1}{12a(-2a-2\epsilon_1)(-2a-\epsilon_1)\epsilon_1^3\epsilon_2(-2\epsilon_1+\epsilon_2)(-\epsilon_1+\epsilon_2)(2a+\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)(2a+3\epsilon_1+\epsilon_2)} \\ & + \frac{1}{4a\epsilon_1^2(-2a+\epsilon_1)(\epsilon_1-\epsilon_2)(2a+\epsilon_1-\epsilon_2)\epsilon_2^3(2a+\epsilon_2)(-2a+\epsilon_1+\epsilon_2)(-2a+2\epsilon_2)} \\ & - \frac{1}{4a\epsilon_1^2(2a+\epsilon_1)(\epsilon_1-\epsilon_2)(-2a+\epsilon_1-\epsilon_2)\epsilon_2^3(-2a+\epsilon_2)(2a+\epsilon_1+\epsilon_2)(2a+2\epsilon_2)} \\ & + \frac{1}{2a(2a-\epsilon_1)\epsilon_1^2(2a-\epsilon_2)(2\epsilon_1-\epsilon_2)\epsilon_2^2(-2a+\epsilon_1+\epsilon_2)(-2a+2\epsilon_1+\epsilon_2)(-\epsilon_1+2\epsilon_2)(-2a+\epsilon_1+2\epsilon_2)} \\ & - \frac{1}{2a(-2a-\epsilon_1)\epsilon_1^2(-2a-\epsilon_2)(2\epsilon_1-\epsilon_2)\epsilon_2^2(2a+\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)(-\epsilon_1+2\epsilon_2)(2a+\epsilon_1+2\epsilon_2)} \\ & + \frac{1}{12a\epsilon_1(2a-2\epsilon_2)(\epsilon_1-2\epsilon_2)(2a-\epsilon_2)(\epsilon_1-\epsilon_2)\epsilon_2^3(-2a+\epsilon_1+\epsilon_2)(-2a+\epsilon_1+2\epsilon_2)(-2a+\epsilon_1+3\epsilon_2)} \\ & - \frac{1}{12a\epsilon_1(-2a-2\epsilon_2)(\epsilon_1-2\epsilon_2)(-2a-\epsilon_2)(\epsilon_1-\epsilon_2)\epsilon_2^3(2a+\epsilon_1+\epsilon_2)(2a+\epsilon_1+2\epsilon_2)(2a+\epsilon_1+3\epsilon_2)}) \end{aligned} \tag{348}$$

It really matches the CFT side under the identification of the parameters! We see that, indeed, the following equation is fulfilled:

$$\langle \Delta, \Lambda^2 | \Delta, \Lambda^2 \rangle = Z_{nek}^{N_f=0}(a, q) \tag{349}$$

So we concluded that the Nekrasov partition function again has a counterpart on the CFT side[37]. It is clear that this is no proof of the AGT conjecture but a hint that this is not just an accident appearing in the lowest order of some expansion.
The main question is if we can define these coherent states in some way. This leads to the idea to use the AGT conjecture to know which object could be interesting on the CFT side and to see that this objects actually exists.

6.2.2 Irregular conformal blocks in $N_f = 0$

Here we want to demonstrate how we can use the AGT conjecture to define interesting objects on the CFT side. We start with a sphere with four punctures. The Gaiotto

[37] We have checked this up to fifth order

construction tells us that we have $SU(2)$ with four flavors. Now sending the masses of the flavors(hypermultiplets) to infinity decouple them from the theory. By doing that we have to keep the renormalization scale fixed. In fact:

$$q\prod_{i=0}^{4}\mu_i = \Lambda \tag{350}$$

Then we define the irregular conformal block, sending $q \to 0$, as:

$$B_\Delta(\Lambda) = \lim_{\Delta_i \to \infty} B(q, \Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta) \tag{351}$$

We take (191) and see what happens to the terms in this certain limit. In fact the inverse of the matrix in the conformal block does not depend one the external dimensions and so nothing happens. The γ functions will change because we can neglect certain terms now caused by the fact that the external conformal dimensions grow infinitly. For the irregular conformal block we get:

$$B(\Lambda, \Delta) = \sum_{|Y|=|Y'|} \Lambda^{4n} K^{-1}_{[1^n],[1^n]} \tag{352}$$

