1. Determinantal identities

Let X be a matrix, I a subset of its rows and J a subset of its columns. We denote $X_{I,J}$ the corresponding minor of X and $|X_{I,J}|$ its determinant.

1.1. Cauchy-Binet identity. Let X be an $n \times m$ matrix and let Y be a $m \times n$ matrix, $n \leq m$.

Theorem 1.1. *We have*

$$\det(XY) = \sum_{J \subseteq [m], |J|=n} |X_{[n],J}||Y_{J,[n]}|.$$

Proof. The formula is multilinear in rows of X and columns of Y. Thus, it suffices to prove it for each row of X and column of Y being a $0-1$ vector with all entries but one equal to zero.

Thus we essentially have two functions $f\colon [n] \to [m]$ and $g\colon [n] \to [m]$, which tell us the location of unit entries in X and Y respectively. If either of the two is not injective, both sides are 0. If the two have not the same image, the two sides are again zero. Indeed, on the right each summand vanishes, while on the left one of the rows is identically zero.

Finally, assume f and g have the same image. Then we have product of two permutation matrices, and the result follows from the fact that signatures of permutations are multiplicative. $\square$

Exercise 1.2. *Show that any real matrix can be realized by a planar acyclic network in a disk with real weights. Deduce from this the Cauchy-Binet identity.*

1.2. Sylvester's identity. Fix $I, J \subset [n]$, $|I| = |J| = k$ and for $i \in [n]/I$, $j \in [n]/J$ let

$$y_{i,j} = |X_{I \cup i, J \cup j}|.$$

Theorem 1.3. *We have*

$$\det(Y) = |X_{I,J}|^{n-k-1}|X|.$$

Exercise 1.4 (Desnanont-Jacobi)**.** *Prove Sylvester's identity for $n - k = 2$.*

We give the proof of the identity for $n - k > 2$ by induction.

Proof. For $n-k = 1$ the statement is trivial and we assume we know it is true for $n-k = 2$ by the above exercise.

Let $[n]/I = i_1 \cup i_2 \cup \bar{I}$ and $[n]/J = j_1 \cup j_2 \cup \bar{J}$ for arbitrary choice of indexes $i_1 < i_2$, $j_1 < j_2$. Then by induction assumption we have

$$|Y||Y_{\bar{I},\bar{J}}| = |Y_{i_1 \cup \bar{I}, j_1 \cup \bar{J}}||Y_{i_2 \cup \bar{I}, j_2 \cup \bar{J}}| - |Y_{i_1 \cup \bar{I}, j_2 \cup \bar{J}}||Y_{i_2 \cup \bar{I}, j_1 \cup \bar{J}}|.$$

By induction assumption

$$|Y_{\bar{I},\bar{J}}| = |X_{I,J}|^{n-k-3}|X_{\bar{I} \cup I, \bar{J} \cup J}| \quad \text{and} \quad |Y_{i \cup \bar{I}, j \cup \bar{J}}| = |X_{I,J}|^{n-k-3}|X_{i \cup \bar{I} \cup I, j \cup \bar{J} \cup J}|.$$

Finally, plugging in those values, we get that the formula we want for $|Y|$ is equivalent to

$$|X_{i_1 \cup \bar{I} \cup I, j_1 \cup \bar{J} \cup J}||X_{i_2 \cup \bar{I} \cup I, j_2 \cup \bar{J} \cup J}| - |X_{i_1 \cup \bar{I} \cup I, j_2 \cup \bar{J} \cup J}||X_{i_2 \cup \bar{I} \cup I, j_1 \cup \bar{J} \cup J}| = |X||X_{\bar{I} \cup I, \cup \bar{J} \cup J}|,$$

which is again the Sylvester's identity for $n - k = 2$. $\square$

1.3. **Plücker relations.** We state the Plücker relations in the coordinate ring of Grassmanian $Gr(n, m)$. A simple trick of appending an identity matrix to any matrix allows to switch between determinantal identities among minors of any matrix and relations among Plücker coordinates of a Grassmanian.

For a subset $J \subset [m]$ of size n we denote X_J the Plücker coordinate with columns labelled by J. Fix a subset of J of size $k < n$. If elements of J are not ordered increasingly, we let $|X_J| = \epsilon(J)|X_{\bar{J}}|$, where $\bar{J}$ is the increasing rearrangement of J and $\epsilon(J)$ is the sign of permutation that accomplishes the rearrangement.

Theorem 1.5. *We have*

$$|X_I||X_J| = \sum |X_{I'}||X_{J'}|,$$

where the summation is taken over all pairs I' and J' obtained from I and J by swapping the chosen k elements of J with any k elements of I.

Proof. Without loss of generality we can assume that we are swapping the first k elements of J. We argue that $|X_I||X_J| - \sum |X_{I'}||X_{J'}|$ is antisymmetric in all columns labelled by I together with the k-th column of J. The proof is by induction on k, case $k = 0$ trivial. It is easy to check that the expression is antisymmetric in pairs of adjacent columns in I. Next, all the terms where k-th element of J was not swapped with the n-th element of I clearly are antisymmetric in those two columns. As for the rest, they form a part of a Plücker relation with value of k one smaller, plus some extra terms again untisymmetric in the two vectors we are interested in.

The only multilinear function antisymmetric in $n + 1$ vectors of size n is constant zero. $\square$

1.4. **Matrix inverse and Laplace expansion.** Assume X is an $n \times n$ matrix.

Theorem 1.6. *We have*

$$X^{-1}_{i,j} = \frac{(-1)^{i+j}|X_{[n]/j,[n]/i}|}{|X|}.$$

Theorem 1.7. *We have*

$$|X| = \sum_{J \subset [n], |J|=k} (-1)^{\epsilon([k])+\epsilon(J)}|X_{[k],J}||X_{[n]/[k],[n]/J}|,$$

where $\epsilon(S) = \sum_{s \in S} s$.

Proof. Follows from Plücker relations. $\square$

1.5. **Binomial relations.** The following two-term relation will play a special role in what follows. Let X be a $n \times (n + 1)$ matrix.

Theorem 1.8. *For any $1 < k, l < n + 1$ we have*

$$|X_{[n],[n+1]/l}||X_{[n]/k,[n]/1}| = |X_{[n],[n+1]/1}||X_{[n]/k,[n]/l}| + |X_{[n],[n]}||X_{[n]/k,[n+1]/\{1,l\}}|.$$

Proof. Add an extra row to X filled with 0-s and 1, apply Sylvester's identity. $\square$

Exercise 1.9. *Let X be an $n \times n$ matrix, $\emptyset \subseteq I, J \subseteq [n]$ two subsets of $[n]$ of the same cardinality $|I| = |J|$. We consider products $|X_{I,J}||X_{[n]/I,[n]/J}|$ of complementary minors of X. Find the dimension of the vector space generated by all such products.*

Exercise 1.10 (Dodgson condensation). *Start with an $n \times n$ matrix $X = X^1$, and assume X^0 is an $(n+1) \times (n+1)$ matrix filled with 1-s. Given matrices X^i and X^{i-1}, we create matrix X^{i+1} of size one smaller than that of X^i according to the rule*

$$x^{i+1}_{j,k} = \det \begin{pmatrix} x^i_{j,k} & x^i_{j,k+1} \\ x^i_{j+1,k} & x^i_{j+1,k+1} \end{pmatrix} / x^{i-1}_{j+1,k+1}.$$

Show that once we get to a 1×1 matrix X^n, its entry is equal to $\det(X)$.

2. Criteria for total positivity

Lemma 2.1 (Fekete). *Assume X is an $n \times m$ matrix, $n \geq m$ such that all minors of size $m-1$ with columns $[m-1]$ are positive and all minors of size m with consecutive rows are positive. Then all minors of X of size m are positive.*

Proof. For $I = i_1 < \ldots < i_k$ let $d(I) = i_k - i_1 - k + 1$ be the *dispersion* of I. We prove the positivity of minors $X_{I,[m]}$ by induction on $d(I)$. The base case $d(I) = 0$ follows from assumption of the lemma.

Assume $d(I) > 0$ and thus we can find J and $1 < l < m+1$ such that $J/j_l = I$. The binomial relation gives us:

$$|X_{J/j_1,[m-1]}||X_{J/j_l,[m]}| = |X_{J/\{j_l,j_{m+1}\},[m-1]}||X_{J/j_1,[m]}| + |X_{J/\{j_l,j_1\},[m-1]}||X_{J/j_{m+1},[m]}|.$$

The terms of size $m-1$ are positive by assumption, and the sets J/j_1 and J/j_{m+1} have smaller dispersion than I, and thus are positive by induction hypothesis. The statement follows. $\square$

Theorem 2.2. *Assume all minors of X having consecutive rows and consecutive columns are positive. Then X is totally positive.*

Proof. First we can use the Fekete lemma to show that all column-solid minors of X are positive. This is done by induction on the size of the minor. Then, we can use the same argument in horizontal direction to show that any minor at all is positive. $\square$

It turns out that in order for a matrix to be TP it suffices to check even smaller set of solid minors.

Theorem 2.3. *Assume all solid minors of X with rows $[k]$ and also all solid minors of X with columns $[k]$ are positive, $k = 1, 2, \ldots$. Then X is totally positive.*

Proof. We prove that if the condition of the theorem holds then all solid minors of X are positive, thus reducing to the previous theorem. Call an entry of the matrix *nice* if all solid minors having it as a NW corner are positive. Originally all entries in the first row and first column are known to be nice. We prove that if (i,j), $(i+1,j)$ and $(i,j+1)$ are nice, then so is $(i+1,j+1)$. We show how it works for $i = j = 1$, the same proof works in general.

