Density estimators for invertible linear processes

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Let $X_1, \dots, X_n$ be observations of a linear process

$$X_j = \varepsilon_j + \sum_{s=1}^{\infty} \varphi_s \varepsilon_{j-s}.$$

Assume that the innovations ε_j have mean zero, finite variance, and density f, and that the coefficients φ_s are summable.
An estimator for the density h of X_j at x is the kernel estimator

$$k_b(x) = \frac{1}{n} \sum_{j=1}^{n} k_b(x - X_j) \quad \text{with} \quad k_b(x) = k(x/b)/b.$$

The more coefficients φ_s are nonzero, the smoother is h. We exploit this by decomposing

$$X_j = Y_j + Z_j \quad \text{with} \quad Z_j = \sum_{s=m}^{\infty} \varphi_s \varepsilon_{j-s}.$$

With f_m denoting the density of $Y_j = \varepsilon_j + \sum_{s=1}^{m-1} \varphi_s \varepsilon_{j-s}$, we have

$$\begin{aligned}\hat{h} - h =& \frac{1}{n} \sum_{j=1}^{n} \Big(k_b(x - X_j) - k_b * f_m(x - Z_j) \Big) \\ &+ \frac{1}{n} \sum_{j=1}^{n} \Big(f_m(x - Z_j) - k_b * h(x) \Big) + k_b * h(x) - h(x).\end{aligned}$$

(For $m = 1$ see Wu/Mielniczuk 2002.)
If only finitely many φ_s are nonzero, the middle term vanishes for m large.

Result: *Assume that at least N coefficients φ_s are nonzero. Let f have finite variation. Choose a kernel of order $N+1$. Then*

$$\|\hat{h} - h\|_1 = O_P(n^{-1/2} b^{-1/2}) + O(b^{N+1}).$$

Test for number of nonzero coefficients.
Then choose optimal bandwidth and order of kernel.
If many φ_s are nonzero, the rate of $\hat{h}$ is close to $n^{-1/2}$.

Suppose at least one φ_s is nonzero. A better estimator for h is given by a *convolution* of two density estimators, using $X_j = \varepsilon_j + Z_j$. Then $h = f * g$ with h, f, g the densities of X_j, ε_j, Z_j.

To estimate ε_j, we assume that the linear process is invertible,

$$\varepsilon_j = X_j - \sum_{s=1}^{\infty} \varrho_s X_{j-s}.$$

Let $\widehat{\varrho}_j$ be an estimator of ϱ_j. Let $p \to \infty$. Estimate ε_j and Z_j by

$$\widehat{\varepsilon}_j = X_j - \sum_{s=1}^{p} \widehat{\varrho}_s X_{j-s} \quad \text{and} \quad \widehat{Z}_j = X_j - \widehat{\varepsilon}_j.$$

Estimate the densities f of ε_j and g of Z_j by kernel estimators

$$\widehat{f}(x) = \frac{1}{n-p} \sum_{j=p+1}^{n} k_b(x - \widehat{\varepsilon}_j), \quad \widehat{g}(x) = \frac{1}{n-p} \sum_{j=p+1}^{n} k_b(x - \widehat{Z}_j).$$

Then estimate $h = f * g$ by the *convolution estimator* $\widehat{h} = \widehat{f} * \widehat{g}$.

Result: *For appropriate choice of kernel and bandwidth, under (mild) conditions on the decay of φ_s and ϱ_s, if f has a moment > 3 and (essentially) finite variation, then $\widehat{h} = \widehat{f} * \widehat{g}$ has the stochastic expansion*

$$\left\| \widehat{h} - h - \mathbb{F} - \mathbb{G} + \sum_{s=1}^{p} (\widehat{\varrho}_s - \varrho_s) \nu_s' \right\|_1 = o_P(n^{-1/2})$$

with

$$\mathbb{F}(x) = \frac{1}{n-p} \sum_{j=p+1}^{n} \Big(f(x - Z_j) - h(x) \Big),$$

$$\mathbb{G}(x) = \frac{1}{n-p} \sum_{j=p+1}^{n} \Big(g(x - \varepsilon_j) - h(x) \Big),$$

and $\nu_s(x) = E[X_0 f(x - Z_s)]$.

If $\widehat{\varrho}_s$ are asymptotically linear (e.g. least squares estimators), then $n^{1/2}(\widehat{h} - h)$ converges weakly in L_1 to a centered Gausian process.

The estimator $\widehat{h} = \widehat{f} * \widehat{g}$ is approximated by a sum of two smoothed empirical estimators:

$$\widehat{f} * \widehat{g} - f * g = f * (\widehat{g} - g) + g * (\widehat{f} - f) + (\widehat{f} - f) * (\widehat{g} - g).$$

This explains the rate $n^{-1/2}$ if the bandwidth is e.g. $(n \log n)^{-1/4}$.

The estimator $\widehat{f} * \widehat{g}(x)$ is equivalent to a local U-statistic:

$$\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} K_b(x - \widehat{\varepsilon}_i - \hat{Z}_j).$$

The estimator $\widehat{f} * \widehat{g}$ can be improved in several ways:
— Use efficient estimators for ϱ_s.
— Use empirical likelihood weights on $\widehat{f}$ and $\widehat{g}$ to exploit $E[\varepsilon_j] = 0$,
— Use the convolution representation of $Z_j = X_j - \varepsilon_j = \sum_{s=1}^{\infty} \varphi_s \varepsilon_{j-s}$ to estimate h by an increasing number $q \to \infty$ of convolutions,

$$\widehat{h}(x) = \int \cdots \int \widehat{f}\Big(x - \widehat{\varphi}_1 z_1 - \cdots - \widehat{\varphi}_q z_q\Big) \widehat{f}(z_1) \cdots \widehat{f}(z_q)\, dz_1 \cdots dz_q.$$