Now we compare this expression to the norm of the Gaiotto state. The norm of the Gaiotto state ($= \sum c_Y L^{-Y}|\Delta >$) is:

$$\left\langle \Lambda^2, \Delta | \Lambda^2, \Delta \right\rangle = \sum_{Y,Y'} c(Y) K^{-1}_{Y,Y'} c(Y') \tag{353}$$

These are two different descriptions of the notion of irregular conformal blocks that should match. So at the end the expansion coefficents of the Gaiotto state are given as:

$$C_Y = \Lambda^{|Y|} K^{-1}_{[1^{|Y|}],Y} \tag{354}$$

So this is the Gaiotto state constructed from the four-point conformal block and, by doing some calculations, it can be shown that it fulfills the condition that was implemented from the M5-brane construction. The upshot is that the expansion coefficents of a Gaiotto state can be given directly. In fact, we could also decouple less flavors from the theory keeping the beta function negative and do the same calculations. The Gaiotto states should be defined from the conformal block description by sending less masses to infinity keeping the scale fixed. Again, the remarkable relation shows that the scalar product of certain Gaiotto states matches the Nekrasov partition function as explained in [4]

7 Toda Field Theory and AGT for SU(N)

The AGT conjecture relates fourdimensional SU(2) gauge theory with 2 supersymmetries to Liouville theory. So the natural question that arises is what happen for higher rank gauge groups? What should be the corresponding CFT. In [34] a possible answer to this question was given. Since Liouville theory arrises as a special case of Toda field theories and is also related to some ADE classification it was natural to assume that SU(N) gauge theory is dual to a Toda field theory. In this section we want to remind you what the Toda Field theory is and also stress that it is much more involved to do some observation in the higher rank gauge group case then in rank 1 case in [1] and that we have to overcome some problems. There is also AGT conjecture for special classical groups like $Sp(1)$ and $SO(4)$ [35] and for generalized quivers or Sicilian quivers studied in Gaiottos $\mathcal{N}=2$ dualities paper [36], but we will restrict to the $SU(N)$ case.

7.1 Toda Field theory

We begin by introduction of the action of Toda field theory:

$$S = \int d^2z\sqrt{g}\left[\frac{1}{8\pi}g^{ad}\left\langle\partial_a\phi,\partial_d\phi\right\rangle + \mu\sum_{i=1}^{N-1}e^{b\langle e_i,\phi\rangle} + \frac{\langle Q,\phi\rangle}{4\pi}R\,\phi\right], \tag{355}$$

Here g_{ad} is the metric on the Riemann surface with curvature R. The e_i are the simple roots of the A_{N-1} Lie algebra and we have a scalar product on the root space. The $N-1$-scalar fields in Toda theory are put into a $N-1$-dimensional vector expanded in the simple roots: $\phi = \sum e_i\phi_i$. If you take $N=2$ then you will deal with one simple root that is 1 so the action reduces to (160) up to normalization.
We will call the class of theories we are dealing with Toda theories although we restrict here to the theories with the roots of the Lie algebra of $SU(N)$. In this case the central charge to obtain a conformal theory is forced to be:

$$c = N-1+12\left\langle Q,Q\right\rangle = (N-1)(1+N(N+1)(b+\frac{1}{b})^2) \tag{356}$$

where we have a certain modification of the charge at infinity Q

$$Q = \left(b+b^{-1}\right)\rho \tag{357}$$

At first sight the theory we are dealing with seems to be almost the same as Liouville theory but in Liouville theory there exists only one conserved current, the Energy Momentum tensor coming from the Virasoro sector. In Toda theory with $N>2$ this is not true as we will have more conserved currents. In fact, there are $N-2$ additional holomorphic currents called $\mathcal{W}^k$-currents and the resulting algebra is called the $\mathcal{W}_{N+1}$-algebra. Here k runs from $2...N+1$. The conformal dimension grows accordingly to the rank of the gauge group plus one.