Using

$$x_{1,2}x_{2,1} + |X_{[1,2],[1,2]}| = x_{1,1}x_{2,2}$$

and assumption of the theorem we conclude that $x_{2,2} > 0$. Next from the identity

$$|X_{[k],[k]}||X_{[2,k+1],[2,k+1]}| = |X_{[k],[2,k+1]}||X_{[2,k+1],[k]}| + |X_{[2,k],[2,k]}||X_{[k+1],[k+1]}|$$

we conclude by induction on k that $|X_{[2,k],[2,k]}|$ is always positive. $\square$

Exercise 2.4. *Minors of the matrix X of the form $|X_{I,I}|$ are called principal. Show that in a non-singular TNN square matrix all principal minors are strictly positive.*

Exercise 2.5. *Let X be a totally nonnegative nonsingular square matrix, and define matrix $Y = X^c$ by $y_{i,j} = (-1)^{i+j}(X^{-1})_{i,j}$. Show that Y is also totally nonnegative.*

2.1. **Density.**

Theorem 2.6. *Totally positive matrices are dense in the class of totally nonnegative matrices.*

Proof. For $q \in (0,1)$ the matrix $Q_n(q) = (q^{(i-j)^2})_{i,j=1}^n$ is strictly totally positive. Indeed, this is equivalent to positivity of $P = (p^{ij})_{i,j=1}^n$, where $p = q^{-2}$. The latter reduces to positivity of a Vandermonde determinant.

Note also that

$$\lim_{q \to 0^+} Q_n(q) = I_n$$

is the identity matrix. Let A be an $n \times m$ totally nonnegative matrix of rank r. Let $B_q = Q_n(q) A Q_m(q)$. By Cauchy-Binet formula all minors of B_q of size at most r are strictly positive, and $\lim_{q\to 0^+} B_n(q) = A$. If $r = \min(n,m)$, we are done. Otherwise add small $\epsilon > 0$ to the first entry of B_q and repeat the argument for the resulting matrix of rank $r+1$. $\square$

Proposition 2.7. *Let A be an $n \times n$ nonsingular matrix. Assume all column-solid minors of A are nonnegative. Then A is totally nonnegative.*

Proof. Let $A_q = Q_n(q)A$. Then by Cauchy-Beniet formula and non-singularity of A, all column-solid minors of A_q are strictly positive. By Theorem 2.2 we conclude A_q is totally positive. Taking limit $q \to 0^+$ we conclude that A is totally nonnegative. $\square$

2.2. **Triangular total positivity.** An upper triangular matrix is *upper totally positive* if all its minors that are not forced to vanish by triangularity condition are strictly positive. Those are the minors $|X_{I,J}|$ where $i_k \leq j_k$ for $I = i_1 < \ldots < i_m$, $J = j_1 < \ldots < j_m$, $k \in [m]$. We write this condition $I \leq J$ and call such minors *dominant.*

Theorem 2.8. *Let X be an $n \times n$ upper triangular matrix satisfying $|X_{[k],J}| > 0$ for all solid minors. Then X is upper totally positive.*

Proof. By Fekete's lemma all minors of the form $|X_{[k],J}|$ are positive, not only the solid ones. Applying it to the minors of the form

$$|X_{[i+k],[i]\cup J}| = |X_{[i],[i]}||X_{[i+1,i+k],J}|$$

with $[i+1, i+k] \leq J$, we conclude that all row-solid dominant minors of X are positive.

Recall the notion of dispersion $d(I)$. Assume that we know $|X_{I,J}| > 0$ for $I \leq J$ with $|I| = |J| \leq k$ and $d(I) < p$. We apply induction on both k and p. We know the statement is true for $d(I) = 0$ and any k.

Assume $d(I) > 0$ and add r_l to I to obtain $R = r_1 < \ldots < r_{k+1} = I \cup r_l$, $1 < l < k+1$. By the binomial relation we have

$$|X_{R/r_l,J}||X_{R/\{r_1,r_{k+1}\},J/j_1}| = |X_{R/r_1,J}||X_{R/\{r_l,r_{k+1}\},J/j_1}| + |X_{R/r_{k+1},J}||X_{R/\{r_1,r_l\},J/j_1}|.$$

Now, $R/r_{k+1} \leq J$, and thus by induction assumption on dispersion $|X_{R/\{r_1,r_{k+1}\},J/j_1}| > 0$. Similarly, the four terms on the right either vanish or are positive. Note that $|X_{R/r_1,J}| = 0$ if $r_{m+1} > j_m$ for some $m \in [l-1]$, otherwise it is positive since it has smaller dispersion. This allows us to conclude that $|X_{I,J}|$, completing the step of induction. $\square$

Theorem 2.9. *Assume X is upper triangular nonsingular matrix with all minors $|X_{[k],J}| \geq 0$ nonnegative. Then X is TNN.*

Proof. It is easy to find an upper TP matrix $R_n(q)$ such that $R_n(q) \to I_n$ as $q \to 0^+$. Then the product $XR_n(q)$ is upper TP by Cauchy-Binet and the previous theorem. Taking limit $q \to 0$ we obtain the needed statement. $\square$

2.3. **LDU factorization.** Let X be a nonsingular $n \times n$ TNN matrix. Then One can write

$$X = LDU,$$

where L is a unipotent lower triangular matrix, D is a diagonal matrix and U is a unipotent upper triangular matrix, all given by

$$l_{i,j} = \frac{|X_{[j-1]\cup i,[j]}|}{|X_{[j],[j]}|}; \quad u_{i,j} = \frac{|X_{[i],[i-1]\cup j}|}{|X_{[i],[i]}|}; \quad d_{i,i} = \frac{|X_{[i],[i]}|}{|X_{[i-1],[i-1]}|}.$$

Recall that the denominators are strictly positive by the exercise above.

Theorem 2.10. *This is a factorization into TNN matrices.*

Proof. By Cauchy-Binet we have

$$|X_{[k],J}| = \sum_I |L_{[k],I}||D_{I,I}||U_{I,J}|.$$

The term $|L_{[k],I}|$ is non-zero only if $I = [k]$, thus

$$|X_{[k],J}| = |L_{[k],[k]}||D_{[k],[k]}||U_{[k],J}|.$$

This implies $|U_{[k],J}| \geq 0$ since the other two terms are positive. By the previous theorem we know that this implies U TNN. Same for L. $\square$

2.4. **Factorization into Chevalley generators.** Let *Chevalley generators* be matrices $e_i(a)$ different from identity matrix only in $(i, i+1)$-st entry, the value of which is a, as well as $f_i(a)$ different from identity matrix only in $(i+1, i)$-st entry, the value of which is a.

Theorem 2.11. *An upper triangular unipotent matrix X can be factored into Chevalley generators $e_i a$ with nonnegative parameters $a \geq 0$. Similarly, a lower triangular unipotent matrix X can be factored into Chevalley generators $f_i a$ with nonnegative parameters $a \geq 0$.*

Proof. We prove the case of upper triangular matrices, the lower triangular case is similar. It is easy to see that zero-nonzero pattern of entries of X has the form of order ideal in North-East oriented ordering of entries. In other words, it looks like a staircase. Let $x_{i,j}$ be one of the corners of this staircase, in other words $x_{i,j} > 0$ but $x_{k,j} = 0$ for $k < i$ and $x_{i,k} = 0$ for $k > j$. Among all such corners we can always find one such that $x_{i+1,j+1}$ either does not exist or is equal to 0.

Now, multiply X on the left by $e_i(-\frac{x_{i,j}}{x_{i+1,j}})$. We claim that the resulting matrix is still upper triangular unipotent TNN matrix. Only TNN is non-trivial. We know that for a non-singular matrix its enough to check that all row-solid minors are nonnegative. The minors not involving i-th row or the ones involving both i-th and $i+1$-st rows did not change after multiplication. Consider now a minor $|X_{I,J}|$ with last row i. We claim that

$$\frac{|X_{I,J}|}{|X_{I/i\cup(i+1),J}|} \geq \frac{x_{i,j}}{x_{i+1,j}}.$$

This follows from expanding the determinant of $X_{I\cup(i+1),J\cup j}$ along the j-th column. Note that if J contains a column to the right of j, both $|X_{I,J}|$ and $|X_{I/i\cup(i+1),J}|$ are zero. If J contains column j itself, we replicate it and thus create a needed matrix. $\square$

3. Networks

3.1. **Lindström lemma.** Unless otherwise specified, we call *network* a graph which is

- directed;
- acyclic;
- imbedded in a disk, *boundary* vertices on the boundary of the disk and *internal* vertices inside;
- planar, that is the only common points of edges are vertices;
- edge-weighted, where weights are either real numbers or formal variables;
- each boundary vertex is either a source or a sink, source and sink vertices do not interlace.

To a network N one can associate the matrix $X(N)$ of *boundary measurements* in the following way. The rows of $X(N)$ correspond to sources of N and columns correspond to sinks. For any multiset S of edges of N let weight $w(S) = \prod_{e\in S} w(e)$ of S be the product of weights of edges in S, counted with multiplicities. Let (i,j)-th entry of $X(N)$ be defined by

$$x_{i,j} = \sum_{p\colon i\to j} w(p),$$

where the sum is over all directed paths from i-th source to j-th sink.

Theorem 3.1. *Assume N has n sources and n sinks. Then*

$$\det(X(N)) = \sum_P w(P),$$

where the sum is taken over all non-crossing families P of paths from sources to sinks.

