Mock theta functions

and

representation theory of

affine Lie superalgebras and superconformal algebras

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Weyl-Kac type character formula for integrable representations of affine Lie algebras and their modular properties:

$$\mathfrak{g} = \overline{\mathfrak{g}} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d$$

Weyl-Kac character formula for integrable highest weight module $L(\Lambda)$:

$$\mathrm{ch}_{L(\Lambda)} = \frac{1}{R} \sum_{w \in W} \varepsilon(w) e^{w(\Lambda+\rho)} = \frac{1}{R} \sum_{w \in \overline{W}} \varepsilon(w) w \Bigg(\underbrace{\sum_{\alpha \in M} e^{t_\alpha(\Lambda+\rho)}}_{\substack{\uparrow \\ \text{theta function}}} \Bigg)$$

where

$$R := \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\mathrm{mult}\alpha} \quad : \ \text{denominator of } \mathfrak{g}$$

$$M := \text{coroot lattice of } \overline{\mathfrak{g}}$$

$$t_\alpha(\lambda) := \lambda + (\lambda|\delta)\alpha - \left\{ \frac{|\alpha|^2}{2}(\lambda|\delta) + (\lambda|\alpha) \right\} \delta$$

$$\rho := \text{Weyl vector}$$

Example: In the case $\mathfrak{g} = \widehat{sl}(2, \mathbf{C})$,

$$\begin{cases} \Pi = \{\alpha_0,\ \alpha_1\} & : \ \text{simple roots} \\ h^\vee = 2 & : \ \text{dual Coxeter number} \\ \Lambda = (m-j)\Lambda_0 + j\Lambda_1 & (m, j \in \mathbf{Z}_{\geqq 0},\ \ 0 \leqq j \leqq m) \\ q := e^{-\delta} & \end{cases}$$

Then

$$q^{\frac{|\Lambda+\rho|^2}{2(m+2)}} \sum_{\alpha \in M} e^{t_\alpha(\Lambda+\rho)} = e^{2\pi i(m+2)t}\theta_{j+1,m+2}(\tau, z)$$

$$\underbrace{q^{\frac{|\Lambda+\rho|^2}{2(m+2)} - \frac{|\rho|^2}{2h^\vee}} \mathrm{ch}_{L(\Lambda)}}_{\substack{\uparrow \\ \text{normalized character}}} = e^{2\pi imt} \frac{[\theta_{j+1,m+2} - \theta_{-(j+1),m+2}](\tau, z)}{[\theta_{1,2} - \theta_{-1,2}](\tau, z)}$$

where

$$\theta_{j,m}(\tau, z) := \sum_{k \in \frac{j}{2m} + \mathbf{Z}} e^{2\pi imkz} q^{mk^2} \quad : \ \text{Jacobi theta function}$$

Weyl-Kac type character formula for partially integrable representations of affine Lie superalgebras:

$$\mathrm{ch}_{L(\Lambda)} = \frac{1}{R} \sum_{w \in W^{\sharp}} \varepsilon(w) w \left(\frac{e^{\Lambda+\rho}}{\prod_{i=1}^{n} \left(1 + e^{-\beta_i}\right)} \right)$$

$$\text{where} \quad \begin{cases} \beta_i & \in & \Pi & = \{\text{simple roots}\} \\ (\Lambda + \rho | \beta_i) & = & 0 & \\ (\beta_i | \beta_j) & = & 0 & \\ \{\beta_i\}_{i=1,\cdots,n} & : & \text{maximal} & (n := \text{atypicality}) \end{cases}$$

$$\left(\begin{array}{l} \text{conjectured by Kac-W (1988)} \\ \text{proved by Gorelik-Kac (arXiv:1406.6860)} \end{array} \right)$$

Also important is the <u>super-character</u>:

$$\mathrm{ch}^{(-)}_{L(\Lambda)} \stackrel{def}{:=} \mathrm{ch}_{L(\Lambda)}\Big|_{\substack{e^{-\alpha_i} \mapsto -e^{-\alpha_i} \text{ if } \alpha_i \text{ is odd} \\ e^{-\alpha_i} \mapsto e^{-\alpha_i} \text{ if } \alpha_i \text{ is even}}}$$

$$= \frac{1}{R^{(-)}} \sum_{w \in W^\sharp} \varepsilon^{(-)}(w) w \left(\frac{e^{\Lambda+\rho}}{\prod_{i=1}^{n} (1 - e^{-\beta_i})} \right)$$

where

$$\varepsilon^{(-)}(r_\alpha) := \begin{cases} 1 & \text{if } \alpha/2 = \text{root} \\ -1 & \text{if } \alpha/2 \neq \text{root} \end{cases}$$

$$R^{(\pm)} := e^\rho \frac{\prod_{\alpha \in \Delta_+^{\text{even}}} (1 - e^{-\alpha})^{\text{mult}\alpha}}{\prod_{\alpha \in \Delta_+^{\text{odd}}} (1 \pm e^{-\alpha})^{\text{mult}\alpha}} \quad : \quad \text{(super)denominator of } \mathfrak{g}$$

Example. In the case $\widehat{sl}(2|1)$:

$$R^{(-)} \cdot \mathrm{ch}^{(-)}_{L((m-1)\Lambda_0)} = e^{2\pi imt}\left\{ \underbrace{\sum_{j\in\mathbf{Z}} \frac{e^{2\pi imj(z_1+z_2)}q^{mj^2}}{1-e^{2\pi iz_1}q^j}}_{\Phi_1^{[m]}} - \underbrace{\sum_{j\in\mathbf{Z}} \frac{e^{-2\pi imj(z_1+z_2)}q^{mj^2}}{1-e^{-2\pi iz_2}q^j}}_{\Phi_2^{[m]}} \right\}$$

$(m \in \mathbf{N})$

|| put || put

Problem: What is the modular property of this function?

Basic affine Lie superalgebras : (Kac: 1978)

- $\widehat{sl}(m|n) \quad (m \neq n)$
- $\widehat{psl}(n|n)$
- $\widehat{osp}(M|N)$
- $\widehat{D}(2,1;a)$
- $\widehat{F}(4)$
- $\widehat{G}(3)$

Importance of modular properties:

$$\text{affine Lie superalgebras} \quad \xrightarrow{\text{quantum reduction}} \quad \text{W-algebras}$$

Examples:

$$\begin{array}{ccc}
\widehat{sl}(2,\mathbf{C}) & \longrightarrow & \text{Virasoro} \\
\widehat{osp}(1|2) & \longrightarrow & \text{super-Virasoro} \\
\widehat{sl}(2|1) & \longrightarrow & \text{N=2 SCA} \\
\widehat{osp}(3|2) & \longrightarrow & \text{N=3 SCA} \\
\widehat{psl}(2|2) & \longrightarrow & \text{N=4 SCA} \\
\widehat{D}(2,1;a) & \longrightarrow & \text{big N=4 SCA} \\
\vdots & &
\end{array}$$

Modular properties of affine Lie superalgebras

$$\downarrow$$

Modular properties of **SCA's**

Example. **Representations of $\widehat{sl}(m|n)$ and $\widehat{osp}(M|N)$ on the Fock space:**

The case $\widehat{sl}(m|n)$ and $\widehat{osp}(2m|2n) = \widehat{D}(m,n)$:

$$\text{fermions} \ : \left\{ \begin{array}{rcl} \psi^{(i)}(z) &=& \displaystyle\sum_{r\in\frac{1}{2}+\mathbf{Z}} \psi_r^{(i)} z^{-r-\frac{1}{2}} \\ \psi^{(i)*}(z) &=& \displaystyle\sum_{r\in\frac{1}{2}+\mathbf{Z}} \psi_r^{(i)*} z^{-r-\frac{1}{2}} \end{array} \right. \qquad (i = 1, \cdots, m)$$

$$\text{bosons} \ : \left\{ \begin{array}{rcl} \varphi^{(i)}(z) &=& \displaystyle\sum_{r\in\frac{1}{2}+\mathbf{Z}} \varphi_r^{(i)} z^{-r-\frac{1}{2}} \\ \varphi^{(i)*}(z) &=& \displaystyle\sum_{r\in\frac{1}{2}+\mathbf{Z}} \varphi_r^{(i)*} z^{-r-\frac{1}{2}} \end{array} \right. \qquad (i = 1, \cdots, n)$$