Let us expand the new conserved currents in modes and as we know that the conformal dimension is the rank of the gauge group plus one we can directly write:

$$\mathcal{W}^{(k)}(z) = \sum_{n=-\infty}^{\infty} z^{-z-k} \tag{358}$$

We can also start to define primary fields in this context and we do this in the same way as for the Virasoro sector (or Verma module)

$$W_0^{(k)} V = w^{(k)} V \,, \qquad W_n^{(k)} V = 0 \quad \text{when } n > 0 \tag{359}$$

So we define the primary fields

$$V_\alpha = e^{\langle \alpha, \phi \rangle} \tag{360}$$

It is more instructive to study a certain example of a $\mathcal{W}$-algebra as done in [34] where the easiest case was considered. This is the A_2-case. We will have an additional holomorphic current of conformal dimension 3. I n this case the Virasoro algebra is untouched:

$$\begin{aligned}
[L_n, L_m] =& (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m} \\
[L_n, W_m] =& (2n-m)W_{n+m} \\
[W_n, W_m] =& \frac{c}{3\cdot 5!}(n^2-1)(n^2-4)n\delta_{n,-m} + \frac{16}{22+5c}(n-m)\Lambda_{n+m} + \\
& (n-m)\left(\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)\right) L_{n+m}
\end{aligned} \tag{361}$$

$$\Lambda_n = \sum_{k=-\infty}^{\infty} : L_k L_{n-k} : + \frac{1}{5} x_n L_n \tag{362}$$

with:

$$x_{2l} = (1+l)(1-l) \,, \qquad x_{2l+1} = (2+l)(1-l) \,. \tag{363}$$

7.2 AGT-W duality

We have seen that if we construct quivers of $SU(3)$ gauge theory we will have different flavor symmetry depending on whether the external leg is a bifundamental or a fundamental. So in the construction with the Riemann surfaces it was also noted that we have to deal with two different punctures on this surfaces. The punctures in the general case of six-dimensional $\mathcal{N} = (2,0)$ theory of type A_{k-1} reduced on a Riemann surface are classified by Young tableaux [28]. In the $SU(2)$ case we inserted Vertex operators in the punctures and so the mass parameter of the hypermultipet is identified with the corresponding conformal dimension. Now the flavor symmetry is labeled by a Young diagram so we have to find a map from the Young diagrams or type of puncture to the conformal dimension or Toda momenta. If we map the puncture corresponding to the $U(1)$ symmetry to the highest weight of the A_2 algebra we can actually see that the

number of parameters on both sides match which would not be the case if we do not constraint the possible Toda momenta for the different types of punctures. So the AGT conjecture gets lifted to higher rank case to the AGTW-conjecture.
Conjecture AGTW
Toda field theory with Lie algebra A_{k-1} *is AGT dual*[38] *to the* $SU(N)$ *theory with* $2N$ *flavors*

[38] In the same way of $SU(2)$

8 Proving AGT

In [1] the AGT conjecture was established by calculating a power series in two characteristic quantities and observing that they match order-by-order in this expansion under identification of the parameters. This is no proof, altough it gives strong evidence that it could hold to all orders. Here we want to present some ideas to proof the AGT conjecture. Firstly we will see that for special cases the proof to the AGT conjecture becomes much easier and can be done. We will constrain the parameters in a way that the quantities become much easier and we can do a careful analysis to match the conformal blocks and the partition function up to all orders. Then we go further and try to give an idea how to prove the AGT conjecture in general with the help of matrix models and β-ensembles.

8.1 Special cases

8.1.1 The c $\to \infty$ limit

In [37] the case of very large central charge is studied and we want to go through the proof briefly. The first important observation is that the conformal block has a nice form in the large central charge limit:

$$B_{\Delta;\Delta_1,\Delta_2,\Delta_3,\Delta_4}(x) = F_1(\Delta+\Delta_1-\Delta_2, \Delta+\Delta_3-\Delta_4; 2\Delta; x) \tag{364}$$

where the function on the right hand side is hypergeometric function that can be given explicitly as:

$$F_1(\Delta+\Delta_1-\Delta_2, \Delta+\Delta_3-\Delta_4; 2\Delta; x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \prod_{k=0}^{n-1} \frac{(\Delta+\Delta_1-\Delta_2+k)(\Delta+\Delta_3-\Delta_4+k)}{2\Delta+k} \tag{365}$$

From the AGT conjecture we have the following conjecture:

$$c = 1 + \frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1 \cdot \epsilon_2} \tag{366}$$

