EXPLICIT TRILINEAR FORMS AND THE TRIPLE PRODUCT L-FUNCTION

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Abstract. We study the central value of the triple product L-function $L(1/2, \Pi)$ where $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ and each π_i is a cuspidal automorphic representation of GL_2 over a number field F. This work verifies a hypothesis of Venkatesh which predicted that the central L-value is bounded by a certain period integral. By bounding the period, Venkatesh showed that under his hypothesis $L(1/2, \Pi)$ satisfies a subconvexity bound in a certain level aspect. A relationship between periods, the L-value in question and local trilinear forms is provided by the work of Ichino. In this paper, we make Ichino's formula explicit by calculating these trilinear forms on various vectors. In the process, we complete the application to subconvexity as well as give a new proof and generalization of Watson's formula.

1. Introduction

The theory of L-functions has played a central role in number theory from its origins in Dirichlet's proof of the density of primes in arithmetic progressions to the present day. In this paper we consider the so-called triple product L-function. We have $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ where each π_i is an automorphic representation of $\mathrm{GL}_2(F)$ for a number field F, and we consider the central value $L(\frac{1}{2}, \Pi)$. (See Section 4.1 for the definition of $L(s, \Pi)$.)

Understanding the value $L(\frac{1}{2}, \Pi)$ began with work of Garrett[10] and was furthered by Gross-Kudla[13], Kudla-Harris[16],[15] and others. An especially beautiful formula was given by Watson[36] for many choices of π_i. Recently, Ichino [17] gave a very general formula extending these results. Although his result is more general than Watson's, it is less explicit. The results of the present paper can be interpreted as making explicit the formula of Ichino. We do this by computing certain local trilinear forms. As a consequence, we also obtain a generalization of Watson's formula.

It is often the case that controlling the growth of the special values over a family of L-functions has number theoretic applications. For example, if $\zeta(s)$ is the Riemann-zeta function then $\zeta(\frac{1}{2} + it)$ can be interpreted as $L(\frac{1}{2}, \pi_t)$ for π_t a family of automorphic representations of GL_1 varying in the "t-aspect." The principle of convexity gives bounds on these values, but for number theoretic applications, an improvement on this bound is usually required. For the case of $\zeta(s)$, convexity gives

$$|L(1/2, \pi_t)| = |\zeta(1/2 + it)| \ll |t|^{1/4},$$

but the expected bound of the Lindelof conjecture (a consequence of the Riemann hypothesis) is that for any $\epsilon > 0$,

$$|L(1/2, \pi_t)| \ll |t|^{\epsilon}.$$

The convexity argument works quite generally for L-functions of the type $L(s, \sigma)$ where σ varies over, for example, automorphic representations of an algebraic group G. Many papers have been dedicated to proving subconvexity results for various choices of G and for families varying in so-called "t-aspect", "eigenvalue aspect" and/or "conductor aspect." We refer the reader to [26] where the cases of $G = \mathrm{GL}_n$ for $n = 1, 2$ are treated *uniformly* in all aspects, and references are given.

Another important application of the results of this paper, and the initial motivation for this work, is to prove a conjecture of Venkatesh [34, Hyp 11.1] which has appliction toward subconvexity for the triple product L-function.

Bernstein and Reznikov, in [2], apply Watson's formula to establish subconvexity in the "eigenvalue aspect." They fix two Maass forms and vary the eigenvalue of the third form. Venkatesh [34] dealt with

Date: April 6, 2012.

subconvexity of the triple product L-function in "level aspect". Using ergodic theory, he establishes bounds for the period integral

$$J(\varphi_1 \otimes \varphi_2 \otimes \varphi_3) = \int_{\mathrm{PGL}_2(F)\backslash \mathrm{PGL}_2(\mathbb{A}_F)} \varphi_1(g)\varphi_2(g)\varphi_3(g)dg \tag{1.1}$$

for varying choices of $\varphi_i \in \pi_i$. In particular, he proved the following.

Theorem 1.1 (Venkatesh). *Let F be a number field and π_1, π_2 automorphic cuspidal representations of $\mathrm{PGL}_2(\mathbb{A}_F)$ with (finite) conductors $\mathfrak{n}_1, \mathfrak{n}_2$ respectively. Let π_3 be another such representation with prime conductor $\mathfrak{p} \nmid \mathfrak{n}_1\mathfrak{n}_2$. Let ϖ be a uniformizer of $F_\mathfrak{p}$. For given $\varphi_i \in \pi_i$ $(i = 1, 2)$ fixed by $\mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$, we put $\varphi = \varphi_1 \otimes \rho(\left(\begin{smallmatrix} \varpi^{-1} & \\ & 1\end{smallmatrix}\right))\varphi_2 \otimes \varphi_3$ where $\varphi_3 \in \pi_3$ and ρ is the right regular action. Then*

$$|J(\varphi)| \ll_{\epsilon,\pi_1,\pi_2} \|\varphi_1\|_{L^4}\|\varphi_2\|_{L^4}\|\varphi_3\|_{L^2} N(\mathfrak{p})^{\epsilon - \frac{(1-4\alpha)(1-2\alpha)}{4(3-4\alpha)}} \tag{1.2}$$

where $\|f\|_{L^p}$ is the standard L^p-norm, and α is any bound towards Ramanujan for GL_2 over F. (We can take $\alpha = 1/9$ by Kim-Shahidi[21].)

Venkatesh conjectured an upper bound for the central L-value in terms of $|J(\varphi)|^2$ which would then give a subconvexity bound for the central L-value $L(\frac{1}{2}, \Pi)$. (See Corollary 1.5.) To describe it properly, however, we need an analogy of the period integral $J(\varphi)$ for a quaternion algebra B over F.

Remark. The subconvexity results for GL_1 and GL_2 L-functions of the paper [26] of Michel and Venkatesh are established by first proving subconvexity for $L(1/2, \Pi)$ where $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ is a triple product just like we study here, but such that one or more of the representations π_i is associated to an Eisenstein series, i.e. *not* cuspidal.

1.1. **Statement of results.** We begin by setting notations. Let $\pi_{i,v}$ denote irreducible admissible representations of $\mathrm{GL}_2(F_v)$ for $i = 1, 2, 3$. Write $\pi_{i,v}^B$ for the corresponding representation of B_v, the division quaternion algebra over F_v, via Jacquet-Langlands. (This is zero if none exists.) Put $\Pi_v = \pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v}$. Globally, we let $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ where each $\pi_i = \bigotimes' \pi_{i,v}$ is a cuspidal automorphic representation, and if B is a quaternion algebra over F, we define Π^B to be the corresponding automorphic representation of $B^\times \times B^\times \times B^\times$. For an algebraic group G over F we denote $[G] = Z(G(\mathbb{A}))G(F)\backslash G(\mathbb{A})$.

Recall that a quaternion algebra B/F is determined by a set of places $\Sigma = \Sigma_B$ of even cardinality where $v \in \Sigma$ if and only if $B(F_v)$ is a division algebra. We call $v \in \Sigma$ *ramified*.

Given a global (or local) representation $\Pi = \bigotimes' \Pi_v$ (or Π_v) as above, there exist certain holomorphic functions $\varepsilon_v(s, \Pi, \psi_v)$ depending on a choice of additive character $\psi_v : F \to \mathbb{C}^\times$. See [32] for details of their definition and properties. In particular, in the global case, if $\psi = \otimes\psi_v$ is trivial on F, the function $\varepsilon(s, \Pi) := \prod_v \varepsilon_v(s, \Pi, \psi)$ is independent of ψ, and the L-function satisfies the functional equation

$$L(1 - s, \Pi) = \varepsilon(s, \Pi)L(s, \widetilde{\Pi}).$$

Moreover, if the central character of Π is trivial (which we will always assume) then $\widetilde{\Pi} \simeq \Pi$ and $\varepsilon_v(\frac{1}{2}, \Pi) = \pm 1$ is independent of ψ_v. Denote the set $\{v \mid \varepsilon_v(\frac{1}{2}, \Pi) = -1\}$ by $\Sigma(\Pi)$. This is a finite set.

Notice that $J : \Pi \to \mathbb{C}$ as given in (1.2) is a $\mathrm{GL}_2(\mathbb{A})$-invariant linear form. Prasad showed that the existence of a $\mathrm{GL}_2(\mathbb{A})$-invariant form on Π is determined by the local epsilon factors. In [29] he proved the theorem below in almost all cases. Loke[25] completed the remaining cases.

Theorem 1.2 (Prasad,Loke). *With the notation as above, the following holds.*

(1) $\dim \mathrm{Hom}_{\mathrm{GL}_2(F_v)}(\Pi_v, \mathbb{C}) + \dim \mathrm{Hom}_{B_v}(\Pi_v^{B_v}, \mathbb{C}) = 1$.

(2) $\dim \mathrm{Hom}_{\mathrm{GL}_2(F_v)}(\Pi_v, \mathbb{C}) = 1$ *if and only if* $\varepsilon(\frac{1}{2}, \Pi_v) = 1$.

In particular,

(3) *When $\varepsilon(\frac{1}{2}, \Pi) = 1$ the global quaternion algebra $B = B_\Pi/F$ whose ramification set is $\Sigma(\Pi)$ is the unique quaternion algebra for which $\mathrm{Hom}_{B^\times(\mathbb{A})}(\Pi^B, \mathbb{C}) \neq 0$.*

(4) *When $\varepsilon(\frac{1}{2}, \Pi) = -1$ one has $\mathrm{Hom}_{B^\times(\mathbb{A})}(\Pi^B, \mathbb{C}) = 0$ for all quaternion algebras B over F.*

We will prove the following in this paper.

Theorem 1.3. *Fix π_1, π_2 cuspidal automorphic representations of* GL_2 *over a number field F with conductors $\mathfrak{n}_1, \mathfrak{n}_2$ respectively. Fix an ideal $\mathfrak{n}$. Let π_3 be cuspidal automorphic with conductor $\mathfrak{n}\mathfrak{p}$ for any fixed ideal $\mathfrak{n}$ and a prime $\mathfrak{p} \nmid \mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}$. Let ϖ be a uniformizer of $F_{\mathfrak{p}}$. Let $S_f = \{\mathfrak{q} \mid \gcd(\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n})\}$, and S_∞ the set of infinite places. Then there is a quaternion algebra B over F such that $\Sigma_B \subset S_f \cup S_\infty$ and there is a finite set of vectors $\mathcal{F}_i^B \subset \pi_i^B$ for $i = 1, 2$ such that*

$$L(\tfrac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3) \ll_{\epsilon,F,R} N(\mathfrak{p})^{1+\epsilon} \left| \int_{[B^\times]} \varphi_1(b)\varphi_2(b \begin{pmatrix} \varpi^{-1} & \\ & 1 \end{pmatrix})\varphi_3(b)db \right|^2 \tag{1.3}$$

as $N(\mathfrak{p}) \to \infty$ and the Langlands parameters of $\pi_{3,\infty}$ remain bounded by R, for some $\varphi_i \in \mathcal{F}_i^B$ ($i = 1, 2$) and $\varphi_3 \in \pi_3^B$ a new vector.

If v is a real place then π_v is isomorphic to either π_{dis}^k a weight k-discrete series representation or a principal series $\pi_{\mathbb{R}}^s$. If v is complex then π_v is isomorphic to a discrete series $\pi_{\mathbb{C}}^{s,k}$. (See Section 6 for definitions.) The condition that the Langlands parameters of $\pi_{3,\infty}$ remain bounded means that the values s, k allowed to appear in $\pi_{3,\infty}$ are restricted to a fixed bounded set R.

We also obtain a generalization of Watson's formula to totally real number fields. In particular, we prove the following. (See Theorem 5.1 for the more general statement.)

Theorem 1.4 (Watson when $F = \mathbb{Q}$)**.** *Suppose that F is a totally real number field and let Π be a cuspidal automorphic representation as above of squarefree level $\mathfrak{N}$. Moreover, we assume that for each $v \mid \infty$, $\pi_{i,v}$ are discrete series representations such that the largest weight is the sum of the two smaller weights. Let B be any quaternion algebra such that $\Pi^B \neq 0$, so $\mathfrak{d}_B = \prod_{\mathfrak{p}\in\Sigma_B} \mathfrak{p}$ divides $\mathfrak{N}$. Let $\varepsilon_v = \varepsilon_v(\frac{1}{2}, \Pi_v)$, and, for finite v, let q_v be the order of the residue field $\mathcal{O}_v$. Then*

$$\frac{\left| \displaystyle\int_{[B^\times]} \varphi(b)db \right|^2}{\displaystyle\prod_{i=1}^{3} \int_{[B^\times]} |\varphi_i(b)|^2 \, db} = |\Delta_F|^{-3/2} \frac{\zeta(2)}{2^3} \frac{L(\frac{1}{2}, \Pi)}{L(1, \Pi, \mathrm{Ad})} \prod_v C_v \tag{1.4}$$

where db is the Tamagawa measure on $[B^\times]$ (defined in Section 2.2), $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$ with φ_i a new vector of π_i^B (chosen so that the sum of the weights at each real place is zero), Δ_F is the discriminant of F, and

$$C_v = \begin{cases} 1 & \text{if } v \nmid \infty\mathfrak{N} \\ \frac{1-\varepsilon_v}{q_v}(1 - \frac{1}{q_v}) & \text{if } v \mid \mathfrak{d}_B \\ \frac{1+\varepsilon_v}{q_v}(1 + \frac{1}{q_v}) & \text{if } v \mid \mathfrak{N},\ v \nmid \mathfrak{d}_B \\ 2 & \text{if } v \mid \infty. \end{cases}$$

The formula as stated in [36] appears slightly different only because it is presented in the language of classical modular forms. Note that this confirms Prasad's theorem in this special case.

1.2. **Application to subconvexity.** From henceforth we will assume without loss of generality that $\varepsilon(\frac{1}{2}, \Pi) = 1$. Under this assumption, Theorems 1.1 and 1.3 imply the following.

Corollary 1.5. *Let $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ where π_i are cuspidal automorphic representations of* PGL_2 *over F with π_1, π_2 fixed and π_3 has prime conductor $\mathfrak{p}$. For all π_3 such that $\Sigma(\Pi) = \emptyset$, one has the following subconvexity bound.*

$$L(\tfrac{1}{2}, \Pi) \ll_{\pi_{3,\infty}} N(\mathfrak{p})^{1-\frac{1}{12}}$$

We describe the limitations of this corollary. We first remark that if v is a nonarchimedean place then $\varepsilon_v(\frac{1}{2}, \Pi_v) = +1$ whenever any one of the representations is unramified. In particular, this implies that $\varepsilon_v(\frac{1}{2}, \pi_v) = +1$ for all but finitely many v. Analogously, if v is archimedean then $\varepsilon(\frac{1}{2}, \Pi_v) = +1$ if any one of $\pi_{i,v}$ is not a discrete series. If all three are discrete series of weights k, l and m where k is the largest weight then $\varepsilon(\frac{1}{2}, \Pi_v) = -1$ if and only if $k < l + m$. If this is the case, we say that Π_v is *balanced*. Otherwise, we say Π_v is *unbalanced*.

Because Theorem 1.1 requires that $\mathfrak{n}_1\mathfrak{n}_2$ be relatively prime to the conductor of π_3, which is required to be a prime $\mathfrak{p}$, we see that in this case the quaternion algebra B for which (1.3) is satisfied ramifies exactly at real primes v such that $\pi_{1,v}$, $\pi_{2,v}$ and $\pi_{3,v}$ are discrete series and for which Π_v is balanced. In other words, the only restriction on Π is that at each real place v either

- at least one of the representations is a principal series, or
- if all three representations are discrete series, the weights are unbalanced.

If Theorem 1.1 could be generalized to arbitrary quaternion algebras, our Theorem 1.3, would be enough to make Corollary 1.5 unconditional.

Theorems 1.3 and 1.4 are obtained via the formula of Ichino and the explicit calculation of certain local trilinear forms. The nonvanishing of the said forms was already known by the work of Gross-Prasad [14], but our contribution is the exact evaluation of each. Ichino's result is discussed in Section 2. The local forms are constructed first via matrix coefficients. In Section 3 these matrix coefficients are completely determined for our particular choice of vectors. Then, in Section 4, we evaluate the trilinear forms using the results on matrix coefficients. Theorem 1.4 and further generalizations are discussed in Section 5, and in the final section we complete the proof of Theorem 1.3.

2. Global trilinear form

In this section, F is a number field with ring of adeles $\mathbb{A} = \mathbb{A}_F$, v a place of F and F_v the corresponding completion. Let $G = \mathrm{GL}_2$ with center Z and $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ where each π_i is a unitary cuspidal automorphic representation of G over F. If ω_i is the central character of π_i then we require that $\omega_1\omega_2\omega_3$ be trivial. We let $L(s, \Pi)$ denote the triple product L-function, $L(s, \Pi, \mathrm{Ad}) = \prod_{i=1}^{3} L(s, \pi_i, \mathrm{Ad})$ with $L(s, \pi_i, \mathrm{Ad})$ the adjoint L-function attached to π_i and $\zeta_F(s)$ the zeta function of the number field F. See Section 4.1 for precise definitions. We use the subscript v to represent the analogous local L and zeta functions. We may assume that B is a quaternion algebra defined over F for which $\Pi^B \neq 0$.