(anti)commutation relations:

$$\left\{ \begin{array}{rcrcr} [\psi_r^{(i)},\, \psi_s^{(j)*}] &=& [\psi_s^{(j)*},\, \psi_r^{(i)}] &=& \delta_{i,j}\delta_{r+s,0} \\ [\varphi_r^{(i)},\, \varphi_s^{(j)*}] &=& -[\varphi_s^{(j)*},\, \varphi_r^{(i)}] &=& -\delta_{i,j}\delta_{r+s,0} \end{array} \right.$$

Define weight and charge for each particle:

$$
\begin{array}{llcc}
 & & \text{weight} & \text{charge} \\
\psi_r^{(i)} & : & \varepsilon_i + r\delta & +1 \\
\psi_r^{(i)*} & : & -\varepsilon_i + r\delta & -1
\end{array}
\qquad\qquad
\begin{array}{llcc}
 & & \text{weight} & \text{charge} \\
\varphi_r^{(i)} & : & \varepsilon'_i + r\delta & +1 \\
\varphi_r^{(i)*} & : & -\varepsilon'_i + r\delta & -1
\end{array}
$$

$$
F : \text{Fock space with} \left\{
\begin{array}{cl}
\text{vacuum} & : \ |0\rangle \\
\text{annihilation} & : \ \psi_r^{(i)}, \ \psi_r^{(i)}, \ \varphi_r^{(i)}, \ \varphi_r^{(i)*} \quad (r > 0) \\
\text{creation} & : \ \psi_r^{(i)}, \ \psi_r^{(i)}, \ \varphi_r^{(i)}, \ \varphi_r^{(i)*} \quad (r < 0)
\end{array}
\right.
$$

$$
F = \bigoplus_{s \in \mathbf{Z}} F_s = \underbrace{\Big(\bigoplus_{s:\,\text{even}} F_s \Big)}_{= F_{\text{even}}} \oplus \underbrace{\Big(\bigoplus_{s:\,\text{odd}} F_s \Big)}_{= F_{\text{odd}}} \ : \ \text{charge decomposition}
$$

- F_s : irreducible $\widehat{gl}(m|n)$-module
- $F_{\text{even}}, \ F_{\text{odd}}$: irreducible $\widehat{osp}(2m|2n)$-module

Characters of these representations:

$$\mathrm{ch}_{F_{\mathrm{even}}} + \mathrm{ch}_{F_{\mathrm{odd}}} = \prod_{r=0}^{\infty} \frac{\prod_{i=1}^{m} \left(1 + e^{\varepsilon_i}\, q^{r+\frac{1}{2}}\right)\left(1 + e^{-\varepsilon_i}\, q^{r+\frac{1}{2}}\right)}{\prod_{i=1}^{n} \left(1 - e^{\varepsilon'_i}\, q^{r+\frac{1}{2}}\right)\left(1 - e^{-\varepsilon'_i}\, q^{r+\frac{1}{2}}\right)} \times \ (\text{weight of } |0\rangle)$$

$$\mathrm{ch}_{F_{\mathrm{even}}} - \mathrm{ch}_{F_{\mathrm{odd}}} = \prod_{r=0}^{\infty} \frac{\prod_{i=1}^{m} \left(1 - e^{\varepsilon_i}\, q^{r+\frac{1}{2}}\right)\left(1 - e^{-\varepsilon_i}\, q^{r+\frac{1}{2}}\right)}{\prod_{i=1}^{n} \left(1 + e^{\varepsilon'_i}\, q^{r+\frac{1}{2}}\right)\left(1 + e^{-\varepsilon'_i}\, q^{r+\frac{1}{2}}\right)} \times \ (\text{weight of } |0\rangle)$$

- $\mathrm{ch}_{F_{\mathrm{even}}}, \quad \mathrm{ch}_{F_{\mathrm{odd}}} \quad : \quad$ modular forms

- $\left\langle \mathrm{ch}^{(-)}_{F_{\mathrm{even}}}, \quad \mathrm{ch}^{(-)}_{F_{\mathrm{odd}}}, \quad \mathrm{ch}^{(-)}_{F^{\mathrm{tw}}_{\mathrm{even}}}, \quad \mathrm{ch}^{(-)}_{F^{\mathrm{tw}}_{\mathrm{odd}}} \right\rangle_{\mathbf{C}} \quad : \quad SL_2(\mathbf{Z})$-invariant

  $\uparrow$

  supercharacters of irreducible $\widehat{osp}(2m, 2n)$-modules

Character of F_s ;

For simplicity, consider the case $n=1$;

$$\mathrm{ch}_{F_s} = \frac{e^{\Lambda_0+s\varepsilon_1'}\,q^{-\frac{s}{2}}}{\varphi(q)^{m+1}}\prod_{j=1}^{\infty}(1+e^{\varepsilon_1-\varepsilon_1'}q^j)(1+e^{\varepsilon_1'-\varepsilon_1}q^{j-1})\sum_{k=(k_2,\cdots,k_m)\in\mathbf{Z}^{m-1}}\frac{e^{\sum_{i=2}^{m}k_i(\varepsilon_i-\varepsilon_1')}\,q^{\frac{1}{2}\sum_{i=2}^{m}k_i(k_i+1)}}{1+e^{\varepsilon_1'-\varepsilon_1}q^{|k|-s}}$$

where $\quad \varphi(q) := \prod_{j=1}^{\infty}(1-q^j)$

In particular in the case $m=2$;

$$\mathrm{ch}_{F_s} = \frac{e^{\Lambda_0+s\varepsilon_1'}\,q^{-\frac{s}{2}}}{\varphi(q)^{3}}\prod_{j=1}^{\infty}(1+e^{\varepsilon_1-\varepsilon_1'}q^j)(1+e^{\varepsilon_1'-\varepsilon_1}q^{j-1})\sum_{k_2\in\mathbf{Z}}\frac{e^{k_2(\varepsilon_2-\varepsilon_1')}q^{\frac{1}{2}k_2(k_2+1)}}{1+e^{\varepsilon_1'-\varepsilon_1}q^{k_2-s}}$$

In particular for $s=0$ (i.e, the space of charge $=0$);

$$\mathrm{ch}_{F_0} = \frac{e^{\Lambda_0}}{\varphi(q)^{3}}\prod_{j=1}^{\infty}(1+e^{\varepsilon_1-\varepsilon_1'}q^j)(1+e^{\varepsilon_1'-\varepsilon_1}q^{j-1})\sum_{k_2\in\mathbf{Z}}\frac{e^{k_2(\varepsilon_2-\varepsilon_1')}q^{\frac{1}{2}k_2(k_2+1)}}{1+e^{\varepsilon_1'-\varepsilon_1}q^{k_2}}$$

Putting $\begin{cases} \alpha_1 := \varepsilon_1 - \varepsilon_1' \\ \alpha_2 := \varepsilon_1' - \varepsilon_2 \end{cases}$, we obtain the character of $\widehat{sl}(2|1)$-module $L(\Lambda_0)$:

$$\mathrm{ch}_{L(\Lambda_0)} = \frac{e^{\Lambda_0}}{\varphi(q)^2} \prod_{j=1}^{\infty} (1 + e^{-\alpha_1} q^{j-1})(1 + e^{\alpha_1} q^j) \sum_{n \in \mathbf{Z}} \frac{e^{-n\alpha_2} q^{\frac{n(n+1)}{2}}}{1 + e^{-\alpha_1} q^n}$$

$$\mathrm{ch}^{(-)}_{L(\Lambda_0)} = \frac{e^{\Lambda_0}}{\varphi(q)^2} \prod_{j=1}^{\infty} (1 - e^{-\alpha_1} q^{j-1})(1 - e^{\alpha_1} q^j) \sum_{n \in \mathbf{Z}} (-1)^n \frac{e^{-n\alpha_2} q^{\frac{n(n+1)}{2}}}{1 - e^{-\alpha_1} q^n}$$

