

# Modelling PCS options via individual indices

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## Abstract

A model for the PCS index is introduced and it is shown how to price a PCS option. It is discussed how to approximate option prices.

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## 1. Introduction

When actuaries realised that a single catastrophe could ruin the whole insurance world one started to look for alternative possibilities to transfer catastrophic risk. Because the daily standard deviation of the trading volume at the US financial markets is about the size of a worst possible catastrophe the financial world could easily take over catastrophe risk, see [14]. A first product introduced at the Chicago Board of Trade in 1992 was the CAT-Future. Models for the ISO index underlying the CAT future were introduced in [7], [1], [10] and [4]. Because of the way the product was constructed it never became popular amongst investors.

In 1995, the CAT future was replaced by the PCS option. This new product meets the criticism against the CAT future. A PCS option works as follows. There are several PCS indices representing catastrophic losses in different areas. Each

of the indices measures catastrophic losses occurred by a well-defined catastrophic event (such as hurricanes, earthquakes etc.) in a predefined region (eastern, western, California, etc.). If a catastrophe occurs, and is identified as a catastrophe by PCS (Property Claim Services) — a statistical agent ‘independent’ of the insurance world — the losses incurred are estimated by PCS. A first estimate is announced within 48 - 72 hours after the occurrence of the catastrophe. Then PCS continues to estimate and re-estimate the losses. The PCS index is then the accumulated loss estimates for all identified catastrophes in the region for catastrophes occurred in the period under consideration, called the occurrence period. A PCS option is a spread on one of the PCS indices, with a maturity date at least 6 months after the occurrence period. That there is a development period after the occurrence period gives the possibility to get a clearer picture on catastrophes occurring late in the occurrence period. The PCS indices are announced daily.

Models for PCS options were introduced in [13] and [2]. A problem for modelling is that even for fairly simple models the pricing problem becomes quite complicated. The simple multiplicative index introduced in the papers mentioned above models catastrophes to be more severe if the index already is large. The motivation for the model comes from the possibility to use the Esscher transform for pricing. This seems reasonable for the price of a share but does not make sense for the PCS index. One problem is that the severity of a catastrophe occurring at time  $t$  depends on the index at time  $t$ . We will in this paper define an index for each single catastrophe and model this individual index similarly as in [2]. The drawback of this model will be that it only becomes possible to calculate prices (numerically) in a simple way if the individual indices can be observed. A survey on securitization and more references can be found in [3] and [6].

The outline of the paper is as follows. In Section 2 we introduce a model of the PCS index under the physical measure. The pricing measure is introduced in

Section 3. In Section 4 we discuss how to approximate the option prices and we give an example. In Section 5 we extend the model to allow some of the parameters to be stochastic. The paper ends with a conclusion and the proofs of the lemmata.

## 2. A model for the PCS index under the physical measure

Let  $0 < T_1 < T_2$  be two real numbers. The occurrence period is the interval  $(0, T_1]$ , and the development period is  $(T_1, T_2]$ . Catastrophes occur according to a point process at times  $0 < \tau_1 < \tau_2 < \dots$ . To each of the catastrophes we model an individual index  $\{L_t^i\}$ . The number of catastrophes in  $(0, t]$  is denoted by  $N_t$ , i.e.  $\{N_t = n\} = \{\tau_n \leq t < \tau_{n+1}\}$ . The PCS index at time  $t$  is then

$$L_t = \sum_{i=1}^{N_{t \wedge T_1}} L_t^i.$$

A PCS option is a spread with underlying  $L_{T_2}$ , i.e. the payoff of the option is

$$F_{T_2} = \min\{\max\{L_{T_2} - K, 0\}, A\} = (L_{T_2} - K)^+ - (L_{T_2} - K - A)^+,$$

where  $K, A \geq 0$ .

We make the following assumptions:

- There is a probability space  $\{\Omega, \mathcal{F}, \mathbb{P}\}$  containing all quantities defined below. The information is contained in a filtration  $\{\mathcal{F}_t\}$ , which is right-continuous but not necessarily complete.
- $N$  is an inhomogeneous Poisson process with rate  $\lambda(t)$ , i.e. the process has independent increments and  $N_t - N_s$  is for  $t \geq s$  Poisson distributed with mean  $\int_s^t \lambda(u) du$ .
- The individual indices  $\{L_{\tau_i+t}^i : t \geq 0\}$  are independent and independent of  $N$ .
- The processes  $\{L_{\tau_i+t}^i : t \geq 0\}$  are identically distributed.

- $L_t^i = 0$  for  $t < \tau_i$ .
- There exists a function  $\sigma(s) \geq 0$  on  $\mathbb{R}_+$ , iid strictly positive variables  $\{Y_i\}$  and independent Brownian motions  $\{W^i\}$  such that

$$L_{\tau_i+t}^i = Y_i \exp\left\{\int_0^t \sigma(s) dW_s^i - \frac{1}{2} \int_0^t \sigma(s)^2 ds\right\}. \quad (1)$$

We assume  $\int_0^\infty \sigma(u)^2 du < \infty$ .