So the large c limit comes from from the limit $\epsilon_1 \to 0$ but then the AGT conjecutre states that also the external conformal dimensions grow infinitely. So we choose the vertex operator in a way that the external dimension stay finite and this leads to the restriction that the $U(1)$ factor drops out because the exponent that appears in the $U(1)$ factor is proportional now to $O(\epsilon_1)$ so in the large central charge limit will give no contribution. The main question is which partitions will survive the limit we are taking in the Nekrasov partition function and if we can bring the results in a systematic approach. To see what we have to do we firstly expand the Nekrasov functions into the instanton counting parameter.
In the linear term only the partition [1][0] will contribute to the Nekrasov partition

function. The other terms will be of order ϵ_1 therefore will vanish within the limit. We have:

$$Z_{[1],[0]} = \frac{(\Delta_0 + \Delta_1 - \Delta_2)(\Delta_0 + \Delta_3 - \Delta_4)}{2\Delta_0} + O(\epsilon_1) \tag{367}$$

Now we compare the term of the Nekrasov function with (365) and see that the result matches indeed in the limit.
We go further by computing the term for $|Y| + |Y'| = 2$ which is the quadratic term in the instanton expansion: The results are all proportional to ϵ_1 except to the term $Z_{[11],[0]}$ which also gets contributions within the limit

$$Z_{[11],[0]} = \frac{(\Delta_0 + \Delta_1 - \Delta_2)(\Delta_0 + \Delta_3 - \Delta_4)(\Delta_0 + \Delta_1 - \Delta_2 + 1)(\Delta_0 + \Delta_3 - \Delta_4) + 1}{4\Delta_0(\Delta_0 + 1)} + O(\epsilon_1) \tag{368}$$

It seems as if we only get contributions from that ones that have one trivial partition $[1^n]$ and the other $[0]$ in the large central charge limit. This is the next step to deduce which Young tableau contributes and then to evaluate the Nekrasov function in general. This we lead to a proof of the AGT conjecture in this special limit. We proceed as follows: We will study from which terms the contribution does come and so we can give explicitly show that the pairs of Young tableaux that contribute reduce from (Y, Y') to $(1^n, 0)$ where n is the cardinality of the Young tableau. Now we analyze when the numerator and the denominator of Nekrasov partition function $Z_{Y,Y'} = N/D$ vanish in the $\epsilon \to 0$ limit. Remember that the numerator is given by:

$$\begin{aligned} N(Y, Y') = &\prod_{(i,j)\in Y} \prod_{\alpha}^{4} (a_1 + \epsilon_1(i-1) + \epsilon_2(j-1) + \mu_\alpha) \times \\ &\prod_{(i',j')\in Y'} \prod_{\alpha}^{4} (a_2 + \epsilon_1(i'-1) + \epsilon_2(j'-1) + \mu_\alpha) \end{aligned} \tag{369}$$

We search for all the terms where the ϵ_2 dependency drops out so that the small ϵ_1 limit gives zero contribution. From the definition of the dictionary we know which terms will vanish in the limit. Remember:

$$\begin{aligned} a = \alpha_0 - \epsilon/2, \mu_1 = -\epsilon/2 + \alpha_1 + \alpha_2, \mu_2 = \epsilon/2 + \alpha_1 - \alpha_2, \\ \mu_3 = -\epsilon/2 + \alpha_3 + \alpha_4, \mu_4 = \epsilon/2 + \alpha_3 - \alpha_4 \end{aligned} \tag{370}$$

The terms $\prod(a + \mu_\alpha + \epsilon_2 + k\epsilon_1), \prod(a + \mu_\alpha + k\epsilon_1), \prod(-a + \mu_\alpha + k\epsilon_1)$ will contribute with powers of ϵ_1^2 and are the reason why the numerator tends to zero in the limit. So we have to set $i = 1, 2$ or $i' = 1$. So the contribution to the partition function is:

$$N(Y, Y') \simeq \epsilon_1^{2(\sharp(i=1)+\sharp(i=2)+\sharp(i'=1))} \tag{371}$$