In coordinates: $\begin{cases} e^{-\alpha_1} = e^{2\pi i z_1} \\ e^{-\alpha_2} = e^{2\pi i z_2} \\ e^{\Lambda_0} = e^{2\pi i t} \end{cases}$ this formula for $\mathrm{ch}^{(-)}_{L(\Lambda_0)}$ is written as follows:

$$\mathrm{ch}^{(-)}_{L(\Lambda_0)} = \frac{e^{2\pi i t}}{\varphi(q)^2} \prod_{j=1}^{\infty} (1 - e^{2\pi i z_1} q^{j-1})(1 - e^{-2\pi i z_1} q^j) \sum_{n \in \mathbf{Z}} (-1)^n \frac{e^{2\pi i n z_2} q^{\frac{n(n+1)}{2}}}{1 - e^{2\pi i z_1} q^n}$$

Zwegers' method (Quick review)

He rewrites this formula as follows:

$$\mathrm{ch}^{(-)}_{L(\Lambda_0)} \;=\; i\, e^{2\pi i t}\, \frac{\vartheta_{11}(\tau, z_1)\vartheta_{11}(\tau, z_2)}{\eta(\tau)^3} \cdot \mu(\tau, z_1, z_2)$$

where

$$\mu(\tau, z_1, z_2) \;:=\; \frac{e^{\pi i z_1}}{\vartheta_{11}(\tau, z_2)} \sum_{n\in\mathbf{Z}} (-1)^n \frac{e^{2\pi i n z_2} q^{\frac{n(n+1)}{2}}}{1 - e^{2\pi i z_1} q^n}$$

$$\vartheta_{11}(\tau, z) \;:=\; e^{\frac{\pi i \tau}{4}} e^{-\pi i (z+\frac{1}{2})} \prod_{n=1}^{\infty} (1-q^n)(1-e^{2\pi i z} q^{n-1})(1-e^{-2\pi i z} q^n)$$

In order to extend $\mu(\tau, z_1, z_2)$ to a modular form, he introduced the function

$$R(\tau, z) := \sum_{n \in \frac{1}{2}+\mathbf{Z}} \left\{ \mathrm{sgn}(n) - E\left(\left(n + \frac{\mathrm{Im}(z)}{y} \right) \sqrt{2y} \right) \right\} (-1)^{n-\frac{1}{2}} q^{-\frac{n^2}{2}} e^{-2\pi i n z}$$

where

$$E(z) \;:=\; 2\int_0^z e^{-\pi t^2} dt \qquad \text{and} \qquad y \;:=\; \mathrm{Im}(\tau)$$

<u>Theorem</u> (Zwegers): The function

$$\widetilde{\mu}(\tau, z_1, z_2) \;:=\; \mu(\tau, z_1, z_2) + \frac{i}{2} R(\tau, z_1 - z_2)$$

satisfies

1) $\widetilde{\mu}(\tau + 1, z_1, z_2) \;=\; e^{-\frac{\pi i}{4}} \widetilde{\mu}(\tau, z_1, z_2)$

2) $\widetilde{\mu}\left(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau} \right) \;=\; -(-i\tau)^{\frac{1}{2}} e^{-\frac{\pi i}{\tau}(z_1 - z_2)^2} \widetilde{\mu}(\tau, z_1, z_2)$

Want to extend his method to general cases; namely

- **higher level**
- **higher rank** **cases.**
- **higher atypicality**

Basic mock theta functions $\Phi_1^{(\pm)[m;s]}$

$$\Phi_1^{(\pm)[m;s]}(\tau, z_1, z_2) \overset{put}{:=} \sum_{j\in\mathbf{Z}} (\pm 1)^j \frac{e^{2\pi imj(z_1+z_2)+2\pi isz_1} q^{mj^2+sj}}{1-e^{2\pi iz_1}q^j} \qquad \begin{pmatrix} m \in \frac{1}{2}\mathbf{N} \\ s \in \frac{1}{2}\mathbf{Z} \end{pmatrix}$$

Note: $\displaystyle\sum_{j\in\mathbf{Z}} (\pm 1)^j \, e^{2\pi im\left(j+\frac{k}{2m}\right)z} q^{m\left(j+\frac{k}{2m}\right)^2} =: \theta_{k,m}^{(\pm)}(\tau, z)$: Jacobi theta functions

"What are the modular properties of these functions? "

- Functions $\Phi_1^{(+)[m;s]}$ take place in supercharacters of $\widehat{sl}(2|1)$-modules.
- These functions play basically important roles in modification of supercharacters for <u>all</u> affine Lie superalgebras.

Put $\quad \Phi_1^{(\pm)[m;s]}(\tau, z_1, z_2, t) \overset{def}{:=} e^{2\pi imt} \, \Phi_1^{(\pm)[m;s]}(\tau, z_1, z_2)$

Want to study the modular properties of $\Phi_1^{(\pm)[m;s]}$

$$\Updownarrow$$

Behavior under the action of $SL_2(\mathbf{Z})$

$$\begin{array}{c} \parallel \\ \left\langle \underset{\substack{\parallel \\ T}}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}, \ \underset{\substack{\parallel \\ S}}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \right\rangle \end{array}$$

$$\Phi_1^{(\pm)[m;s]}|_T(\tau, z_1, z_2, t) \ := \ \Phi_1^{(\pm)[m;s]}(\tau+1, z_1, z_2, t)$$

$$\Phi_1^{(\pm)[m;s]}|_S(\tau, z_1, z_2, t) \ := \ \frac{1}{\tau}\,\Phi_1^{(\pm)[m;s]}\Big(-\frac{1}{\tau}, \ \frac{z_1}{\tau}, \ \frac{z_2}{\tau}, \ t-\frac{z_1 z_2}{\tau}\Big)$$

$$\begin{cases} \Phi_1^{(\pm)[m;s]}\big|_T \; = \; \text{easy} \\ \Phi_1^{(\pm)[m;s]}\big|_S \; = \; \textbf{?} \end{cases}$$

For simplicity, we discuss here about the function

$$\Phi_1^{[m;s]}(\tau, z_1, z_2, t) \;\; \overset{put}{:=} \;\; \Phi_1^{(+)[m;s]}(\tau, z_1, z_2, t)$$

and consider

$$\Phi_1^{[m;s]}\big|_S \, - \, \Phi_1^{[m;s]} \;\; = \;\; ?$$

Note: Let $\begin{cases} m \in \frac{1}{2}\mathbf{N} \\ s \in \mathbf{Z} \end{cases}$. Then

$$\Phi_1^{[m;s]} - \Phi_1^{[m;s]}|_S \;=\; \text{holomorphic function w.r.to } z_1 \text{ and } z_2$$

$(\because)$ $\Phi_1^{[m;s]}$ is meromorphic w.r.to z_1 with only simple poles:

$$\{\text{poles of } \Phi_1^{[m;s]} \quad \text{w.r.to } z_1\} = \mathbf{Z} + \tau\,\mathbf{Z}$$

$$\{\text{poles of } \Phi_1^{[m;s]}|_S \text{ w.r.to } z_1\} = \mathbf{Z} + \tau\,\mathbf{Z}$$

By simple calculation, we see that

$$\begin{array}{rcl} \operatorname{Res}_{z_1=j+n\tau}\Phi_1^{[m;s]} & = & \dfrac{-1}{2\pi i}\, e^{2\pi i j n m}\, e^{-2\pi i n m z_2}\, e^{2\pi i m t} \\ & & \qquad\qquad || \\ \operatorname{Res}_{z_1=j+n\tau}\Phi_1^{[m;s]}|_S & = & \dfrac{-1}{2\pi i}\, e^{2\pi i j n m}\, e^{-2\pi i n m z_2}\, e^{2\pi i m t} \end{array}$$

So $\Phi_1^{[m;s]} - \Phi_1^{[m;s]}|_S$ is holomorphic w.r.to z_1. $\square$

Note: Actually the following claim was proved in the above :

Let $\left\{ \begin{array}{rcl} m & \in & \frac{1}{2}\mathbf{N} \\ s,\ s_1 & \in & \mathbf{Z} \end{array} \right.$. Then

$$\Phi_1^{[m;s]} - \Phi_1^{[m;s_1]}|_S \quad = \quad \text{holomorphic function w.r.to } z_1 \text{ and } z_2$$