The variable  $Y_i$  denotes the first estimate of the losses from the  $i$ -th catastrophe. After the first estimate the loss is re-estimated continuously. This is of course an idealisation. In reality, after a refined estimate within one month after the catastrophe the individual index will only be changed about every two months. We can take this into account by the choice of  $\sigma(u)$ . Typically,  $\sigma(u)$  will be large for small  $u$  and small for larger  $u$ . The fluctuations from the Brownian motions are modelled exponentially. That is, the index will fluctuate stronger for a larger individual index. It seems reasonable that the estimation error is relative and not absolute. Note that  $\{L_{\tau_i+t}^i : t \geq 0\}$  is a martingale. This should be the case, otherwise PCS's estimate would be biased. The Poisson assumption for  $N$  is standard in actuarial mathematics. It has moreover the advantage that the distribution of  $L_{T_2}$  becomes compound Poisson, as will be seen below.

That catastrophes will be independent is also an idealisation and is used here as a first approximation to reality. For earthquakes this assumption seems reasonable. With an earthquake usually a series of earthquakes occurs that can be considered as a single event. Different events will then be so far apart that it is difficult to believe that they are dependent. On the other hand, frequently used models in earthquake modelling state that the interoccurrence time is positively correlated to the severity of an earthquake. However, because most likely at most one earthquake occurs affecting the index this dependence may be neglected. In practice information from

earthquake prediction should be reflected in the choice of the parameters of our model.

The severity of hurricanes depends on climatic variables. A phenomena like El Niño shows that there is dependence for windstorm losses. On the other side, it is unlikely that in a period several events hit heavily populated areas. One could also assume that whether such a phenomenon is present or not is known by climatic researchers beforehand, and can therefore widely taken into the modelling through the distribution of  $Y_i$  and the choice of  $\lambda(t)$ .

That  $W^i$  is independent of  $Y_i$  seems reasonable. The first estimate should in some sense reflect the actual loss and the quality of the work of PCS should not be influenced by how large the actual loss or the first estimate are. Clearly, the estimation error should be relative and not absolute.

The insurance world will besides the PCS option also have the insurance losses in their portfolio. In order that the PCS option is of any value for an insurer it has to reflect the actual losses. We assume that, if PCS would estimate the losses for a sufficient time the loss index  $L^i$  would become the actual loss  $X_i$ , i.e.  $X_i = L_\infty^i$ . Clearly, the losses  $X_i$  will have an influence on the price of the security. This fact will be taken into account when defining the pricing measure in Section 3 below.

For modelling catastrophic losses one typically assumes that the distribution of  $X_i$  is heavy-tailed, see [9]. Because the lognormal distribution is heavy-tailed, this is automatically fulfilled for our model. Data moreover imply that the distribution tail of  $X_i$  is regularly varying. It is therefore natural to assume that  $Y_i$  possesses a regularly varying distribution tail. Note that this does not imply that  $X_i$  has a regularly varying distribution tail, but the asymptotic behaviour is at least similar. Even if the distribution tail is not regularly varying our model is consistent with data.

### 3. The pricing measure

In the market there exist the following securities. A riskless zero coupon bond with maturity  $T_2$  and with price  $B_t = e^{-\delta(T_2-t)}$  for some  $\delta > 0$  (or alternatively a bank account), insurance contracts to cover the catastrophe risk and the PCS option. Then there is a pricing measure  $\mathbb{P}^*$  such that the price of the PCS option becomes

$$F_t = B_t \mathbb{E}^*[F_{T_2} | \mathcal{F}_t] = B_t \mathbb{E}^*[\min\{\max\{L_{T_2} - K, 0\}, A\} | \mathcal{F}_t].$$

Alternatively, the option price can be written as

$$F_t = B_t \int_K^{K+A} (1 - F_{L,t}(y)) dy, \quad (2)$$

where  $F_{L,t}(y) = \mathbb{P}^*[L_{T_2} \leq y | \mathcal{F}_t]$  is the conditional distribution function of  $L_{T_2}$  under  $\mathbb{P}^*$ . The actual price is  $200F_t$  because a basic point is worth \$200. Because the insurance losses are part of the same market the aggregate premium of all the insurance contracts is obtained as

$$\Pi = \mathbb{E}^*[L_\infty].$$

If the insurance premiums are known then the above equation can be seen as a side condition the Radon-Nikodym derivative  $d\mathbb{P}^*/d\mathbb{P}$  has to fulfil.

The information at time  $\infty$  is contained in the first estimates  $Y_i$  and the Brownian motions  $W^i$  determining the development of the individual indices. Thus the Radon-Nikodym derivative  $d\mathbb{P}^*/d\mathbb{P}$  must be a functional of  $Y_i$  and  $W^i$ .