2.1. Ichino's formula.

The central identity which we will use to relate the period on the right side of (1.3) to the L-value on the left is due to Ichino[17] and arises when studying the space $\mathrm{Hom}_{B^\times(\mathbb{A})}(\Pi^B, \mathbb{C})$ of $B^\times(\mathbb{A})$-invariant linear forms where $B^\times$ is embedded diagonally into $B^\times \times B^\times \times B^\times$. These are the so-called trilinear forms. Theorem 1.2 tells us that this space is at most 1-dimensional.

It is more convenient to work with bilinear forms:

$$\mathrm{Hom}_{B^\times(\mathbb{A})\times B^\times(\mathbb{A})}(\Pi^B \otimes \widetilde{\Pi}^B, \mathbb{C}) \tag{2.1}$$

It is elementary to see that Theorem 1.2 carries over to this setting. In particular, since $\dim \mathrm{Hom}_{B_v^\times}(\Pi_v^B \otimes \widetilde{\Pi}_v^B, \mathbb{C}) \leq 1$, the space in (2.1) is again at most 1-dimensional. Moreover, letting db be a Haar measure, there is a choice of invariant form:

$$I(\varphi \otimes \widetilde{\varphi}) = \int_{[B^\times]} \varphi_1\varphi_2\varphi_3(b)db \int_{[B^\times]} \widetilde{\varphi}_1\widetilde{\varphi}_2\widetilde{\varphi}_3(b)\ db. \tag{2.2}$$

Locally, there is also an obvious choice of $B^\times(F_v)$-invariant form. Let

$$\langle \cdot, \cdot \rangle_v : \pi_{i,v}^B \otimes \widetilde{\pi}_{i,v}^B \to \mathbb{C} \tag{2.3}$$

be any $B^\times(F_v)$-invariant pairing. Note that this is unique up to nonzero scalar. For $\varphi_v = \varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi_{3,v} \in \Pi_v^B$ and $\widetilde{\varphi}_v \in \widetilde{\Pi}_v^B$ defined similarly, this gives a *matrix coefficient*

$$\langle \Pi_v^B(g_v)\varphi_v, \widetilde{\varphi}_v\rangle_v = \langle \pi_{1,v}^B(g_v)\varphi_{1,v}, \widetilde{\varphi}_{1,v}\rangle_v \langle \pi_{2,v}^B(g_v)\varphi_{2,v}, \widetilde{\varphi}_{2,v}\rangle_v \langle \pi_{3,v}^B(g_v)\varphi_{3,v}, \widetilde{\varphi}_{3,v}\rangle_v$$

such that

$$I_v'(\varphi_v \otimes \widetilde{\varphi}_v) = \int_{F_v^\times \backslash B^\times(F_v)} \langle \Pi_v^B(g_v)\varphi_v, \widetilde{\varphi}_v\rangle_v db_v$$

is $B^\times(F_v)$-invariant. We assume that db_v is any measure such that $\mathrm{vol}(F_v^\times \backslash B^\times(\mathcal{O}_v)) = 1$ for almost all v, so that $db = C \prod db_v$ for some constant C. (We will choose measures in Section 2.2 so that $C = \Delta_F^{-3/2}\zeta_F(2)^{-1}$.)

Ignoring possible issues with convergence, I and $\prod_v I_v'$ are both elements of (2.1), and so they must differ by a constant. This is formalized by Ichino in the following.

Theorem 2.1 (Ichino)**.** *Let I, I_v be as above and*

$$I_v(\varphi_v \otimes \widetilde{\varphi}_v) = \zeta_{F_v}(2)^{-2} \frac{L_v(1, \Pi_v, \mathrm{Ad})}{L_v(\frac{1}{2}, \Pi_v)} I'_v(\varphi_v \otimes \widetilde{\varphi}_v). \tag{2.4}$$

Then $\frac{I_v(\varphi_v \otimes \widetilde{\varphi}_v)}{\langle \varphi_v \otimes \widetilde{\varphi}_v \rangle_v} = 1$ for almost all v, and

$$\frac{I(\varphi \otimes \widetilde{\varphi})}{\prod_{j=1}^{3} \int_{[B^\times]} \varphi_j(b) \widetilde{\varphi}_j(b) db} = \frac{C}{2^3} \cdot \zeta_F(2)^2 \cdot \frac{L(\frac{1}{2}, \Pi)}{L(1, \Pi, \mathrm{Ad})} \prod_v \frac{I_v(\varphi_v \otimes \widetilde{\varphi}_v)}{\langle \varphi_v, \widetilde{\varphi}_v \rangle_v} \tag{2.5}$$

whenever the denominators are nonzero.

Note that on each side of (2.5) the dependence on the choice of pairings (2.2) and (2.3) is removed by dividing by (on the left side) by $\prod \int_{B^\times} \varphi_i \otimes \widetilde{\varphi}_i db$ or (on the right side) by $\langle \varphi_v, \widetilde{\varphi}_v \rangle_v$.

In the case that the central characters are trivial, Π is self dual and $\overline{\Pi} \simeq \widetilde{\Pi}$. Using the composition of the injection $\Pi \hookrightarrow \Pi \times \widetilde{\Pi}$ given by $\varphi \mapsto (\varphi, \overline{\varphi})$ with (2.2) yields a quadratic form, namely

$$I(\varphi) := I(\varphi_v \otimes \overline{\varphi}_v) = \left| \int_{[G]} \varphi_1(g) \varphi_2(g) \varphi_3(g) dg \right|^2, \tag{2.6}$$

such that $I(\varphi) = |J(\varphi)|^2$ in the notation of Theorem 1.1. Hence Ichino's result provides us with the necessary tools to prove Theorem 1.3. Indeed, we will derive exact formulas for $I_v(\varphi_v) := I_v(\varphi \otimes \overline{\varphi})$ from which sufficient lower bounds will be obtained. (See, for example, Corollary 4.2.)

As described in the introduction, the functional equation implies that $L(\frac{1}{2}, \Pi)$ must be zero unless $\varepsilon(\frac{1}{2}, \Pi) = 1$. Thus Theorem 1.2, together with (2.5), implies one direction of the fact conjectured by Jacquet and proved by Harris and Kudla in [15] and [16]. We record their result here as it will be used in Section 6.

Theorem 2.2 (Harris-Kudla)**.** *The central value $L(\frac{1}{2}, \Pi) \neq 0$ if and only if there exists some B and some $\varphi \in \Pi^B$ such that $\int_{[B^\times]} \varphi(b) db \neq 0$.*

By Theorem 1.2, the quaternion algebra B is that for which $\Sigma_B = \Sigma(\Pi)$.

2.2. **Measures.** Let F_v be a p-adic field with $\mathcal{O}_v$ and q as above, and let B_v be a quaternion algebra over F_v. If $B_v^\times = \mathrm{GL}_2(F_v)$, we choose the (multiplicative) Haar measure db_v (or dg_v) to be that for which the maximal compact subgroup $K_v = \mathrm{GL}_2(\mathcal{O}_v)$ has volume 1. We abuse notation and also denote by db_v (or by dg_v) the measure on $\mathrm{PGL}_2(F_v)$ consistent with the choice above and the exact sequence

$$1 \to F_v^\times \to \mathrm{GL}_2(F_v) \to \mathrm{PGL}_2(F_v) \to 1. \tag{2.7}$$

This means that the image of K_v in $\mathrm{PGL}_2(F_v)$ again has volume 1.

If B_v is division then it contains a unique maximal order R_v. We denote by db_v the Haar measure on $B_v^\times$ for which $R_v^\times$ has measure $(q-1)^{-1}$. Again, we write db_v for the measure on $PB_v^\times$ compatible with the analogous exact sequence to (2.7). We remark that $R_v^\times$ has index 2 in $F_v^\times \backslash B_v^\times$, and so $\mathrm{vol}(F_v^\times \backslash B_v^\times) = \frac{2}{q-1}$.

In the real case, let dx be the standard measure on $\mathbb{R}$ such that the volume of $[0,1] = 1$. Define the subgroups

$$N_\infty = \{n(x) = \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) \mid x \in \mathbb{R}\}, \quad A_\infty = \{a(y) = \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) \mid a \in \mathbb{R}^\times\},$$
$$K_\infty = \{\kappa_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \mid \theta \in [0, 2\pi)\}$$

of $\mathrm{GL}_2(\mathbb{R})$ or $\mathrm{PGL}_2(\mathbb{R})$. The we define the Haar measure via

$$\frac{1}{2\pi} \frac{dydx}{|y|^2} d\theta, \qquad \frac{1}{4\pi}(1 - y^{-2}) d\theta_1 dy d\theta_2 \tag{2.8}$$

in the $N_\infty A_\infty K_\infty$ and $K_\infty A_\infty K_\infty$ coordinate systems respectively. Note that in $K_\infty A_\infty K_\infty$ coordinates we take $a(y)$ only for $|y| \geq 1$.

We recall the notion of Tamagawa measure. As described in [27], for a connected linear algebraic group G defined over a number field F, a measure ω defined over F determines measures $\omega_\mathfrak{p}$ such that for almost all $\mathfrak{p}$, the volume of $G(\mathcal{O}_\mathfrak{p}) = q^{-\dim G} \# G(\mathfrak{p})$ where $G(\mathfrak{p})$ represents the group obtained from G by reducing modulo $\mathfrak{p}$.

For semisimple groups G, the Tamagawa measure can now be defined on $G(F)\backslash G(\mathbb{A})$ by giving $G(F)$ the counting measure as a discrete subgroup of $G(\mathbb{A})$ and defining

$$dg_{\mathbb{A}} = |\Delta_F|^{-\dim G/2} \prod_v \omega_v$$

where Δ_F is the discriminant of F. In particular, for the case at hand of a quaternion algebra B defined over F, and $G = PB^\times$, it can be checked that $\omega_v = \zeta_{F_v}^{-1}(2) db_v$. Therefore, the measure $db_{\mathbb{A}}$ on $\mathbb{A}^\times\backslash B^\times(\mathbb{A})$ which corresponds to the Tamagawa measure db on $[B^\times]$ is

$$db_{\mathbb{A}} = C \prod db_v, \qquad C = |\Delta_F|^{-3/2} \zeta_F^{-1}(2).$$

This constant C is exactly the constant appearing in Theorem 2.1 for our choice of measures.

By way of comparison, we recall that Ichino defines measures $d^\times b_v$ on $PB_v^\times$ by choosing a nontrivial additive character $\psi = \otimes\psi_v : \mathbb{A}/F \to \mathbb{C}^\times$ and letting $d^\times b_v$ be the Haar measure that is self dual with respect to the Fourier transform. For places v that are unramified over $\mathbb{Q}$ and for which ψ_v is level zero, the measures $d^\times b_v$ and db_v agree. More precisely, $db_{\mathbb{A}} = C' \prod d^\times b_v$ with $C' = \zeta_F(2)^{-1}$.

We remark that the Tamagawa number of $PB^\times$, i.e. the volume of $[B^\times]$ under Tamagawa measure, is 2.

3. Matrix coefficients: nonarchimedean places

We fix the following notation for this section. Let F be a p-adic field with ring of integers $\mathcal{O}_F$ and uniformizer ϖ. Let q be the order of the residue field $\mathcal{O}_F/(\varpi)$. Let $v = \mathrm{ord}_\varpi$ be the additive valuation, and define $|x| = q^{-v(x)}$. Let

$$G = \mathrm{GL}_2(F), \quad K = \mathrm{GL}_2(\mathcal{O}_F), \quad Z = \{\left(\begin{smallmatrix} z & \\ & z \end{smallmatrix}\right) \in G\},$$

$$P = NA \quad \text{where} \quad N = \{\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right) \in G\}, \quad A = \{\left(\begin{smallmatrix} a_1 & \\ & a_2 \end{smallmatrix}\right) \in G\}.$$

Let $\sigma_n = \left(\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right)$, $e = \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$, $w = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$.

We assume that $\chi_1, \chi_2 : F^\times \to \mathbb{C}^\times$ are unramified characters, meaning that $\chi_i|_{\mathcal{O}_F^\times} = 1$ for $i = 1, 2$. The character $\chi_1 \otimes \chi_2$ gives a representation of P. We consider $\pi = \pi(\chi_1, \chi_2)$ which is the (admissible) unitary induction from P to G of this representation. Hence π is the right regular action of G on the space of functions $f : G \to \mathbb{C}$ (i.e. $[\pi(g)f](h) = f(hg)$) such that

$$f(\left(\begin{smallmatrix} a_1 & b \\ & a_2 \end{smallmatrix}\right) g) = \chi_1(a_1)\chi_2(a_2) \left|\frac{a_1}{a_2}\right|^{1/2} f(g) \tag{3.1}$$

for all $g \in G$, and there exists a compact open subgroup $L \subset G$ which acts trivially on f. We will follow the common abuse of notation in using π to denote both the space and the action. Note that π need not be irreducible.

The contragradient $\widetilde{\pi}$ of π is equal to $\pi(\chi_1^{-1}, \chi_2^{-1})$. Let dk be the Haar measure on G for which K has volume one. Then the pairing

$$\langle \cdot, \cdot \rangle : \pi \times \widetilde{\pi} \to \mathbb{C} \qquad \langle f, \widetilde{f} \rangle = \int_K f(k)\widetilde{f}(k) dk \tag{3.2}$$

is bilinear and G-invariant. (See [1], [6] for details.) Using this one defines the matrix coefficient associated to $f, \widetilde{f}$:

$$\Phi_{f,\widetilde{f}}(g) = \langle \pi(g)f, \widetilde{f} \rangle.$$

Note that in this section we do not necessarily assume that π is unitary.

We will compute explicitly these matrix coefficients for particular choices of f and $\widetilde{f}$—namely for those vectors that are fixed by

$$K_0 := K_0(\varpi) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K \mid v(c) \geq 1\}.$$

This is a two dimensional subspace. To perform this calculation, we derive general formulas in Section 3.1 and then apply these results to the unramified and special representations in Sections 3.2 and 3.3 respectively.

First, we record some standard results regarding the decomposition of G and volumes of certain subsets. Recall that our measure on G (or on $Z\backslash G$) is that for which K (or its image) has volume 1.

Lemma 3.1. *Let $\Omega_0 = K$ and $\Omega_n = K\sigma_n K$ for $n \geq 1$. Then $Z\backslash G = \bigcup_{n\geq 0} \Omega_n$ and for $n \geq 1$,*

$$K\sigma_n K = K_0\sigma_n K_0 \bigcup K_0 w\sigma_n K_0 \bigcup K_0\sigma_n w K_0 \bigcup K_0 w\sigma_n w K_0. \tag{3.3}$$

Moreover, all of these unions are disjoint.

Let $X_n = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0 \mid v(c) = n\}$ and $Y_n = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0 \mid v(b) = n\}$. The following table is valid.

X	K	K_0	X_n	Y_n	$K\sigma_n K$	$K_0\sigma_n K_0$
$\mathrm{vol}(X)$	1	$(1+q)^{-1}$	$\frac{1-\frac{1}{q}}{1+\frac{1}{q}} q^{-n}$	$\frac{1-\frac{1}{q}}{1+\frac{1}{q}} q^{-n-1}$	$q^n(1+\frac{1}{q})$	$\frac{q^{n-1}}{1+\frac{1}{q}}$

X	$K_0 w\sigma_n K_0$	$K_0\sigma_n w K_0$	$K_0 w\sigma_n w K_0$
$\mathrm{vol}(X)$	$\frac{q^{n-2}}{1+\frac{1}{q}}$	$\frac{q^{n}}{1+\frac{1}{q}}$	$\frac{q^{n-1}}{1+\frac{1}{q}}$

TABLE 1. Volumes of various subsets of $\mathrm{GL}_2(F)$, F a p-adic field

These results can all be obtained by interpreting the action of K on the left or right as row or column operations respectively. For example, to get the decomposition $Z\backslash G = \cup_{n\geq 0}\Omega_n$, we may assume, by multiplying by an element of the center that $\min\{v(a), v(b), v(c), v(d)\} = 0$. Then we alter $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ by adding any integer multiple of one row or column to the other, and by multiplying any row or column by any element of $\mathcal{O}^\times$. From this description it is clear that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = k\sigma_n k'$ for some $k, k' \in K$ and $n = v(\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))$.

In a similar fashion we obtain the decomposition

$$K\sigma_n K = \bigcup_{a\in\mathcal{O}/\mathfrak{p}^n} \left(\begin{smallmatrix} \varpi^n & a \\ & 1 \end{smallmatrix}\right) K \cup \bigcup_{d\in\mathcal{O}/\mathfrak{p}^{n-1}} w \left(\begin{smallmatrix} \varpi^n & d\varpi \\ & 1 \end{smallmatrix}\right) K.$$

Similarly decompositions can be given for K_0 double cosets, and from these decompositions the stated volumes are apparent.

3.1. **Matrix coefficient associated to Iwahori fixed vectors.** Recall that $\pi = \pi(\chi_1, \chi_2)$ is the induced representation defined in (3.1). The Iwasawa decomposition $G = NAK$ implies that a vector $f \in \pi$ is uniquely determined by its restriction to K. With this in mind we define f_0 and f_1 to be the vectors whose restrictions to K are the characteristic function on K_0 and $K_0 w K_0$ respectively. Let $\widetilde{f}_0$ and $\widetilde{f}_1$ be the analogously defined vectors in $\widetilde{\pi}$.

We call the space π^{K_0} of vectors fixed by K_0 the *space of Iwahori fixed vectors*. Note that every such vector is a linear combination of f_0 and f_1 because $K = K_0 \cup K_0 w K_0$ and $K_0 w K_0 = (N \cap K)K w K_0$. Moreover, if $f \in \pi^{K_0}$ then $f = af_0 + bf_1$ where $a = f(e)$ and $b = f(w)$.