Now let us analyze the denominator. To get the $SU(2)$ partition function we have to set $a_1 = -a_2 = a$. This is the reason we get terms in the denominator terms dependent and independent of a because some of them depend on $a_1 + a_2$ thus the dependence drops out for these terms. Again we want so get terms where the ϵ_2-dependency drops out and the limit contributes in powers of ϵ_1. From the evaluation of the partition function in the former chapters we know that the dependency drops out exactly if $k_i(Y) = j$ which is the left edge of a row[39] and so we conclude that the number of columns contributes with $\epsilon_1^{\sharp(i=1)+\sharp(i'=1)}$ where the second term comes from the second Young tableau Y'. Similiarly we can analyze the a dependent terms and see that they contribute to the denominator by $\epsilon_1^{\sharp(i=2)+\sharp(i=1)}$. From this it follows that the partition function for a pair of Young tableaux is:

$$Z_(Y,Y') = N(Y,Y')/D(Y,Y') = \epsilon_1^{\sharp(i=2)+\sharp(i'=1)} \tag{372}$$

This contributes if and only if we can set $\sharp(i=2)=0$ and $\sharp(i'=1)=0$. This means that the partitions that can contribute are only the ones of the form $(1^{|Y|},0)$ and so we can go further and compute the partition functions explicitly
We calculate now:

$$\begin{aligned} Z_{[1^n],[0]} =& \frac{(-)^n}{n!(\epsilon_1\epsilon_2)^n}\prod_{i=1}^{n}\frac{\prod_{\alpha=1}^{4}(a+\mu_\alpha+(i-1)\epsilon_1)}{(2a+\epsilon+(n-i)\epsilon_1)(2a+(n-i)\epsilon_1)} \\ =& \frac{(-)^n}{n!(\epsilon_1\epsilon_2)^n}\prod_{i=1}^{n}\frac{\epsilon^{2n}\epsilon_1^{2n}(\Delta_0+\Delta_1-\Delta_2+(i-1))(\Delta_0+\Delta_3-\Delta_4+(i-1))+O(\epsilon_1)}{(2\alpha_0+(n-i)\epsilon_1)(2\alpha_0-\epsilon+(n-i)\epsilon_1)} \\ \to& \frac{1}{n!}\prod_{i=1}^{n}\frac{(\Delta_0+\Delta_1-\Delta_2+(i-1))(\Delta_0+\Delta_3-\Delta_4+(i-1))}{2\Delta_0+(n-i)} \end{aligned} \tag{373}$$

Here it was important that ϵ_2 stay finite and in the small ϵ_1 limit the term in the denominator in front of ϵ exactly cancels the sign dependency. To complete the calculation we built up the sum where the summation index is now $n = |Y|$, which is the only free parameter in the pair of Young diagrams and we rename $i-1 \to k$:

$$\sum_{n=0}^{\infty} x^n Z_{[1^n].[0]} \to \sum_{n=0}^{\infty} x^n \frac{1}{n!}\prod_{k=0}^{n-1}\frac{(\Delta_0+\Delta_1-\Delta_2+k)(\Delta_0+\Delta_3-\Delta_4+k)}{2\Delta_0+k} \tag{374}$$

This is (365) and this completes the proof of the AGT conjecture in the large central charge limit with finite external conformal dimensions.

8.1.2 Proving the asymptotically free case in the pure gauge limit

We turn now to general proofs. The pure gauge AGT conjecture first formulated in [4] relates the instanton partition function to the irregular conformal block which can be identified with the norm of the Gaiotto state which is a highest weight state in the Verma module. One possibility to proof such relations is to show that every side of an equation fulfills the same recursive relation. Then under the identification of the parameters the conjecture is proven. The important equation is a recursive formula of Zamolodchikov.

[39]depending how you draw the Young diagram

In the limit of $N_f = 0$ the formula states that the irregular conformal block has the following expansion:

$$\mathcal{B}_{\Delta,\Delta_1,\Delta_2,\Delta_3,\Delta_4}(x) = \mathcal{H}_{\Delta,\Delta_1,\Delta_2,\Delta_3,\Delta_4}(q) \tag{375}$$

with $q = (x) = e^{-\pi\frac{K(1-x)}{K(x)}}$ and $K(x) \int_0^1 \frac{dt}{\sqrt{(1-t^2)(a-xt^2)}}$. The nice observation of Zamolodchikov was that the function on the rhs can be given as a power series, where the terms have a recursive relation:

$$\mathcal{H}_{\Delta,\Delta_1,\Delta_2,\Delta_3,\Delta_4}(q) = 1 + \sum_{n=1}^{\infty} (16q)^n \, \mathcal{H}^n_{\Delta,\Delta_1,\Delta_2,\Delta_3,\Delta_4} \tag{376}$$

with:

$$\mathcal{H}^n_{\Delta,\Delta_1,\Delta_2,\Delta_3,\Delta_4} = \delta_{n,0} + \sum_{1\leq rs\leq n} \frac{A_{rs}}{\Delta - \Delta_{rs}} \mathcal{H}^{n-rs}_{\Delta_{rs}+rs,\Delta_1,\Delta_2,\Delta_3,\Delta_4} \tag{377}$$