In principle, any price  $F_t \in (0, AB_t)$  would be possible. In contrast to the usual option pricing theory the index  $L_t$  is not traded. That is, arbitrage pricing theory does not apply. If the insurance market was liquid (i.e. it is possible to sell and buy parts of the portfolio without any restriction) one could consider the market value of the insurance contracts  $\Pi_t = \mathbb{E}^*[L_\infty | \mathcal{F}_t]$  as the price of an asset. If there would be enough reinsurance contracts (meaning that it is possible to reinsure the whole

portfolio even though the agent does not hold the whole risk) it would be possible to duplicate  $\Pi_t$  by reinsurance strategies. This market value should be close to the price of the index  $\mathbb{E}^*[L_{T_2} | \mathcal{F}_t]$ , i.e. a portfolio of PCS options. These prices are not the same because  $L_{T_2} = \mathbb{E}[L_\infty | \mathcal{F}_{T_2}]$ , i.e. the physical measure is used for evaluation. If we assume that the insurance risk can be traded the PCS index could be approximated by an insurance portfolio. Therefore we use an equivalent measure for evaluation. An equivalent measure would also appear in a utility approach. Note that  $\{L_t\}$  is not a martingale under  $\mathbb{P}^*$ , but  $\mathbb{E}^*[L_{T_2} | \mathcal{F}_t]$  is.

The measure  $\mathbb{P}^*$  represents the market's view of the securities in the market. In order to determine possible measures  $\mathbb{P}^*$  we have to define the type of model a risk neutral representative agent sees. The author believes that under  $\mathbb{P}^*$  the 'same type' of model as under  $\mathbb{P}$  should appear. We therefore look for measures  $\mathbb{P}^*$  under which  $N$  remains a Poisson process, and  $W^i$ ,  $N$  and  $\{Y_i\}$  remain mutually independent.

If  $N$  would not be a Poisson process it would be possible to obtain information under  $\mathbb{P}^*$  on the occurrence of the next catastrophe. This seems to be strange. If  $Y_i$  and  $W^i$  would be dependent after the change of measure the agents would agree upon that PCS estimates differently for different first estimates. This is only possible, if the general believe in the market would change because of a catastrophe. This could for example be the case if new research discovers some until present unknown influence on catastrophes. Then the parameters of the model have to be changed. Modelling the stochastic quantities independent under  $\mathbb{P}^*$  means that we do not incorporate the parameter risk in our model.

The problem of changing the measure for a compound Poisson process such that the process remains compound Poisson is investigated in [8]. The Radon-Nikodym derivative must therefore be of the form

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left\{\sum_{k=1}^{N_{T_1}} \beta(Y_i, W^i, \tau_i) - \int_0^t \lambda(s) \mathbb{E}[\exp\{\beta(Y, W, s)\} - 1] ds\right\},$$

where  $\beta(Y, W, t)$  is a functional depending on the first estimate  $Y$ , the Brownian motion  $W$  and the occurrence time of the catastrophe. We drop the index if we speak about a generic variable. In order that the formula makes sense we assume that  $\mathbb{E}[\exp\{\beta(Y, W, s)\}] < \infty$ . In order that  $Y_i$  and  $W^i$  remain independent the functional has to be of the form

$$\beta(Y, W, t) = \beta(Y, t) + \int_0^\infty \gamma(s, t) dW_s.$$

The dependence on time indicates that the agents do value catastrophes occurring early and late in the occurrence period differently. For a PCS option, however, the development period is chosen in such a way that  $L_{T_2 - \tau_i}^i$  should be close to the real loss. Therefore in our model we expect  $\int_{T_2 - T_1}^\infty \sigma(s)^2 ds$  to be small. If the latter quantity was zero there would be no reason to use time dependence. Hence it is at least not a large loss of generality to assume that  $\beta(Y, W, t)$  does not depend on  $t$ . We therefore choose

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left\{\left(\sum_{k=1}^{N_t} \beta(Y_i) + \int_0^\infty \gamma(s) dW_s^i\right) - \int_0^t \lambda(s) \mathbb{E}[\Gamma \exp\{\beta(Y)\} - 1] ds\right\}, \quad (3)$$

where  $\Gamma = \exp\{\frac{1}{2} \int_0^\infty \gamma(t)^2 dt\}$ . In particular, we have to assume that  $\int_0^\infty \gamma(t)^2 dt < \infty$  and  $\mathbb{E}[\exp\{\beta(Y)\}] < \infty$ .

We obtain the following law for the process under  $\mathbb{P}^*$ .

**Lemma 1.** *Under the measure  $\mathbb{P}^*$  the claim number process  $N$  is a (inhomogeneous) Poisson process with rate  $\lambda^*(t) = \Gamma \mathbb{E}[\exp\{\beta(Y)\}] \lambda(t)$ . The first estimates  $Y_i$  are iid with distribution function*

$$dF_Y^*(y) = \frac{e^{\beta(y)} dF_Y(y)}{\mathbb{E}[\exp\{\beta(Y)\}]},$$

and the process  $W^i$  becomes an Itô process satisfying

$$W_t^i = \widetilde{W}_t^i + \int_0^t \gamma(s) dt,$$



where  $\widetilde{W}^i$  are independent standard Brownian motions under  $\mathbb{P}^*$  independent of  $\{Y_i\}$  and  $N$ .