Note: For $\Phi_1^{(\pm)[m;s]}$ and $\Phi_1^{(\pm)[m;s']}$ $\left(\begin{array}{ccc} m & \in & \frac{1}{2}\mathbf{N} \\ s,\ s_1 & \in & \mathbf{Z} \\ s',\ s_1' & \in & \frac{1}{2}+\mathbf{Z} \end{array}\right)$, the following holds:

- $\Phi_1^{(+)[m;s]} - \Phi_1^{(+)[m;s_1]}|_S$: holomorphic
- $\Phi_1^{(-)[m;s]} - \Phi_1^{(+)[m;s']}|_S$: holomorphic
- $\Phi_1^{(+)[m;s']} - \Phi_1^{(-)[m;s]}|_S$: holomorphic
- $\Phi_1^{(-)[m;s']} - \Phi_1^{(-)[m;s_1']}|_S$: holomorphic

Quasi-elliptic properties of $\Phi_1^{(\pm)[m;s]}$:

$$\left\{\begin{array}{lll} \Phi_1^{(\pm)[m;s]}(\tau, \ z_1+a, \ z_2+b, \ t) & = & ? \\ \Phi_1^{(\pm)[m;s]}(\tau, \ z_1+a\tau, \ z_2+b\tau, \ t) & = & ? \end{array}\right. \qquad (a, b \in \mathbf{Z})$$

Note : $\left\{\begin{array}{lcl} m & \in & \frac{1}{2}\mathbf{N} \\ a, \ b & \in & \mathbf{Z} \\ a+b & \in & 2\mathbf{Z} \end{array}\right.$ or $\left\{\begin{array}{lcl} m & \in & \mathbf{N} \\ a, \ b & \in & \mathbf{Z} \end{array}\right.$

$$\Downarrow$$

$$\Phi_1^{(\pm)[m;s]}(\tau, \ z_1+a, \ z_2+b, \ t) \ = \ e^{2\pi i s a}\, \Phi_1^{(\pm)[m;s]}(\tau, z_1, z_2, t)$$

Note: Let $m \in \frac{1}{2}\mathbf{N}$, $s \in \frac{1}{2}\mathbf{Z}$. Then

1) $$\Phi_1^{(\pm)[m;s]}(\tau, z_1, z_2, t) - e^{4\pi i m z_1}\, \Phi_1^{(\pm)[m;s]}(\tau,\ z_1,\ z_2 + 2\tau,\ t)$$
$$= \ e^{2\pi i m t} \sum_{k=0}^{2m-1} e^{\pi i (k+s)(z_1 - z_2)}\, q^{-\frac{(k+s)^2}{4m}}\, \theta^{(\pm)}_{k+s,m}(\tau, z_1 + z_2)$$

2) $$\Phi_1^{(\pm)[m;s]}(\tau, z_1, z_2, t) - e^{-4\pi i m z_2}\, \Phi_1^{(\pm)[m;s]}(\tau,\ z_1 - 2\tau,\ z_2,\ t)$$
$$= \ e^{2\pi i m t} \sum_{k=0}^{2m-1} e^{\pi i (k+s)(z_1 - z_2)}\, q^{-\frac{(k+s)^2}{4m}}\, \theta^{(\pm)}_{k+s,m}(\tau, z_1 + z_2)$$

where

$$\theta^{(\pm)}_{j,m}(\tau, z) \ := \ \sum_{k \in \mathbf{Z}} (\pm 1)^k\, e^{2\pi i m \left(k + \frac{j}{2m}\right) z}\, q^{m \left(k + \frac{j}{2m}\right)^2}$$

<u>Proof of 1)</u> : Enough to show in the case $t = 0$.

$$\Phi_1^{(\pm)[m;s]}(\tau, z_1, z_2, 0) \;=\; \sum_{j\in\mathbf{Z}} (\pm 1)^j \; \frac{e^{2\pi ijm(z_1+z_2)}e^{2\pi isz_1}q^{mj^2+sj}}{1-e^{2\pi iz_1}q^j}$$

$$\Phi_1^{(\pm)[m;s]}(\tau, z_1, z_2+2\tau, 0) \;=\; \sum_{j\in\mathbf{Z}} (\pm 1)^j \; \frac{e^{2\pi ijm(z_1+z_2)} \overbrace{q^{2mj}}^{(e^{2\pi iz_1}q^j)^{2m}e^{-4\pi imz_1} \,\|} e^{2\pi isz_1}q^{mj^2+sj}}{1-e^{2\pi iz_1}q^j}$$

$$= \; e^{-4\pi imz_1}\sum_{j\in\mathbf{Z}} (\pm 1)^j \; \frac{e^{2\pi ijm(z_1+z_2)}e^{2\pi isz_1}q^{mj^2+sj}(e^{2\pi iz_1}q^j)^{2m}}{1-e^{2\pi iz_1}q^j}$$

Making (1st eqn) $- \, e^{4\pi imz_1} \times$ (2nd eq), one has

$$\Phi_1^{(\pm)[m;s]}(\tau, z_1, z_2, 0) - e^{4\pi imz_1}\Phi_1^{(\pm)[m;s]}(\tau, z_1, z_2+2\tau, 0)$$

$$= \sum_{j\in\mathbf{Z}} e^{2\pi ijm(z_1+z_2)} e^{2\pi isz_1} q^{mj^2+sj} \underbrace{\frac{1-(e^{2\pi iz_1}q^j)^{2m}}{1-e^{2\pi iz_1}q^j}}_{\displaystyle \begin{array}{c} \| \\ \sum_{k=0}^{2m-1}(e^{2\pi iz_1}q^j)^k \end{array}}$$

$$= \sum_{k=0}^{2m-1} e^{\pi i(k+s)(z_1-z_2)} q^{-\frac{(k+s)^2}{4m}} \underbrace{\sum_{j\in\mathbf{Z}} (\pm 1)^j e^{2\pi im\left(j+\frac{k+s}{2m}\right)(z_1+z_2)} q^{m\left(j+\frac{k+s}{2m}\right)^2}}_{\displaystyle \begin{array}{c} \| \\ \theta^{(\pm)}_{k+s,m}(\tau, z_1+z_2) \end{array}}$$

$\square$

<u>Change of variables</u> :

$$\begin{cases} z_1 = w - u \\ z_2 = -w - u \end{cases} \quad \text{i.e,} \quad \begin{cases} u = -\dfrac{z_1 + z_2}{2} \\ w = \dfrac{z_1 - z_2}{2} \end{cases}$$

$$\phi_1^{(\pm)[m;s]}(\tau, u, w, t) \overset{put}{:=} \Phi_1^{(\pm)[m;s]}(\tau,\ z_1,\ z_2,\ t)$$

Note :

$$\begin{cases} m \in \frac{1}{2}\mathbf{N} \\ a,\ b \in \mathbf{Z} \end{cases} \quad \text{or} \quad \begin{cases} m \in \mathbf{N} \\ a,\ b \in \frac{1}{2}\mathbf{Z} \\ a + b \in \mathbf{Z} \end{cases}$$

$$\Downarrow$$

$$\phi_1^{(\pm)[m;s]}(\tau,\ u+a,\ w+b,\ t) = e^{2\pi i s(b-a)}\, \phi_1^{(\pm)[m;s]}(\tau, u, w, t)$$

Note: Let $m \in \frac{1}{2}\mathbf{N}$, $s \in \frac{1}{2}\mathbf{Z}$. Then

- $\phi_1^{(\pm)[m;s]}(\tau, u, w, t) - e^{4\pi i m(w-u)} \phi_1^{(\pm)[m;s]}(\tau,\ u-\tau,\ w-\tau,\ t)$

$$= e^{2\pi i m t} \sum_{k=0}^{2m-1} e^{2\pi i (k+s) w} q^{-\frac{(k+s)^2}{4m}} \theta^{(\pm)}_{-(k+s),m}(\tau, 2u)$$

- $\phi_1^{(\pm)[m;s]}(\tau, u, w, t) - e^{4\pi i m(w+u)} \phi_1^{(\pm)[m;s]}(\tau,\ u+\tau,\ w-\tau,\ t)$

$$= e^{2\pi i m t} \sum_{k=0}^{2m-1} e^{2\pi i (k+s) w} q^{-\frac{(k+s)^2}{4m}} \theta^{(\pm)}_{-(k+s),m}(\tau, 2u)$$

$$\Downarrow$$

- $\phi_1^{(\pm)[m;s]}(\tau, u, w, t) - e^{8\pi i m(w-\tau)} \phi_1^{(\pm)[m;s]}(\tau,\ u,\ w-2\tau,\ t)$

$$= e^{2\pi i m t} \sum_{k=0}^{4m-1} e^{2\pi i (k+s) w} q^{-\frac{(k+s)^2}{4m}} \theta^{(\pm)}_{-(k+s),m}(\tau, 2u)$$

