Homework Set Ten

Due Thursday, July 7.

Question 1. Suppose that $x \in \mathbb{R}$ has the decimal expansion

$$x = n + \sum_{j=1}^{\infty} \frac{a_j}{10^j} \qquad \text{with} \qquad n \in \mathbb{Z}, \text{ and } 0 \leq a_j \leq 9.$$

If there exists $m, \ell \in \mathbb{N}$ such that $a_{k+\ell} = a_k$ for all $k > m$ we say that the decimal expansion of x is *periodic*. If x is periodic with m and ℓ as above, we write

$$x = n.a_1 \cdots a_m \overline{a_{m+1} \cdots a_{m+\ell}}.$$

(a) Find integers a, b with $\gcd(a, b) = 1$ such that $\frac{a}{b} = 1.234\overline{56789}$.

(b) Suppose that $0 < c, d \leq 9$. Find integers a, b such that $\frac{a}{b} = 0.\overline{cd}$.

(c) Find integers a, b with $\gcd(a, b) = 1$ such that $\frac{a}{b} = 0.\overline{9}$.

(d) (BONUS) Prove that if $x \in \mathbb{R}$ is periodic then $x \in \mathbb{Q}$.

Question 2. Find the value for each of the following periodic continued fractions and express this value in the form $\frac{a+b\sqrt{D}}{c}$, where $a, b, c, D \in \mathbb{Z}$.

(a) $[1, 2, 3, \overline{4, 5}]$

(b) $[1, 1, \overline{2, 3}]$

(c) $[3, 2, \overline{1, 2}]$

(d) $[a, b, b, \dots]$ for $a, b \in \mathbb{N}$.

Question 3. For each of the following numbers, find the periodic continued fraction and state the period. Be sure your solution proves that your answer is correct.

(a) $\dfrac{2 + \sqrt{3}}{5}$

(b) $\dfrac{1 + 2\sqrt{5}}{7}$

Question 4. Using continued fractions, find a (nontrivial) solution to the equation $x^2 - Dy^2 = 1$ for the following D:

(a) $D = 19$

(b) $D = 41$

(c) $D = a^2 + 1$, for $a \in \mathbb{N}$.

(d) (BONUS) $D = a^2 - 1$, for $a \in \mathbb{N}$, $a > 1$

(e) (BONUS) $D = a(a + 1)$, for $a \in \mathbb{N}$