Proof. Clearly

$$\det(X(N)) = \sum_Q \epsilon(Q)w(Q),$$

where the sum is over all families of paths Q from sources to sinks and $\epsilon(Q)$ denotes the sign of the permutation induced by Q. We construct a sign-reversing involution on families Q which would cancel out all terms that have a crossing. Indeed, assume Q is a family of paths that has crossings. Acyclic network Q induces a partial order on vertices of N, where sources are maximal elements and sinks are minimal. Fix an extension of this partial order. Among all vertices where paths in $Q = (q_1, \ldots, q_n)$ cross choose v largest in this order. Among the paths passing through v choose q_i and q_j for which the sinks i and j in which they start are minimal. Build a new family of paths Q by swapping between q_i and q_j the parts that comve after they cross in v. We claim that v is still the largest crossing vertex in Q'. This is trivial since the set of all vertices of crossing in Q' is the same as in Q. Thus this gives an involution, which is clearly sign reversing, and we are done. □

Corollary 3.2. *Assume edge weights of N are nonnegative real numbers. Then the matrix $X(N)$ is TNN.*

Proof. Any minor of $X(N)$ counts non-crossing families of paths from a subset of sources to a subset of sinks. □

Theorem 3.3. *Any non-singular square TNN matrix can be realized by a network with nonnegative edge weights.*

Proof. We have shown that any such matrix factors into Chevalley generators and torus. Concatinating the corresponding simple networks we obtained the needed network. □

Exercise 3.4. *Show that any TNN matrix can be realized by a network with nonnegative edge weights.*

Exercise 3.5. *Show that the network on the* ****** Figure** *can realize any TNN non-singular $n \times n$ matrix.*

4. Pfaffians

Call two pairs (i,j) and (k,l) *crossing* if when drawn as two arcs above line containing vertices they must cross. For a complete matching π on vertices $[2n]$ let

$$\epsilon(\pi) = (-1)^{\text{number of crossings in } \pi}.$$

Let $A = (a_{i,j})_{1\le i<j\le 2n}$ be a skew-symmetric matrix. Define *pfaffian* $pf(A)$ as follows:

$$pf(A) = \sum_{\pi} \epsilon(\pi) \prod_{(i,j)\in\pi} a_{i,j},$$

where the sum is taken over all complete matchings π on $[2n]$.

Example 4.1. For $n=2$ we have $pf(A) = a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3}$.

Theorem 4.1. *We have* $(pf(A))^2 = \det(A)$.

Proof. Let $E_{2n} \subset S_{2n}$ be the set of permutations where every cycle is even. Let $\sigma = \sigma_1 \dots \sigma_l \in S_{2n}/E_{2n}$ be such that σ_1 is odd with smallest element as small as possible. Then $\sigma \mapsto \sigma' = \sigma_1^{-1} \dots \sigma_n$ is an involution that changes sign of $a_\sigma = a_{1,\sigma(1)} \dots a_{2n,\sigma(2n)}$. As permutations however σ and σ' have the same sign, thus corresponding terms in $\det(A)$ cancel out. Therefore

$$\det(A) = \sum_{\sigma\in E_{2n}} \epsilon(\sigma) a_\sigma.$$

There exists a natural bijection between pairs (π, π') of complete matchings on $[2n]$ and $\sigma \in E_{2n}$ obtained by taking union $\pi \cup \pi'$ and orienting each cycle from smallest element to edge of π. We claim that

$$\epsilon(\pi)\epsilon(\pi') \prod_{(i<j)\in\pi\cup\pi'} a_{i,j} = \epsilon(\sigma) a_\sigma.$$

Once known, the claim implies the theorem. Define $e(\sigma) = |\{i \mid \sigma(i) < i\}|$. The claim is equivalent to equality

$$\epsilon(\pi)\epsilon(\pi') = \epsilon(\sigma)(-1)^{e(\sigma)}.$$

Assume it is true for some σ. We argue that it is true for conjugate of σ by s_i for any i. If i and $i+1$ are consecutive in the same cycle. both sides change sign. If not, both sides stay the same. It remains now for each cycle type to present a σ for which the equality holds. One can choose σ such that each cycle acts on consecutive indexes, both sides in such case are equal to 1. $\square$

Exercise 4.2. *Assume* $a_{i,j} = 1$ *for any* $i, j \in [2n]$, $i < j$. *Show that* $pf(A) = 1$.

4.1. **Planar networks a la Stembridge.** Assume now we have a planar network N in the same class as before with $2n$ sources and any numbe of sinks. Create a skew-symmetric matrix $A(N)$ from it as follows:

$$a_{i,j} = \sum_{p,q} w(p)w(q),$$

where the sum is taken over all pairs of noncrossing paths from sources i and j to any pair of sinks. ****** example**

Theorem 4.3 (Stembridge)**.** *We have*

$$pf(A(N)) = \sum_P w(P),$$

where the sum is taken over all non-crossing families of paths P from all sources to any subset of sinks.

Proof. The proof resembles the proof of Linström lemma. We build a sign reversing involution that gets rid of the terms in $pf(A)$ such that the corresponding path families have a crossing. Assume path family P has crossings and v is the largest one in previously fixed order. Among all paths that go through v assume p_i and p_j start at smallest sources i and j. Swap their parts that follow v and also swap i and j in the corresponding complete matching π. We claim that in each pair $(k, l) \in \pi$ the corresponding pair of paths still does not intersect. This follows from the fact that there are no crossing points on parts of p_i and p_j preceding v. We also argue that this involution is sign-reversing. Indeed, if there is any k between i and j, then the path p_k would intersect either p_i or p_j in a vertex bigger than v, or would path through v. In both case we get contradiction with choices made.

Once we are left only with non-crossing families, we apply the exercise above. $\square$

Exercise 4.4. *Assume $n = 2$ and N is a network with nonnegative edge weights. Show that $a_{1,3}a_{2,4} - a_{1,4}a_{2,3} \geq 0$. Deduce that there are skew-symmetric matrices with all pfaffians nonnegative which do not come from a network with nonnegative edge weights.*

Exercise 4.5. *Can any 4×4 skew symmetric matrix A with $a_{i,j}$, $a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3}$ and $a_{1,3}a_{2,4} - a_{1,4}a_{2,3}$ positive (or nonnegative) be realized by a planar network?*

Exercise 4.6. *Assume N has nonnegative edge weights. Show that for $I \leq J$ of even size we have $|A(N)_{I,J}| \geq 0$.*

Problem 4.7. Give semialgebraic description of skew symmetric matrices that come from networks with nonnegative edge weights.

5. Immanants

5.1. **Immanants.** Let $X = (x_{i,j})_{i,j=1}^{n}$ be a square matrix, and let $f : S_n \to \mathbb{R}$ be a function on symmetric group. If f is constant on conjugacy classes of S_n we call it a *class function*, or a *character*. The functions of the form

$$Imm_f(X) = \sum_{w \in S_n} f(w) x_{1,w(1)} \dots x_{n,w(n)}$$

we call *immanants* of X. If f happens to be a class function we call Imm_f *character immanant* of X. Historically the term was immanant used only in cases when f is an irreducible character of S_n, but we shall use it more widely. In the case when f is an irreducible character we call Imm_f *irreducible immanant*.

Example 5.1. The determinant is an irreducible immanant corresponding to alternating character of S_n.

Example 5.2. The permanent is an irreducible immanant corresponding to trivial character of S_n.

An immanant Imm_f is called *totally positive* if it takes nonnegative values on totally nonnegative matrices X.

Problem 5.1. Describe the cone of totally positive immanants. Is it polyhedral (probably not).

Exercise 5.2. *Describe explicitely the cone of totally positive immanants when X is a*

(1) 2×2
(2) 3×3

matrix.

5.2. **Goulden-Jackson argument.** We know by a result of Brenti that each totally nonnegative matrix $X = X(N)$ is representible by a planar network N with nonnegative edge weights. Let us treat those edge weights as variables, and collect the terms in $Imm_f(X)$ according to monomials in those edge variables. Each such monomial determines a E multiset of edges in N that can be viewed as union of n paths from sources to sinks, possibly in more than one way. The coefficient of such monomial is equal to

$$\sum_{w} f(w) n_w,$$

where $n(w)$ denotes the number of different ways to split E into n paths that determine permutation w between sources and sinks of N. In order for Imm_f to be a totally positive immanant it is *sufficient* for each such coefficient to be nonnegative. Thus, we are interested in all possible vectors of n_w, $w \in S_n$ as we vary E. Each such vector determines a wall in the cone, and all immanants inside this cone are totally positive.

Now, each E can be reduced to sequential concatination of simple building blocks of the following form. Choose an interval of $[i, j] \in [1, n]$ and connect sources i through j and sinks i through j to the same internal vertex. For $k \notin [i, j]$ connect k-th source to k-th sink. By abuse of notation, let

$$[i, j] = \sum_{w \in S_{[i,j]}} w$$

be the sum of all permutations in the parabolic subgroup permuting elements i through j. For example, $[1,2] = 1 + s_1$, while $[2,4] = 1 + s_2 + s_3 + s_2s_3 + s_3s_2 + s_2s_3s_2$. Assume E is obtained by concatinating building blocks corresponding to $[i_1, j_1], \ldots, [i_k, j_k]$. Then we have the following simple proposition.

Proposition 5.3. *The n_w is the coefficient of w in $[i_1, j_1] \ldots [i_k, j_k]$.*

The cone bounded by all such hyperplanes is contained in the cone of total positivity. We call it the *cone of polynomial total positivity.* Note that a priori this cone can have infinitely many walls, and thus not be polyhedral.

Conjecture 5.4 (Stembridge)**.** *The cone of polynomial total positivity is polyhedral.*

Example 5.3. For $n = 3$ the walls of the cone are determined by 1 (identity), $[1,2]$, $[2,3]$, $[1,2][2,3]$, $[2,3][1,2]$, $[1,3]$. One can deduce this from the relation

$$[1,2][2,3][1,2] = [1,3] + [1,2].$$

For $n = 4$ the cone has 24 walls, for $n = 5$ it has 121 walls, beyond that the conjecture is not varified.

For pairs of permutations $u, v \in S_n$ consider immanants of the form

$$Imm_{u,v}(X) = x_{1,u(1)}x_{2,u(2)} \ldots x_{n,u(n)} - x_{1,v(1)}x_{2,v(2)} \ldots x_{n,v(n)}.$$

Theorem 5.5. *$Imm_{u,v}$ is polynomial TP if and only if $u \leq v$ in Bruhat order.*

Exercise 5.6. *Call a subset $I \subseteq S_n$ an ideal if whenever $v \in I$ and $u < v$ in Bruhat order, then $u \in I$. Denote $[I] = \sum_{w \in I} w$, call such elements of the group algebra ideals. Show that product of any two ideals $[I][J]$ is a nonnegative combination of some ideals.*

Proof. Once we know the statement of the exercise, we can deduce the theorem. Indeed, each $[i,j]$ is an ideal. Therefor if $u < v$, reapeated application of the exercise implies that coefficient of u in $\prod_k [i_k, j_k]$ is at least as large as that of v. By the argument above, this implies one direction of the statement.