Functions $G_1^{(\pm,\pm)[m;s,s']}$ **:**

Notation in this section : $\quad m \in \frac{1}{2}\mathbf{N}$ and $\begin{cases} s, \;\; s_1 \in & \mathbf{Z} \\ s', \;\; s'_1 \in & \frac{1}{2}+\mathbf{Z} \end{cases}$

Put

$$G_1^{(+,+)[m;s,s_1]}(\tau,u,w,t) \overset{put}{:=} \phi_1^{(+)[m;s]} - \underbrace{\phi_1^{(+)[m;s_1]}|_S}_{\begin{array}{c}\| \\ \frac{1}{\tau}\,\phi_1^{(+)[m;s_1]}\Big(-\frac{1}{\tau},\;\frac{u}{\tau},\;\frac{w}{\tau},\;t-\frac{u^2-w^2}{\tau}\Big)\end{array}}$$

$$G_1^{(-,+)[m;s,s']}(\tau,u,w,t) := \phi_1^{(-)[m;s]} - \phi_1^{(+)[m;s']}|_S$$

$$G_1^{(-,-)[m;s',s'_1]}(\tau,u,w,t) := \phi_1^{(-)[m;s']} - \phi_1^{(-)[m;s'_1]}|_S$$

Lemma 1. Write simply $G_1^{[m;s,s_1]} := G_1^{(+,+)[m;s,s_1]}$. Then

1) $G_1^{[m;s,s_1]}$: holomorphic function

2) $G_1^{[m;s,s_1]}(\tau,\, u,\, w+2,\, t) - G_1^{[m;s,s_1]}(\tau, u, w, t)$

$$= e^{2\pi imt}(-i\tau)^{-\frac{1}{2}}\frac{-i}{\sqrt{2m}} \sum_{\substack{k\in\mathbf{Z} \\ s_1\leqq k<s_1+4m}} \sum_{j\in\mathbf{Z}/2m\mathbf{Z}} e^{\frac{\pi ijk}{m}} e^{\frac{2\pi im}{\tau}(w+\frac{k}{2m})^2}\theta_{j,m}(\tau, 2u)$$

3) $G_1^{[m;s,s_1]}(\tau, u, w, t) - e^{8\pi im(w-\tau)}G_1^{[m;s,s_1]}(\tau,\, u,\, w-2\tau,\, t)$

$$= e^{2\pi imt} \sum_{\substack{k\in\mathbf{Z} \\ s\leqq k<s+4m}} e^{2\pi ikw}\, q^{-\frac{k^2}{4m}}\, \theta_{-k,m}(\tau, 2u)$$

4) $G_1^{[m;s,s_1]}$ **is determined uniquely by the above 3 conditions.**

Proof of 4): Assume that there exist 2 functions satisfying conditions 1), 2), 3), and let $g(w)$ be their difference. Want to show $g(w) = 0$. By 2) and 3),

- $g(w+2) \ = \ g(w)$
- $g(w) \ = \ e^{8\pi i m(w-\tau)} \, g(w-2\tau) \qquad$ i.e, $\quad g(w-2\tau) \ = \ e^{8\pi i m(\tau - w)} g(w)$

Consider the Jacobi's theta function

$$\vartheta_{11}(\tau, w) := e^{\frac{\pi i \tau}{4}} e^{-\pi i (w + \frac{1}{2})} \prod_{n=1}^{\infty} (1 - q^n)(1 - e^{2\pi i w} q^{n-1})(1 - e^{-2\pi i w} q^n)$$

This function satisfies

$$\vartheta_{11}(\tau, w+2) \ = \ \vartheta_{11}(\tau, w), \qquad \vartheta_{11}(\tau, w-2\tau) \ = \ e^{4\pi i (w-\tau)} \vartheta_{11}(\tau, w)$$

Put $f(w) \ := \ g(w)\vartheta_{11}(\tau, w)^{2m}$. Then

$$\begin{cases} f(w+2) & = \ f(w) \\ f(w-2\tau) & = \ f(w) \\ f(w) & : \ \text{holomorphic} \\ f(w) & = \ 0 \ \text{ at some point} \end{cases} \quad \Longrightarrow \quad f(w) = 0 \quad \Longrightarrow \quad g(w) = 0$$

$\square$

Zwegers' function $R^{(\pm)}_{j;m}(\tau, w) \qquad (m \in \frac{1}{2}\mathbf{N}, \;\; j \in \frac{1}{2}\mathbf{Z})$ **:**

$$R^{(\pm)}_{j;m}(\tau, w) \;:=\; \sum_{\substack{n \in j+\mathbf{Z} \\ n \equiv j \bmod 2m}} (\pm 1)^{\frac{n-j}{2m}} \Big\{ \mathrm{sgn}\Big(n - \frac{1}{2} - j + 2m\Big)$$

$$- E\bigg(\Big(n - 2m\frac{\mathrm{Im}(w)}{\mathrm{Im}(\tau)}\Big)\sqrt{\frac{\mathrm{Im}(\tau)}{m}}\bigg)\bigg\} e^{-\frac{\pi i n^2 \tau}{2m} + 2\pi i n w}$$

where $\qquad E(z) \;;= \displaystyle\int_0^z e^{-\pi t^2} dt$

Properties of $R_{j;m}(\tau, w)$ **:**

- $R^{(\pm)}_{j;m}(\tau, w+1) \;=\; (-1)^{2j} R^{(\pm)}_{j;m}(\tau, w)$
- $R^{(\pm)}_{j;m}(\tau, w - \tau) = \pm e^{2\pi i m(\tau - 2w)} \Big\{ R^{(\pm)}_{j;m}(\tau, w) - 2q^{-\frac{j^2}{4m}} e^{2\pi i j w} \Big\}$

Zwegers' coordinates (a,b) **:**

$$\begin{cases} y \;=\; \mathrm{Im}(\tau) \\ w \;=\; a\tau - b \quad (a,b\in\mathbf{R}) \end{cases} \qquad \text{then} \qquad \frac{\mathrm{Im}(w)}{\mathrm{Im}(\tau)} = a$$

1) $$\frac{\partial}{\partial a} + \tau\frac{\partial}{\partial b} \;=\; -2iy\,\frac{\partial}{\partial \overline{w}}$$

2) $$\left(\frac{\partial}{\partial a} + \tau\frac{\partial}{\partial b}\right) R^{(\pm)}_{j;m}(\tau,w) \;=\; 4\sqrt{my}\,e^{-4\pi m y a^2}\,\theta^{(\pm)}_{j,m}(-\overline{\tau},\,2\overline{w})$$

3) $$\left(\frac{\partial}{\partial a} + \tau\frac{\partial}{\partial b}\right) R^{(\pm)}_{j;m}\Big(-\frac{1}{\tau},\frac{w}{\tau}\Big) \;=\; \frac{4\tau}{|\tau|}\sqrt{my}\,e^{-4\pi m b^2\frac{y}{|\tau|^2}}\,\theta^{(\pm)}_{j,m}\Big(\frac{1}{\overline{\tau}},\,\frac{2\overline{w}}{\overline{\tau}}\Big)$$