The proof is given in Section 7.

The stochastic integral in (1) can be written as

$$\int_0^t \sigma(s) dW_s^i = \int_0^t \sigma(s)\gamma(s) ds + \int_0^t \sigma(s) d\widetilde{W}_s^i.$$

Hence the market adds a drift to the re-estimates. This drift can be seen as the price for the risk an investor takes over.

#### 4. The option price

The PCS index  $L_{T_2}$  can for  $t < T_2$  be written as

$$L_{T_2} = \sum_{k=1}^{N_t \wedge T_1} L_t^k \exp\left\{ \int_{t-\tau_k}^{T_2-\tau_k} \sigma(s) dW_s^k - \frac{1}{2} \int_{t-\tau_k}^{T_2-\tau_k} \sigma(s)^2 ds \right\} + \sum_{k=N_t \wedge T_1 + 1}^{N_{T_1}} L_{T_2}^k. \quad (4)$$

In the following discussion we suppose that the present time is  $t$  and that  $L_t^k$  is observable for all catastrophes occurred until time  $t$ . If  $t < T_1$  then the second sum  $\sum_{k=N_t+1}^{N_{T_1}} L_{T_2}^k$  is compound Poisson distributed. The first sum is a sum of independent lognormally distributed random variables. Its distribution can be calculated numerically if  $N_{t \wedge T_1}$  is small, i.e. in practice smaller than three. This will usually not be a problem, because there will typically not be more than three catastrophes in the occurrence period contributing to the index. For the compound Poisson distribution there are no closed expressions. Its distribution can theoretically be approximated by Panjer recursions, see [11] or [12], or by approximations. Convolution of the two distributions, however, is a numerical problem unless  $N_t \in \{0, 1\}$ . We therefore discuss here approximations only. Another problem with Panjer recursion is, that the distribution of the summands is an average over  $\tau \in (t, T_1)$  of the distribution of the individual index of a catastrophe occurring at time  $\tau$ .

Most approximations to distribution functions are based on moments. Let us therefore calculate the conditional moments of  $L_{T_2}$  under  $\mathbb{P}^*$  given the information up to time  $t$ . Because the expressions for the sum over claims occurred and claims occurring in the future are independent it is enough to determine the moments of the two sums in (4) separately.

The summands in

$$\sum_{k=1}^{N_{t \wedge T_1}} L_t^k \exp \left\{ \int_{t-\tau_k}^{T_2-\tau_k} \sigma(s) dW_s^k - \frac{1}{2} \int_{t-\tau_k}^{T_2-\tau_k} \sigma(s)^2 ds \right\}$$

are independent. The  $n$ -th moment of  $L_t^k \exp \left\{ \int_{t-\tau_k}^{T_2-\tau_k} \sigma(s) dW_s^k - \frac{1}{2} \int_{t-\tau_k}^{T_2-\tau_k} \sigma(s)^2 ds \right\}$  is  $m_n^k = (L_t^k)^n \exp \left\{ \frac{1}{2} n(n-1) \int_{t-\tau_k}^{T_2-\tau_k} \sigma(s)^2 ds + n \int_{t-\tau_k}^{T_2-\tau_k} \sigma(s) \gamma(s) ds \right\}$ . For the compound Poisson sum we obtain from the general formula the cumulants

$$\kappa_n^0 = \int_t^{T_1} \lambda^*(s) \mathbb{E}^*[Y^n] \exp \left\{ \frac{1}{2} n(n-1) \int_0^{T_2-s} \sigma(u)^2 du + n \int_0^{T_2-s} \sigma(u) \gamma(u) du \right\} ds.$$

Note that the moments are  $m_1^0 = \kappa_1^0$ ,  $m_2^0 = \kappa_2^0 + \kappa_1^{0^2}$ ,  $m_3^0 = \kappa_3^0 + 3\kappa_1^0 \kappa_2^0 + \kappa_1^{0^3}$ ,  $m_4^0 = \kappa_4^0 + 4\kappa_3^0 \kappa_1^0 + 3\kappa_2^{0^2} + 6\kappa_2^0 \kappa_1^{0^2} + \kappa_1^{0^4}$ , etc.

The conditional moments of  $L_{T_2}$  can now be obtained from

$$m_n = \sum_{\substack{k_0, \dots, k_n \\ k_0 + \dots + k_n = n}} \binom{n}{k_0, \dots, k_n} m_{k_0}^0 \cdots m_{k_n}^n,$$

where  $m_0^i = 1$  and  $\binom{n}{k_0, \dots, k_n}$  is the multinomial coefficient.

Alternatively, the cumulants  $\kappa_n^k$  of  $L_t^k \exp \left\{ \int_{t-\tau_k}^{T_2-\tau_k} \sigma(s) dW_s^k - \frac{1}{2} \int_{t-\tau_k}^{T_2-\tau_k} \sigma(s)^2 ds \right\}$  could be calculated. The cumulants of the distribution of interest are then just the sums of the cumulants.