For the other direction, note that if $u \not\leq v$, one can always find a product $\prod_k [i_k, j_k]$ such that v is largest term occuring in it. Then on a network of corresponding combinatorial type the value of the immanant above is negative. $\square$

5.3. **Irreducible immanants.** Let us see now how the Goulden-Jackson argument can be used to prove positivity of immanants.

Theorem 5.7 (Stembridge)**.** *The irreducible immanants Imm_λ are totally positive.*

The original proof used a result of Greene on Young's seminormal form of elements $[i,j]$. We give a quick and dirty argument using Kazhdan-Lusztig theory.

Proof. We have

$$\sum_w f(w) n_w = \sum_w \chi_\lambda(w) n_w = \chi_\lambda([i_1, j_1] \ldots [i_k, j_k]).$$

In other words, we want to show that irreducible characters of group algebra elements of the form $[i_1, j_1] \ldots [i_k, j_k]$ are nonnegative. It is well known that $C'_{w_0} = [1, n]$, and thus each $[i,j]$ is equal to C'_w for w longest element in $S_{[i,j]}$. The group algebra $\mathbb{C}S_n$ has nonnegative structure constants with respect to Kazhdan-Lusztig basis, which implies that all matrix entries of any C'_w in (Kazhdan-Lusztig realization of) any irreducible representation are nonnegative. A product of several matrices with nonnegative entries has nonnegative trace. □

Note that if one takes the alternating representation of the symmetric group, one recovers the usual Lindstrom lemma argument.

Exercise 5.8. *Show that Imm_λ take nonnegative values for two-row partitions λ if it is known that all minors of the matrix of size at most 2 are nonnegative.*

Exercise 5.9. *Let p and q be two entries in a standard tableau T of shape $\lambda \vdash n$. If p lies in row r column c and q lies in row r' column c', denote*

$$\delta(p,q) = c' - r' - c + r.$$

Symmetric group S_n acts on tableaux of shape λ by permuting the entries. Consider the vector space spanned by standard tableaux T_i of shape λ. Let $\delta_i = \delta(k, k+1)$ taken in T_i. Define matrices $\rho_\lambda(s_k)$ as follows:

$$\rho_\lambda(s_k)|_{i,j} = \begin{cases} 1/\delta_i & \text{if } i = j; \\ 0 & \text{if } i \neq j \text{ and } T_j \neq s_k T_i; \\ 1 - 1/\delta_i^2 & \text{if } i < j \text{ and } T_j = s_k T_i; \\ 1 & \text{if } i > j \text{ and } T_j = s_k T_i. \end{cases}$$

(1) *Show that this gives a representation of the symmetric group S_n.*
(2) *Show that all matrix entries of elements $[i,j]$ in this representation are nonnegative.*

5.4. **Temperley-Lieb immanants.** *Temperley-Lieb algebra TL_n* is a $\mathbb{C}$-algebra generated by $t_1, \dots, t_{n-1}$ subject to relations

$$t_i^2 = 2t_i; \quad t_i t_j = t_j t_i \ \text{ if } \ |i-j| > 1; \quad t_i t_j t_i = t_i \ \text{ if } \ |i-j| = 1.$$

For a permutation $w \in S_n$ let $w = s_{i_1} \dots s_{i_l}$ be a reduced decomposition, and denote $t_w = t_{i_1} \dots t_{i_l}$.

Exercise 5.10. *Show that t_w does not depend on the reduced decomposition chosen for w (as long as w is 321-avoiding). Show that as w ranges over 321-avoiding permutations in S_n, the t_w form a linear basis of TL_n.*

Proposition 5.11. *The map $\theta\colon s_i \mapsto t_i - 1$ determines a homomorphism $\theta\colon \mathbb{C}S_n \to TL_n$.*

Proof. Directly verify relations. □

Elements of TL_n can be represented by *Kauffman diagrams* ****** figure** . A diagram for t_w can be obtained by uncrossing vertically all crossings in the wiring diagram of w. This gives a bijection between 321-avoiding permutations in S_n and non-crossing matchings on $2n$ vertices. If we have a Kauffman diagram containing loops, each loop can be removed at the cast of adding factor 2 to the expression.

For a 321-avoiding permutation w and any permutation v let

$$f_w(v) = \text{ coefficient of } t_w \text{ in } \theta(v) = (t_{i_1} - 1) \dots (t_{i_l} - 1)$$

for a reduced decomposition $v = s_{i_1} \dots s_{i_l}$. Rhoades and Skandera have defined *Temperley-Lieb immanants*

$$Imm_w = Imm_{f_w}.$$

Example 5.4. Let $w = s_1$. Then $f_w(1) = 0$, $f_w(s_1) = 1$, $f_w(s_2) = 0$, $f_w(s_1 s_2) = -1$, $f_w(s_2 s_1) = -1$, $f_w(s_1 s_2 s_1) = 1$. Thus

$$Imm_{s_1}(X) = x_{1,2}x_{2,1}x_{3,3} - x_{1,2}x_{2,3}x_{3,1} - x_{1,3}x_{2,1}x_{3,2} + x_{1,3}x_{2,2}x_{3,1}.$$

5.5. **Decomposition of minor products.** Consider $2n$ points arrange around a disk so that n of them are on the left and n on the right. For a subset $S \subseteq [2n]$ of cardinality $|S| = n$ let $\Theta(S)$ be a set of non-crossing matchings on $[2n]$ *compatible* with S in the following sense: each edge of a matching has one endpoint in S and one not in S. Equivalently, one can think of points in S colored white and points of the complement $[2n]/S$ colored black. Then a matching is compatible with this coloring if each edge has endpoints of different color. By abuse of notation we also denote $\Theta(S)$ the set of 321-avoiding permutations in S_n which are in bijection with matchings in $\Theta(S)$.

Theorem 5.12 (Rhoades-Skandera)**.** *For two subsets $I, J \subseteq [n]$ of the same cardinality and $S = I \cup \{2n+1-i \mid i \notin J\}$ we have*

$$|X_{I,J}||X_{[n]/I,[n]/J}| = \sum_{w \in \Theta(S)} Imm_w(X).$$

Proof. Let us fix a permutation $v \in S_n$ with a reduced decomposition $v = s_{i_1} \cdots s_{i_l}$. The coefficient of the monomial $x_{1,v(1)} \cdots x_{n,v(n)}$ in the expansion of the product of two minors $|X_{I,J}||X_{[n]/I,[n]/J}|$ equals

$$\begin{cases} (-1)^{\mathrm{inv}(I)+\mathrm{inv}([n]/I)} & \text{if } v(I) = J, \\ 0 & \text{if } v(I) \neq J, \end{cases}$$

where $\mathrm{inv}(I)$ is the number of inversions $i < j$, $v(i) > v(j)$ such that $i, j \in I$.

On the other hand, by the definition of Imm_w, the coefficient of $x_{1,v(1)} \cdots x_{n,v(n)}$ in the right-hand side of the identity equals the sum $\sum (-1)^r\, 2^s$ over all diagrams obtained from the wiring diagram of the reduced decomposition $s_{i_1} \cdots s_{i_l}$ by replacing each crossing "$\times$" with either a *vertical uncrossing* " $)($ " or a *horizontal uncrossing* " $\asymp$ " so that the resulting diagram is S-compatible, where r is the number of horizontal uncrossings " $\asymp$ " and s is the number of internal loops in the resulting diagram.

Let us pick directions of all strands and loops in such diagrams so that the initial vertex in each strand belongs to S (and, thus, the end-point is not in S). There are 2^s ways to pick directions of s internal loops. Thus the above sum can be written as the sum $\sum (-1)^r$ over such *directed Temperley-Lieb* diagrams.

****** example**

Let us construct a sign reversing partial involution ι on the set of such directed Temperley-Lieb diagrams. If a diagram has a *misaligned uncrossing*, i.e., an uncrossing of the form " $)($ ", " $)($ ", " $\asymp$ ", or " $\asymp$ ", then ι switches the leftmost such uncrossing according to the rules $\iota :\ )(\ \leftrightarrow\ \asymp$ and $\iota :\ )(\ \leftrightarrow\ \asymp$. Otherwise, when the diagram involves only *aligned uncrossings* " $)($ ", " $)($ ", " $\asymp$ ", " $\asymp$ ", the involution ι is not defined.

****** example**

Since the involution ι reverses signs, this shows that the total contribution of all diagrams with at least one misaligned uncrossing is zero. Let us show that there is at most one S-compatible directed Temperley-Lieb diagram with all aligned uncrossings. If we have a such diagram, then we can direct the strands of the wiring diagram for $v = s_{i_1} \ldots s_{i_l}$ so that each segment of the wiring diagram has the same direction as in the Temperley-Lieb diagram. In particular, the end-points of strands in the wiring diagram should have different colors. Thus each strand starting at an element of J should finish at an element of $I^\wedge$, or, equivalently, $v(I) = J$. The directed Temperley-Lieb diagram can be uniquely recovered from this directed wiring diagram by replacing the crossings with uncrossings, as follows: $\to$, $\to$, $\to$, $\to$. Thus the coefficient of $x_{1,v(1)} \cdots x_{n,v(n)}$ in the right-hand side of the needed identity is zero, if $v(I) \neq J$, and is $(-1)^r$, if $v(I) = J$, where r is the number of crossings of the form “ ” or “ ” in the wiring diagram. In other words, r equals the number of crossings such that the right end-points of the pair of crossing strands have the same color. This is exactly the same as the expression for the coefficient in the left-hand side of the needed identity. $\square$

Corollary 5.13. *TL immanants form a basis for the space of product of complementary minors.*

Proof. They form a generating set, and their number is equal to the dimension. $\square$

5.6. **TL immanants are TP.** Above we have computed Imm_{s_1} in 3×3 matrices. Recall that to check that this immanant is totally positive it suffices to verify six linear inequalities for f_{s_1}. ****** check inequalities explicitely** .