Proof of 2) :

$$\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)R^{(\pm)}_{j;m}(\tau,w) = \left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)$$

$$\left(\sum_{n\equiv j \bmod 2m}(\pm 1)^{\frac{n-j}{2m}}\left\{\operatorname{sgn}\left(n-\frac{1}{2}-j+2m\right)-E\left((n-2ma)\sqrt{\frac{y}{m}}\right)\right\}e^{-\frac{\pi i n^2\tau}{2m}+2\pi i n w}\right)$$

$$= -\sum_{n\equiv j \bmod 2m}(\pm 1)^{\frac{n-j}{2m}}\underbrace{\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)E\left((n-2ma)\sqrt{\frac{y}{m}}\right)}_{\displaystyle \underbrace{E'\left((n-2ma)\sqrt{\frac{y}{m}}\right)}_{2\,e^{-\pi(n-2ma)^2\frac{y}{m}}}\cdot\left(-2m\sqrt{\frac{y}{m}}\right)}\quad e^{-\frac{\pi i n^2\tau}{2m}+2\pi i n w}$$

$$\overset{\substack{n=j+2mk\\ \downarrow}}{=} 4\sqrt{my}\;e^{-4\pi m a^2 y}\underbrace{\sum_{k\in\mathbf{Z}}(\pm 1)^k e^{2\pi i m(k+\frac{j}{2m})^2(-\tau+2iy)}\,e^{2\pi i m(k+\frac{j}{2m})\cdot 2(w-2iay)}}_{\displaystyle \theta^{(\pm)}_{j,m}\big(\underbrace{-\tau+2iy}_{-\overline{\tau}},\ \underbrace{2(w-2iay)}_{2\,\overline{w}}\big)}$$

$\square$

Let $\quad m \in \frac{1}{2}\mathbf{N}, \quad j \in \frac{1}{2}\mathbf{Z}.$

- $$\left(\frac{\partial}{\partial a} + \tau\frac{\partial}{\partial b}\right)\left((-i\tau)^{-\frac{1}{2}} e^{\frac{2\pi i m w^2}{\tau}} R^{(+)}_{j;m}\Big(-\frac{1}{\tau}, \frac{w}{\tau}\Big)\right)$$
$$= \ 4i\sqrt{\frac{y}{2}}\ e^{-4\pi m a^2 y} \sum_{k\in\mathbf{Z}/2m\mathbf{Z}} e^{\frac{\pi i j k}{m}} \times \begin{cases} \theta^{(+)}_{k,m}(-\overline{\tau},\ 2\overline{w}) & \text{if} \quad j \in \mathbf{Z} \\ \theta^{(-)}_{k,m}(-\overline{\tau},\ 2\overline{w}) & \text{if} \quad j \in \frac{1}{2} + \mathbf{Z} \end{cases}$$

- $$\left(\frac{\partial}{\partial a} + \tau\frac{\partial}{\partial b}\right)\left((-i\tau)^{-\frac{1}{2}} e^{\frac{2\pi i m w^2}{\tau}} R^{(-)}_{j;m}\Big(-\frac{1}{\tau}, \frac{w}{\tau}\Big)\right)$$
$$= \ 4i\sqrt{\frac{y}{2}}\ e^{-4\pi m a^2 y} \sum_{k\in\mathbf{Z}/2m\mathbf{Z}} e^{\frac{\pi i j}{m}(k+\frac{1}{2})} \times \begin{cases} \theta^{(+)}_{k+\frac{1}{2},m}(-\overline{\tau},\ 2\overline{w}) & \text{if} \quad j \in \mathbf{Z} \\ \theta^{(-)}_{k+\frac{1}{2},m}(-\overline{\tau},\ 2\overline{w}) & \text{if} \quad j \in \frac{1}{2} + \mathbf{Z} \end{cases}$$

Functions $a_j^{[m;s]}(\tau, w)$ **:**

$$a_j^{[m;s_1]}(\tau, w) \overset{put}{:=} -R_{j;m}^{(+)}(\tau, w) - \frac{i}{\sqrt{2m}}(-i\tau)^{-\frac{1}{2}} e^{\frac{2\pi i m w^2}{\tau}} \sum_{\substack{k \in \mathbf{Z} \\ s_1 \leqq k < s_1 + 2m}} e^{-\frac{\pi i j k}{m}} R_{k;m}^{(+)}\Big(-\frac{1}{\tau}, \frac{w}{\tau}\Big)$$

Lemma 2.

1) $a_j^{[m;s_1]}(\tau, w) - e^{8\pi i m(w-\tau)} a_j^{[m;s_1]}(\tau, w - 2\tau) = -2\left\{q^{-\frac{j^2}{4m}} e^{2\pi i j w} + q^{-\frac{(j+2m)^2}{4m}} e^{2\pi i (j+2m) w}\right\}$

2) $a_j^{[m;s_1]}(\tau, w+2) - a_j^{[m:s_1]}(\tau, w) = \dfrac{2i}{\sqrt{2m}}(-i\tau)^{-\frac{1}{2}} \displaystyle\sum_{\substack{k \in \mathbf{Z} \\ s_1 \leqq k < s_1 + 4m}} e^{-\frac{\pi i j k}{m}} e^{\frac{2\pi i m}{\tau}(w + \frac{k}{2m})^2}$

3) $a_j^{[m;s_1]}(\tau, w)$: <u>holomorphic</u>

<u>Proof of 3)</u> :

$$\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)a_j^{[m;s_1]}(\tau,w) \;=\; -\underbrace{\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)R^{(+)}_{j;m}(\tau,w)}_{\substack{\Vert \\ 4\sqrt{my}\,e^{-4\pi ma^2y}\theta^{(+)}_{j,m}(-\overline{\tau},2\overline{w})}}$$

$$-\frac{i}{\sqrt{2m}}\sum_{\substack{k\in\mathbf{Z}\\ s_1\leqq k<s_1+2m}} e^{-\frac{\pi ijk}{m}}\underbrace{\left(\frac{\partial}{\partial a}+\tau\frac{\partial}{\partial b}\right)\left((-i\tau)^{-\frac{1}{2}}e^{\frac{2\pi imw^2}{\tau}}R^{(+)}_{j;m}\Big(-\frac{1}{\tau},\frac{w}{\tau}\Big)\right)}_{\substack{\Vert \\ 4i\sqrt{\frac{y}{2}}\,e^{-4\pi ma^2y}\sum\limits_{n\in\mathbf{Z}/2m\mathbf{Z}} e^{\frac{\pi ikn}{m}}\theta^{(+)}_{n,m}(-\overline{\tau},2\overline{w})}}$$

$$=\;4\sqrt{my}e^{-4\pi ma^2y}\left\{-\theta^{(+)}_{j,m}(-\overline{\tau},2\overline{w})+\frac{1}{2m}\underbrace{\sum_{n\in\mathbf{Z}/2m\mathbf{Z}}\Bigg(\sum_{\substack{k\in\mathbf{Z}\\ s_1\leqq k<s_1+2m}} e^{\frac{\pi ik}{m}(n-j)}\Bigg)\theta^{(+)}_{n,m}(-\overline{\tau},2\overline{w})}_{\substack{\Vert \\ 2m\,\theta^{(+)}_{j,m}(-\overline{\tau},2\overline{w})}}\right\}$$

$$=\;0 \qquad\qquad \square$$

Functions $\phi^{(\pm)[m;s]}_{1,\mathrm{add}}$ and $G^{[m;s,s_1]}_{1,\mathrm{add}}$:

Put

$$\phi^{(\pm)[m;s]}_{1,\mathrm{add}}(\tau,u,w,t) \overset{put}{:=} \frac{-1}{2}\, e^{2\pi imt} \sum_{\substack{k\in s+\mathbf{Z} \\ s\leqq k<s+2m}} R^{(\pm)}_{j;m}(\tau,w)\theta^{(\pm)}_{-j,m}(\tau,2u)$$

For simplicity, consider the case "+" and $s, s_1 \in \mathbf{Z}$ and put

$$G^{[m;s,s_1]}_{1;\mathrm{add}} \overset{put}{:=} \phi^{(+)[m;s]}_{1;\mathrm{add}} - \phi^{(+)[m;s_1]}_{1;\mathrm{add}}\Big|_S$$

Lemma 3.

$$G^{[m;s,s_1]}_{1;\mathrm{add}}(\tau,u,w,t) = \frac{1}{2}\, e^{2\pi imt} \sum_{\substack{j\in\mathbf{Z} \\ s\leqq j<s+2m}} a^{[m;s_1]}_j(\tau,w)\,\theta^{(+)}_{-j,m}(\tau,2u)$$

Proof of Lemma 3 : Suffices to show in the case $t=0$.