Using the coset decompositions of Lemma 3.1 it is immediate that

$$\Phi_{i,j}(g) := \Phi_{f_i,\widetilde{f}_j}(g) = \begin{cases} \displaystyle\int_{K_0} f_i(kg)dk & \text{if } j = 0 \\ \displaystyle q\int_{K_0} f_i(wkg)dk & \text{if } j = 1. \end{cases} \tag{3.4}$$

Our goal is to determine $\Phi_{i,j}(g)$ for all $g \in G$. By the G-invariance of the inner form if $f \in \pi^{K_0}$ and $\widetilde{f} \in \widetilde{\pi}^{K_0}$ and $k \in K_0$ then

$$\Phi_{f,\widetilde{f}}(kg) = \langle \pi(kg)f, \widetilde{f}\rangle = \langle \pi(g)f, \widetilde{\pi}^{-1}(k)\widetilde{f}\rangle = \Phi_{f,\widetilde{f}}(g) \qquad \text{and}$$

$$\Phi_{f,\widetilde{f}}(gk) = \langle \pi(g)\pi(k)f, \widetilde{f}\rangle = \Phi(g).$$

Hence $\Phi_{i,j}(g)$ depends only on the double coset $K_0 g K_0$. So by Lemma 3.1 we need to perform the calculation for each of only four K_0-double coset representatives.

Proposition 3.2. *The values of $\Phi_{i,j}$ are given by $\frac{q^{-n/2}}{1+\frac{1}{q}}$ times the values in the following table. (Note that the values for $\Phi_{i,j}(w\sigma_n)$ hold only when $n > 0$. Otherwise, the table is valid for all $n \geq 0$.)*

g	σ_n	$w\sigma_n$	$\sigma_n w$	$w\sigma_n w$
$\Phi_{0,0}(g)$	$\alpha_1^n \cdot \frac{1}{q}$	$\frac{\alpha_1^{n-1}-\alpha_2^{n-1}}{\alpha_2^{-1}-\alpha_1^{-1}}(1-\frac{1}{q})$	0	$\alpha_2^n \cdot \frac{1}{q}$
$\Phi_{1,0}(g)$	0	α_2^n	$\alpha_1^n \frac{1}{q}$	$\frac{\alpha_2^n-\alpha_1^n}{1-\alpha_1\alpha_2^{-1}}(1-\frac{1}{q})$
$\Phi_{0,1}(g)$	$\frac{\alpha_1^n-\alpha_2^n}{1-\alpha_1^{-1}\alpha_2}(1-\frac{1}{q})$	α_1^n	$\alpha_2^n \frac{1}{q}$	0
$\Phi_{1,1}(g)$	α_2^n	0	$\frac{\alpha_2^{n+1}-\alpha_1^{n+1}}{\alpha_2-\alpha_1}(1-\frac{1}{q})$	α_1^n

TABLE 2. Values of matrix coefficients of Iwahori fixed vectors I

Proof. We prove the formula in the case that $g \in K_0\sigma_n K_0$. Let $\alpha_i = \chi_i(\varpi)$.

Let $g, g' \in G$. We define an equivalence relation such that $g \sim g'$ if there exists $k \in K_0$ such that $g = g'k$. Since the Iwahori fixed vectors are, by definition, invariant by K_0, $f_i(g)$ depends only on the equivalence class of g. Realizing that K_0 acts on G is by column operations it is easy to see that for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0$,

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_n = \left(\begin{smallmatrix} a\varpi^n & b \\ c\varpi^n & d \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} \varpi^n & b' \\ 0 & 1 \end{smallmatrix}\right) \tag{3.5}$$

where b and b' have the same valuation.[1] Applying this to (3.4) (for $j = 0$) yields

$$\Phi_{0,0}(\sigma_n) = \int_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in K_0} f_0(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_n)dk = \int_{K_0} f_0\left(\left(\begin{smallmatrix} \varpi^n & b \\ 0 & 1 \end{smallmatrix}\right)\right)dk = \mathrm{vol}(K_0)\alpha_1^n q^{-n} = \frac{q^{-n/2}}{1+\frac{1}{q}}\alpha_1^n \cdot \frac{1}{q},$$

and

$$\Phi_{1,0}(\sigma_n) = \int_{K_0} f_1(k\sigma_n)dk = \int_{K_0} f_1\left(\left(\begin{smallmatrix} \varpi^n & b \\ 0 & 1 \end{smallmatrix}\right)\right)dk = 0.$$

The equivalence class of an element of G of the form $w\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_n$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0$. Using the relation $\left(\begin{smallmatrix} 0 & 1 \\ 1 & x \end{smallmatrix}\right) = \left(\begin{smallmatrix} -x^{-1} & 1 \\ 0 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ x^{-1} & 1 \end{smallmatrix}\right)$, it is not hard to show that

$$w\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\sigma_n \sim \begin{cases} w\sigma_n ww\left(\begin{smallmatrix} 1 & b\varpi^{-n} \\ 0 & 1 \end{smallmatrix}\right) \in \left(\begin{smallmatrix} 1 & \\ & \varpi^n \end{smallmatrix}\right)wK_0 & v(b) \geq n \\ w\sigma_n w\left(\begin{smallmatrix} -b^{-1}\varpi^n & 1 \\ & b\varpi^{-n} \end{smallmatrix}\right) \in \left(\begin{smallmatrix} \varpi^{n-m} & * \\ & \varpi^m \end{smallmatrix}\right)K_0 & m = v(b) < n. \end{cases} \tag{3.6}$$

Hence, (using (3.4) in the case $j = 1$) we have

$$\begin{aligned} \Phi_{0,1}(\sigma_n) =& q\int_{K_0} f_0(wk\sigma_n)dk \\ =& q\sum_{m=0}^{n-1} \mathrm{vol}(Y_m) f_0(\left(\begin{smallmatrix} \varpi^{n-m} & \\ & \varpi^m \end{smallmatrix}\right)) \\ =& \frac{1-\frac{1}{q}}{1+\frac{1}{q}}\sum_{m=0}^{n-1} q^{-m}\alpha_1^{n-m}\alpha_2^m q^{m-n/2} \\ =& \frac{q^{-n/2}}{1+\frac{1}{q}}(1-\frac{1}{q})\alpha_1^n\sum_{m=0}^{n-1}(\alpha_1^{-1}\alpha_2)^m = \frac{q^{-n/2}}{1+\frac{1}{q}}\frac{\alpha_1^n-\alpha_2^n}{1-\alpha_1^{-1}\alpha_2}(1-\frac{1}{q}), \end{aligned}$$

[1]In the calculations below we will abuse notion by writing b instead of b' which differs from b by a unit (and similarly for c, c'.) Since our character χ is unramified, this is harmless.

and

$$
\begin{aligned}
\Phi_{1,1}(\sigma_n) =& q\int_{K_0} f_1(wk\sigma_n)dk\\
=& q\sum_{m=n}^{\infty} \mathrm{vol}(Y_m) f_1(\left(\begin{smallmatrix}1 & \\ & \varpi^n\end{smallmatrix}\right) w)\\
=& \frac{1-\frac{1}{q}}{1+\frac{1}{q}}\sum_{m=n}^{\infty}\alpha_2^n q^{-m} q^{n/2} = \frac{q^{-n/2}}{1+\frac{1}{q}}\alpha_2^n.
\end{aligned}
$$

These results give the values in the first column of Table 2. The calculations for the other columns are similar. The only nuances are that the decompositions analogous to (3.5) and (3.6) may depend instead on $v(c)$ and take a slightly different shape. We leave the details to the reader. □

3.2. **Application to unramified representations.** Strictly speaking, the calculation of the local trilinear form could be carried out at this point using the results of Proposition 3.2 with respect to the basis $\{f_0, f_1\}$ of Iwahori fixed vectors. However, it turns out that the calculations are drastically simplified by using a different basis. In this section we describe this basis and the corresponding matrix coefficients in the case that $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$. This implies that $\pi = \pi(\chi_1, \chi_2)$ is irreducible. It is called an *unramified principal series.*

The vector $\phi_0 = f_0 + f_1$, which we call the *normalized new vector*, is obviously fixed by K and, in fact, $\pi^K = \mathbb{C}\phi_0$. Note that $\phi_0(e) = 1$. Using bilinearity, $\Phi_{\phi_0,\widetilde{\phi}_o} = \sum_{i,j}\Phi_{i,j}$. It is easy to see that $\Phi_{\phi_0,\widetilde{\phi}_0}$ is K-biinvariant, so using only the first (or any other) column of Table 2 above, we obtain the well-known formula of Macdonald. (See, for example, [7, Theorem 4.6.6].)

Proposition 3.3. *Let $\pi = \pi(\chi_1, \chi_2)$ be an unramified admissible representation of $\mathrm{GL}_2(F)$. If $\phi_0, \widetilde{\phi}_0$ are the normalized new vectors of π and $\widetilde{\pi}$ respectively then the function $\Phi_{\phi_0,\widetilde{\phi}_0}$ is K-biinvariant and*

$$
\Phi_{\phi_0,\widetilde{\phi}_0}(\sigma_n) = \frac{q^{-n/2}}{1+\frac{1}{q}}\left(\alpha_1^n \frac{1-\frac{\alpha_1^{-1}\alpha_2}{q}}{1-\alpha_1^{-1}\alpha_2} + \alpha_2^n\frac{1-\frac{\alpha_1\alpha_2^{-1}}{q}}{1-\alpha_1\alpha_2^{-1}}\right).
$$

for $n \geq 0$ and $\alpha_i = \chi_i(\varpi)$.

Recall that $\pi = \pi(\chi_1, \chi_2)$. For $\pi' = \pi(\chi_2, \chi_1)$, there is exists an intertwining operator

$$
M : \pi \to \pi'. \tag{3.7}
$$

The map M is given by the formula

$$
(Mf)(g) = \int_F f(w\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right) g)dx \tag{3.8}
$$

whenever the integral converges. It is defined elsewhere by analytic continuation.

Lemma 3.4. *Let $\pi = \pi(\chi_1, \chi_2)$, and $\pi' = \pi(\chi_2, \chi_1)$. Let ϕ_0 (respectively ϕ_0') be the normalized new vector of π (resp. π') as above. Let $\phi_1 = f_0 - \frac{1}{q}f_1$, and let ϕ_1' be the similarly defined vector in π_i'. Then*

$$
M\phi_0 = \frac{1-\frac{\alpha_1\alpha_2^{-1}}{q}}{1-\alpha_1\alpha_2^{-1}}\phi_0' \quad \textit{and} \quad M\phi_1 = -\frac{1-\frac{\alpha_1^{-1}\alpha_2}{q}}{1-\alpha_1^{-1}\alpha_2}\phi_1'. \tag{3.9}
$$

Given this result we abuse notation and call ϕ_0, ϕ_1 *eigenvectors* of the M.

Proof. As was remarked at the beginning of Section 3.1, if $f \in V$ then

$$
Mf = af_0' + bf_1' \quad \text{where} \quad a = Mf(e) \text{ and } b = Mf(w).
$$

Using the identity $\left(\begin{smallmatrix}0 & 1\\ 1 & x\end{smallmatrix}\right) = \left(\begin{smallmatrix}-x^{-1} & 1\\ 0 & x\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & 0\\ x^{-1} & 1\end{smallmatrix}\right)$ one deduces that

$$
f_0(w\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)) = \begin{cases} 0 & \text{if } x \in \mathcal{O}_F\\ \chi_1^{-1}(-x)\chi_2(x)\,|x|^{-1} & \text{if } x \notin \mathcal{O}_F,\end{cases} \tag{3.10}
$$

and

$$f_0(w\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)w) = \begin{cases} 1 & \text{if } x \in \varpi\mathcal{O}_F \\ 0 & \text{if } x \notin \varpi\mathcal{O}_F. \end{cases} \tag{3.11}$$

Therefore, by combining (3.8) and (3.10) we have

$$\begin{aligned} Mf_0(e) =& \int_F f_0(w\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)) \\ =& \sum \int_{\varpi^n\mathcal{O}_F^\times} f_0(w\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right))dx \\ =& \sum_{n=-1}^{-\infty} \mathrm{vol}(\varpi^n\mathcal{O}_F)(\alpha_1^{-1}\alpha_2)^n \\ =& (1-\frac{1}{q})\sum_{n=1}^{\infty}(\alpha_1\alpha_2^{-1})^n = (1-\frac{1}{q})\frac{\alpha_1\alpha_2^{-1}}{1-\alpha_1\alpha_2^{-1}}. \end{aligned}$$

Similarly, (3.8) and (3.11) combine to give $Mf_0(w) = \mathrm{vol}(\varpi\mathcal{O}_F) = \frac{1}{q}$. So[2]

$$Mf_0 = (1-\frac{1}{q})\frac{\alpha_1\alpha_2^{-1}}{1-\alpha_1\alpha_2^{-1}}f_0' + \frac{1}{q}f_1'. \tag{3.12}$$

A similar calculation for f_1 shows that

$$Mf_1 = f_0' + \frac{1-\frac{1}{q}}{1-\alpha_1\alpha_2^{-1}}f_1'.$$

Using the linearity of M, the result follows. $\square$

We have now completed all of the computations necessary to prove the following extension of Proposition 3.3 using the basis $\{\phi_0, \phi_1\}$ of eigenvectors of M.

Proposition 3.5. *Let $\pi = \pi(\chi_1, \chi_2)$ such that $\chi_1\chi_2^{-1} \neq \pm|\cdot|$. Let ϕ_0, ϕ_1 be the eigenvectors in the sense of Lemma 3.4, and $\widetilde{\phi}_0, \widetilde{\phi}_1$ the analogous vectors in $\widetilde{\pi} = \pi(\chi_1^{-1}, \chi_2^{-1})$. The values of the matrix coefficients $\Psi_{i,j}(g) = \Phi_{\phi_i,\widetilde{\phi}_j}(g)$ are $\frac{q^{-n/2}}{1+\frac{1}{q}}$ times the values given by Table 3 in which*

$$A = \frac{1-\frac{\alpha_1\alpha_2^{-1}}{q}}{1-\alpha_1\alpha_2^{-1}}, \qquad \frac{1-\frac{\alpha_1^{-1}\alpha_2}{q}}{1-\alpha_1^{-1}\alpha_2}.$$

Note that A is the eigenvalue of ϕ_0 and $-B$ is the eigenvalue of ϕ_1.

(i,j)	$\Phi_{\phi_i,\widetilde{\phi}_j}(\sigma_n)$	$\Phi_{\phi_i,\widetilde{\phi}_j}(w\sigma_n)$	$\Phi_{\phi_i,\widetilde{\phi}_j}(\sigma_n w)$	$\Phi_{\phi_i,\widetilde{\phi}_j}(w\sigma_n w)$
$(0,1)$	$\frac{1}{q}(\alpha_1^n - \alpha_2^n)A$	$-(\alpha_1^n - \alpha_2^n)A$	$\frac{1}{q}(\alpha_1^n - \alpha_2^n)A$	$-(\alpha_1^n - \alpha_2^n)A$
$(1,0)$	$(\alpha_1^n - \alpha_2^n)B$	$(\alpha_1^n - \alpha_2^n)B$	$-\frac{1}{q}(\alpha_1^n - \alpha_2^n)B$	$-\frac{1}{q}(\alpha_1^n - \alpha_2^n)B$
$(1,1)$	$\frac{1}{q}(\alpha_1^n A + \alpha_2^n B)$	$-(\alpha_1^n A + \alpha_2^n B)$	$-\frac{1}{q^2}(\alpha_1^n A + \alpha_2^n B)$	$\frac{1}{q}(\alpha_1^n A + \alpha_2^n B)$
$(0,0)$	$\alpha_1^n B + \alpha_2^n A$	$\alpha_1^n B + \alpha_2^n A$	$\alpha_1^n B + \alpha_2^n A$	$\alpha_1^n B + \alpha_2^n A$

Table 3. Values of matrix coefficients of Iwahori fixed vectors II

[2]In all of these calculations, we see that the integral converges if and only if $|\alpha_1\alpha_2^{-1}| < 1$. However, via analytic continuation, the results are the same for all $\left|\chi_1\chi_2^{-1}\right| \neq 1$.