It turns out that all Temperley-Lieb immanants are totally positive. The proof below gives an explicite interpretation to $Imm_w(X)$ in terms of weighted network N realizing $X = X(N)$.

Let E be a multiset of edges in a network N such that

- E can be decomposed into union of paths from n sources of N to its n sinks;
- no vertex in N is contained in more than two of such paths.

For a non-crossing matching w on $[2n]$ we say that E is *of type* w if after uncrossing vertically all vertices in E the resulting matching is w ****** figure, example** . Denote $mult(E)$ the number of closed loops formed after such uncrossing.

Theorem 5.14 (Rhoades-Skandera)**.** *If $X = X(N)$ is the matrix associated to weighted network N, then*

$$Imm_w(X) = \sum_E 2^{mult(E)} wt(E),$$

where the sum is taken over all E of type w. In particular, Temperley-Lieb immanants of totally nonnegative matrices are nonnegative.

Proof. The function f_w is clearly linear, and thus to give interpretation to the immanants we need to compute $f_w([i_1, j_1] \ldots [i_k, j_k])$. Now, if $j - i > 1$, then it is easy to check that $\theta([i,j]) = 0$, while if $j - i = 1$, then $\theta([i,j]) = t_i$. Thus, only the terms without triple crossings will contribute. The value of $f_w([i_1, j_1] \ldots [i_k, j_k])$ is then either 0 or the desired power of 2, depending on whether the resulting product of t_i-s produces (constant times) t_w or some other basis element of TL_n. $\square$

The following is an example of how TL immanants can be used to prove inequalities.

Corollary 5.15. *For a TNN matrix X we have*

$$x_{i,i} |X_{[n]/i,[n]/i}| \geq |X|.$$

Proof. One of the immanants accuring in the decomposition of this product is the determinant, the rest are nonnegative. $\square$

Exercise 5.16. *Show that TL immanants of Jacobi-Trudi matrices are Schur positive.*

Exercise 5.17. *Assume some pairs of neighboring rows and columns of a TNN matrix are equal. Find a necessary and sufficient consition for a particular TL immanant* Imm_w *not to vanish constantly on such matrices.*

Exercise 5.18. *Use TL immanants to prove Plücker relations.*

5.7. **Monomial immanants.** To any symmetric function f one can associate a (possibly virtual) character χ_f via

$$\chi_f(w) = \langle f, p_{\rho(w)} \rangle.$$

When $f = s_\lambda$ this is the irreducible character labeled by λ.Denote

$$\eta_\lambda = \chi_{m_\lambda},$$

where m_λ is the monomial symmetric function labeled by λ, and call the corresponding immanants *monomial immanants.*

Conjecture 5.19 (Stembridge)**.** *Monomial immanants of TNN matrices are nonnegative.*

Theorem 5.20. *The monomial immanant* $\eta_{(2,1^{n-2})}$ *is TP.*

Proof. Note that $m_{(2,1^{n-2})} = e_{n-1}p_1 - ne_n$. We claim that

$$\eta_{(2,1^{n-2})} = -n|X| + \sum_{i=1}^{n} x_{i,i}|X_{[n]/i,[n]/i}|.$$

Indeed, the character corresponding to e_n is alternating character, and thus the corresponding immanant is just a determinant. Next, $\langle e_{n-1}p_1, p_\rho \rangle = 0$ unless ρ has a part of size 1. Thus permutations without fixed point do not appear in the immanant of $e_{n-1}p_1$. It remains to note that p_λ-s form an orthogonal basis for symmetric functions and

$$\langle p_{\rho'} p_{1^k}, p_{\rho'} p_{1^k} \rangle = k \langle p_{\rho'} p_{1^{k-1}}, p_{\rho'} p_{1^{k-1}} \rangle.$$

Now, using Theorem 5.12 one checks that for each i

$$x_{i,i}|X_{[n]/i,[n]/i}| - |X|$$

is a nonnegative combination of Temperley-Lieb immanants, and thus is nonnegative when evaluated on TNN matrices. □

Exercise 5.21. *Assume λ has two columns. Show that* Imm_{η_λ} *is a nonnegative combination of Temperley-Lieb immanants. Deduce that such immanants are TP.*

Exercise 5.22. *Assume* $\lambda = r^l$ *is a rectangular shape. Show that*

$$Imm_{\eta_\lambda}(X) = \sum |X_{I_1,I_2}||X_{I_2,I_3}| \ldots |X_{I_r,I_1}|,$$

where the sum ranges over all ordered partitions $(I_1, \ldots, I_r)$ *of* $[n]$ *into disjoint subsets of size* l.

6. Edrei-Thoma theorem

6.1. **The tower of symmetric group algebras.** Let S_n be a symmetric group on n symbols. There exists a natural embedding $S_n \hookrightarrow S_{n+1}$ such that the elements of S_n acts on first n symbols and fixes the $(n+1)$-st one. Recall that irreducible characters π_λ of the symmetric group S_n are indexed by partitions λ of n. Let λ be a partition of n, and μ a partition of $n+1$. Then by Frobenius reciprocity

$$\langle Res_{S_n}^{S_{n+1}} \pi_\mu, \pi_\lambda \rangle = \langle \pi_\mu, Ind_{S_n}^{S_{n+1}} \pi_\lambda \rangle.$$

Denote this number by $\Xi(\lambda, \mu)$. One can form a graph then by connecting partitions λ and μ by an edge of multiplicity $\Xi(\lambda, \mu)$. Gluing such graphs together for all n we obtain the *Bratelli diagram*, or *branching graph* of the tower of algebras

$$\mathbb{C}[S_0] \hookrightarrow \mathbb{C}[S_1] \hookrightarrow \mathbb{C}[S_2] \hookrightarrow \dots.$$

Here for convinience we introduce the group algebra $\mathbb{C}[S_0] = \mathbb{C}$ of the nonexisting group S_0. It turns out that

$$\Xi(\lambda, \mu) = \begin{cases} 1 & \text{if } \lambda \text{ is obtained from } \mu \text{ by deleting a single corner box;} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, all edges of the Bratelli diagram in this case are multiplicity-free. The resulting graph $\mathbb{Y}$ on partitions is called *Young's lattice*, and is shown on the figure. ****** figure**

We denote

$$\mathbb{C}[S_\infty] = \varinjlim \mathbb{C}[S_n]$$

the inductive limit of algebras $\mathbb{C}[S_n]$. $\mathbb{C}[S_\infty]$ is an example of a *locally semisimple algebra*, see [Ker].

6.2. **Characters of S_∞.** A *character* of a discrete group G is any function $\chi\colon G \to \mathbb{C}$ with the following properties:

- (positive definitness) for any $g_1, \dots, g_n \in G$ and any $z_1, \dots, z_n \in \mathbb{C}$ we have
$$\sum_{i,j=1}^{n} \chi(g_i g_j^{-1}) z_i \bar{z}_j \geq 0;$$
- (centrality) for any $g, h \in G$ we have $\chi(gh) = \chi(hg)$;
- (normalization) for unity $e \in G$ we have $\chi(e) = 1$.

Denote $\pi(G)$ the set of characters of G endowed with pointwise convergence topology. Clearly $\pi(G)$ is convex and compact. Hense each character $\chi \in \pi(G)$ is the barycenter of a measure supported by the set of extreme points $\mathfrak{E}(G)$.

Exercise 6.1. *Show that when G is a finite group, $\mathfrak{E}(G)$ consists of normalized characters of irreducible representations of G.*

In fact, it can be shown that space $\pi(G)$ is always a simplex, see [Ker].

We wish to describe the set $\pi(S_\infty)$ of characters of infinite symmetric group. This will be accomplished in three steps as follows:

(1) characters can be identified with harmonic functions on Young's lattice $\mathbb{Y}$;
(2) extreme harmonic functions correspond homomorphisms from the ring Λ of symmetric functions to $\mathbb{R}$, taking nonnegative values on Schur functions (*totally positive homomorphisms*);

(3) totally positive homomorphisms are in bijection with *totally positive sequences* and are classified by the Edrei-Thoma theorem.

6.3. **Characters as harmonic functions.** Let us restrict a character χ of $\mathbb{C}[S_\infty]$ to a character χ_n of $\mathbb{C}[S_n]$:

$$\chi_n = Res^{\mathbb{C}[S_\infty]}_{\mathbb{C}[S_n]}\chi.$$

The resulting function by the above exercise is a nonnegative combination of irreducible characters of S_n:

$$\chi_n = \sum_{\lambda \vdash n} \varphi(\lambda)\pi_\lambda.$$

If we are restricting to $\mathbb{C}[S_{n-1}]$, the result should not depend on whether we have the intermidiate step of restricting to $\mathbb{C}[S_n]$. Therefore

$$Res^{\mathbb{C}}_{\mathbb{C}[S_{n-1}]}[S_n](\sum_{\lambda \vdash n} \varphi(\lambda)\pi_\lambda) = \sum_{\mu \vdash n-1} \varphi(\mu)\pi_\mu,$$

and thus

$$\varphi(\mu) = \sum_{\mu \prec \lambda} \varphi(\lambda),$$

where $\mu \prec \lambda$ denotes the covering relation in $\mathbb{Y}$. Such functions φ we call *harmonic.*

Theorem 6.2. [Ker, Theorem 1] *Normalized ($\varphi(\emptyset) = 1$) harmonic functions $\varphi : \mathbb{Y} \to \mathbb{R}_+$ are in bijection with characters $\chi : \mathbb{C}[S_\infty] \to \mathbb{C}$, given by*

$$\chi(w) = \sum_{\lambda \vdash n} \varphi(\lambda)\pi_\lambda(a).$$

Proof. We have already seen that each character corresponds to a harmonic function. On the other hand, the three conditions one must check for a function to be a character hold once they hold inside some $\mathbb{C}[S_n] \hookrightarrow \mathbb{C}[S_\infty]$. By the above exercise, the latter is true for any character coming from a harmonic function. □

6.4. **Extreme harmonic functions as totally positive homomorphisms.** Let us multiply partitions the way we multiply Schur functions:

$$\lambda \cdot \mu = \sum c^\nu_{\lambda,\mu}\nu \quad \text{if} \quad s_\lambda s_\mu = \sum c^\nu_{\lambda,\mu} s_\nu.$$

Extend functions $\varphi : \mathbb{Y} \to \mathbb{R}$ by linearily to $\mathbb{C}\mathbb{Y}$.