$$G^{[m;s,s_1]}_{1;\mathrm{add}}(\tau,u,w,0)$$

$$= \underbrace{\phi^{(+)[m;s]}_{1;\mathrm{add}}(\tau,u,w,0)}_{\displaystyle \| \atop \displaystyle \frac{-1}{2}\sum_{s\leqq j<s+2m} R^{(+)}_{j,m}(\tau,w)\theta^{(+)}_{-j,m}(\tau,2u)} - \frac{1}{\tau}e^{-\frac{2\pi i m}{\tau}(u^2-w^2)} \underbrace{\phi^{(+)[m;s_1]}_{1;\mathrm{add}}\Big(-\frac{1}{\tau},\frac{u}{\tau},\frac{w}{\tau},0\Big)}_{\displaystyle \| \atop \displaystyle \frac{-1}{2}\sum_{s_1\leqq k<s_1+2m} R^{(+)}_{k,m}\Big(\frac{-1}{\tau},\frac{w}{\tau}\Big) \underbrace{\theta^{(+)}_{-k,m}\Big(\frac{-1}{\tau},\frac{2u}{\tau}\Big)}_{\displaystyle \| \atop \displaystyle \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m}}e^{\frac{2\pi i m u^2}{\tau}}\sum_{j\in\mathbf{Z}/2m\mathbf{Z}} e^{-\frac{\pi i j k}{m}}\theta^{(+)}_{-j,m}(\tau,2u)}}$$

$$= \frac{-1}{2}\sum_{s\leqq j<s+2m} \underbrace{\Big\{ R^{(+)}_{j,m}(\tau,w) + \frac{i}{\sqrt{2m}}(-i\tau)^{\frac{1}{2}} \sum_{s_1\leqq k<s_1+2m} e^{-\frac{\pi i j k}{m}} R^{(+)}_{k,m}\Big(\frac{-1}{\tau},\frac{w}{\tau}\Big) \Big\}}_{\displaystyle \| \atop \displaystyle -a^{[m;s_1]}_j(\tau,w)} \theta^{(+)}_{-j,m}(\tau,2u)$$

$\square$

Then, by Lemma 2 and Lemma 3, we get the following:

Lemma 4. **(properties of $G^{[m;s,s_1]}_{1,\mathrm{add}}$)**

- $G^{[m]}_{1,\mathrm{add}}$: holomorphic function

- $G^{[m;s,s_1]}_{1,\mathrm{add}}(\tau, u, w+2, t) - G^{[m;s,s_1]}_{1,\mathrm{add}}(\tau, u, w, t)$

$$= e^{2\pi imt}(-i\tau)^{-\frac{1}{2}}\frac{i}{\sqrt{2m}} \sum_{\substack{k\in\mathbf{Z} \\ s_1 \leqq k < s_1+2m}} \sum_{j\in\mathbf{Z}/2m\mathbf{Z}} e^{\frac{\pi ijk}{m}} e^{\frac{2\pi im}{\tau}(w+\frac{k}{2m})^2}\theta_{j,m}(\tau, 2u)$$

- $G^{[m;s,s_1]}_{1,\mathrm{add}}(\tau, u, w, t) - e^{8\pi im(w-\tau)}G^{[m;s,s_1]}_{1,\mathrm{add}}(\tau, u, w-2\tau, t)$

$$= \ -e^{2\pi imt} \sum_{\substack{k\in\mathbf{Z} \\ s \leqq k < s+2m}} e^{2\pi ikw}\, q^{-\frac{k^2}{4m}}\, \theta_{-k,m}(\tau, 2u)$$

By Lemma 1 and Lemma 4, we have

$$\begin{array}{ccc} G_1^{[m;s,s_1]} & = & -\ G_{1,\mathrm{add}}^{[m;s,s_1]} \\ \| & & \| \\ \phi_1^{[m;s]} - \phi_1^{[m;s_1]}|_S & & -\ (\phi_{1,\mathrm{add}}^{[m;s]} - \phi_{1,\mathrm{add}}^{[m;s_1]}|_S) \end{array}$$

i.e,

$$\underbrace{\phi_1^{[m;s]} + \phi_{1,\mathrm{add}}^{[m;s]}}_{\| \ put \atop \widetilde{\phi}_1^{[m;s]}} = \Big(\underbrace{\phi_1^{[m;s_1]} + \phi_{1,\mathrm{add}}^{[m;s_1]}}_{\| \ put \atop \widetilde{\phi}_1^{[m;s_1]}}\Big)\Big|_S$$

Thus, putting $\quad \widetilde{\phi}_1^{[m;s]} \overset{put}{:=} \phi_1^{[m;s]} + \phi_{1,\mathrm{add}}^{[m;s]}$, we have

$$\widetilde{\phi}_1^{[m;s]} = \widetilde{\phi}_1^{[m;s_1]}|_S$$

Theorem 1. Let $m \in \frac{1}{2}\mathbf{N}$, $s, s_1 \in \mathbf{Z}$. Then

1) $\widetilde{\phi}_1^{[m;s_1]}|_S = \widetilde{\phi}_1^{[m;s]}$, $\widetilde{\phi}_1^{[m;s]}|_T = \widetilde{\phi}_1^{[m;s]}$

2) $\widetilde{\phi}_1^{[m;s]}(\tau, u+a, w+b, t) = \widetilde{\phi}_1^{[m;s]}(\tau, u, w, t)$

3) $\widetilde{\phi}_1^{[m;s]}(\tau, u+a\tau, w+b\tau, t) = q^{m(b^2-a^2)} e^{4\pi i m(-au+bw)} \widetilde{\phi}_1^{[m;s]}(\tau, u, w, t)$

where $a, b \in \frac{1}{2}\mathbf{Z}$ s.t. $a+b \in \mathbf{Z}$.

4) $\widetilde{\phi}_1^{[m;s_1]} = \widetilde{\phi}_1^{[m;s]}$ i.e, $\widetilde{\phi}_1^{[m;s]}$ does not depend on $s \in \mathbf{Z}$.

<u>Proof of 3)</u> follows from 2) and transformation property:

$$\widetilde{\phi}_1^{[m;s]}(\tau, u, w, t) \;=\; \frac{1}{\tau}\, e^{-\frac{2\pi i m}{\tau}(u^2-w^2)}\, \widetilde{\phi}_1^{[m;s]}\Big(-\frac{1}{\tau},\, \frac{u}{\tau},\, \frac{w}{\tau},\, t\Big)$$

Letting $\begin{cases} u \;\longrightarrow\; u+a\tau \\ w \;\longrightarrow\; w+b\tau \end{cases}$, we have

$$\widetilde{\phi}_1^{[m;s]}(\tau, u+a\tau, w+b\tau, t) = \frac{1}{\tau}\, \overbrace{e^{-\frac{2\pi i m}{\tau}\left[(u+a\tau)^2-(w+b\tau)^2\right]}}^{\substack{e^{-\frac{2\pi i m}{\tau}(u^2-w^2)}e^{4\pi i m(-au+bw)}q^{m(b^2-a^2)} \\ \|}}\; \underbrace{\widetilde{\phi}_1^{[m;s]}\Big(-\frac{1}{\tau}, \frac{u}{\tau}+a, \frac{w}{\tau}+b, t\Big)}_{\substack{\| \\ \widetilde{\phi}_1^{[m;s]}\left(-\frac{1}{\tau},\, \frac{u}{\tau},\, \frac{w}{\tau},\, t\right)}}$$

$$=\; e^{4\pi i m(-au+bw)}q^{m(b^2-a^2)}\, \underbrace{\frac{1}{\tau}\, e^{-\frac{2\pi i m}{\tau}(u^2-w^2)}\, \widetilde{\phi}_1^{[m;s]}\Big(-\frac{1}{\tau},\, \frac{u}{\tau},\, \frac{w}{\tau},\, t\Big)}_{\substack{\| \\ \widetilde{\phi}_1^{[m;s]}(\tau, u, w, t)}} \qquad \square$$

Translating these to $\Phi_1^{[m;s]}$:

Corollary. Let $m \in \frac{1}{2}\mathbf{N}$, $s \in \mathbf{Z}$, and put

$$\Phi_{1,\mathrm{add}}^{[m;s]}(\tau, z_1, z_2, t) \overset{put}{:=} \frac{-1}{2}\, e^{2\pi imt} \sum_{s \leqq j < s+2m} R_{j;m}^{(+)}\Big(\tau, \frac{z_1 - z_2}{2}\Big)\theta_{j,m}^{(+)}(\tau, z_1 + z_2)$$

$$\widetilde{\Phi}_1^{[m]} \overset{put}{:=} \Phi_1^{[m;s]} + \Phi_{1,\mathrm{add}}^{[m;s]}$$

Then

- $\widetilde{\Phi}_1^{[m;s]}$ does not depend on $s \in \mathbf{Z}$.
- $\widetilde{\Phi}_1^{[m;s]}|_S = \widetilde{\Phi}_1^{[m;s]}, \qquad \widetilde{\Phi}_1^{[m;s]}|_T = \widetilde{\Phi}_1^{[m;s]}$
- $\widetilde{\Phi}_1^{[m;s]}(\tau,\ z_1 + a,\ z_2 + b,\ t) = \widetilde{\Phi}_1^{[m;s]}(\tau, z_1, z_2, t)$
- $\widetilde{\Phi}_1^{[m;s]}(\tau, z_1 + a\tau, z_2 + b\tau, t) = q^{-mab} e^{-2\pi im(bz_1 + az_2)} \widetilde{\Phi}_1^{[m;s]}(\tau, z_1, z_2, t)$

where $a, b \in \mathbf{Z}$.