Proof. In the case of $\Psi_{1,0} = \Phi_{\phi_1,\widetilde{\phi}_0}$, we see that

$$\Psi_{1,0}(kg) = \langle \pi(kg)\phi_1, \widetilde{\phi}_0\rangle = \langle \pi(g)\phi_1, \widetilde{\pi}(k^{-1})\widetilde{\phi}_0\rangle = \langle \pi(g)\phi_1, \widetilde{\phi}_0\rangle = \Psi_{1,0}(g)$$

for all $k \in K$. So $\Phi(\sigma_n) = \Phi(w\sigma_n)$ and $\Phi(w\sigma_n w) = \Phi(\sigma_n w)$. Now we use the bilinearity of matrix coefficients to write

$$\Phi_{\phi_1,\widetilde{\phi}_0} = \Phi_{0,0} + \Phi_{0,1} - \frac{1}{q}(\Phi_{1,0} + \Phi_{1,1}),$$

and calculate the value on σ_n using Table 2:

$$\begin{aligned}
\Psi_{1,0}(\sigma_n) &= \Phi_{0,0}(\sigma_n) + \Phi_{0,1}(\sigma_n) - \frac{1}{q}(\Phi_{1,0}(\sigma_n) + \Phi_{1,1}(\sigma_n)) \\
&= \alpha_1^n \cdot \frac{1}{q} + \frac{\alpha_1^n - \alpha_2^n}{1-\alpha_1^{-1}\alpha_2}(1-\frac{1}{q}) - \frac{1}{q}\,(0+\alpha_2^n) \\
&= (\alpha_1^n - \alpha_2^n)\left(\frac{1}{q} + \frac{1-\frac{1}{q}}{1-\alpha_1^{-1}\alpha_2}\right) \\
&= (\alpha_1^n - \alpha_2^n)\left(\frac{(\frac{1}{q} - \frac{\alpha_1^{-1}\alpha_2}{q}) + (1-\frac{1}{q})}{1-\alpha_1^{-1}\alpha_2}\right) = (\alpha_1^n - \alpha_2^n)B.
\end{aligned}$$

For $w\sigma_n w$:

$$\begin{aligned}
\Psi_{1,0}(w\sigma_n w) &= \Phi_{0,0}(w\sigma_n w) + \Phi_{0,1}(w\sigma_n w) - \tfrac{1}{q}(\Phi_{1,0}(w\sigma_n w) + \Phi_{1,1}(w\sigma_n w)) \\
&= \alpha_2^n \cdot \tfrac{1}{q} + 0 - \tfrac{1}{q}\left(\tfrac{\alpha_2^n - \alpha_1^n}{1-\alpha_1\alpha_2^{-1}}(1-\tfrac{1}{q}) + \alpha_1^n\right) \\
&= -\tfrac{1}{q}(\alpha_1^n - \alpha_2^n)\left(1 - \tfrac{1-\frac{1}{q}}{1-\alpha_1\alpha_2^{-1}}\right) = -\tfrac{1}{q}(\alpha_1^n - \alpha_2^n)B.
\end{aligned}$$

The computation of $\Psi_{1,1}(\sigma_n) = \Phi_{\phi_1,\widetilde{\phi}_1}(\sigma_n)$ is similar:

$$\begin{aligned}
\Psi_{1,1}(\sigma_n) =& \Phi_{0,0}(\sigma_n) - \tfrac{1}{q}(\Phi_{0,1}(\sigma_n) + \Phi_{1,0}(\sigma_n)) = \tfrac{1}{q^2}\Phi_{1,1}(\sigma_n) \\
=& \tfrac{1}{q}\alpha_1^n - \tfrac{1}{q}(1-\tfrac{1}{q})\tfrac{\alpha_1^n - \alpha_2^n}{1-\alpha_1^{-1}\alpha_2} + \tfrac{1}{q^2}\alpha_2^n \\
=& \tfrac{1}{q}\left(\alpha_1^n\left(1 - \tfrac{1-\frac{1}{q}}{1-\alpha_1^{-1}\alpha_2}\right) + \alpha_2^n\left(\tfrac{1}{q} + \tfrac{1-\frac{1}{q}}{1-\alpha_1^{-1}\alpha_2}\right)\right) \\
=& \tfrac{1}{q}(\alpha_1^n A + \alpha_2^n B).
\end{aligned}$$

The formulas for $\Psi_{1,1}(w\sigma_n)$, $\Psi_{1,1}(\sigma_n w)$, $\Psi_{1,1}(w\sigma_n w)$ and for $\Phi_{\phi_0,\widetilde{\phi}_1}$ are derived in the same fashion. The final row is just a restatement of Proposition 3.3. $\square$

In order to prove Theorems 1.3 and 5.1 we will need to know something about the matrix coefficient $\Psi_{2,2} = \Phi_{\phi_2,\widetilde{\phi}_2}$ where $\phi_2 = \left(\begin{smallmatrix}\varpi^{-1} & \\ & 1\end{smallmatrix}\right)\phi_0$. (In particular, see Corollary 1.5.) By way of the following lemma and a messy calculation, one could give a formula for $\Psi_{2,2}$ similar to those of Proposition 3.5.

Lemma 3.6. *Let $\pi = \pi(\chi,\chi^{-1})$, and $\alpha = \chi(\varpi)$. Let $\phi_2 = \left(\begin{smallmatrix}\varpi^{-1} & \\ & 1\end{smallmatrix}\right)\phi_0 \in \pi$ where ϕ_0 is as above. Then $\phi_2 \in \pi^{K_0}$. More precisely,*

$$\phi_2 = \frac{1}{1+\frac{1}{q}}\left((\alpha+\alpha^{-1})q^{-1/2}\phi_0 + (\alpha^{-1}q^{1/2} - \alpha q^{-1/2})\phi_1\right)$$

Proof. Let $k = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in K_0$. Then

$$\begin{aligned}
\phi_2(gk) &= \phi_0(g\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\left(\begin{smallmatrix}\varpi^{-1} & \\ & 1\end{smallmatrix}\right)) = \phi_0(g\left(\begin{smallmatrix}a\varpi^{-1} & b\\ c\varpi^{-1} & d\end{smallmatrix}\right)) \\
&= \phi_0(g\left(\begin{smallmatrix}\varpi^{-1} & \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}a & b\varpi\\ c\varpi^{-1} & d\end{smallmatrix}\right)) = \pi(\left(\begin{smallmatrix}\varpi^{-1} & \\ & 1\end{smallmatrix}\right))\phi_{\mathfrak{p},0}(g) = \phi_{\mathfrak{p},2}(g)
\end{aligned}$$

since $\left(\begin{smallmatrix}a & b\varpi\\ c\varpi^{-1} & d\end{smallmatrix}\right) \in K$ and $\phi_{\mathfrak{p},0}$ is fixed by K.

So $\phi_2 = \phi_2(e)f_0 + \phi_2(w)f_1$. Writing this in terms of the basis $\{\phi_0, \phi_1\}$ and simplifying leads to the desired result. $\square$

3.3. **Application to the special representations.** We remark that the results of this section are not new. The matrix coefficient of a Steinberg representation was given by Godement-Jacquet in [12] for GL_n, and by Borel [5] in general[3]. However, the present proof is included since it follows so neatly from the results of the previous section.

If $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$ then the resulting induced representation is reducible. Indeed, writing

$$\pi = \pi(\chi\,|\cdot|^{1/2}, \chi\,|\cdot|^{-1/2}) \quad \text{and} \quad \widetilde{\pi} = \pi(\chi^{-1}\,|\cdot|^{-1/2}, \chi^{-1}\,|\cdot|^{1/2}),$$

this is reflected in the exact sequences

$$0 \longrightarrow \sigma_\chi \longrightarrow \pi \longrightarrow \mathbb{C}_\chi \longrightarrow 0, \tag{3.13}$$

$$0 \longrightarrow \mathbb{C}_{\chi^{-1}} \longrightarrow \widetilde{\pi} \longrightarrow \sigma_{\chi^{-1}} \longrightarrow 0 \tag{3.14}$$

where $\mathbb{C}_\mu$ is the 1-dimensional space on which G acts by $\mu(\det g)$ and σ_μ is a *special representation.*

g	σ_n	$w\sigma_n$	$\sigma_n w$	$w\sigma_n w$
$\Phi_{0,0}(g)$	$\frac{\alpha^n q^{-n-1}}{1+\frac{1}{q}}$	$\frac{\alpha^n(q^{-1}-q^{-n})}{1+\frac{1}{q}}$	0	$\frac{\alpha^n q^{-1}}{1+\frac{1}{q}}$
$\Phi_{1,0}(g)$	0	$\frac{\alpha^n}{1+\frac{1}{q}}$	$\frac{\alpha^n q^{-n-1}}{1+\frac{1}{q}}$	$\frac{\alpha^n(1-q^{-n})}{1+\frac{1}{q}}$
$\Phi_{0,1}(g)$	$\frac{\alpha^n(q^{-1}-q^{-n-1})}{1+\frac{1}{q}}$	$\frac{\alpha^n q^{-n}}{1+\frac{1}{q}}$	$\frac{\alpha^n q^{-1}}{1+\frac{1}{q}}$	0
$\Phi_{1,1}(g)$	$\frac{\alpha^n}{1+\frac{1}{q}}$	0	$\frac{\alpha^n(1-q^{-n-1})}{1+\frac{1}{q}}$	$\frac{\alpha^n q^{-n}}{1+\frac{1}{q}}$

Table 4. Values of matrix coefficients for $\pi = \pi(\chi\,|\cdot|^{1/2}, \chi\,|\cdot|^{-1/2})$

We write $\alpha = \chi(\varpi)$. So $\alpha_1 = \alpha q^{1/2}$ and $\alpha_2 = \alpha q^{-1/2}$. Using these values of α_1, α_2, we rewrite Table 2 including the factor $\frac{q^{-n/2}}{1+\frac{1}{q}}$ as Table 4.

Our standard model of σ_χ will be as a subset of π as in (3.13). We will see that $\widetilde{\sigma}_\chi \simeq \sigma_{\chi^{-1}}$ which could be considered as a subset of $\pi(\chi^{-1}\,|\cdot|^{1/2}, \chi^{-1}\,|\cdot|^{-1/2})$. However, for the purposes of computing the inner form

$$\langle\cdot,\cdot\rangle : \sigma_\chi \times \widetilde{\sigma}_\chi \to \mathbb{C} \tag{3.15}$$

it is better to consider $\widetilde{\sigma}_\chi$ as the quotient $\widetilde{\pi}/\mathbb{C}_{\chi^{-1}}$ as in (3.14). As a matter of notation, given these models of σ_χ and $\sigma_{\chi^{-1}}$ we use letters f, ϕ to denote elements of π (or σ_χ), $\widetilde{f}, \widetilde{\phi}$ to denote elements of $\widetilde{\pi}$, and $\bar{f}, \bar{\phi}$ to denote elements of $\sigma_{\chi^{-1}}$. The following is a well-known result.

Lemma 3.7. *The pairing on $\pi \times \widetilde{\pi}$ defined in* (3.2) *descends to a well defined pairing on $\sigma_\chi \times \sigma_{\chi^{-1}}$. That is to say, if $f \in \sigma_\chi$ and $\bar{f} \in \sigma_{\chi^{-1}}$ then*

$$\langle f, \bar{f}\rangle = \int_K f(k)\widetilde{f}(k)dk$$

for any $\widetilde{f} \in \widetilde{\pi}$ whose image in $\sigma_{\chi^{-1}}$ is $\bar{f}$.

Note that $\widetilde{\phi_0} = \widetilde{f}_0 + \widetilde{f}_1 \in \widetilde{\pi}$ is K-invariant so it's image in $\sigma_{\chi^{-1}}$ must be zero. Therefore, as a corollary to Lemma 3.7, we have that

$$\int_K f(k)dk = \int_K f\widetilde{\phi_0}(k)dk = \langle f, 0\rangle = 0 \tag{3.16}$$

[3] What we call the special representation in this paper is a twist of the Steinberg representation of GL_2.

for all $f \in \sigma_\chi$. In other words, if $f \in \sigma_\chi$ then $\int_K f(k)dk = 0$. In fact, the converse is also true. Namely if $f \in \pi$ is such that $\int_K f(k)dk = 0$ then $f \in \sigma_\chi$.

The space $\sigma_\chi^{K_0}$ is well known to be 1-dimensional. The vector $\phi = f_0 - \frac{1}{q} f_1 \in \pi$ meets the criterion above, hence is an element of σ_χ. We call it the *normalized new vector*. In order to choose a new vector in $\sigma_{\chi^{-1}}$ consistent with the choice of ϕ, note that by

$$M : \widetilde{\pi} = \pi(\chi^{-1} |\cdot|^{-1/2}, \chi^{-1} |\cdot|^{1/2}) \to \pi(\chi^{-1} |\cdot|^{1/2}, \chi^{-1} |\cdot|^{-1/2}) = \pi',$$

$\sigma_{\chi^{-1}}$ is isomorphic to a subspace of π'.

So we define the normalized new vector $\bar{\phi}$ of $\sigma_{\chi^{-1}}$ to be the image of any vector $\widetilde{\phi}$ for which $M(\widetilde{\phi}) = f_0' - \frac{1}{q} f_1'$. By Lemma 3.4, $M(\widetilde{f}_0 - \frac{1}{q}\widetilde{f}_1) = (1 + \frac{1}{q})\widetilde{\phi}_1$. A simple calculation then shows that $(1 + \frac{1}{q})\widetilde{\phi}_1$ and $\widetilde{f}_0$ have the same image in $\sigma_{\chi^{-1}}$.

We now have all of the necessary ingredients to prove the generalization of Proposition 3.3 to the case of π a special representation.

Proposition 3.8. *Let χ be an unramified character of F and $\alpha = \chi(\varpi)$. If $\phi, \bar{\phi}$ are new vectors of a special representation σ_χ and its contragradient $\sigma_{\chi^{-1}}$ respectively, the value of the matrix coefficient $\Phi = \Phi_{\phi_1,\bar{\phi}_1}/\Phi_{\phi_1,\bar{\phi}_1}(e)$ is determined by the Table 5 in which $n \geq 1$.*

g	w	σ_n	$w\sigma_n$	$\sigma_n w$	$w\sigma_n w$
$\Phi(g)$	$-q^{-1}$	$\alpha^n q^{-n}$	$-\alpha^n q^{1-n}$	$-\alpha^n q^{-1-n}$	$\alpha^n q^{-n}$

Table 5. Values of matrix coefficient associated to the new vector of a special representation

Proof. We take $\phi, \bar{\phi}$ as above. Then $\bar{\phi}$ is the image of $\widetilde{f}_0$. So, by Lemma 3.7,

$$\Phi_{\phi_1,\bar{\phi}_1}(g) = \langle \sigma_\chi(g)\phi, \bar{\phi} \rangle = \int_K (f_0 - \tfrac{1}{q} f_1)(kg)\widetilde{f}_0(k)dk = \Phi_{0,0}(g) - \tfrac{1}{q}\Phi_{1,0}(g).$$

The result can now be deduced from Table 4. □

Remark. For all of our applications we are interested in determining $\frac{I(\phi \otimes \widetilde{\phi})}{\langle \phi, \widetilde{\phi} \rangle}$. Since $\langle \phi, \widetilde{\phi} \rangle = \Phi_{\phi,\overline{\phi}}(e)$, it is convenient to normalize Φ in this way.

4. Local trilinear forms

In this section we compute the term $I_v(\varphi_v)$ appearing on the right hand side of (2.5). To this end, we tabulate the local L and zeta factors that appear in Ichino's formula in Section 4.1. In Section 4.2 we calculate the forms $I_v'(\varphi_v)$ using our results above.

We now assume that all local representations are unitary. In the unramified case this means we only consider $\pi(\chi_1, \chi_2)$ and that $\chi_i(\varpi)$ is either complex with absolute value 1 for $i = 1, 2$, or else $\chi_1(\varpi) = \chi_2^{-1}(\varpi) = q^{-\lambda}$ for some real λ which we can assume to be in the interval $(0, \frac{1}{2})$. (The latter case are the so-called *complementary series*.) In the case of special representations σ_χ, χ we may assume without loss of generality that $\chi^2 = 1$.

4.1. Local L-factors.

Let F be a nonarchimedean local field with uniformizer ϖ, and let q be the order of the residue field. We let $\chi, \nu, \mu : F^\times \to \mathbb{C}^\times$ be unramified characters (perhaps with subscripts), and denote $\gamma = \chi(\varpi)$, $\beta = \nu(\varpi)$ and $\alpha = \mu(\varpi)$ (with subscripts if χ, ν, μ have subscripts). Then $L_v(s, \chi) = (1 - \gamma q^{-s})^{-1}$, and the local zeta function is $\zeta_{F_v}(s) = L_v(s, 1)$.

Corresponding to each irreducible admissible representation π of $\mathrm{GL}_2(F)$ there is a representation $\rho : W_F \to \mathrm{GL}_2(\mathbb{C})$ of the Weil group. Any representation ρ of the W_F gives rise to an L-factor as in [33]. Let ρ_i correspond to π_i in this manner. Let $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$. We define

$$L(s, \Pi) = L(s, \rho_1 \otimes \rho_2 \otimes \rho_3) \qquad \text{and}$$

$$L(s, \Pi, \mathrm{Ad}) = L(s, \oplus_i \mathrm{Ad}(\rho_i)) = \prod_{i=1}^{3} L(s, \mathrm{Ad}(\rho_i)) = \prod_{i=1}^{3} L(s, \pi_i, \mathrm{Ad})$$

where $\rho_1 \otimes \rho_2 \otimes \rho_3 : W_F \to \mathrm{GL}_8(\mathbb{C})$ is the standard tensor product representation and $\mathrm{Ad}(\rho_i) : W_F \to \mathrm{GL}_3(\mathbb{C})$ is the adjoint representation.