Theorem 6.3. *A (normalized) harmonic function $\varphi : \mathbb{Y} \to \mathbb{R}_+$ is extreme if and only if for any $f, g \in \mathbb{C}\mathbb{Y}$ we have*

$$\varphi(f)\varphi(g) = \varphi(f \cdot g).$$

Proof. First, we argue that any extreme harmonic function is a totally positive homomorphism. The definition of a harmonic function implies that for any $f \in \mathbb{C}\mathbb{Y}$ we have

$$\varphi(\square \cdot f) = \varphi(\square)\varphi(f).$$

Indeed, it is enough to check this for the linear basis of partitions. The statement follows from the normalization and harmonic conditions

$$\varphi(\square \cdot \mu) = \varphi(\sum_{\mu \prec \lambda} \lambda) = \varphi(\mu) = \varphi(\square)\varphi(\mu).$$

Assume now $\lambda \vdash n$ and $\mu \vdash m$ are any two elements of $\mathbb{Y}$. We argue that $\varphi(\lambda \cdot \mu) = \varphi(\lambda)\varphi(\mu)$. There are two cases to consider, either one of $\varphi(\lambda)$ or $\varphi(\mu)$ vanishes or not. If say $\varphi(\lambda) = 0$, then

$$0 \leq \varphi(\lambda \cdot \mu) \leq \varphi(\lambda \cdot \mu) + \varphi(\lambda \cdot (\square^m - \mu)) = \varphi(\lambda) = 0,$$

and the statement follows. We use the fact that $\square^m - \mu$ is a nonnegative combination of elements of $\mathbb{Y}$.

Consider now the case when both $\varphi(\lambda)$ or $\varphi(\mu)$ are strictly positive. Then one can define two new functions

$$\varphi_1(\mu) = \frac{\varphi(\lambda \cdot \mu)}{\varphi(\lambda)} \quad \text{and} \quad \varphi_2(\mu) = \frac{\varphi((\square^n - \lambda) \cdot \mu)}{\varphi(\square^n - \lambda)},$$

extending by linearity to $\mathbb{C}\mathbb{Y}$. Those functions are again normalized harmonic functions on $\mathbb{Y}$. Since

$$\varphi = \varphi(\lambda)\varphi_1 + \varphi(\square^n - \lambda)\varphi_2$$

is a nonnegative linear combination of φ_1 and φ_2, it can only be extreme if either $\varphi = \varphi_1$ or $\varphi = \varphi_2$. Each of those relations implies $\varphi(\lambda \cdot \mu) = \varphi(\lambda)\varphi(\mu)$.

Now we argue that if φ is a totally positive homomorphism, it has to be extreme. Indeed, assume it is not. Then it can be written as a barycenter of some measure ρ supported on the boundary:

$$\varphi = \int_{\mathfrak{E}} \varphi_\theta d\rho(\theta).$$

Then for any $f \in \mathbb{C}\mathbb{Y}$ we have

$$Var(\varphi(f)) = \int_{\mathfrak{E}} \varphi_\theta(f)^2 d\rho(\theta) - (\int_{\mathfrak{E}} \varphi_\theta(f) d\rho(\theta))^2 = \varphi(f^2) - \varphi(f)^2 = 0,$$

which can hold only if ρ is concentrated in one point. $\square$

Edrei-Thoma theorem. We have shown that the points of the boundary $\mathfrak{E}(S_\infty)$ can be identified with homomorphisms from the ring Λ of symmetric functions to $\mathbb{R}$ that take nonnegative values on Schur functions (or *totally positive homomorphisms*). Ring λ is a polynomial ring over the basis of complete homogenous symmetric functions $h_n = s_{(n)}$. Thus to specify a homomorphism $\varphi\colon \Lambda \to \mathbb{R}$ it suffices to specify its value on every h_n, $n = 1, 2, \ldots$.

Let $\mathbf{a} = (a_0 = 1, a_1, a_2, \ldots)$ be a sequence of real numbers, and let $\varphi_{\mathbf{a}}$ be the corresponding homomorphism given by $\varphi_{\mathbf{a}}(h_n) = a_n$, $n = 1, 2, \ldots$. Construct an infinite upper triangular Toeplits matrix $X_{\mathbf{a}}$ as follows:

$$x_{i,j} = \begin{cases} a_{j-i} & \text{if } j \geq i; \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.4. *Homomorphism $\varphi_{\mathbf{a}}$ is totally positive if and only if the infinite Toeplitz matrix $X_{\mathbf{a}}$ is totally nonnegative.*

Proof. The minors of $X_{\mathbf{a}}$ are exactly the values of $\varphi_{\mathbf{a}}$ on skew Schur functions. Because of the Littlewood-Richardson rule, $\varphi_{\mathbf{a}}$ takes nonnegative values on skew Schur functions if and only if it takes nonnegative values on Schur functions. $\square$

In the case $X_{\mathbf{a}}$ is totally nonnegative the sequence a is called a *totally positive sequence*, and the associated formal power series

$$a(t) = a_0 + a_1 t + a_2 t^2 + \dots$$

is called a *totally positive function*.

Lemma 6.5. *The set of totally positive functions is closed under multiplication (they form a semigroup).*

Proof. Follows from the fact that totally nonnegative matrices form a semigroup. □

Exercise 6.6. *Check that the functions $1+\beta t$, $1/(1-\alpha t)$ and $e^{\gamma t}$ are totally positive for $\alpha, \beta, \gamma \geq 0$.*

Exercise 6.7. *Give an explicite combinatorial interpretation to minors of $e^{\gamma t}$.*

Theorem 6.8 (Edrei-Thoma)**.** *Every totally positive function $a(t)$ can be written in the form*

$$a(t) = e^{\gamma t} \prod_i \frac{1+\beta_i t}{1-\alpha_i t},$$

where $\alpha_i, \beta_i, \gamma \geq 0$ and $\sum_i (\alpha_i + \beta_i) < \infty$.

The proof will be given in the next section.

Exercise 6.9. *Assume $a(t) = e^{\gamma t} \prod_i \frac{1+\beta_i t}{1-\alpha_i t}$ and let $\varphi_{\mathbf{a}}$ be the corresponding totally positive homomorphism.*

1. *Find $\sum_{n \geq 0} \varphi_{\mathbf{a}}(e_n) t^n$ (where $\varphi_{\mathbf{a}}(e_0) = 1$).*
2. *Show that $\varphi_{\mathbf{a}}(p_n) = \sum_i \alpha_i^n + (-1)^{n+1} \sum_i \beta_i^n$ for $n \geq 2$.*
3. *Express $\varphi_{\mathbf{a}}(s_\lambda)$ as a generating function of monomials associated to some kind of semistandard tableaux.*

Corollary 6.10. [Ker, Theorem 8] *Consider the space of pairs of sequences*

$$\Delta = \Big\{ \alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq 0), \;\; \beta = (\beta_1 \geq \beta_2 \geq \ldots \geq 0) \;\mid\; \sum (\alpha_i + \beta_i \leq 1) \Big\}$$

with pointwise convergence topology (Thoma simplex). Then the set of extreme characters $\mathfrak{E}(S_\infty)$ is homeomorphic to Δ. The character $\chi_{\alpha,\beta} \in \mathfrak{E}(S_\infty)$ corresponding to a pair $(\alpha, \beta) \in \Delta$ is determined by its values

$$\chi_{\alpha,\beta}(w_n) = \sum_i \alpha_i^n + (-1)^{n+1} \sum_i \beta_i^n, \quad n = 2, 3, \ldots$$

on one-cycle permutations by the formula

$$\chi_{\alpha,\beta}(w) = \prod_i \chi_{\alpha,\beta}(w_{n_i}),$$

where $w = w_{n_1} w_{n_2} \ldots$ is a decomposition of $w \in S_\infty$ into the product of one-cycle permutations.

Proof. The multiplicativity in terms of cycle structure of $w \in S_\infty$ follows from the multiplicativity of characters in $\mathfrak{E}(S_\infty)$ as in Theorem 6.3. Indeed, for a symmetric function $f \in \Lambda$ let $\chi(f) = \langle \sum_\lambda \varphi(\lambda) s_\lambda, f \rangle$, where we use the inner product of Λ. Then $\chi(fg) = \chi(f)\chi(g)$, as it is easily checked on the linear basis of Schur functions using multiplicativity of φ. It remains to note that $\chi(p_{\rho(w)}) = \chi(w)$, where $\rho(w)$ denotes the cycle type of $w \in S_\infty$.

The formula for a one-cycle permutation follows from the Murnaghan-Nakayama rule and the exercise above. $\square$

In fact, one can give an explicite interpretation to α_i and β_i via the following theorem. Denote $d(\lambda)$ the dimension of irreducible symmetric group representation corresponding to λ.

Theorem 6.11. *Consider a sequence of Young diagrams $\lambda^n \vdash n$, $n = 1, 2, \ldots$. The following conditions are equivalent:*

(1) *The limits*

$$\lim_{n\to\infty} \frac{\chi^{\lambda^n}(w)}{d(\lambda^n)} = \chi(w)$$

exist for all $w \in S_\infty$.

(2) *The limits of the relative row and column lengths*

$$\lim_{n\to\infty} \frac{\lambda^n_k}{n} = \alpha_k \qquad \lim_{n\to\infty} \frac{(\lambda^n)_k}{n} = \beta_k$$

exist for all $k = 1, 2, \ldots$. The limiting character consides with character $\chi_{\alpha,\beta}$ above.