With the formulae:

$$\begin{aligned}\Delta_{rs} =& \frac{Q^2}{4} - \frac{1}{4}(rb + sb^{-1})^2 \\ A_{rs} =& \frac{1}{2} \prod_{p=1-q}^{r} \prod_{q=1-s}^{s} \left(\mu - \frac{pb + qb^{-1}}{2}\right), (p,q) \neq (0,0), (r,s)\end{aligned} \tag{378}$$

If we rewrite the irregular block for $N_f = 0$ in the following way, we can get an easy recursive relation for the irregular conformal block.

$$\left\langle \Delta, \Lambda^2 | \Delta, \Lambda^2 \right\rangle = \sum_{n=0}^{\infty} \Lambda^{4n} \left\langle \Delta, n | \Delta, n \right\rangle \tag{379}$$

From (377) we conclude:

$$\langle \Delta, n | \Delta, n \rangle = \delta_0^n + \sum_{1\leq rs\leq n} \frac{A_{rs}}{\Delta - \Delta_{rs}} \langle \Delta_{rs} + rs, n - rs | \Delta_{rs} + rs, n - rs \rangle \tag{380}$$

In [38] it was shown that the Nekrasov partition function (with $a_1 = -a_2 = a$) fulfills the same relation as (380). The instanton partition function can be written as a contour integral and every pair of Young diagrams corresponds to a choice of integration contour. One has to analyze the pole structure and calculate the residua of this function. This yields a long expression that can be simplified by some algebra and leads to an equation how the residue of a factorized partition function corresponds to the original one. Fortunately, we can calulate the rhs directly and the lhs is the Nth-contribution in the instanton partition function expansion. As we go to the $SU(2)$ and manipulate the equation further[40] we see that the Nekrasov partition function fulfil the following equation:

$$Z_n(\Delta, b) = \delta_{0,n} + \sum_{1\leq rs\leq n} \frac{A_{rs}}{\Delta - \Delta_{rs}} Z_{n-rs}(\Delta_{rs} + rs, b) \tag{381}$$

[40]We have to match the large casimir condition and that is reason for the δ-term in the recursive relation

By Comparing the equation for Z_n with (380) the AGT conjecture in the pure gauge limit is proven.

8.2 Proof via Topological string theory

In [39] a way to relate the AGT conjecture naturally to topological string theory and matrix models was sketched. The bridge that appears to be one of the most interessting ways to prove the conjecture IS Matrix models. The reason for this is that we can express the Liouville 3-point functions in a matrix model representation. Let us begin by recalling basic facts of matrix models and the relation to Calabi-Yau manifolds in the topological B-model. For a more detailed discussion of matrix models and topological string we refer to [40].
The partition function of the matrix model with one matrix (corresponding to the rank 1 case) is given as:

$$Z = \int_{N\times N} d\Phi \exp \frac{1}{g_s} \mathrm{Tr} W(\Phi) \tag{382}$$

g_s is the coupling constant and the integral is a contour integral in the space of matrices. This matrix model describes the B-model on a Calabi-Yau of the following form:

$$uv + F(x,z) = 0 \tag{383}$$

where $F(x,z) = 0$ is the spectral curve of the matrix model called Σ embedded into $\mathbb{C}^2$. The function $F(x,z)$ is given as:

$$F(x,z) = x^2 - W'(z)^2 + f(z) = 0 \tag{384}$$

This is a double cover of the z-plane. Saddle points of this integral correspond to the critical points of the potential $W(z)$ and determine the function $f(z)$. The connection between matrix models and CY-geometries were explained in several papers by Vafa and Dijkgraaf in 2002 [41–43]. Consider the following geometry given by:

$$uv + x^2 - W'(z)^2 \tag{385}$$

The singularities at $u = v = x = W'(z) = 0$ are resolved into a one dimensional projective space. We wrap N B-branes over this spaces so we resolve the singularities and this is the geometry after the so called *geometric transition*. For large N in the matrix model it computes the closed topological string amplitudes on the resolved geometry. If one defines $W'(z)^2 - f(z) = P(z)$ and tunes the functions, one can bring the equation $x^2 = P(z)$ to hyperelliptic curve, where the higher genus corrections come from a field on the spectral cover $\phi(z)$. This field has a natural interpretation in CFT.
In a series of papers from the mid 90s it was shown that the following partition function has another interpretation:

$$Z = \int d^N z \prod_{I<J} (z_I - z_J)^2 \exp \sum_I \frac{1}{g_s} W(Z_I) \tag{386}$$