For the triple product, we are especially interested in the following three cases:

$$\Pi^1 = \pi(\mu_1, \mu_2) \otimes \pi(\nu_1, \nu_2) \otimes \sigma_\chi, \quad \Pi^2 = \pi(\mu_1, \mu_2) \otimes \sigma_\nu \otimes \sigma_\chi, \quad \Pi^3 = \sigma_\mu \otimes \sigma_\nu \otimes \sigma_\chi.$$

The triple product L-function in each of these cases is

$$\begin{aligned} L_v(s, \Pi^1) &= \prod_{\epsilon,\delta \in \{1,2\}} L_v(s + \tfrac{1}{2}, \mu_\epsilon \nu_\delta \chi) = \prod_{\epsilon,\delta \in \{1,2\}} \frac{1}{1 - \alpha_\epsilon \beta_\delta \gamma q^{-s-1/2}}, \\ L_v(s, \Pi^2) &= \prod_{\epsilon \in \{1,2\}} L_v(s, \mu_\epsilon \nu \chi) L_v(s+1, \mu^\epsilon \nu \chi) = \prod_{\epsilon \in \{1,2\}} \frac{1}{(1 - \alpha_\epsilon \beta \gamma q^{-s})(1 - \alpha^\epsilon \beta \gamma q^{-s-1})}, \\ L_v(s, \Pi^3) &= L_v(s + \tfrac{3}{2}, \mu\nu\chi) L_v(s + \tfrac{1}{2}, \mu\nu\chi)^2 = \frac{1}{(1 - \alpha\beta\gamma q^{-s-3/2})(1 - \alpha\beta\gamma q^{-s-1/2})^2}. \end{aligned}$$

The adjoint L-functions agree with that given by Gelbart-Jacquet [11]. For the cases at hand,

$$\begin{aligned} L_v(s, \pi(\chi_1, \chi_2), \mathrm{Ad}) =& \frac{L_v(s, 1)}{(1 - \gamma_1 \gamma_2^{-1} q^{-s})(1 - \gamma_1^{-1} \gamma_2 q^{-s})}, \\ L_v(s, \sigma_\chi, \mathrm{Ad}) =& L_v(s+1, 1) = \frac{1}{1 - q^{-s-1}}. \end{aligned}$$

If $\chi_1 = \chi_2^{-1}$, $L(s, \pi(\chi_1, \chi_2)$ is equal to the symmetric square L-function precisely because, in this case, π has trivial central character.

For the discrete series representations π_{dis}^k of weight k and the principal series representations $\pi_{\mathbb{R}}^t$ of $\mathrm{GL}_2(\mathbb{R})$ we have the following L-functions. First, let

$$\zeta_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2), \qquad \zeta_{\mathbb{C}}(s) = (2\pi)^{-s} \Gamma(s)$$

where $\Gamma(s)$ is the standard Γ-function. Then

$$\begin{aligned} L(s, \pi_{\mathrm{dis}}^{k_1} \otimes \pi_{\mathrm{dis}}^{k_2} \otimes \pi_{\mathrm{dis}}^{k_3}) &= \zeta_{\mathbb{C}}(s + 1/2) \zeta_{\mathbb{C}}(s + k_1 - 3/2) \zeta_{\mathbb{C}}(s + k_2 - 1/2) \zeta_{\mathbb{C}}(s + k_3 - 1/2), \\ L(s, \pi_{\mathrm{dis}}^k \otimes \pi_{\mathrm{dis}}^k \otimes \pi_{\mathbb{R}}^t) &= \zeta_{\mathbb{C}}(s + t + k - 1) \zeta_{\mathbb{C}}(s + t) \zeta_{\mathbb{C}}(s - t + k - 1) \zeta_{\mathbb{C}}(s - t), \\ L(s, \pi_{\mathbb{R}}^{t_1} \otimes \pi_{\mathbb{R}}^{t_2} \otimes \pi_{\mathbb{R}}^{t_3}) &= \prod_{\epsilon,\delta,\gamma \in \{\pm 1\}} \zeta_{\mathbb{R}}(s + \epsilon t_1 + \delta t_2 + \gamma t_3) \end{aligned}$$

$$L(s, \pi_{\mathrm{dis}}^k, \mathrm{Ad}) = \zeta_{\mathbb{C}}(s + k - 1) \zeta_{\mathbb{R}}(s + 1), \qquad L(s, \pi_{\mathbb{R}}^t, \mathrm{Ad}) = \zeta_{\mathbb{R}}(s + 2t) \zeta_{\mathbb{R}}(s) \zeta_{\mathbb{R}}(s - 2t).$$

4.2. **Explicit computations: F nonarchimedean.** Recall that we want to make Ichino's formula (2.5) explicit, i.e. calculate $I_v(\varphi)$ which is a product of L-factors times

$$I'_v(\varphi) = \int_{Z(F) \backslash G(F)} \langle \Pi(g)\varphi, \overline{\varphi} \rangle = \int_{Z(F) \backslash G(F)} \Phi_1(g) \Phi_2(g) \Phi_3(g) dg.$$

If all of the local data is unramified, Ichino calculated that the trilinear form $I(\varphi) = 1$—the normalization of (2.4) is chosen precisely to give this result. So in all of our computations at least one of the representations will be special.

We assume from now on that $\pi_3 = \sigma_\chi$, and we denote the normalized matrix coefficient associated to a new vector of σ_χ by Φ^3 as in Proposition 3.8. So our object is to calculate

$$I'_v(\varphi_1 \otimes \varphi_2 \otimes \varphi_3) = \int_{Z(F) \backslash G(F)} \Phi^1(g) \Phi^2(g) \Phi^3(g) dg$$

where $\Phi^i = \Phi_{\varphi_i, \overline{\varphi}_i}$ for various choices of φ_i.

From Tables 1 and 4 we read off that (for all $n \geq 0$)

$$\mathrm{vol}(K_0 g K_0) \Phi(g) = \pm \frac{\gamma^n}{q(1 + \frac{1}{q})}$$

for $g \in \{\sigma_n, w\sigma_n, \sigma_n w, w\sigma_n w\}$ with the minus sign in the cases $\sigma_n w$ or $w\sigma_n$ and the positive sign otherwise.

All of the matrix coefficients we are considering are K_0-biinvariant, so we combine this with Lemma 3.1 to obtain the formula

$$\begin{aligned}(4.1)\quad \int_{Z(F)\backslash G(F)} \Phi_1(g)\Phi_2(g)\Phi(g)dg &= \frac{\Phi_1(e)\Phi_2(e)-\Phi_1(w)\Phi_2(w)}{1+q}\\ &+\sum_{n=1}^{\infty}\frac{\gamma^n}{1+q}\Big(\Phi_1\Phi_2(\sigma_n)-\Phi_1\Phi_2(w\sigma_n)-\Phi_1\Phi_2(\sigma_n w)+\Phi_1\Phi_2(w\sigma_n w)\Big).\end{aligned}$$

In computing the forms, we will use (many times) the following identities. Let

$$A=\frac{1-\frac{\alpha^2}{q}}{1-\alpha^2}\qquad\text{and}\qquad B=\frac{1-\frac{\alpha^{-2}}{q}}{1-\alpha^{-2}}. \tag{4.2}$$

Then

$$A+B=\frac{1-\frac{\alpha^2}{q}}{1-\alpha^2}+\frac{1-\frac{\alpha^{-2}}{q}}{1-\alpha^{-2}}=\frac{-\alpha^{-1}+\frac{\alpha}{q}+\alpha-\frac{\alpha^{-1}}{q}}{\alpha-\alpha^{-1}}=1+\tfrac{1}{q}, \tag{4.3}$$

$$A\alpha+B\alpha^{-1}=\frac{1-\frac{\alpha^2}{q}}{\alpha^{-1}-\alpha}+\frac{1-\frac{\alpha^{-2}}{q}}{\alpha-\alpha^{-1}}=\frac{-1+\frac{\alpha^2}{q}+1-\frac{\alpha^{-2}}{q}}{\alpha-\alpha^{-1}}=\tfrac{1}{q}(\alpha+\alpha^{-1}), \tag{4.4}$$

$$A\alpha^{-1}+B\alpha=\frac{\alpha^{-1}-\frac{\alpha}{q}}{1-\alpha^2}+\frac{\alpha-\frac{\alpha^{-1}}{q}}{1-\alpha^{-2}}=\frac{-\alpha^{-2}+\frac{1}{q}+\alpha^2-\frac{1}{q}}{\alpha-\alpha^{-1}}=\alpha+\alpha^{-1} \tag{4.5}$$

$$A\alpha^2+B\alpha^{-2}=(A\alpha+B\alpha^{-1})(\alpha+\alpha^{-1})-(A+B)=\tfrac{1}{q}(\alpha^2+\alpha^{-2})+\tfrac{1}{q}-1 \tag{4.6}$$

$$A\alpha^{-2}+B\alpha^{2}=(A\alpha^{-1}+B\alpha)(\alpha+\alpha^{-1})-(A+B)=\alpha^2+\alpha^{-2}+1-\tfrac{1}{q}. \tag{4.7}$$

4.2.1. *Only one of the representations is special.* In this section we treat the case when two of the representations are unramified, and only one is special. For the application to subconvexity this is the case of most interest since it corresponds to the place $\mathfrak{p}$ in Corollary 1.5. Let $\pi_1=\pi(\mu,\mu^{-1})$ and $\pi_2=\pi(\nu,\nu^{-1})$ and $\pi=\sigma_\chi$. Set $\alpha=\mu(\varpi)$, $\beta=\nu(\varpi)$ and $\gamma=\chi(\varpi)$. Finally, let $\{\phi_0^i,\phi_1^i\}$ be the basis of eigenvectors of π_i as in Proposition 3.5.

Remark. Note that we have fixed π_1 and π_2 to have trivial central character. This has the effect of making the formulas somewhat more manageable. However, it is true that the results below extend to the more general situation in which only the product of central characters has trivial central character, i.e. $\pi_1=\pi(\chi_1,\chi_2)$, $\pi_2=\pi(\nu_1,\nu_2)$ and $\chi_1\chi_2\nu_1\nu_2=1$.

We combine the results of the previous sections to prove the following.

Proposition 4.1. *Let Φ^3 be the normalized matrix coefficient corresponding to a new vector $\phi^3\in\sigma_\chi$ as in Proposition 3.8, and define $\Psi^i_{j,k}=\Phi_{\phi^i_j,\overline{\phi}^i_k}$ for $i=1,2$ as in Proposition 3.5. Then*

$$\int_{Z\backslash G}\Psi^1_{0,0}\Psi^2_{j,k}\Phi^3(g)dg=L(\tfrac12,\Pi^1)\cdot\tfrac{1}{q^2}(1-\tfrac1q)\cdot C \tag{4.8}$$

where

$$C=\begin{cases}1+\frac{1}{q^2}-\frac{\alpha^2+\alpha^{-2}}{q} & \text{if } j=k=1,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. To show directly that $C=0$ when j and k are not both 1 is not hard using (4.1). However, we give a more conceptual proof. Suppose that $\ell:\pi_1\times\pi_2\times\sigma_\chi\to\mathbb{C}$ is any G-invariant linear form. Note that if $\phi\in\sigma_\chi$ then Lemma 3.7 implies $\int_K\pi(k)\phi dk=0$.

We claim that $\ell(\phi_0^1,\phi_0^2,\phi)=0$ for all $\phi\in\sigma_\chi$. Since ϕ_0^1 and ϕ_0^2 are K-invariant (and ℓ is G-invariant),

$$\ell(\phi_0^1,\phi_0^2,\pi(k)\phi)=\ell(\pi(k)^{-1}\phi_0^1,\pi(k)^{-1}\phi_0^2,\phi)=\ell(\phi_0^1,\phi_0^2,\phi)$$

for all $k\in K$. Integrating over K, we find that

$$\ell(\phi_0^1,\phi_0^2,\phi)=\ell(\phi_0^1,\phi_0^2,\int_K\pi(k)\phi dk)=\ell(\phi_0^1,\phi_0^2,0)=0.$$

Since the matrix coefficients are *bi*linear forms, fixing the vectors in one of the coordinates gives a linear form in the other. For example, if we fix $\widetilde{\phi}^1 \otimes \widetilde{\phi}^2 \otimes \widetilde{\phi} \in \widetilde{\Pi}$ then

$$\ell : \pi_1 \times \pi_2 \times \pi \to \mathbb{C} \qquad \ell(\phi^1, \phi^2, \phi) = \int_{Z\backslash G} \Phi_{\phi^1, \widetilde{\phi}^1} \Phi_{\phi^2, \widetilde{\phi}^2} \Phi_{\phi, \widetilde{\phi}}(g) dg$$

is a linear form on Π. From this point of view, it is immediate that in each of the cases of Proposition 4.1 that are claimed to be zero, as a form on either Π or $\widetilde{\Pi}$ (or both) the claim applies.

To complete the proof of Proposition 4.1, we need to do the calculation of (4.8). Let $A_i, -B_i$ be the eigenvalues of the intertwining operator M on π_i. By (4.1), the integral (4.8) is equal to $\frac{1}{q}$ times

$$\frac{1}{q} + (1+\tfrac{1}{q})^{-1} \sum_{n=1}^{\infty} \left[\begin{array}{l} \left(\frac{\alpha\beta\gamma}{q}\right)^n B_1 A_2 + \left(\frac{\alpha^{-1}\beta^{-1}\gamma}{q}\right)^n A_1 B_2 \\ \qquad + \left(\frac{\alpha^{-1}\beta\gamma}{q}\right)^n A_1 A_2 + \left(\frac{\alpha\beta^{-1}\gamma}{q}\right)^n B_1 B_2 \end{array} \right]$$

$$= \tfrac{1}{q} + (1+\tfrac{1}{q})^{-1} \left[\frac{B_1 A_2 \frac{\alpha\beta\gamma}{q}}{1 - \frac{\alpha\beta\gamma}{q}} + \frac{A_1 B_2 \frac{\alpha^{-1}\beta^{-1}\gamma}{q}}{1 - \frac{\alpha^{-1}\beta^{-1}\gamma}{q}} + \frac{A_1 A_2 \frac{\alpha^{-1}\beta\gamma}{q}}{1 - \frac{\alpha^{-1}\beta\gamma}{q}} + \frac{B_1 B_2 \frac{\alpha\beta^{-1}\gamma}{q}}{1 - \frac{\alpha\beta^{-1}\gamma}{q}} \right]$$

Simplifying the expression inside the parentheses yields $L(\frac{1}{2}, \Pi)$ times

$$\begin{aligned} &\frac{\gamma}{q}(A_1\alpha^{-1} + B_1\alpha)(A_2\beta + B_2\beta^{-1}) - \frac{\gamma^4}{q^4}(A_1 + B_1)(A_2 + B_2) \\ &\qquad + \frac{\gamma^3}{q^3}[(A_2 + B_2)(A_1\alpha^{-1} + B_1\alpha)(\beta + \beta^{-1}) + (A_1\alpha + B_1\alpha^{-1})(A_2\beta + B_2\beta^{-1})] \\ &\qquad - \frac{\gamma^2}{q^2}\left[(A_1 + B_1)(A_2 + B_2) + (A_1 + B_1)(A_2\beta^2 + B_2\beta^{-2}) + (A_2 + B_2)(A_1\alpha^{-2} + B_1\alpha^2)\right] \end{aligned}$$

Applying formulas (4.3)-(4.7), this becomes $\frac{1}{q}(1 + \frac{1}{q})L(\frac{1}{2}, \Pi)$ times

$$-\tfrac{1}{q}(1 + \tfrac{1}{q})(1 + \tfrac{1}{q^2}) + \gamma(\alpha + \alpha^{-1})(\beta + \beta^{-1})(\tfrac{1}{q} + \tfrac{1}{q^3}) - \tfrac{1}{q}(\alpha^2 + \alpha^{-2} + \tfrac{\beta^2 + \beta^{-2}}{q}) \tag{4.9}$$

So, combining the above, we find that (4.8) is equal to $\frac{1}{q^2}L(\frac{1}{2}, \Pi)$ times

$$L(\tfrac{1}{2}, \Pi)^{-1} + (4.9).$$

Since $L(\frac{1}{2}, \Pi)^{-1}$ is equal to

$$(1 + \tfrac{1}{q^2})^2 - \tfrac{1}{q}(1 + \tfrac{1}{q^2})\gamma(\alpha + \alpha^{-1})(\beta + \beta^{-1}) + \tfrac{1}{q^2}(\alpha^2 + \alpha^{-2} + \beta^2 + \beta^{-2}), \tag{4.10}$$

the value of C in the statement of the proposition follows. $\square$

Corollary 4.2. *Let the notation be as in Proposition 4.1, and set* $\phi = \phi_0^1 \otimes \phi_2^2 \otimes \phi^3$ *where* $\phi_2^2 = \begin{pmatrix} \varpi^{-1} & \\ & 1 \end{pmatrix} \phi_0^2$ *as in Lemma 3.6. Then*

$$\frac{I_v(\phi)}{\langle \phi, \overline{\phi} \rangle} = \frac{1}{q}(1 + \tfrac{1}{q})^{-1} \tag{4.11}$$

In particular, $\frac{I_v(\phi)}{\langle \phi, \overline{\phi} \rangle} \gg \frac{1}{q}$ *where the implicit constant can be taken to be independent of* Π *and* q.