6.5. **Aissen-Schoenberg-Whitney argument.**

Exercise 6.12. *Show that if $a(t)$ is totally positive, then so is $(a(-t))^{-1}$.*

Lemma 6.13. *Every totally positive function $a(t)$ can be written in the form*

$$a(t) = e^{b(t)} \prod_i \frac{1+\beta_i t}{1-\alpha_i t},$$

where $\alpha_i, \beta_i \geq 0$, $\sum_i(\alpha_i + \beta_i) < \infty$ and $b(t)$ is an entire function.

Proof. First we show that one can write $a(t)$ as

$$a(t) = c(t) \prod_i \frac{1+\beta_i t}{1-\alpha_i t},$$

where α_i, β_i are as above, $c(t)$ is totally positive and both $c(t)$ and $(c(-t))^{-1}$ are entire. Note that by total positivity of $X_{\mathbf{a}}$, we have

$$\frac{x_{1,2}}{x_{2,2}} \geq \frac{x_{1,3}}{x_{2,3}} \geq \frac{x_{1,4}}{x_{2,4}} \geq \ldots \geq 0,$$

and therefore the limit

$$\alpha = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exists. We claim that $(1-\alpha t)a(t)$ is a totally positive function. Indeed, it suffices to check that row-solid minors of $X_{\mathbf{a}}$ remain nonnegative. Consider the minor of $X_{\mathbf{a}}$ with rows i through $i+k-1$ and columns $j_1 < j_2 < \ldots < j_k$. We have

$$\lim_{l \longrightarrow \infty} \frac{\det \begin{pmatrix} x_{i,j_1} & x_{i,j_2} & \ldots & x_{i,j_k} & x_{i,l} \\ x_{i+1,j_1} & x_{i+1,j_2} & \ldots & x_{i+1,j_k} & x_{i+1,l} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{i+k-1,j_1} & x_{i+k-1,j_2} & \ldots & x_{i+k-1,j_k} & x_{i+k-1,l} \\ x_{i+k,j_1} & x_{i+k,j_2} & \ldots & x_{i+k,j_k} & x_{i+k,l} \end{pmatrix}}{x_{i+k,l}} =$$

$$\lim_{l \longrightarrow \infty} \det \begin{pmatrix} x_{i,j_1} & x_{i,j_2} & \ldots & x_{i,j_k} & x_{i,l}/x_{i+k+1,l} \\ x_{i+1,j_1} & x_{i+1,j_2} & \ldots & x_{i+1,j_k} & x_{i+1,l}/x_{i+k+1,l} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{i+k-1,j_1} & x_{i+k-1,j_2} & \ldots & x_{i+k-1,j_k} & x_{i+k-1,l}/x_{i+k,l} \\ x_{i+k,j_1} & x_{i+k,j_2} & \ldots & x_{i+k,j_k} & 1 \end{pmatrix} =$$

$$\det \begin{pmatrix} x_{i,j_1} & x_{i,j_2} & \ldots & x_{i,j_2} & \alpha^k \\ x_{i+1,j_1} & x_{i+1,j_2} & \ldots & x_{i+1,j_2} & \alpha^{k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{i+k-1,j_1} & x_{i+k-1,j_2} & \ldots & x_{i+k-1,j_2} & \alpha \\ x_{i+k,j_1} & x_{i+k,j_2} & \ldots & x_{i+k,j_2} & 1 \end{pmatrix} =$$

$$\det \begin{pmatrix} x_{i,j_1} - \alpha x_{i+1,j_1} & x_{i,j_2} - \alpha x_{i+1,j_2} & \ldots & x_{i,j_k} - \alpha x_{i+1,j_k} & 0 \\ x_{i+1,j_1} - \alpha x_{i+2,j_1} & x_{i+1,j_2} - \alpha x_{i+2,j_2} & \ldots & x_{i+1,j_k} - \alpha x_{i+2,j_k} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{i+k-1,j_1} - \alpha x_{i+k,j_1} & x_{i+k-1,j_2} - \alpha x_{i+k,j_2} & \ldots & x_{i+k-1,j_k} - \alpha x_{i+k,j_k} & 0 \\ x_{i+k,j_1} & x_{i+k,j_2} & \ldots & x_{i+k,j_k} & 1 \end{pmatrix} =$$

$$\det \begin{pmatrix} x_{i,j_1} - \alpha x_{i+1,j_1} & x_{i,j_2} - \alpha x_{i+1,j_2} & \ldots & x_{i,j_k} - \alpha x_{i+1,j_k} \\ x_{i+1,j_1} - \alpha x_{i+2,j_1} & x_{i+1,j_2} - \alpha x_{i+2,j_2} & \ldots & x_{i+1,j_k} - \alpha x_{i+2,j_k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i+k-1,j_1} - \alpha x_{i+k,j_1} & x_{i+k-1,j_2} - \alpha x_{i+k,j_2} & \ldots & x_{i+k-1,j_k} - \alpha x_{i+k,j_k} \end{pmatrix}.$$

and the claim follows. This way we can keep factoring factors of the form $1/(1-\alpha_i t)$ from $a(t)$, possibly infinitely many times. At the limit however we get an entire totally positive function. Similarly, we can keep factoring factors of the form $1/(1-\beta_i t)$ from $(a(-t))^{-1}$ until we obtain an entire function at the limit. Combining both, we get the desired function $c(t)$.

Now we show that $c(t) = e^{b(t)}$ for an entire $b(t)$. Define $d(t) = e^{-\int (c(t))^{-1} c'(t) dt}$ where $c'(t)$ denotes $\frac{d}{dt}(c(t))$. Clearly, $d(t)$ is an entire function. We may pick the constant of integration so that $d(0) = 1$. However,

$$\frac{d}{dt}(c(t)d(t)) = c'(t)d(t) - c(t)(c(t))^{-1}c'(t)d(t) = 0.$$

Thus $c(t)d(t)$ is a constant. But $c(0)d(0) = 1$, so the result holds with

$$b(t) = -\int (c(t))^{-1} c'(t) dt$$

which is clearly entire. $\square$

6.6. **Edrei-Thoma argument.** Denote $M(r)$ the maximum of $|f(t)|$ on $|t| = r$. Then the *order* of an entire function $f(t)$ is the smallest τ such that

$$M(r) \leq e^{r^{\tau+\epsilon}}$$

for any given $\epsilon > 0$ and r sufficiently large.

Given an entire function $f(t)$ and a point z, let $t_1, t_2, \dots$ be the sequence of all points such that $f(t_i) = z$. The *exponent of convergence* (also called *index of convergence*) of z is the number

$$\rho_z = \inf\{\rho \geq 0 \mid \sum_i |t_i|^{-\rho} < \infty\}.$$

Theorem 6.14 (Nevanlinna). *If $\rho_z \leq \rho$ and $\rho_w \leq \rho$ for two distinct values z, w of an entire function f, then the order of f is at most ρ.*

Now we are ready to proof the Edrei-Thoma theorem.

Proof. Let $a(t) = a_0 + a_1 t + a_2 t^2 \dots$ be a totally positive function and $b(t) = b_1 t + b_2 t^2 + \dots$ be an entire function such that $a(t) = e^{b(t)}$. We show that $b_2 = b_3 = \dots = 0$. First we show that $b_3 = b_5 = \dots = 0$.

The function $f(t) = a_0 + a_2 t + a_4 t^2 + \dots$ is also totally positive and entire. Hense it can be written as

$$f(t) = e^{g(t)} \prod (1 + \theta_i t),$$

where $g(t)$ is entire, $\theta_i \geq 0$, $\sum \theta_i < \infty$. Equality $a(t) + a(-t) = 2f(t^2)$ can be rewritten then as

$$e^{b(t)} + e^{b(-t)} = 2e^{g(t^2)} \prod (1 + \theta_i t^2).$$

Set $h(t) = e^{b(t) - b(-t)}$. Then

$$h(t) + 1 = 2e^{g(t^2) - b(-t)} \prod (1 + \theta_i t^2).$$

We see that $h(t)$ is entire and takes value -1 at a sequence of points of exponent of convergence not exceeding 2. By Nevanlinna's theorem, the order of $h(t)$ is no larger than 2, which immidiately implies $b_3 = b_5 = \dots = 0$.

Now, both $e^{b(t)}$ and $e^{-b(-t)}$ are totally positive, and $e^{b(t)} e^{-b(-t)} = e^{2b_1 t}$. Then $e^{b(t)}$ grows no faster than $\frac{e^{2b_1 t}}{e^{-b(0)}}$, and thus $b_2 = b_4 = \dots = 0$. $\square$

7. Cells in $GL_n(\mathbb{R})_{\geq 0}$

For a real parameter $a \in \mathbb{R}$ and an integer k, we define $e_k(a) = (x_{i,j})_{i,j=1}^n \in GL_n(\mathbb{R})$ to be the matrix given by

$$x_{i,j} = \begin{cases} 1 & \text{if } i = j \\ a & \text{if } j = i+1 \text{ and } i = k \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, define $f_k(a) \in GL_n(\mathbb{R})$ to be the transpose of $e_k(a)$.

Finally, define $h_k(a)$ by

$$x_{i,j} = \begin{cases} 1 & \text{if } i = j \neq k \\ a & \text{if } i = j = k \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 7.1. *The following identities hold:*

$$\begin{aligned} h_{k+1}(a)e_k(b) &= e_k(b/a)h_{k+1}(a) \\ h_k(a)e_k(b) &= e_k(ab)h_k(a) \\ h_k(a)e_j(b) &= e_j(b)h_k(a) if\ k \neq j, j+1 \\ h_{k+1}(a)f_k(b) &= f_k(ab)h_{k+1}(a) \\ h_k(a)f_k(b) &= f_k(b/a)h_k(a) \\ h_k(a)f_j(b) &= f_j(b)h_k(a) if\ k \neq j, j+1 \\ e_i(a)f_j(b) &= f_j(b)e_i(a) if\ i \neq j \\ e_i(a)f_i(b) &= f_i(b/(1+ab))h_i(1+ab)h_{i+1}(1/(1+ab))e_i(a/(1+ab)) \end{aligned}$$

Proof. Direct computation. □

Consider the monoid $G_{\geq 0}$ generated by the e_i, f_i and h_i with nonnegative parameters.

