Total positivity, SS 17

Exercise Sheet 5

to be discussed on 06.07.2017

This sheet contains five regular exercises.

Exercise 1. (Tableaux counting)

Find all semi-standard Young tableaux over $\{1, 2, \cdots, 12\}$ of shape $\lambda = (4, 4, 3, 1)$ and type $(4, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0)$.

Exercise 2. (The power of van der Monde)

Let $V(x_1, \cdots, x_n) := \det V_n(x_1, \cdots, x_n)$ be the van der Monde determinant. A polynomial $f(x_1, \cdots, x_n) \in \mathbb{Z}[x_1, \cdots, x_n]$ is called alternating, if for any $\sigma \in \mathfrak{S}_n$, $\sigma \cdot f = (-1)^{\ell(\sigma)} f$. Prove that $f(x_1, \cdots, x_n) \in \mathbb{Z}[x_1, \cdots, x_n]$ is alternating if and only if there exists a symmetric polynomial $g(x_1, \cdots, x_n) \in \mathbb{Z}[x_1, \cdots, x_n]$ such that

$$f(x_1, \cdots, x_n) = V(x_1, \cdots, x_n) g(x_1, \cdots, x_n).$$

Exercise 3. (Schur via Young)

Let $\lambda = (a_1, a_2)$ be a partition. Find all monomials in the Schur function $s_\lambda(x_1, x_2)$ using semi-standard Young tableaux.

Exercise 4. (An example of one line proof)

Prove that $e_n(x_1, \cdots, x_n) = \det(h_{1-i+j})_{i,j=1}^n$.

Exercise 5. (Move, move, move)

Consider the following words for the symmetric group $\mathfrak{S}_{n+1}$:

$$\mathbf{i}^{\min} = (1, 2, 1, 3, 2, 1, \cdots, n, n-1, \cdots, 2, 1);$$

$$\mathbf{i}^{\max} = (n, n-1, n, n-2, n-1, n, \cdots, 1, 2, \cdots, n-1, n).$$

1. Prove that $\mathbf{i}^{\min}$ and $\mathbf{i}^{\max}$ are words for the same permutation. What is this permutation?
2. Prove that both $\mathbf{i}^{\min}$ and $\mathbf{i}^{\max}$ are reduced.
3. For $n = 3$, find an explicit sequence $\mathbf{i}^{\min} = \mathbf{i}_0$, $\mathbf{i}_1$, $\cdots$, $\mathbf{i}_{m-1}$, $\mathbf{i}_m = \mathbf{i}^{\max}$ such that for any $k = 1, \cdots, m$, $\mathbf{i}_k$ is obtained by applying either a 2-move or a 3-move to $\mathbf{i}_{k-1}$.
4. (Optional) Answer the question in (3) for arbitrary n.