Similar argument works for $\Phi_1^{(\pm)[m;s]}$ $\left(m \in \frac{1}{2}\mathbf{N},\ \ s \in \frac{1}{2}\mathbf{Z}\right)$:

Theorem 2. Let $m \in \frac{1}{2}\mathbf{N},\ \ s \in \mathbf{Z},\ \ s' \in \frac{1}{2} + \mathbf{Z}$. Then

1) S-transformation:

- $\widetilde{\Phi}_1^{(+)[m;s]}\big|_S = \widetilde{\Phi}_1^{(+)[m;s]}$
- $\widetilde{\Phi}_1^{(-)[m;s]}\big|_S = \widetilde{\Phi}_1^{(+)[m;s']}$
- $\widetilde{\Phi}_1^{(+)[m;s']}\big|_S = \widetilde{\Phi}_1^{(-)[m;s]}$
- $\widetilde{\Phi}_1^{(-)[m;s']}\big|_S = \widetilde{\Phi}_1^{(-)[m;s']}$

2) T-transformation:

$$\widetilde{\Phi}_1^{(\pm)[m;s]}\big|_T = \begin{cases} \widetilde{\Phi}_1^{(\pm)[m;s]} & \text{if} \quad m+s \in \mathbf{Z} \\ \widetilde{\Phi}_1^{(\mp)[m;s]} & \text{if} \quad m+s \in \frac{1}{2}+\mathbf{Z} \end{cases}$$

Corollary. $$\left.\begin{array}{rcl} m & \in & \frac{1}{2}\mathbf{N} \\ s,\ s' & \in & \frac{1}{2}\mathbf{Z} \\ s-s' & \in & \mathbf{Z} \end{array}\right\} \implies \widetilde{\Phi}^{(\pm)[m;s]} = \widetilde{\Phi}^{(\pm)[m;s']}$$

Elliptic transformation properties for $\widetilde{\Phi}_1^{(\pm)[m;s]}$ $\quad (m \in \frac{1}{2}\mathbf{N},\ \ s \in \frac{1}{2}\mathbf{Z})$:

Theorem 3. Let $\left\{\begin{array}{rcl} m & \in & \frac{1}{2}\mathbf{N} \\ a,\ b & \in & \mathbf{Z} \\ a+b & \in & 2\mathbf{Z} \end{array}\right.$ or $\left\{\begin{array}{rcl} m & \in & \mathbf{N} \\ a,\ b & \in & \mathbf{Z} \end{array}\right.$. Then

- $\widetilde{\Phi}_1^{(\pm)[m;s]}(\tau,\ z_1+a,\ z_2+b,\ t) \ = \ e^{2\pi isa}\,\widetilde{\Phi}_1^{(\pm)[m;s]}(\tau, z_1, z_2, t)$

- $\widetilde{\Phi}_1^{(\pm)[m;s]}(\tau,\ z_1+a\tau,\ z_2+b\tau,\ t) \ = \ (\pm 1)^a\, e^{-2\pi im(bz_1+az_2)}\, q^{-mab}\,\widetilde{\Phi}_1^{(\pm)[m;s]}(\tau, z_1, z_2, t)$

Note : Relation with Zwegers' functions μ and $\widetilde{\mu}$:

$$\mu(\tau, z_1, z_2) = \frac{e^{\pi i z_1}}{\vartheta_{11}(\tau, z_2)} \sum_{n\in\mathbf{Z}} (-1)^n \frac{e^{2\pi i n z_2} q^{\frac{n(n+1)}{2}}}{1 - e^{2\pi i z_1} q^n}$$

$$= \frac{1}{\vartheta_{11}(\tau, z_2)} \underbrace{\sum_{n\in\mathbf{Z}} (-1)^n \frac{e^{2\pi i n[z_1+(z_2-z_1)]}\, e^{2\pi i(\frac{1}{2}z_1)}\, q^{\frac{1}{2}n^2+\frac{1}{2}n}}{1 - e^{2\pi i z_1} q^n}}_{\substack{\| \\ \Phi_1^{(-)[\frac{1}{2};\,\frac{1}{2}]}(\tau,\ z_1,\ z_2-z_1,\ 0)}}$$

and

$$\widetilde{\mu}(\tau, z_1, z_2) = \frac{1}{\vartheta_{11}(\tau, z_2)}\ \widetilde{\Phi}_1^{(-)[\frac{1}{2};\,\frac{1}{2}]}(\tau,\ z_1,\ z_2-z_1,\ 0)$$

Modified supercharacter of $\widehat{sl}(2|1)$-module $L((m-1)\Lambda_0)$:

$$R^{(-)} \cdot \mathrm{ch}^{(-)}_{L((m-1)\Lambda_0)} \;=\; \Phi^{[m;0]}_1(\tau, z_1, z_2, t) - \Phi^{[m;0]}_2(\tau, z_1, z_2, t)$$

where

$$\Phi^{[m;s]}_2(\tau, z_1, z_2, t) \;:=\; \Phi^{[m;s]}_1(\tau, -z_2, -z_1, t)$$

Make this to be a modular form, replacing $\Phi^{[m;0]}_i$ with $\widetilde{\Phi}^{[m;0]}_i$:

$$\begin{aligned} R^{(-)} \cdot \widetilde{\mathrm{ch}}^{(-)}_{L((m-1)\Lambda_0)} \;:=\;& \widetilde{\Phi}^{[m;0]}_1(\tau, z_1, z_2, t) - \widetilde{\Phi}^{[m;0]}_2(\tau, z_1, z_2, t) \\ & \qquad \| \; \mathit{put} \\ & \qquad \widetilde{\Phi}^{[m;0]}(\tau, z_1, z_2, t) \\ & \qquad \| \; \mathit{put} \\ & \qquad \widetilde{\Phi}^{[m]}(\tau, z_1, z_2, t) \end{aligned}$$

Properties of $\widetilde{\Phi}^{[m]}$:

- $\widetilde{\Phi}^{[m]}|_S = \widetilde{\Phi}^{[m]}, \qquad \widetilde{\Phi}^{[m]}|_T = \widetilde{\Phi}^{[m]}$

- $\widetilde{\Phi}^{[m]}(\tau,\ z_1 + a,\ z_2 + b,\ t) \ = \ \widetilde{\Phi}^{[m]}(\tau, z_1, z_2, t)$

- $\widetilde{\Phi}^{[m]}(\tau,\ z_1 + a\tau,\ z_2 + b\tau,\ t) \ = \ q^{-mab}\, e^{-2\pi i m(bz_1 + az_2)}\, \widetilde{\Phi}^{[m]}(\tau, z_1, z_2, t)$

- $\widetilde{\Phi}^{[m]}(\tau,\ z_2,\ z_1,\ t) \ = \ \widetilde{\Phi}^{[m]}(\tau,\ z_1,\ z_2,\ t)$

- $\widetilde{\Phi}^{[m]}(\tau,\ -z_1,\ -z_2,\ t) \ = \ -\,\widetilde{\Phi}^{[m]}(\tau,\ z_1,\ z_2,\ t)$

where $\ a, b \in \mathbf{Z}$.