The "dual" formulation states that the Vandermonde determinant is nothing but the N-point function of vertex operators at the positions of the eigenvalues. Now the potential is coupled to to the current $\partial\phi$. The complete formula states:

$$Z =< N|\exp\left(\int_{-\infty}^{\infty} dz e^{i\phi(z)}\right)\exp\left(\oint_{\infty} dz\frac{1}{g_s}W(z)\partial\phi(z)\right)|0> \tag{387}$$

From this expression one can deduce the relation between the Liouville field and the matrix field, where the Liouville field is interpretated as the string field living on the spectral cover:

$$\phi(z) = \frac{1}{g_s}W(z) + 2\mathrm{Tr}\log(z-\Phi) \tag{388}$$

From this, one can argue that the spectral curve looks like

$$x^2 = g_s^2\left\langle\partial\phi(z)^2\right\rangle \tag{389}$$

This fact comes from the observation that the energy momentum tensor which generates the matrix transformation is the derivative of the Liouville field. By rewriting the terms in free fermions ψ_1, ψ_2 we see that the fermions build up $U(2)$ affine current algebra. The $U(1)$ part decouples so we are left with a $SU(2)$ algebra. By bosonization $\partial\phi_1 = \psi_1\psi_1^*$ and $\partial\phi_2 = \psi_2\psi_2^*$ of the free fermions we derive that the partition function can be written as:

$$Z = \left\langle\exp\left(\int dzJ_+(z)\right)\exp\left(\oint_{\infty} dzW(z)J_3(z)/g_s\right)\right\rangle_N \tag{390}$$

By the subscript we mean that the term is inserted as $\langle N|.,.|0\rangle$ and with the $J_i,\ i = +,-,3$ the $SU(2)$ currents given as:

$$J_+ = e^{i\phi_-},\ J_3 = \partial\phi_- \tag{391}$$

where the subscript on the ϕ field means that we have taken the difference of the bosonized fields. If we now take the potential to zero, the CY-geometry will give rise to a A_1-singularity as you directly see from (385). In this special case the CFT description of matrix models corresponds to the following action:

$$S = \int d^2z\partial\phi\bar{\partial}\phi + \int dz e^{i\phi(z)} \tag{392}$$

The nice conclusion is that the action is the Liouville action in the case of vanishing background charge Q or (hopefully) equivalently Ω-deformation $\epsilon_1 + \epsilon_2 = 0$.
To really conclude that the AGT conjecture is true we have to go further and turn on the deformation parameters. We define $\beta = -\epsilon_1/\epsilon_2$. The main question is how the geometry changes now. Let us introduce a complex structure on the four dimensional real space and take the local CY geometry to be:

$$\mathcal{N}_\Sigma \oplus \mathbb{C}^2 \tag{393}$$

Considering the canonical line bundle of $\mathbb{C}^2$ over a Riemann surface Σ. This is analgous to the T^2 fibration we have seen in the evaluation of Nekrasov partition function. Now let $F_{\mathbb{C}^2}$ be the curvature of this bundle and assume that it can be expressed in terms of the curvature of the Riemann surface.

$$F_{\mathbb{C}^2} = -\epsilon \cdot R_\Sigma \tag{394}$$

The problem is that we do not have a Ricci-flat space anymore and supersymmetry is broken.However we can twist the normal bundle again to get rid of the curvature term. There is a rotation generator in the normal bundle. If we take the trace of it, we have a $U(1)$ symmetry. Then denote J_R as the generator acting on the field and we have to add a term to the action of the fields, a term like