We next describe some possible approximation methods.

**The normal approximation** The simplest approximation uses a normal distribution with the same mean value and the same variance as  $L_{T_2}$ . The option price

is

$$B_t \left( \frac{\sigma}{\sqrt{2\pi}} \left( e^{-(K-m_1)^2/(2\sigma^2)} - e^{-(K+A-m_1)^2/(2\sigma^2)} \right) + (K+A-m_1)\Phi(-(K+A-m_1)/\sigma) - (K-m_1)\Phi(-(K-m_1)/\sigma) \right),$$

where  $\sigma^2 = m_2 - m_1^2$  denotes the variance. Because the exact distribution is heavy-tailed we do not expect the normal approximation to perform well.

**The translated gamma approximation** The distribution is approximated by  $k + Z$ , where  $k \in \mathbb{R}$  and  $Z$  is  $\Gamma(g, a)$  distributed. The parameters are chosen such that the first three moments coincide. Then

$$g = \frac{4(m_2 - m_1^2)^3}{(m_3 - 3m_2m_1 + 2m_1^3)^2}, \quad a = \frac{2(m_2 - m_1^2)}{m_3 - 3m_2m_1 + 2m_1^3}, \quad k = m_1 - \frac{g}{a}.$$

The option price is then approximated by

$$B_t \left( \int_{K-k}^{K+A-k} \int_y^\infty \frac{a^g}{\Gamma(g)} z^{g-1} e^{-az} dz dy \right). \quad (5)$$

Experience shows that this approximation works very well as long as the interval  $[K, K+A]$  is not far out in the tail. The reason is that the skewness is caught well by the approximation.

**The Edgeworth approximation** The idea of the Edgeworth approximation is to consider a variable  $Z = (L_{T_2} - m_1)/\sqrt{m_2 - m_1^2}$ , and then to approximate its moment generating function by  $\mathbb{E}[e^{rZ}] \approx e^{r^2/2}(1 + a_3r^3/6 + a_4r^4/24 + a_3^2r^6/72)$ , where  $a_3$  and  $a_4$  are the third and fourth cumulant of  $Z$ , respectively. Here

$$a_3 = \frac{m_3 - 3m_2m_1 + m_1^3}{(m_2 - m_1^2)^{3/2}}, \quad a_4 = \frac{m_4 - 4m_3m_1 + 6m_2m_1^2 - 3m_1^4}{(m_2 - m_1^2)^2} - 3.$$

The corresponding distribution function is

$$F_{L,t}(y) = \Phi(y) - \frac{a_3}{6}\Phi^{(iii)}(y) + \frac{a_4}{24}\Phi^{(iv)}(y) + \frac{a_3^2}{72}\Phi^{(vi)}(y),$$

where  $\Phi(y)$  is the distribution function of the standard normal distribution and  $\Phi^{(n)}(y)$  is its  $n$ -th derivative. From this the price can be calculated from (2). The motivation for the approximation is to make a Taylor approximation to  $\log \mathbb{E}[e^{rZ}]$  around  $r = 0$ . Theoretically, such an approximation is not possible because the radius of convergence is zero. But it turns out that the approximation works well. The disadvantage is that the fourth moment has to exist. Data, however suggest that this is (under the physical measure) not the case if  $t < T_1$ . For  $t \geq T_1$  moments of all orders exist, and therefore the approximation is applicable.

**The corrected log-normal approximation** An alternative to the approach above is to expand  $\log \mathbb{E}[e^{rL_{T_2}} \mid \mathcal{F}_t] - \log \mathbb{E}[e^{r\tilde{Z}}]$  into a Taylor series, where  $\tilde{Z}$  is a log-normal distribution with the same mean value and variance as  $L_{T_2}$ . The approach is similar to the Edgeworth approximation, but instead of the normal distribution the log-normal distribution and its derivatives appear. We do not work out the details here. Because this approximation has a heavy tail one can expect it to work well.

A problem with the approximations is that between two and four moments have to exist. Typically, only the first moment will be finite. In order to get around this problem one could also approximate  $Y_i$  by  $Y_i \wedge m(K + A)$  for some value  $m$ . The value  $m$  is chosen such that

$$\mathbb{P}^* \left[ \int_0^t \sigma(s) d\tilde{W}_s^i + \int_0^t (\sigma(s)\gamma(s) - \frac{1}{2}\sigma(s)^2) ds < -\log m \right]$$

becomes small enough for all  $t \in [T_2 - T_1, T_2]$ . By doing so we assure that all moments exist. The upper bound does not have a large influence because if  $Y_i > m(K + A)$  it is very likely that  $L_{T_2} \geq (K + A)$ . If this is not the case one can at least expect that  $L_{T_2}$  is close to  $K + A$ .

