General Linear Groups, WS 18/19

Exercise Sheet 2

Exercise 1. Let V be a $\mathbb{K}$-vector space. The following statements are equivalent:

1. V admits a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_k$;
2. there exist complete orthogonal idempotents $\pi_1, \cdots, \pi_k \in \mathrm{End}(V)$ such that for any $1 \leq i \leq k$, $V_i = \mathrm{im}(\pi_i)$.

Exercise 2. Show that the rank function is lower semi-continuous: that is to say, for any sequence of matrices $(A_k)_{k\geq 1}$ in $\mathcal{M}_{n,m}(\mathbb{K})$ of the same rank r and converging to a matrix $B \in \mathcal{M}_{n,m}(\mathbb{K})$. Then $\mathrm{rank}(B) \leq r$.

Exercise 3. Prove that $\chi : \mathcal{M}_n(\mathbb{K}) \to \mathbb{K}_{\leq n}[t]$, sending a matrix A to its characteristic polynomial $\chi_A(t)$, is continuous.

Exercise 4. Let $A, B \in \mathcal{M}_n(\mathbb{C})$. We denote $[A, B] := AB - BA$.

1. Prove that a matrix $A \in \mathcal{M}_n(\mathbb{C})$ is nilpotent if and only if for any $k > 0$, $\mathrm{Tr}(A^k) = 0$.
2. Prove that if $[A, [A, B]] = 0$ then

$$\mathrm{Tr}([A, B]^m) = \mathrm{Tr}([AB, [A, B]^{m-1}]).$$

   Deduce that: if $[A, [A, B]] = 0$ then $[A, B]$ is nilpotent.

Exercise 5. (Hilbert scheme of points on the plane)

For $n \geq 1$, we define

$$\mathrm{Hilb}_n := \{I \subseteq \mathbb{C}[x, y] \text{ is an ideal } \mid \dim_{\mathbb{C}} \mathbb{C}[x, y]/I = n\}.$$

Recall that a sub-vector space $I \subseteq \mathbb{C}[x, y]$ is an ideal, if for any $P \in \mathbb{C}[x, y]$, $Q \in I$, $PQ \in I$. Then I is a $\mathbb{C}$-vector space and $\mathbb{C}[x, y]/I$ is not only a ring but a $\mathbb{C}$-vector space, this allows us to define the dimension.

We define $\mathrm{Com} := \{(A, B) \in \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C}) \mid AB = BA\}$.

1. Let $P \in \mathbb{C}[x, y]$, $(A, B) \in \mathrm{Com}$. Prove that $\mathrm{GL}_n(\mathbb{C})$ acts on Com via: for $g \in \mathrm{GL}_n(\mathbb{C})$,

$$g \cdot (A, B) := (gAg^{-1}, gBg^{-1}).$$

2. We call a pair of matrices $(A, B) \in \mathrm{Com}$ cyclic, if there exists $\mathbf{v} \in \mathbb{C}^n$ such that

$$\{P(A, B)\mathbf{v} \mid P \in \mathbb{C}[x, y]\} = \mathbb{C}^n.$$

   Let Com^0 denote the set of cyclic pairs in Com. Prove that $\mathrm{Com}^0 \neq \emptyset$; it is a $\mathrm{GL}_n(\mathbb{C})$-set, and it is open in $\mathrm{Com} \subseteq \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C}) \cong \mathbb{C}^{2n^2}$. (Hint: for the last point, show that $\mathrm{Com} \setminus \mathrm{Com}^0$ is the zero set of some $n \times n$ determinants.)

3. Prove that if (A, B) cyclic, then $I(A, B) := \{P \in \mathbb{C}[x, y] \mid P(A, B) = 0\}$ is contained in Hilb_n.

4. Consider the map
$$\pi : \mathrm{Com}^0 \to \mathrm{Hilb}_n, \quad (A, B) \mapsto I(A, B).$$
Prove that for any $I \in \mathrm{Hilb}_n$, $\pi^{-1}(I)$ is a $\mathrm{GL}_n(\mathbb{C})$-orbit. (Hint: let $V := \mathbb{C}[x, y]/I \cong \mathbb{C}^n$. Consider two endomorphisms μ_x and μ_y of V given by multiplication by x and y.)

N.B. This map π *is used to endow* Hilb_n *with a topology structure.*