Proof. Apply Lemma 3.6 to write

$$\phi_2^2 = a\phi_0^2 + b\phi_1^2 \qquad \text{where} \qquad a = \frac{\beta q^{-1/2} + \beta^{-1}q^{-1/2}}{1 + \frac{1}{q}}, \quad b = \frac{\beta^{-1}q^{1/2} - \beta q^{-1/2}}{1 + \frac{1}{q}}.$$

First, assume that $\overline{\beta} = \beta^{-1}$. By the bilinearity of matrix coefficients,

$$\Psi_{2,2} = \Phi_{a\phi_0 + b\phi_1, \overline{a\phi_0 + b\phi_1}} = |a|^2 \Psi_{1,1} + a\overline{b}\Psi_{0,1} + \overline{a}b\Psi_{1,0} + |b|^2 \Psi_{1,1}. \tag{4.12}$$

Recall that $I_v = \zeta_{F_v}(2)^{-2}\frac{L_v(1,\Pi^1,\mathrm{Ad})}{L_v(\frac{1}{2},\Pi^1)}I'_v$. Therefore, (4.8) implies that

$$\begin{aligned}\frac{I_v(\phi)}{\langle\phi,\widetilde{\phi}\rangle} =&\zeta_{F_v}(2)^{-2}\frac{L_v(1,\Pi^1,\mathrm{Ad})}{L_v(\frac{1}{2},\Pi^1)}\int_{Z\backslash G}\Psi^1_{0,0}\Psi^2_{2,2}\Phi^3(g)dg\\ =&\zeta_{F_v}(2)^{-2}\frac{L_v(1,\Pi^1,\mathrm{Ad})}{L_v(\frac{1}{2},\Pi^1)}|b|^2\int_{Z\backslash G}\Psi^1_{0,0}\Psi^2_{1,1}\Phi^3(g)dg = \tfrac{1}{q^2}(1-\tfrac{1}{q})^{-1}.\end{aligned}$$

In the final step we have used that

$$\zeta_{F_v}(2)^{-2}L_v(1,\Pi^2,\mathrm{Ad}) = \frac{1+\frac{1}{q}}{(1-\frac{1}{q})}(1+\tfrac{1}{q^2}-\tfrac{\alpha^2+\alpha^{-2}}{q})^{-1}(1+\tfrac{1}{q^2}-\tfrac{\beta^2+\beta^{-2}}{q})^{-1}.$$

Although the result is the same if π_2 is a complementary series (i.e. $\beta = q^\lambda$ for some $\lambda\in(0,\frac{1}{2})$) the method is slightly different because the inner product is not the same. If $\phi,\phi'\in\pi_2$, then

$$\langle\phi,\phi'\rangle = \int_K \phi(k)(M\phi')(k)dk.$$

(Compare with (3.2).) Thus, since $\overline{\phi}_2=\phi_2$,

$$\begin{aligned}\Psi_{2,2} =&\langle\pi_2(\cdot)\phi_2,\phi_2\rangle = \int_K\phi_2(k\cdot)(M\phi_2)(k)dk\\ =&a^2A_2\Psi_{0,0}+abA_2\Psi_{1,0}-abB_2\Psi_{0,1}-b^2B_2\Psi_{1,1}.\end{aligned}$$

The remainder of proof goes through as above, except one replaces $|b|^2$ with $-b^2B_2/A_2$. (We must divide by A_2 because $\langle\phi_2,\overline{\phi}_2\rangle = \Psi_{2,2}(e) = A_2$.) As it turns out, this is exactly the same expression in terms of β as was $|b|^2$. □

4.2.2. *Two of the representations are special.* We now consider the case $\Pi^2 = \pi(\mu,\mu)\otimes\sigma_\nu\otimes\sigma_\chi$. The following result will be used in Theorem 5.1.

Proposition 4.3. *Let Π^2 be as above, and denote by Φ^2 and Φ^3 the normalized matrix coefficients corresponding to the new vectors $\phi^2\in\sigma_\nu$ and $\phi^3\in\sigma_\chi$ as in Proposition 3.8. Let $\Psi^1_{0,0}=\Phi_{\phi^1_0,\overline{\phi}^1_0}$ be the matrix coefficient corresponding to $\phi^1_0\in\pi(\mu,\mu^{-1})$ as in Proposition 3.5. Then*

$$\int_{Z\backslash G}\Psi^1_{i,j}\Phi^2\Phi^3(g)dg = \tfrac{1}{q}(1-\tfrac{1}{q})L(\tfrac{1}{2},\Pi^2)(1-\tfrac{\alpha^2}{q})(1-\tfrac{\alpha^{-2}}{q}). \tag{4.13}$$

Therefore, if we set $\phi=\phi^1_0\otimes\phi^2\otimes\phi^3$ then $\frac{I_v(\phi)}{\langle\phi,\overline{\phi}\rangle}=\frac{1}{q}$.

Proof. The evaluation of (4.13) is obtained by a calculation analogous to the proof of (4.8). Since the method is identical, we leave the details to the reader.

Note that

$$\zeta_{F_v}(2)^{-2}L_v(1,\Pi^2,\mathrm{Ad}) = (1-\tfrac{\alpha^2}{q})^{-1}(1-\tfrac{\alpha^{-2}}{q})^{-1}(1-\tfrac{1}{q})^{-1}.$$

Thus, given (4.13), the conclusion that $\frac{I_v(\phi)}{\langle\phi,\overline{\phi}\rangle}=\frac{1}{q}$ is immediate. □

4.2.3. *All three representations are special.* This is the case that corresponds to the primes dividing $\mathfrak{N}$ in Theorem 1.4. We remark that in this case and that of Section 4.3, the results are already known by [18]. However, the proofs given here differs from that of [18], and we include it to highlight the analogy with the calculations of the previous sections.

We denote

$$\varepsilon = \varepsilon(\tfrac{1}{2},\sigma_\chi\otimes\sigma_\mu\otimes\sigma_\nu) = -(\chi\mu\nu)(\varpi) = -\alpha\beta\gamma.$$

Proposition 4.4. *Let $\pi_1=\sigma_\mu$ and $\pi_2=\sigma_\nu$. Denote the matrix coefficient associated to the new vector $\phi^i\in\pi_i$ as in Proposition 3.8 by Φ^i and set $\phi=\phi^1\otimes\phi^2\otimes\phi^3$. Then*

$$\int_{Z\backslash G}\Phi^1\Phi^2\Phi^3(g)dg = (1+\varepsilon)L_v(\tfrac{1}{2},\Pi^3)\tfrac{1}{q}(1-\tfrac{1}{q})(1+\tfrac{\varepsilon}{q})^2, \tag{4.14}$$

and $\frac{I_v(\phi)}{\langle\phi,\widetilde{\phi}\rangle}=\frac{1-\varepsilon}{q}(1+\frac{1}{q})$.

Proof. We use Proposition 3.8 and apply (4.1):

$$\frac{1-\frac{1}{q^2}}{1+q}+\frac{(1-q^2)(1-\frac{1}{q^2})}{1+q}\sum_{n=1}^{\infty}\left(-\frac{\varepsilon}{q^2}\right)^n=\frac{1-\frac{1}{q^2}}{1+q}\left(1-\frac{(1-q^2)(\frac{\varepsilon}{q^2})}{1+\frac{\varepsilon}{q^2}}\right)$$

Equation (4.14) is immediate. The value of $I_v(\phi)$ now follows by multiplying by the appropriate normalizing factors as given in Section 4.1. $\square$

Proposition 4.5. *Let $\Pi=\sigma_\mu\otimes\sigma_\nu\otimes\sigma_\chi$ and Π^B the admissible representation of B associated to Π via Jacquet-Langlands where B is the unique quaternion division algebra over F. Let $\varepsilon=-(\mu\nu\chi)(\varpi)$. If $\phi\in\Pi^B$ and $\widetilde{\phi}\in\widetilde{\Pi}^B$ then*

$$\frac{I_v(\phi)}{\langle\phi,\overline{\phi}\rangle}=(1-\varepsilon)\tfrac{1}{q}(1-\tfrac{1}{q}). \tag{4.15}$$

Proof. The Jacquet-Langlands lift of σ_χ is the character $\chi_B: B^\times\to\mathbb{C}^\times$ given by $\chi_B(\beta)=\chi\circ N_B(\beta)$ where N_B is the reduced norm. Hence $\Pi^B\simeq\eta_B$ where $\eta_B=\mu_B\nu_B\chi_B$, and

$$\begin{aligned}\int_{F^\times\backslash B^\times}\Phi_{\phi,\overline{\phi}}(\beta)d\beta&=\int_{F^\times\backslash B^\times}\langle\Pi^B(\beta)\phi,\overline{\phi}\rangle d\beta\\&=\int_{F^\times\backslash B^\times}\eta_B(\beta)\langle\phi,\overline{\phi}\rangle d\beta\\&=\langle\phi,\overline{\phi}\rangle\begin{cases}\mathrm{vol}(F^\times\backslash B^\times)&\text{if }\eta_B\text{ is trivial,}\\0&\text{otherwise.}\end{cases}\end{aligned}$$

Recall (see Section 2.2) that $\mathrm{vol}(F^\times\backslash B^\times)=\frac{2}{q-1}$. To obtain I_v, one multiplies this by the factor $\zeta_{F_v}(2)^{-2}\frac{L_v(1,\Pi_v,\mathrm{Ad})}{L_v(\frac{1}{2},\Pi_v)}$. Using the values given in Section 4.1 and simplifying, the result follows. $\square$

Our factor ε is indeed the local factor $\varepsilon(\frac{1}{2},\Pi)$. So Propositions 4.4 and 4.5 provide an explicit realization of Prasad's result. More precisely, the integrated matrix coefficient provides a trilinear form on Π (respectively Π^B) that is invariant by the diagonal action of G (resp. $B^\times$.) It is nonzero on $B^\times$ if and only if $\varepsilon=-1$. On G, $\phi=\phi_1^1\otimes\phi_1^2\otimes\phi_1^3$ is a test vector if and only if $\varepsilon=+1$.

4.3. **Explicit computations:** $F=\mathbb{R}$. Although the proof given here differs from that of Ichino and Ikeda, the result is Proposition 7.2 of [18].

Let $K=\{\kappa_\theta=\left(\begin{smallmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{smallmatrix}\right)\mid\theta\in[0,2\pi)\}$, $N=\{n(x)=\left(\begin{smallmatrix}1&x\\&1\end{smallmatrix}\right)\mid x\in\mathbb{R}\}$ and $A=\{a(y)=\left(\begin{smallmatrix}y&\\&1\end{smallmatrix}\right)\mid |y|\geq 1\}$. Then $\mathrm{PGL}_2(\mathbb{R})$ can be given by the coordinates KAK. Recall (2.8) gives the Haar measure in this case.

It is well known (see [23, Prop. 14.1]) that for the weight k discrete series, which we denote by π^k_{dis}, the matrix coefficient corresponding to a new vector ϕ is given by

$$\Phi(g)=\frac{(2\sqrt{\det g})^k}{(a+d+i(b-c))^k}\qquad g=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right),\det g>0 \tag{4.16}$$

and $\Phi(g)=0$ if $\det g<0$. In the KAK coordinates (and $y\geq 1$) this is

$$\Phi(\kappa_{\theta_1}a(y)\kappa_{\theta_2})=2^k e^{2\pi ik(\theta_2-\theta_1)}\frac{y^{k/2}}{(y+1)^k}. \tag{4.17}$$

As a matter of notation, let $\phi^-=\pi(\left(\begin{smallmatrix}-1&\\&1\end{smallmatrix}\right))\phi$. In particular, if $\phi\in\pi^k_{\mathrm{dis}}$ has weight m then ϕ^- has weight $-m$. Moreover, if ϕ is a new vector then the matrix coefficient associated to ϕ^- is $\overline{\Phi}$.

Proposition 4.6. *For $i=1,2,3$, let $\pi_i=\pi^{k_i}_{\mathrm{dis}}$ be such that $k=k_1=k_2+k_3$, and let $\phi_i\in\pi_i$ be a new vector. Set $\phi=\phi_1\otimes\phi_2^-\otimes\phi_3^-\in\Pi_\infty=\pi^{k_1}_{\mathrm{dis}}\otimes\pi^{k_2}_{\mathrm{dis}}\otimes\pi^{k_3}_{\mathrm{dis}}$. Then*

$$\int_{\mathrm{PGL}_2(\mathbb{R})}\Phi_{\phi,\overline{\phi}}(g)dg=\frac{2\pi}{k-1},$$

and $\frac{I_v(\phi)}{\langle\phi,\overline{\phi}\rangle}=2.$

Proof. Using the description above and the KAK coordinates, we have

$$\begin{aligned}\int_{\mathrm{PGL}_2(\mathbb{R})}\Phi_{\phi,\bar\phi}(g)dg =&\frac{2^{k_1+k_2+k_3-2}}{\pi}\int_0^{2\pi}\int_1^{\infty}\int_0^{2\pi}\frac{y^{(k_1+k_2+k_3)/2}}{(1+y)^{k_1+k_2+k_3}}(1-y^{-2})d\theta_1 dy d\theta_2\\ =&2^{2k}\pi\int_1^{\infty}\frac{y^k-y^{k-2}}{(1+y)^{2k}}dy\\ =&2^{2k}\pi\left[-\frac{y^{k-1}}{(k-1)(1+y)^{2k-2}}\right]_1^{\infty}=2\pi/(k-1).\end{aligned}$$

Since $\langle\phi,\phi\rangle=\Phi(e)=1$, to get $I_v(\phi)$ one multiplies this result by $\zeta_{\mathbb{R}}(2)^{-2}\frac{L_v(1,\Pi_\infty,\mathrm{Ad})}{L_v(\frac12,\Pi_\infty)}$ which is easily seen to be equal to $(k-1)/\pi$. Hence, $\frac{I_v(\phi)}{\langle\phi,\phi\rangle}=2$. □

5. Generalizing Watson's formula

We now work globally. Let F be a totally real number field, and π_1,π_2,π_3 cuspidal automorphic representations of $\mathrm{PGL}_2(F)$. It is clear from the description in Section 4.3 that if $\varphi=\varphi_1\otimes\varphi_2\otimes\varphi_3$ for some $\varphi_i=\bigotimes_v\varphi_{i,v}\in\pi_i$ then $I(\varphi)=0$ unless at each infinite place the weights $k_{i,v}$ of φ_i sum to zero.

We say that $\Pi=\pi_1\otimes\pi_2\otimes\pi_3$ is *almost balanced* if at each real place v, the largest weight is the sum of the two smaller weights.

Theorem 5.1. *Let π_1,π_2,π_3 cuspidal automorphic representations of PGL_2 over a totally real number field F such that $\Pi=\pi_1\otimes\pi_2\otimes\pi_3$ is almost balanced. Assume that the conductor $\mathfrak{N}_i$ of π_i is squarefree for each $i=1,2,3$. Let $\mathfrak{N}=\gcd(\mathfrak{N}_1,\mathfrak{N}_2,\mathfrak{N}_3)$. For $\{j,k,l\}=\{1,2,3\}$, write*

$$\mathfrak{N}_j=\mathfrak{N}\mathfrak{n}_j\mathfrak{n}_{jk}\mathfrak{n}_{jl}\qquad \textit{where}\quad \mathfrak{n}_{jk}=\mathfrak{n}_{kj}=\gcd(\mathfrak{N}_j,\mathfrak{N}_k)/\mathfrak{N}.$$

Let $\mathfrak{M}$ be the product of all primes dividing $\mathfrak{N}_1\mathfrak{N}_2\mathfrak{N}_3$. Let B be the global quaternion algebra such that $\dim\mathrm{Hom}_B(\Pi^B,\mathbb{C})=1$. (So the discriminant of B divides $\mathfrak{N}$.)

Define the vector $\varphi=\varphi_1\otimes\varphi_2\otimes\varphi_3\in\Pi^B$ as follows. For $v\mid\infty$, let k_v be the maximum weight, and let $\varphi_{i,v}$ be twists of the normalized new vectors such that their weights add to zero as in Proposition 4.6. At a finite prime $\mathfrak{p}\mid\mathfrak{n}_j$, let $\varphi_{i,\mathfrak{p}}$ be the normalized new vector for $i=j,k$, and the twists of the normalized new vector by $\left(\begin{smallmatrix}\varpi^{-1}&\\&1\end{smallmatrix}\right)$ as in Lemma 3.6 (where it is denoted as ϕ_2.) In all other cases we take $\varphi_{i,v}$ to be the normalized new vector. Let $\varphi_i=\otimes_v\varphi_{i,v}$.

Under this choice of φ and for db the Tamagawa measure,

$$\frac{\left|\int_{[B^\times]}\varphi(b)db\right|^2}{\prod_{i=1}^3\int_{[B^\times]}|\varphi_i(b)|^2\,db}=\frac{\zeta_F(2)}{2^3\,|\Delta_F|^{3/2}}\frac{L(\frac12,\Pi)}{L(1,\Pi,\mathrm{Ad})}\prod_{v\mid\infty\mathfrak{M}}C_v \tag{5.1}$$

where, for finite places,

$$C_{\mathfrak{p}}=\begin{cases}\frac{2}{N(\mathfrak{p})}(1+\frac{\varepsilon_{\mathfrak{p}}}{N(\mathfrak{p})}) & \textit{if } \mathfrak{p}\mid\mathfrak{N}\\ \frac{1}{N(\mathfrak{p})}(1+\frac{1}{N(\mathfrak{p})})^{-1} & \textit{if } \mathfrak{p}\mid\mathfrak{n}_j\\ \frac{1}{N(\mathfrak{p})} & \textit{if } \mathfrak{p}\mid\mathfrak{n}_{jk}\end{cases}$$

and for infinite places, $C_v=2$ if all three representations $\pi_{i,v}$ are discrete series and $C_v=1$ otherwise. Here $\varepsilon_{\mathfrak{p}}=\varepsilon_{\mathfrak{p}}(\frac12,\Pi_{\mathfrak{p}})$.

Remark. It is not strictly necessary to choose the local components of φ to be *normalized*. Indeed, if we replace φ_v by any nonzero constant multiple, since (5.1) is self-normalizing, the same formula still holds.