**Example 1.** As an example suppose  $T_1 = T_2/2 = 1$ ,  $\lambda(t) = 1$ ,  $\mathbb{P}[Y > y] = y^{-5} \wedge 1$ ,

K/A	4/0.25	4.25/0.25	4.5/0.25	4.75/0.25	5/0.25	5.25/0.25	5.75/0.25
MC	8.35	7.41	6.58	5.86	5.20	4.62	4.12
tG	8.16	7.31	6.55	5.88	5.26	4.71	4.23

Table 1: *Translated gamma approximations to undiscounted option prices and Monte Carlo simulation values*

and  $\sigma(s) = e^{-s}$ . We choose  $\beta(y) = \frac{1}{2} \log y$  and  $\gamma(s) = 0.06 e^{-0.01s}$ . Then the expected value of  $L_t$  for  $t > T_1$  under the physical measure is 1.25.

We start by calculating the parameters under  $\mathbb{P}^*$ . The density of the claim size distribution is proportional to

$$\exp\left\{\frac{1}{2} \log(y)\right\} y^{-6} \mathbb{1}_{y>1} = y^{-5.5} \mathbb{1}_{y>1}.$$

Thus  $\mathbb{P}^*[Y > y] = y^{-4.5} \wedge 1$  and  $\mathbb{E}[\exp\{\beta(y)\}] = 5/4.5 = 10/9$ . The parameter  $\Gamma$  becomes 1.5. Thus  $\lambda^*(t) = 5/3$ . The Brownian motion is then  $W_t^i = \widetilde{W}_t^i + 6(1 - e^{-0.01t})$ .

For the premium  $\Pi$  we obtain

$$\Pi = \int_0^{T_1} \frac{5}{3} \frac{4.5}{3.5} \exp\left\{\int_0^\infty e^{-u} 0.06 e^{-0.01u} du\right\} ds = \frac{45}{21} e^{6/101} = 2.27401.$$

We calculate the price at time zero. The distribution of  $L_{T_2}$  is compound Poisson.

The cumulants are then

$$\begin{aligned} \kappa_n^0 &= \frac{22.5}{13.5 - 3n} \int_0^1 \exp\left\{\frac{n(n-1)}{4}(1 - e^{-2(2-s)}) + \frac{6n}{101}(1 - e^{-1.01(2-s)})\right\} ds \\ &= \frac{22.5}{13.5 - 3n} \int_1^2 \exp\left\{\frac{n(n-1)}{4}(1 - e^{-2s}) + \frac{6n}{101}(1 - e^{-1.01s})\right\} ds \\ &= \frac{22.5}{13.5 - 3n} e^{n(101n-77)/404} \int_1^2 \exp\left\{-\frac{n(n-1)}{4}e^{-2s} - \frac{6n}{101}e^{-1.01s}\right\} ds. \end{aligned}$$

These expressions have to be calculated numerically. We find  $\kappa_1^0 = 2.24327$ ,  $\kappa_2^0 = 5.26574$  and  $\kappa_3^0 = 23.5908$ . The parameters of the translated gamma approximation

become  $g = 4\kappa_2^{03}/\kappa_3^{02} = 1.04942$ ,  $a = 2\kappa_2^0/\kappa_3^0 = 0.446422$  and  $k = \kappa_1^0 - g/a = -0.107472$ .

Table 1 gives prices of the PCS option calculated by Monte Carlo simulations for different  $K$  and  $A = 0.25$  and the price obtained by the translated gamma approximation. Discounting is not taken into account (i.e.  $B_0 = 1$ ). The value given in (5) is multiplied by 200, because a basic point is worth 200.

The approximation works quite well. The difference between the approximation and the Monte-Carlo price is less than 3%. This is because the translated gamma approximation catches the skewness of the distribution quite well. ■

## 5. Doubly stochastic occurrences

A drawback of the model presented in Section 2 is that phenomena like El Niño not are taken into account. In this section we show how the model can be changed in order to model periods with more or more severe catastrophes. The problem is that we introduce two more random variables. The model therefore becomes quite easily over-specified, and estimation from data becomes a problem. Of course, a better fit is obtained, but choosing the correct model may be a problem because of overparametrisation.

To keep the model simple we choose an intensity of the form  $\Lambda\lambda(t)$ , where  $\Lambda$  is stochastic and  $\lambda(t)$  is a given function. This means that we multiply an average intensity  $\lambda(t)$  by a constant that is different every period. If in a period the catastrophe intensity is larger then it is larger all the period.

Let so  $\Lambda \in (0, \infty)$  and  $\theta \in \mathbb{R}^d$  be two random variables. These two variables may be dependent. We change the assumptions of Section 2 in the following way:

- The variables  $(\Lambda, \theta)$  have distribution function  $H(\ell, \vartheta)$ .

- Given  $(\Lambda, \theta)$ , the occurrence process  $N$  is conditionally an inhomogeneous Poisson process with rate  $\Lambda\lambda(t)$ .
- Given  $(\Lambda, \theta)$ , the first estimates are conditionally iid with distribution function  $F_\theta(y)$ .

Through this construction the occurrence times become dependent, the first estimates (and thus the aggregate claims) become dependent, and the number of claims and their sizes may be dependent.

