Total positivity, SS 17

Exercise Sheet 3

to be discussed on 01.06.2017

This sheet contains five regular exercises.

Recall the following notations from the lecture: for $i = 1, \dots, n-1$, $x_i(t) := I_n + t\mathbf{E}_{i,i+1} \in \mathrm{GL}_n(\mathbb{R})$; $y_i(t) = x_i(t)^T$; and for $1 \leq k \leq n$, $h_k(t) := I_n + (t-1)\mathbf{E}_{k,k} \in \mathrm{GL}_n(\mathbb{R})$.

Exercise 1. (LDU without computer) Compute (by hand!) the LDU decomposition for the following matrix:

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}.$$

Exercise 2. (Loewner-Whitney) Consider the following matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}.$$

1. Prove that A is TP.
2. Factorise A into a product of $x_i(t)$, $y_i(t)$ and $h_k(t)$.

Exercise 3. (Exchange relations) Prove the following identities: for $a, b, c \in \mathbb{R}$,

1. if $a + c \neq 0$, then

$$x_i(a)x_{i+1}(b)x_i(c) = x_{i+1}\left(\frac{bc}{a+c}\right) x_i(a+c) x_{i+1}\left(\frac{ab}{a+c}\right),$$

$$y_i(a)y_{i+1}(b)y_i(c) = y_{i+1}\left(\frac{bc}{a+c}\right) y_i(a+c) y_{i+1}\left(\frac{ab}{a+c}\right);$$

2. if $1 + ab \neq 0$, then

$$x_i(a)y_i(b) = y_i\left(\frac{b}{1+ab}\right) h_i(1+ab) h_{i+1}\left(\frac{1}{1+ab}\right) x_i\left(\frac{a}{1+ab}\right).$$

Exercise 4. (Copycat)

1. Give a family of upper totally positive matrix $U_n(q) \in \mathrm{GL}_n(\mathbb{R})$ depending on $q \in (0,1)$ such that $\lim_{q \to 0^+} U_n(q) = I_n$. Prove your statement.
2. Let $A \in \mathrm{GL}_n(\mathbb{R})$ be an upper triangular invertible matrix. Assume that for any $k = 1, 2, \dots, n$ and $J \in \binom{[n]}{k}$, $\det A[1, \dots, k|J] \geq 0$. Prove that A is upper totally non-negative.

Exercise 5. (Explicit LDU) Assume that $A \in \mathrm{GL}_n(\mathbb{R})$ admits an LDU decomposition $A = LDU$ with $L = (l_{i,j})$, $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ and $U = (u_{i,j})$. Then the entries of these matrices are given by: for $1 \leq i < j \leq n$ and $1 \leq k \leq n$,

$$l_{j,i} = \frac{\det A[1, \ldots, i-1, j | 1, \ldots, i-1, i]}{\det A[1, \ldots, i | 1, \ldots, i]},$$

$$u_{i,j} = \frac{\det A[1, \ldots, i-1, i | 1, \ldots, i-1, j]}{\det A[1, \ldots, i | 1, \ldots, i]},$$

$$d_k = \frac{\det A[1, \ldots, k | 1, \ldots, k]}{\det A[1, \ldots, k-1 | 1, \ldots, k-1]}.$$