Homework Set Four

Due Monday, May 23.

Question 1. For an integer n, let $\sigma(n)$ denote the sum of divisors of n. That is, $\sigma(n) = \sum_{d|n} d$. Find the smallest n such that there exists another $m \neq n$ satisfying $\sigma(n) = \sigma(m)$.

Question 2. A positive integer N is a *perfect number* if $\sigma(n) = 2n$ (that is, n is the sum of its proper divisors). For example, $6 = 1 + 2 + 3$ is perfect. Prove that if $2^m - 1 = p$ is prime, then $2^{m-1}p$ is perfect.

Question 3.

(a) If m and n are integers with $\gcd(m, n) = 1$, prove that $\sigma(mn) = \sigma(m)\sigma(n)$.

(b) If p is prime and $k \geq 1$, prove that

$$\sigma\left(p^k\right) = \frac{p^{k+1} - 1}{p - 1}.$$

Question 4. Prove that if $2^n - 1$ is prime, then n is prime.

Question 5. Suppose N is an even perfect number.

(a) Explain why $N = 2^k m$ for some $k \geq 1$ and odd m. Then use the results of the previous problem to show $\sigma(N) = (2^{k+1} - 1)\sigma(m)$.

(b) Show that 2^{k+1} divides $\sigma(m)$. (Hint: Use the definition of a perfect number to deduce $\sigma(N) = 2^{k+1}m$.)

(c) Show that there is an integer r such that $m = (2^{k+1} - 1)r$ and $\sigma(m) = 2^{k+1}r$.

(d) Assume there is an integer r such that $m = (2^{k+1} - 1)r$ as in part (c). Show that if $r > 1$ this would imply

$$\sigma(m) \geq 1 + 2^{k+1}r. \tag{1}$$

(Hint: Consider some distinct divisors of m.)

(e) Show why we cannot have (1), and conclude that $r = 1$.

(f) Show that $\sigma(m) = m + 1$ and that m must then be prime.

(g) Use the previous questions to prove the following theorem.

Theorem. If N is an even perfect number, then $N = 2^{p-1}(2^p - 1)$ for some prime p.