The chosen model is Bayesian. One should therefore try to choose distributions such that conditioned on the observations the posterior distribution is of the same type, see [5]. This means that  $\Lambda$  should follow a gamma distribution. For our setup a nice model is for instance  $\theta$  follows a gamma distribution and  $Y_i$  is conditioned on  $\theta$  exponentially distributed with parameter  $\theta$ . Then the unconditional distribution of  $Y_i$  is the Pareto law. This approach is well-known in actuarial mathematics, where it is called ‘credibility’.

The natural extension of the Radon-Nikodym derivative (3) to the present model is

$$\begin{aligned} \frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp \left\{ h(\Lambda, \theta) + \left( \sum_{k=1}^{N_t} \beta(Y_k) + \int_0^\infty \gamma(s) dW_s^i \right) \right. \\ \left. - \Lambda \int_0^t \lambda(s) \mathbb{E}[\Gamma \exp\{\beta(Y)\} - 1 \mid \theta] ds \right\} (\mathbb{E}[e^{h(\Lambda, \theta)}])^{-1}. \end{aligned}$$

Here  $h(\ell, \vartheta)$  is a function such that  $\mathbb{E}[e^{h(\Lambda, \theta)}] < \infty$ . Under the measure  $\mathbb{P}^*$  we have a process of the same type.

**Lemma 2.** *Under the measure  $\mathbb{P}^*$  the variables  $(\Lambda, \theta)$  have distribution function*

$$dH^*(\ell, \vartheta) = \frac{e^{h(\ell, \vartheta)} dH(\ell, \vartheta)}{\mathbb{E}[e^{h(\Lambda, \theta)}]}.$$

Given  $(\Lambda, \theta)$  the claim number process  $N$  is conditionally a Poisson process with rate  $\lambda^*(t) = \Lambda \Gamma \mathbb{E}[\exp\{\beta(Y)\} \mid \theta] \lambda(t)$ . The first estimates  $Y_i$  have conditional distribution function

$$dF_{Y^*}^*(y) = \frac{e^{\beta(y)} dF_\theta(y)}{\mathbb{E}[\exp\{\beta(Y)\} \mid \theta]},$$

and the process  $W^i$  becomes an Itô process satisfying

$$W_t^i = \widetilde{W}_t^i + \int_0^t \gamma(s) dt,$$

where  $\widetilde{W}^i$  are independent standard Brownian motions under  $\mathbb{P}^*$ , independent of  $\Lambda, \theta, \{Y_i\}$  and  $N$ .

The proof is given in Section 7.

## 6. Conclusion

In this paper we proposed a model for the PCS index based on individual indices for each single catastrophe. Aiming to obtain the same type of model under the pricing measure  $\mathbb{P}^*$ , we found the Radon-Nikodym derivative and calculated the parameters of the PCS index under the pricing measure. Calculation of prices can then be done by approximation or by simulation.

In practice the individual indices are not published. Denote by  $\mathcal{F}_t^L$  the smallest right continuous filtration which  $L_t$  is adapted to. By conditioning on  $\mathcal{F}_t^L$  and by the independence assumptions made in the model the value  $L_t^k$  has then to be replaced by  $\mathbb{E}[L_t^k \mid \mathcal{F}_t^L]$ . Note that the PCS index only is announced daily. This simplifies the problem. We can write  $L_t^k = L_{\tau_k}^k Z_1^k \cdots Z_{(t-\tau_k)/\Delta}^k$ , where  $\Delta$  is one day measured in the used time unit. One has therefore only a finite number of random variables to condition upon. It should also be noted that typically the number of catastrophes covered by the index is small.

The same result may be applied to price some sorts of catastrophe bonds with triggering events based on the PCS index. Suppose the catastrophe bond promises



to pay back one unit if the PCS index does not exceed  $K$ , and the payoff is linearly reduced thereafter until the PCS index reaches  $K + A$ . Then the payoff is

$$\max\{\min\{1 - (L_{T_2} - K)/A, 1\}, 0\} = 1 - \min\{\max\{L_{T_2} - K, 0\}, A\}/A.$$

The price can therefore easily be obtained from the discounting factor  $B_t$  and the price of a PCS option with strike price  $K$  and cap  $A$ .

This paper does not address the problem of statistical inference. Before applying the theory the quantities involved have to be estimated. The parameters of the physical model are not a big problem because one has the PCS indices following the law  $\mathbb{P}$  since 1995. The parameters of the Radon-Nykodym derivative have to be estimated from the PCS index and the actually observed prices. This may be the main problem in practice. On one hand it is possible that market prices are misspecified, and therefore are observed with errors. On the other hand the product is not intensively traded. One therefore may have missing data.