Note that Theorem 1.4 follows from Theorem 5.1 in the case that $\mathfrak{N}_1=\mathfrak{N}_2=\mathfrak{N}_3=\mathfrak{N}$. Moreover, assuming that $F=\mathbb{Q}$ this is (the adelic version of) Watson's result [36, Theorem 3]. To verify this, note that the left hand side of (1.4) (or (5.1)) is essentially the adelic version of the left hand side of Watson's formula. Indeed, the only difference is that the volume of the fundamental domain in Watson's case is

$$2\zeta_{\mathbb{Q}}(2)\prod_{p\in\Sigma_B}(1-p)\prod_{p\mid N,p\notin\Sigma_B}(1+p)$$

whereas here, since we have chosen the Tamagawa measure, the volume is 2. Adjusting the formulas accordingly, the right hand sides agree.

5.1. **Setup and proof of Ichino's theorem.** Before giving the proof of Theorem 5.1 we recall the notation and proof of Theorem 2.1. Let

$$\mathrm{GSp}_{2n} = \{g \in \mathrm{GL}_{2n} \mid gJ^tg = \nu(g)J, \nu(g) \in \mathbb{G}_m\}$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

and I_n is the $n \times n$ identity matrix. Let V be the quadratic space associated to the quaternion algebra B. Let ι denote the main involution of B, and define the symmetric bilinear form $(x, y) = \mathrm{tr}_{B/F}(xy^\iota)$. Let

$$\mathrm{GO}(V) = \{g \in \mathrm{GL}(V) \mid (gx, gy) = \nu(g)(x, y), \nu(g) \in \mathbb{G}_m\}.$$

The subgroups Sp_{2n} and $O(V)$ are those for which $\nu(g) = 1$.

Let $\psi = \otimes\psi_v$ be a nontrivial additive character of $\mathbb{A}/F$. The Weil representation ω on $\mathrm{Sp}_{2n}(\mathbb{A}) \times O(V)(\mathbb{A})$ can be extended to the adelic points of

$$G(\mathrm{Sp}_{2n} \times O(V)) = \{(g, h) \in \mathrm{GSp}_{2n} \times \mathrm{GO}(V) \mid \nu(g) = \nu(h)\}.$$

5.1.1. *Shimizu's theta lifting.* For a certain space of Schwartz functions $S(V(\mathbb{A})^n)$ one defines the associated theta function

$$\theta(g, h, \phi) = \sum_{x \in V(F)^n} \omega(g, h)\phi(x).$$

When $n = 1$, π is an irreducible unitary cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$, and π^B is the representation of $B^\times(\mathbb{A})$ given by the Jacquet-Langlands correspondence, this theta function can be employed to give a realization of the Jacquet-Langlands transfer.

$$\theta : \pi \otimes S(V(\mathbb{A})) \to \pi^B \otimes \widetilde{\pi}^B$$

given by

$$\theta(h; f, \phi) = \int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})} f(gg')\theta(gg', h; \phi)dg$$

Here, $dg = \prod dg_v$ is the Tamagawa measure on $\mathrm{SL}_2(\mathbb{A})$, and $h \in \mathrm{GO}(V)(\mathbb{A})$ and $g' \in \mathrm{GL}_2(\mathbb{A})$ satisfies $\nu(g') = \nu(h)$. This map is well-defined, equivariant and surjective. Moreover, there exist corresponding local maps

$$\theta_v : \pi_v \otimes S(V(F_v)) \to \pi_v^B \otimes \widetilde{\pi}_v^B \qquad \text{such that} \qquad \theta = \otimes\theta_v.$$

We remark that $\theta(f, \phi)$ is an automorphic form on $\mathrm{GO}(V)$, but it can be regarded as an automorphic form on $B^\times(\mathbb{A}) \times B^\times(\mathbb{A})$ via the map

$$B^\times \times B^\times \to \mathrm{GO}(V) \qquad (b_1, b_2) \cdot x = b_1 x b_2^{-1}.$$

For $f = \otimes f_v \in \pi$, we define the Whittaker function of f with respect to $\bar{\psi}$ to be

$$W_f^\psi(g) = \int_{F\backslash\mathbb{A}} f(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) g)\psi(x)dx \tag{5.2}$$

where dx is the Tamagawa measure on $\mathbb{A}$. Using this we define

$$\widetilde{\mathcal{B}}_v(f_v, \phi_v) = \int_{N(F_v)\backslash\mathrm{SL}_2(F_v)} \omega(g_v, 1)\phi_v(1)W_{f_v}^{\psi_v}(g_v)dg_v. \tag{5.3}$$

5.1.2. *Local/global forms on $\pi^B \otimes \widetilde{\pi}^B$.* We continue to consider the case $n=1$. It is a result of Waldspurger (see [35]) that if $\theta_v(f_v, \phi_v) = \varphi_v \otimes \widetilde{\varphi}_v$, defining

$$\mathcal{B}_v^\natural(\varphi \otimes \widetilde{\varphi}_v) = \zeta_v(2) L(1, \pi_v, \mathrm{Ad})^{-1} \widetilde{\mathcal{B}}_v(f_v, \phi_v)$$

gives a B_v-invariant pairing on $\pi_v^B \otimes \widetilde{\pi}_v^B$. Moreover, for a fixed element of $\pi^B \otimes \widetilde{\pi}^B$, $\mathcal{B}_v^\natural(\varphi_v \otimes \widetilde{\varphi}_v) = 1$ for all but finitely many places v.

Waldspurger also proved that

$$\mathcal{B}(\varphi \otimes \widetilde{\varphi}) = \int_{[B^\times]} \varphi(b)\widetilde{\varphi}(b) db = 2\zeta_F(2)^{-1} L(1, \pi, \mathrm{Ad}) \prod_v \mathcal{B}_v^\natural(\varphi_v \otimes \widetilde{\varphi}_v). \tag{5.4}$$

5.1.3. *Zeta integrals of Garret and Piatetski-Shapiro–Rallis.* Now let $n=3$. This allows us to simultaneously consider $\theta : \pi_i \otimes S(V(\mathbb{A})) \to \pi_i^B \otimes \widetilde{\pi}_i^B$ for $i = 1,2,3$. From this point onward, we may assume that $f = f_1 \otimes f_2 \otimes f_3 \in \Pi$ and $\phi_v = \phi_{1,v} \otimes \phi_{2,v} \otimes \phi_{3,v} \in S(V(F_v)^3)$ such that $\theta_v(f_{i,v} \otimes \phi_{i,v}) = \varphi_{i,v} \otimes \widetilde{\varphi}_{i,v}$.

Let

$$H = \{(g_1, g_2, g_3) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2 \mid \nu(g_1) = \nu(g_2) = \nu(g_3)\}.$$

This can be considered as a subgroup of GSp_6. Let $dh = \prod dh_v$ be the Tamagawa measure on $\mathbb{A}^\times \backslash H(\mathbb{A})$. Writing $h = (g_1, g_2, g_3)$, the Whittaker function of f with respect to $\bar{\psi}$ can be defined by

$$W_f^\psi(h) = W_{f_1}^\psi(g_1) W_{f_2}^\psi(g_2) W_{f_3}^\psi(g_3)$$

where W_{f_i} is defined via (5.2). Let $N_0 \subset H$ be the group

$$N_0 = \{(n(x_1), n(x_2), n(x_3)) \in H \mid x_1 + x_2 + x_3 = 0\}.$$

Let $P \subset \mathrm{GSp}_6$ be the Siegel parabolic. Explicitly,

$$P = \left\{ \begin{pmatrix} A & * \\ 0 & \nu A^{-1} \end{pmatrix} \middle| A \in \mathrm{GL}_3, \nu \in \mathbb{G}_m \right\}.$$

We define a character $\chi_v : P(F_v) \to \mathbb{C}^\times$,

$$\chi_v\left(\begin{pmatrix} A & * \\ 0 & \nu A^{-1} \end{pmatrix} \right) = |\det(A)|_v^2 \, |\nu|_v^{-3}.$$

Now let $I_v(s) = \mathrm{Ind}_{P(F_v)}^{\mathrm{GSp}_6(F_v)}(\chi_v^s)$.

For $\phi_v \in S(V(F_v)^3)$ as above, let

$$F_{\phi_v}(g, 0) = |\nu(g)|_v^{-3} \, \omega(\left(\begin{smallmatrix} I_n & 0 \\ 0 & \nu(g)^{-1} I_n \end{smallmatrix} \right) g, 1)\phi_v(0).$$

This is an element of $I_v(0)$ which can be extended to $F_{\phi_v}(g, s)$, a standard section of $I_v(s)$. The local zeta integral of Garret[10] and Piatetski-Shapiro and Rallis[28], is

$$Z_v(s, W_{f_v}^{\psi_v}, F_{\phi_v}) = \int_{F_v^\times N_0(F_v) \backslash H(F_v)} F_{\phi_v}(\eta h_v, s) W_{f_v}^{\psi_v}(h_v) dh_v \tag{5.5}$$

where η is a representative of the open orbit of H in $P \backslash \mathrm{GSp}_6$.

Harris and Kudla, in [16], give the integral representation of $L(\frac{1}{2}, \Pi)$

$$\int_{[B^\times]} \varphi(b) db \int_{[B^\times]} \widetilde{\varphi}(b) db = I(\theta(f, \phi)) = \zeta_F(2)^{-2} L(\frac{1}{2}, \Pi) \prod_v \underbrace{\zeta_{F_v}(2)^2 L_v(1/2, \Pi_v)^{-1} Z_v(0, W_{\phi_v}^{\psi_v}, F_{\phi_v})}_{Z_v^\natural(0, W_{f_v}^{\psi_v}, F_{\phi_v})}. \tag{5.6}$$

The sum is taken over all places; indeed, there is a finite set of places S such that for $v \notin S$, $Z_v^\natural(0, W_{f_v}^{\psi_v}, F_{\phi_v}) = 1$. (This is recorded as Theorem 4.1 in [17].)

5.1.4. *Putting it all together.* With this notation established, we can now recall the proof of Theorem 2.1 which consists of three ingredients. The first is the result (5.4) of Waldspurger which implies that

$$\mathcal{B}(\varphi \otimes \widetilde{\varphi}) = \prod_{j=1}^{3} \int_{[B^\times]} \varphi_j(b)\widetilde{\varphi}_j(b)db = 2^3 \zeta_F(2)^{-3} L(1, \Pi, \mathrm{Ad}) \prod_v \mathcal{B}_v^\natural(\varphi_v \otimes \widetilde{\varphi}_v). \tag{5.7}$$

Recall that $\mathcal{B}_v^\natural(\varphi_v \otimes \widetilde{\varphi}_v) = 1$ for all but finitely many v. (This follows from Proposition 3.1 and Lemma 3.2 of [17].) Note that to ease notation we are writing

$$\mathcal{B}_v^\natural(\varphi_v \otimes \widetilde{\varphi}_v) = \prod_{i=1}^{3} \mathcal{B}_v^\natural(\varphi_{i,v} \otimes \widetilde{\varphi}_{i,v}).$$

The second result is (5.6) which combined with (5.7) implies that

$$\frac{I(\varphi \otimes \widetilde{\varphi})}{\mathcal{B}(\varphi \otimes \widetilde{\varphi})} = \frac{\zeta_F(2)}{2^3} \frac{L(1/2, \Pi)}{L(1, \Pi, \mathrm{Ad})} \prod_{v \in S} \frac{Z_v^\natural(0, W_{f_v}^{\psi_v}, F_{\phi_v})}{\mathcal{B}_v^\natural(\varphi_v \otimes \widetilde{\varphi})}. \tag{5.8}$$

The final ingredient is Proposition 5.1 of [17] which is equivalent to the fact that

$$Z_v^\natural(0, W_{f_v}^{\psi_v}, F_{\phi_v}) = \gamma_v \zeta_{F_v}(2)^{-2} \frac{L_v(1, \Pi_v, \mathrm{Ad})}{L_v(1/2, \Pi_v)} \int_{F_v^\times \backslash B_v} \mathcal{B}_v^\natural(\Pi(b_v)\varphi_v \otimes \widetilde{\varphi}_v)d^\times b_v \tag{5.9}$$

where $\gamma_v = 1$ if B_v is split and $\gamma_v = -1$ otherwise. Note that the measure $d^\times b_v$ is the that for which the Fourier transform with respect to ψ_v is self-dual. On the other hand, the measure db_v used to define I_v (see (2.4)) is that for which $F_v^\times \backslash B_v^\times(\mathcal{O}_v)$ has a given volume, as discussed in Section 2.2. We have $d^\times b_v = c_v db_v$ where $c_v = 1$ for almost all v and $\prod_v c_v = |\Delta_F|^{-3/2}$.

Since $\mathcal{B}_v^\natural : \Pi^B \otimes \widetilde{\Pi}^B \to \mathbb{C}$ is a pairing, it must differ from $\langle \cdot, \cdot \rangle_v$ by a constant. Hence,

$$\frac{Z_v^\natural(0, W_{f_v}^{\psi_v}, F_{\phi_v})}{\mathcal{B}_v^\natural(\varphi_v \otimes \widetilde{\varphi}_v)} = \gamma_v c_v \zeta_{F_v}(2)^{-2} \frac{L_v(1/2, \Pi_v)}{L_v(1, \Pi_v, \mathrm{Ad})} \frac{I_v'(\varphi_v \otimes \widetilde{\varphi}_v)}{\langle \varphi_v, \widetilde{\varphi}_v \rangle_v} = \gamma_v c_v \frac{I_v(\varphi_v \otimes \widetilde{\varphi}_v)}{\langle \varphi_v, \widetilde{\varphi}_v \rangle_v} \tag{5.10}$$

where I_v is as in (2.4).

Putting this all together, (5.8) becomes

$$\frac{I(\varphi \otimes \widetilde{\varphi})}{\mathcal{B}(\varphi \otimes \widetilde{\varphi})} = \frac{\zeta_F(2)}{2^3} \frac{L(1/2, \Pi)}{L(1, \Pi, \mathrm{Ad})} \prod_v \frac{Z_v^\natural(0, W_{f_v}^{\psi_v}, F_{\phi_v})}{\mathcal{B}_v^\natural(\varphi_v \otimes \widetilde{\varphi}_v)} = \frac{\zeta_F(2)}{2^3 |\Delta_F|^{3/2}} \frac{L(1/2, \Pi)}{L(1, \Pi, \mathrm{Ad})} \prod_v \frac{I_v(\varphi \otimes \widetilde{\varphi})}{\langle \varphi_v, \widetilde{\varphi} \rangle_v}. \tag{5.11}$$

This is exactly (2.5) for our particular choice of measures.

5.2. **Proof of Theorem 5.1.**

Proof. By combining (5.11) in the case that $\widetilde{\varphi} = \overline{\varphi}$ with the local calculations of Corollary 4.2, Proposition 4.3, Proposition 4.4, Proposition 4.5 and Proposition 4.6, all of the local factors are accounted for except in the case that $v \mid \infty$ and one of the factors $\pi_{i,v}$ is not a discrete series. So it remains to establish the local factors for $v \mid \infty$ in the following cases.

Case 1: All three representations $\pi_{i,v}$ are principal series.

Case 2: One of the representations is a principal series and the other two are discrete series (of the same weight k.)

To complete the proof, note that it suffices to compute the left hand side of (5.10). Indeed, we may choose ψ to be the character that is level zero at all finite places and which satisfies $\psi_v(x) = e^{2\pi i x}$ at real places. Then the constant $c_v = |\Delta_{F_v}|^{-3/2}$ for all v. In particular, for $v \mid \infty$, $c_v = 1$. Since globally γ_v is irrelevant, it therefore remains to show, in each of the two remaining cases, that

$$\frac{Z_v^\natural(0, W_{f_v}^{\psi_v}, F_{\phi_v})}{\widetilde{\mathcal{B}}_v^\natural(f_v \otimes \phi_v)} = 1$$

for some choice of f_v and ϕ_v for which $\theta_v(f_v, \phi_v) = \varphi_v \otimes \overline{\varphi}_v$.

In fact, Watson[36] gives choices f_v and ϕ_v so that $\theta_v(f_v, \phi_v)$ is equal to the function defined in the statement of Theorem 5.1. For this choice, [36, Theorem 2] says that

$$Z_v^\natural(0, W_{f_v}^{\psi_v}, F_{\phi_v}) = \begin{cases} 1 & v \mid \infty, \text{ Case 1} \\ 2^{-2k-2} & v \mid \infty, \text{ Case 2.} \end{cases} \tag{5.12}$$

Suppose that we are in the split case: $\theta_v(f_v, \phi_v) = \varphi_v \otimes \widetilde{\varphi}_v \in \pi_v \otimes \widetilde{\pi}_v$. Then

$$\widetilde{\mathcal{B}}_v(f_{i,v} \otimes \phi_{i,v}) = \int_{F_v^\times} W_{\varphi_v}((\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}))W_{\widetilde{\varphi}_v}((\begin{smallmatrix} -a & \\ & 1 \end{smallmatrix}))d^\times a$$

where $d^\times a$ is Tamagawa measure for $F\backslash\mathbb{A}$. (See [35, Lemme 5].) Hence, by Watson's choice of $(f_{i,v}, \phi_{i,v})$ in the real case, and [36, Lemma 5],

$$\widetilde{\mathcal{B}}_v(f_{i,v}, \phi_{i,v}) = L_v(1, \pi_i, \mathrm{Ad})\zeta_{F_v}(2)^{-1} \begin{cases} 1 & \text{if } \pi_{i,v} \text{ is a principal series} \\ 2^{-k-1} & \text{if } \pi_{i,v} \text{ is a discrete series of weight } k \end{cases} \tag{5.13}$$

Combining (5.12) and (5.13) it is immediate that $\frac{Z_v^\natural}{\mathcal{B}_v^\natural} = 1$. $\square$

Remark. The result of [36, Theorem 2] in Case 1 was previously computed by Ikeda[19].