$$\epsilon \int_\Sigma J_R \wedge \omega_\Sigma \tag{395}$$

The generator gets wedged with the spin connection on the Riemann surface, but what should be J_R? The geometry normal to the Riemann surface has a holomorphic two-form and the $U(1)$ symmetry can be seen as the freedom to muliply a phase. The phase can be given explicilty as $\theta = \epsilon_1 + \epsilon_2/\sqrt{\epsilon_1\epsilon_2}$. Then by implementing the transformation on the fermions we can deduce the correct form of the term we have to add to the free Liouville action. It reproduces the missing term of the Liouville action correctly and introduces a background charge at infinity.
If we now go back to the affine $SU(2)$ current algebra and turn on the background charge we see that the vertex operator should have the following form:

$$J_+ = e^{b\phi} \tag{396}$$

It reproduces the known answer if we take $b = i \to Q = 0$. In this case the Vandermonde determinant gets corrected slightly to be raised by the power of 2β and the potential gets a prefactor of β. So this results in the derivation of the AGT conjecture. The relation between gauge theories and the Liouville theory comes from the bridge which is a β-deformed matrix model with penner type potential which was shown by engeeniering the gauge theory and matching the matrix model description in the topological string with the CFT side [39]

9 Speculations

What could be a further conclusion of this AGT duality? One interesting property of the Nekrasov partition function is that it computes the refined topological string partition function in a certain limit:

$$Z^{nek} = Z^{top}_{ref} \tag{397}$$

In [44] a generalization of the holomorphic anomaly equation [45] was conjectured to the refined case where the Ω-background is turned on. In fact, the holomorphic anomaly gives the possibilty to study the free energies of the topological string for higher genera amplitudes. In the refined case the free energies depend on two numbers associated to the genus of the Riemann surface and the number of punctures. The holomorphic anomaly equation is a recursive formula in the free energies so, in principle, computable. The refined holomorphic anomaly equation was used to calculate higher genera contributions to the refined topological string. However if the Nekrasov partition function in four dimensions is the "same" as the topological string partition function then one might ask what is the topological string partition function in the dual AGT frame? Is there a corresponding equation to the generalized holomorphic equation in Liouville or Toda theory? We have performed calculations in this direction but nothing meaningful have appeared until now. An interesting question related to this is the worldsheet description of the Ω-background which should be clarified further.
There could also be a relation between M-theory and quantum deformed Virasoro algebras coming from the observation that the five-dimensional Nekrasov partition function defined on $\mathbb{R}^4 \times S^1$ is the same as the irregular conformal blocks in the quantum deformed Virasoro algebra. These irregular conformal blocks are again norms of certain coherent states living in Verma module over the deformed Virasoro algebra. [46]
Other speculations could be adressed in the direction to super Liouville theory. Is there for any Liouville theory a dual gauge theory ? Even for the complicated coset models of $\mathcal{W}$-theories there seems to be a correspdoning gauge theory living on an orbifold. [47], [48], [49], [50], [51]
Further issues are dealing with surface operators. What is their dual interpretation in the conformal field theory side? E.g. [52] There is ongoing research in proving AGT by studying matrix models in much more detail as we have done here. The generalization to β-ensembles lead to the possibility to compute also higher genus terms with matrix models techniques and is a good candidate for the proof of AGT. There is also some other higher dimensional dualiy conjecture.Instead of splitting $2+4=6$ there could be another possibility like $3+3=6$. Recently a paper [53] was published where the authors constructed a supersymmetric theory in three dimensions $\mathcal{T}_M$ to every 3-manifold M of a certain type. One motivation was a work of this year stating that that the Partition function of bosonic $SL(2,\mathbb{Z})$ Chern Simons theory on a three-manifold is equivalent to the Partition function of superconformal theory in 3-dimensions.

10 Conclusions and Summary

The aim of this master thesis was to study the AGT conjecture. The conjecture relates quantities in four dimensional $\mathcal{N} = 2$ gauge theories to two-dimensional conformal field theory, in particular to Liouville theory. We presented how one can solve $\mathcal{N} = 2$ field theories in the infrared limit via elliptic curves and how to construct these curves from branes in typ II string theory. We have seen how we can compute the non-perturbative effects by using Lorentz symmetries, localization and equivariant cohomology. We have studied a way to classify gauge theories and their non-Lagrangian regions in the moduli space of gauge couplings and formulated drastic observations. We have seen that these theories can be understood from a higher dimensional point of view. We have formulated the AGT conjecture for conformal theories and for asymptotically free theories and proved them in very special cases.

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