## 7. Proofs of the results

**Proof of Lemma 1.** Let  $n \in \mathbb{N}$  and  $\Sigma$  be a Borel set of  $(\tau_1, \dots, \tau_n)$ . Let  $B_i$  be real Borel sets and  $A_i$  be Borel sets of  $C_{[0, \infty)}(\mathbb{R})$ , the space of continuous real

functions endowed with the usual metric. Then

$$\begin{aligned}
& \mathbb{P}^*[N_{T_1} = n, (\tau_1, \dots, \tau_n) \in \Sigma, Y_i \in B_i, W^i \in A_i] \\
&= \mathbb{E}\left[\frac{d\mathbb{P}^*}{d\mathbb{P}}; N_{T_1} = n, (\tau_1, \dots, \tau_n) \in \Sigma, Y_i \in B_i, W^i \in A_i\right] \\
&= \mathbb{E}\left[\exp\left\{\left(\sum_{i=1}^n \beta(Y_i) + \int_0^\infty \gamma(s) dW_s^i\right)\right\};\right. \\
&\quad \left. N_{T_1} = n, (\tau_1, \dots, \tau_n) \in \Sigma, Y_i \in B_i, W^i \in A_i\right] \\
&\quad \times \exp\left\{-\int_0^{T_1} \lambda(s) \mathbb{E}[\Gamma \exp\{\beta(Y)\} - 1] ds\right\} \\
&= \frac{(\Gamma \mathbb{E}[\exp\{\beta(Y)\}] \int_0^{T_1} \lambda(s) ds)^n}{n!} \exp\left\{-\Gamma \mathbb{E}[\exp\{\beta(Y)\}] \int_0^{T_1} \lambda(s) ds\right\} \\
&\quad \times \mathbb{P}[(\tau_1, \dots, \tau_n) \in \Sigma \mid N_{T_1} = n] \\
&\quad \times \prod_{i=1}^n \frac{\mathbb{E}[\exp\{\beta(Y_i)\}; Y_i \in B_i]}{\mathbb{E}[\exp\{\beta(Y_i)\}]} \frac{\mathbb{E}[\exp\{\int_0^\infty \gamma(s) dW_s^i\}; W^i \in A_i]}{\Gamma}.
\end{aligned}$$

This shows that  $N$ ,  $\{Y_i\}$  and  $\{W^i\}$  remain independent. It also shows that  $N_{T_1}$  follows a Poisson distribution with parameter  $\int_0^{T_1} \lambda^*(t) dt$ , and that  $Y_i$  follows distribution  $F_Y^*(y)$ . Conditioned on  $N_{T_1} = n$ , the occurrence points have the same distribution as the order statistics of  $n$  iid variables  $\tilde{\tau}_i$  with density

$$\frac{\lambda(t)}{\int_0^{T_1} \lambda(s) ds} = \frac{\lambda^*(t)}{\int_0^{T_1} \lambda^*(s) ds},$$

see also [12, Thm. 12.2.1]. Thus  $N$  is an inhomogeneous Poisson process with rate  $\lambda^*(t)$ . The law of  $W^i$  follows from the Girsanov theorem.  $\square$

**Proof of Lemma 2.** We use the same notation as in the proof of Lemma 1. Let  $\mathcal{L}$  and  $\mathcal{T}$  be Borel sets. Calculation of

$$\mathbb{E}\left[\frac{d\mathbb{P}^*}{d\mathbb{P}}; N_{T_1} = n, (\tau_1, \dots, \tau_n) \in \Sigma, Y_i \in B_i, W^i \in A_i \mid \Lambda, \theta\right]$$

yields as in the proof of Lemma 1

$$\begin{aligned}
& \frac{(\Lambda \Gamma \mathbb{E}[\exp\{\beta(Y)\} \mid \theta] \int_0^{T_1} \lambda(s) ds)^n}{n!} \exp\left\{-\Lambda \Gamma \mathbb{E}[\exp\{\beta(Y)\} \mid \theta] \int_0^{T_1} \lambda(s) ds\right\} \\
& \times \mathbb{P}[(\tau_1, \dots, \tau_n) \in \Sigma \mid N_{T_1} = n] \\
& \times \prod_{i=1}^n \frac{\mathbb{E}[\exp\{\beta(Y_i)\}; Y_i \in B_i \mid \theta]}{\mathbb{E}[\exp\{\beta(Y_i)\} \mid \theta]} \frac{\mathbb{E}[\exp\{\int_0^\infty \gamma(s) dW_s^i\}; W^i \in A_i]}{\Gamma} \\
& \times \frac{e^{h(\Lambda, \theta)}}{\mathbb{E}[e^{h(\Lambda, \theta)}]}.
\end{aligned}$$

The assertion follows now readily from

$$\begin{aligned}
& \mathbb{P}^*[\Lambda \in \mathcal{L}, \theta \in \mathcal{T}, N_{T_1} = n, (\tau_1, \dots, \tau_n) \in \Sigma, Y_i \in B_i, W^i \in A_i] \\
& = \mathbb{E}\left[\mathbb{E}\left[\frac{d\mathbb{P}^*}{d\mathbb{P}}; N_{T_1} = n, (\tau_1, \dots, \tau_n) \in \Sigma, Y_i \in B_i, W^i \in A_i \mid \Lambda, \theta\right]; \Lambda \in \mathcal{L}, \theta \in \mathcal{T}\right].
\end{aligned}$$

□

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