6. Proof of Theorem 1.3

We return to the notation of Section 2 in which F is any fixed number field and $\mathbb{A} = \mathbb{A}_F$ its ring of adeles. Let π_1, π_2 be cuspidal automorphic representations of GL_2 over F with trivial central character and fixed conductors $\mathfrak{n}_1$ and $\mathfrak{n}_2$ respectively. Let π_3 be a cuspidal autormorphic representation of GL_2 over F with trivial central character and conductor $\mathfrak{n}\mathfrak{p}$ where $\mathfrak{p}$ is prime and does not divide $\mathfrak{n}\mathfrak{n}_1\mathfrak{n}_2$. We denote $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$. Let ϖ be a normalizer of $F_\mathfrak{p}$. Note that the Langlands parameters of $\pi_{3,\infty}$ are allowed to vary (within a bounded set.) Moreover, although $\mathfrak{n}$ is fixed, the local components $\pi_{3,\mathfrak{q}}$ for $\mathfrak{q} \mid \mathfrak{n}$ need not be.

At the real places $\pi_{i,v}$ corresponds to either π_{dis}^k, a *weight k-discrete series* with $k \geq 2$ even, or an irreducible (weight zero) principal series $\pi_\mathbb{R}^s = \pi(|\cdot|^s, |\cdot|^{-s})$ defined in the same way as (3.1). At a complex place $\pi_{i,v}$ is an irreducible principal series $\pi_\mathbb{C}^{s,k}$.

Each of these is a $(\mathfrak{g}, K)$-module where $\mathfrak{g}$ is the complexified Lie algebra of $G_v = \mathrm{GL}_2(F_v)$ and $K = O(2)$ or $U(2)$ depending on whether v is real or complex respectively. The irreducible representations of K are called *weights*. In the real case these are integers and, as is well-known they are given by the characters $\kappa_\theta \mapsto e^{in\theta}$. In the complex case, the weights are nonnegative integers k which correspond to representations of dimension $k+1$. In the notation above for $\pi_\mathbb{C}^{k,s}$ and $\pi_\mathbb{R}^s$, the minimal weight is encoded by the parameter k. The parameter s is a complex number. The assumption that π_∞ be restricted to a bounded set means that the s and k are bounded for each $v \mid \infty$.

We will show that for the quaternion algebra B such that $\Sigma_B = \Sigma(\Pi)$ there exists a finite collection of vectors $\mathcal{F}_1^B \subset \pi_1^B$ and $\mathcal{F}_2^B \subset \pi_2^B$ such that for φ_3 a new vector,

$$L(\tfrac{1}{2}, \Pi) \ll_{\epsilon, F, \pi_{3,\infty}, \mathfrak{n}} N(\mathfrak{p})^{1+\epsilon} \left| \int_{[B^\times]} \varphi_1(b)\varphi_2(b \left(\begin{smallmatrix} \varpi^{-1} & \\ & 1 \end{smallmatrix}\right))\varphi_3(b)db \right|^2$$

for some $\varphi_i \in \mathcal{F}_i^B$. As a first step, we prove the following.

Proposition 6.1. *Let $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ be as in Theorem 1.3. Let B be the quaternion algebra such that $\Sigma_B = \Sigma(\Pi)$. There exist vectors $\varphi_i \in \pi_i^B$ such that for $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$,*

$$L(\tfrac{1}{2}, \Pi) \ll N(\mathfrak{p})^{1+\epsilon} \prod_{v \mid \mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}\infty} \left(\frac{I_v(\varphi_v)}{\langle \varphi_v, \varphi_v \rangle} \right)^{-1} \left| \int_{[B^\times]} \varphi(b)db \right|^2. \tag{6.1}$$

where the implied constant is dependant on ϵ, Π_∞, $N(\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n})$ and $\varphi_{i,v}$ where $i = 1, 2$ and $v \mid \infty\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}$.

Proof. We apply Theorem 2.1. Write $\pi_i^B = \otimes_v \pi_{i,v}$. Clearly, we may assume that $L(\frac{1}{2}, \Pi) \neq 0$. Hence Theorem 2.2 guarantees that we may choose some $\varphi_i = \otimes_v \varphi_{i,v}$ for $i = 1, 2, 3$ such that, writing $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$, $I(\varphi) \neq 0$ and $I_v(\varphi_v) \neq 0$ for all v. In particular, we may let $\varphi_{i,v}$ be the (normalized) new vector for all $v \nmid \infty\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}$. For such places v, $I_v(\varphi_v) = 1$. Note that twisting φ_2 by $\left(\begin{smallmatrix}\varpi^{-1} & \\ & 1\end{smallmatrix}\right)$ means that $I_{\mathfrak{p}}(\varphi_{\mathfrak{p}})$ is as in Corollary 4.2, hence $I_{\mathfrak{p}}(\varphi_{\mathfrak{p}}) \sim N(\mathfrak{p})^{-1}$.

To deal with the term

$$\frac{L(1,\Pi,\mathrm{Ad})}{\prod_{j=1}^3 \int_{[B^\times]} |\varphi_j(b)|^2\, db} = \prod_{j=1}^3 \frac{L(1,\pi_j,\mathrm{Ad})}{\int_{[B^\times]} |\varphi_j(b)|^2\, db}$$

in (2.5), we use the bound of Iwaniec[20] which says that if $\varphi \in \pi_i$ is the normalized new vector then

$$\frac{L(1,\pi_i,\mathrm{Ad})}{\int_{[G]} |\varphi(g)|^2 dg} \ll N(\pi_i)^\epsilon \tag{6.2}$$

as $N(\pi_i) \to \infty$. Here, $N(\pi_i)$ denotes the conductor of π_i and the implied constant depends continuously on the Langlands parameters of $\pi_{i,\infty}$.

After solving for $L(\frac{1}{2},\Pi)$ in (2.5) and applying (6.2), (6.1) follows. Since we are fixing $\mathfrak{n}_1, \mathfrak{n}_2$ and $\mathfrak{n}$, all of the terms in $[N(\pi_1)N(\pi_2)N(\pi_3)]^\epsilon$ except for $N(\mathfrak{p})^\epsilon$ can be absorbed into the implied constant. ☐

As a matter of terminology, we call φ_v a *test vector* if $I_v(\varphi_v) \neq 0$. Given test vectors for all v we can construct the global vector $\varphi = \otimes_v \varphi_v$ such that $I(\varphi) \neq 0$. Therefore, given Proposition 6.1, Theorem 1.3 will follow provided we can show the following.

- Show that for each $v \mid \mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}\infty$ there exists a finite sets of vectors such that as $\pi_{3,\infty}$ and $\pi_{3,\mathfrak{p}}$ vary a test vector can be chosen from the said finite sets.
- Show that the values of the corresponding linear forms is uniformly bounded.

In order to bound the terms $I_v(\varphi_v)/\langle\varphi_v,\varphi_v\rangle_v$, we need to know that if there is *some* nonzero trilinear form then integration of the matrix coefficient is such a form.

Lemma 6.2. *If $\varepsilon(\frac{1}{2},\Pi) = 1$ then the trilinear form*

$$\ell : \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \mapsto \int_{Z\backslash G} \Phi_{\varphi_1,\overline{\varphi_1}}(g)\Phi_{\varphi_2,\overline{\varphi_2}}(g)\Phi_{\varphi_3,\overline{\varphi_3}}(g)dg$$

is nonzero. Similarly, if $\varepsilon(\frac{1}{2},\Pi) = -1$ then integration over the division algebra $F^\times\backslash B^\times$ gives a nonzero trilinear form on Π^B.

Proof. The case that Π is tempered is proved by Sakellaridis-Venkatesh in [30, Proposition 6.4.1]. If Π is non-tempered, we may assume that one of the representations is a principal series. In this case, let π_3 be the principal series representation. One may also consider the trilinear form

$$\ell_{\mathrm{RS}} : \mathcal{W}(\pi_1)\otimes\mathcal{W}(\pi_2)\otimes\pi_3 \to \mathbb{C}$$
$$(W_1,W_2,f_3) \mapsto \int_K \int_{F^\times} W_1(a(y)k)W_2(a(y)k)f_3(a(y)k)\,|y|^{-1}\,d^\times y dk$$

where $\mathcal{W}(\pi_i)$ is the Whittaker model attached to π_i.

It is precisely this latter form that is employed by [9] where specific test vectors are given. By [26, Lemma 3.4.2] $\ell = |\ell_{\mathrm{RS}}|^2$ (up to a nonzero constant). ☐

Now we bound $I_v(\varphi_v)/\langle\varphi_v,\varphi_v\rangle_v$ for archimedean primes v.

Lemma 6.3. *Let $\pi_{i,\infty} = \prod_{v\mid\infty}\pi_{i,v}$ be the archimedean parts of automorphic representations π_i for $i = 1,2,3$ such that π_1 and π_2 are fixed, and $\pi_{3,\infty}$ is bounded. Assuming that $L(\frac{1}{2},\Pi) \neq 0$, let B be the quaternion algebra such that $\Sigma_B = \Sigma(\Pi)$. Then there exists a* finite *collection of vectors $\mathcal{F}_i^B$ in $\pi_{i,\infty}^B$ for $i = 1,2$ such that if $\varphi_{3,\infty} \in \pi_{3,\infty}^B$ is a new vector then*

$$\prod_{v\mid\infty} I_v(\varphi_{1,v}\otimes\varphi_{2,v}\otimes\varphi_{2,v}) \geq \delta$$

for some choice of $\varphi_{i,\infty} \in \mathcal{F}_i$ and some $\delta > 0$.

Proof. If v is real, for each of the finitely many choices k for which $\pi_{3,v}$ could be π^k_{dis}, Theorem 2.2 guarantees we may chose vectors $\varphi_{1,v} \in \pi^B_{1,v}$ and $\varphi_{2,v} \in \pi^B_{2,v}$ such that $\varphi = \varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi_{3,v}$ is a test vector.

Now assume that $\pi_{3,v} = \pi^s_{\mathbb{R}}$ is a principal series. Loke[25, Cor. 2.2] gives a choice of vectors $\varphi_{i,v} \in \pi_{i,v}$ $(i = 1, 2)$ independent of s such that for $\varphi_{3,v}$ equal to the normalized new vector, the unique $(\mathfrak{g}, K)$-invariant linear form, $\varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi_{3,v}$ is a test vector, hence the matrix coefficient is nonzero.

Because the matrix coefficient attached to $\varphi_{3,v}$ is a continuous function with respect to its Langlands parameter[4], under the boundedness condition there must be a lower bound on the values of the matrix coefficient. More explicitly, given Loke's choice of $\varphi_{1,v}, \varphi_{2,v}$ we have a nonzero continuous map

$$\lambda : \Omega \to \mathbb{C}^\times \qquad s \mapsto I_v(\varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi^s_{3,v})$$

where $\varphi^s_{3,v} \in \pi^s_{\mathbb{R}}$ is the normalized new vector and $s \in \Omega \subset \mathbb{C}$. (The map is nonzero by Lemma 6.2) By the boundedness assumption implies we may assume Ω is compact, hence its image is uniformly bounded away from zero.

If v is a complex place, Loke[25, Cor. 4.6] again proved that there must be a test vector independent of s. The continuity argument of above again applies with Ω replaced by $\Omega \times \{k_1, \cdots, k_m\}$ where k_i are the distinct nonnegative integers corresponding the possible weights. In this case, Loke does not give a test vector independent of s. However, a test vector for a given value s, by continuity, will also be a test vector for some open subset $U_s \subset \Omega$. Since Ω is compact, there exist finitely many values s_i such that $\cup_i U_{s_i} = \Omega$.

The finite set $\mathcal{F}^B_{i,\infty}$ is obtained by taking the union of all of the vectors obtained above. ☐

In each of Corollary 4.2, Proposition 4.3, Proposition 4.4 and Proposition 4.5 the value of $I_v(\varphi_v)/\langle \varphi_v, \varphi_v \rangle_v$ is constant even if one of the representations is varied. That is to say, suppose $v = \mathfrak{q}$ is a finite prime such that $\mathfrak{q}^2 \nmid \mathfrak{n}_i$ for $i = 1, 2$ and $\mathfrak{q}^2 \nmid \mathfrak{n}$. In particular, $\pi_{3,\mathfrak{q}} = \pi(\mu, \mu^{-1})$ or σ_μ for some unramified character μ if v is finite. Hence there is a choice of φ_v such that $I_v(\varphi_v)/\langle \varphi_v, \varphi_v \rangle_v$ is nonzero and independent of μ. Hence, this completes the proof of Theorem 1.3 in the case that $\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}$ are squarefree.

Although we believe that this phenomenon–the existence a test vector such that $I_v(\varphi_v)$ depends only on the level of $\pi_{3,v}$–should hold more generally, it is not a priori evident. However, the following lemma allows us to conclude that $\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}$ may be taken arbitrarily.

Lemma 6.4. *Fix irreducible admissible representations $\pi_{1,v}, \pi_{2,v}$ of $\mathrm{GL}_2(F_v)$ that are local components of global automorphic representations. There exist finite collections $\mathcal{F}_{1,v} \subset \pi_{1,v}$ and $\mathcal{F}_{2,v} \subset \pi_{2,v}$ and a constant $\delta > 0$ such that if $\pi_{3,v}$ is any other such representation of fixed level such that $\Pi_v = \pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v}$ admits a $\mathrm{GL}_2(F_v)$-invariant linear form,*

$$I_v(\varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi_{3,v}) \geq \delta$$

for $\varphi_{3,v}$ the new vector and some $\varphi_{i,v} \in \mathcal{F}_{i,v}$.

Proof. Note that since $\pi_{i,v}$ is the local component of an automorphic representation it is unitary. Moreover, as demonstrated in [31], $c(\pi(\chi, \chi')) = c(\chi) + c(\chi')$. Therefore, that principal series representations of fixed level and are parametrized by characters $\chi, \chi' : F_v^\times \to \mathbb{C}$ of bounded conductor or which there are only finitely many.

Let ϖ be a uniformizer, q_v the order of the residue field, and let

$$U^{(i)} = \begin{cases} \{1 + u\varpi^i \mid u \in \mathcal{O}^\times\} & \text{if } i \geq 1 \\ \mathcal{O}^\times & \text{if } i = 0. \end{cases}$$

As is well known, for χ to be a character of level n means that

$$\chi(u\varpi^k) = |\varpi|^{ks}\, \bar{\chi}(u)$$

where $\bar{\chi}$ is a character of the finite group $\mathcal{O}^\times/U^{(n)}$. If $\pi(\chi, \chi^{-1})$ comes from an automorphic representations, we may assume $s = s_\chi$ is a complex number with $\mathrm{Re}(s) \in [0, \lambda]$ where λ is any bound towards Ramanujan. Note we may also assume that $\mathrm{Im}(s)$ is bounded. In other words, s is restricted to a compact set.

Let $\pi_{3,v} \simeq \pi(\chi_0, \chi_0^{-1})$ be such that $I_v \neq 0$, and denote the new vector by $\varphi_{3,v}$. Given $\varphi_{1,v}, \varphi_{2,v}$ such that $\varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi_{3,v}$ is a test vector, as in the proof of Lemma 6.3, $\varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi^s_{3,v}$ is also a test vector for $\pi_{3,v} \simeq \pi(\mu, \mu^{-1})$ where $\bar{\mu} = \bar{\chi}$ and s_μ is in some open set containing s_χ. Since the set of possible s_χ is

[4]The local L-factors are also continuous since they are products of gamma functions.

compact, and the number of characters $\bar{\chi}$ is finite, we find that a finite collection of choices $\varphi_{1,v}, \varphi_{2,v}$ suffice to give a set of test vectors for $\pi_{3,v}$ any principal series representation.

A completely analogous argument gives a finite set of choices for $\varphi_{1,v}, \varphi_{2,v}$ in the case that $\pi_{3,v} \simeq \sigma_\chi$. In this case, however, the argument is simplified by the fact that χ must be unitary, i.e. $\mathrm{Re}(s_\chi) = 0$.

Finally, if $\pi_{3,v}$ is supercuspidal then its Kirillov model associated to a nontrivial additive character ψ is completely determined by the level of $\pi_{3,v}$ and the epsilon factor $\varepsilon(\frac{1}{2}, \pi_{3,v}, \psi)$. (See, for example, [8, 37.3 Theorem].) Again, the associated parametrizing set of such representations in compact, so the same argument as above applies.

The existence of the constant δ is established in exactly the same fashion as in Lemma 6.3. Namely, values assumed by I_v on the given set of test vectors is the image of a compact set (corresponding to the finite number of characters of $\mathcal{O}^\times/U^{(n)}$ and the admissible values of s_χ.) $\square$

This completes the proof of Theorem 1.3.

Remark. In [9] explicit test vectors are given for all cases except that where all three representations $\pi_{i,v}$ are supercuspidals. This can be used to make the choices of $\mathcal{F}_{1,v}, \mathcal{F}_{2,v}$ in Lemma 6.4 explicit. However, note that the argument above is still needed to ensure the existence of δ.

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