Theorem 7.2. *$G_{\geq 0}$ coinsides with $GL_n(\mathbb{R})_{\geq 0}$.*

Proof. We already proved this theorem when we prooved that elements of $GL_n(\mathbb{R})_{\geq 0}$ factor into Chevalley generators and torus elements with nonnegative parameters. □

Lemma 7.3. *Any element of $GL_n(\mathbb{R})_{\geq 0}$ can be rewritten in the form $\prod f_{i_j}(a_j) \prod h_{i_j}(b_j) \prod e_{i_j}(c_j)$.*

Proof. We proved that before. One can also use the relations above to sort factors in that order. □

Theorem 7.4. *The following identities hold:*

$$\begin{aligned} e_i(a)e_{i+1}(b)e_i(c) &= e_{i+1}(bc/(a+c))e_i(a+c)e_{i+1}(ab/(a+c)) \\ f_i(a)f_{i+1}(b)f_i(c) &= f_{i+1}(bc/(a+c))f_i(a+c)f_{i+1}(ab/(a+c)) \end{aligned}$$

Proof. Direct computation. □

Let U be the upper unitriangular subgroup of $GL_n(\mathbb{R})$, and let $U_{\geq 0}$ be its totally nonnegative part. Let $w \in S_n$ and $w = s_{i_1} \ldots s_{i_l}$ be a reduced word for w. Consider the map $\mathbb{R}^l_{>0} \to U_{\geq 0}$ given by

$$(a_1, \ldots, a_l) \mapsto e_{i_1}(a_1) \ldots e_{i_l}(a_l).$$

Denote $U(w)$ the image of this map.

Theorem 7.5. (1) *as suggested by notation, $U(w)$ depends only on w and not on the reduced decomposition chosen;*
(2) *if $u \neq w$ then $U(u) \cap U(w) = \emptyset$;*
(3) *the map is an injection.*

The first statement of the theorem follows trivially from relations satisfied by Chevalley generators, and the fact that any two reduced decompositions are connected by a sequence of braid and commutation moves. We postpone the other two statements until we describe Bruhat decomposition and some of its properties.

7.1. **Bruhat decomposition.** Let B^+ and B^- be the upper and the lower triangular Borel subgroups of $GL_n(\mathbb{R})$, respectively. For each $w \in S_n$ call the set B^-wB^- the *Bruhat cell* B_w^- associated to w. Similarly one can also define Bruhat cells $B_u^+ = B^+uB^+$. Finally, define *double Bruhat cells* $B_{u,w} = B_u^+ \cap B_w^-$.

Proposition 7.6. *Any element of $GL_n(\mathbb{R})$ belongs to one of the Bruhat cells.*

Proof. One can always transform an element of $GL_n(\mathbb{R})$ into a permutation matrix using only lower (resp., only upper) triangular row and column operations. □

We shall use the following characterization of Bruhat cells in terms of vanishing/nonvanishing of minors. Call a submatrix $X_C = X_{I(C),J(C)}$ a w-NE-ideal if the following condition holds: if $(i, w(i)) \in C$ and j is such that $j < i$ but $w(j) > w(i)$, then $(j, w(j)) \in C$. Call a submatrix $X_D = X_{I(D),J(D)}$ a *shifted w-NE-ideal* if $I(D) \leq I(C)$ and $J(C) \leq J(D)$ in termwise order for some w-NE-ideal C, but $D \neq C$.

Proposition 7.7. *For $X \in GL_n(\mathbb{R})$ we have $X \in B_w^-$ if and only if the following two conditions are satisfied:*

- $|X_C| \neq 0$ *for w-NE-ideals C,*
- $|X_C| = 0$ *for shifted w-NE ideals C.*

Proof. The statemetn is easily varified for $X = w$. It is also easy to see that the conditions are preserved by left and right multiplication by B^-. □

Of course, one can similarly characterize cells B_w^+ via w-SW-ideals.

Corollary 7.8. *Bruhat cells B_w^- are disjoint with each other. Same for B_w^+.*

Proof. Assume two cells B_w^- and B_u^- are not disjoint. Then $w(1) = u(w)$, since otherwise we would have a single cell which is a w-NE-ideal and shifted u-NE-ideal, or the other way around. Similarly, we also must have $w(2) = u(2)$, since otherwise we can obtain a contradiction for C with rows $1, 2$ and columns $w(1)$ and the larger of $w(2), u(2)$. Proceeding like that we conclude that $w = u$. □

Now let us go back to products of Chevalley generators.

Proposition 7.9. *We have $U(w) \subset B_w^-$.*

Proof. The proof is by induction on length of w. For $w = I$ the statement is clear. Now observe that as we multiply by $e_{i_j}(a_j)$ say on the left, we swap two rows in w in order to make it $w' = s_{i_j}w$. Examining what are the new w'-NE-ideals and shifted w'-NE-ideals, one can easily see that corresponding minors can be expressed in terms of original minors of w-NE-ideals and shifted w-NE-ideals in a predictable way. In particular, in each case it suffices to know the original vanishing/nonvanishing pattern to conclude the desired statement. □

Now we can finish the proof of Theorem 7.5.

Proof. For the second statement, use Proposition 7.9 and the fact that Bruhat cells are disjoint. For the third statement, assume that we have

$$e_{i_1}(a_1)\dots e_{i_l}(a_l) = e_{i_1}(a_1')\dots e_{i_l}(a_l'),$$

but the two sets of parameters are not the same. Without loss of generality we can assume that the difference occures already in the first factor, say $a_1 > a_1'$. Then

$$e_{i_1}(a_1 - a_1')\dots e_{i_l}(a_l) = e_{i_2}(a_2')\dots e_{i_l}(a_l'),$$

which contradicts the second statement of the theorem. □

By analogy with $U = U^+$ we can consider its transpose U^- and sets $U^-(u)$ within it. Denote $G(u,w) = U^-(u)T_{>0}U^+(w)$, where $T_{>0}$ is the positive torus (that is diagonal matrices with positive entries).

Theorem 7.10. *We have $G(u,w) = GL_n(\mathbb{R})_{\geq 0} \cap B_{u,w}$.*

Proof. It is easy to see that $G(u,w) \subseteq GL_n(\mathbb{R})_{\geq 0} \cap B_{u,w}$. Since the latter sets are disjoint, the statement follows. □

Exercise 7.11. *Show that $GL_n(\mathbb{R})_{>0} = G(w_0, w_0)$.*

7.2. **Topology of cells.**

Theorem 7.12. *The closure $\overline{U(w)}$ consists of $U(v)$ for all $v \leq w$ in Bruhat order.*

Proof. It is clear that every such $U(v)$ is in the closure, from the definition of Bruhat order as a subword order on reduced words. Indeed, we can just let some of the parameters a_j in the product $e_{i_1}(a_1)\dots e_{i_l}(a_l)$ approach zero, effectively ommiting some of the factors.

The non-trivial part is to show that no element of $U(v)$, $v \not\leq w$ belongs to $\overline{U(w)}$. Recall the following characterization of the Bruhat order from [BB]. For each $(i,j) \in [n]\times[n]$ let $N_w(i,j) = |\{k \mid k \leq i, w(k) \geq w(i)\}|$. Then $u \leq w$ if and only if for every (i,j) we have $N_u(i,j) \leq N_w(i,j)$. Thus, if $v \not\leq w$, then there exists (i,j) such that $N_v(i,j) > N_w(i,j)$. Take the minimal v-NE-ideal X_C containing cell (i,j). Then as we know, $|X_C| \neq 0$ for $X \in U(v)$. On the other hand, if $X \in U(w)$ then X_C is not of full rank, since it is obtained by row and column operations from a matrix of smaller rank $N_w(i,j)$. Thus $|X_C| = 0$, which should remain true for all points in the closure $\overline{U(w)}$, contradiction. □

Conjecture 7.13 (Fomin, proved by Hersh)**.** *Each of the closed sets $\overline{U(w)}$ is homeomorphic to a ball.*

Example 7.1. Take the whole $U_{\geq 0}$ for $n = 3$. It consists of matrices

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

with $x, y, z, xz - y \geq 0$. This is a cone since we can simultaniously rescale all the variables $(x,y,z) \mapsto (ax, a^2y, az)$ for $a > 0$. Cutting the cone by plane $x + z = 1$ we get the set

$$\{(x,y) \in \mathbb{R}^2_{\geq 0} \mid x \leq 1, y \leq x(1-x)\}.$$

This set naturally decomposes into one 2-dimensional, two 1-dimensional and two 0-dimensional cells. Their closure order is exactly the Bruhat order of S_3, with the smallest element corresponding to empty set.

7.3. **Frenkel-Moore relation.** Assume we are given a reduced product of Chevlley generators $e_{i_1}(a_1)\dots e_{i_k}(a_k)$. Represent it diagramatically: draw the wiring diagram for $w = s_{i_1}\dots s_{i_k}$, and assign a parameter a_j to the crossing in the diagram corresponding to the factor s_{i_j}. We can consider equivalence classes of reduced words up to commutation relations $s_is_j = s_js_i$, $|i-j| > 1$. This would correspond to making the wiring diagrams obtained by arbitrary stratchings of the wires equivalent. In what follows we shall often refer to such equivalence classes when we speak about wiring diagrams.

Now, we can start applying braid moves to products $e_{i_1}(a_1)\dots e_{i_k}(a_k)$. Simultaniously, we apply Yang-Baxter moves to the corresponding wiring diagrams, simultaneously changing the parameters a_j according to the rule

$$e_i(a)e_j(b)e_(c) \mapsto e_j(bc/(a+c))e_i(a+c)e_j(bc/(a+c)), \quad |i-j| = 1.$$

**** **example**

The following is a corollary of the Theorem 7.5.

Corollary 7.14. *Assume after several such steps we return to the original commutation class of the product (equivalently, to the original wiring diagram). Then the parameters at crossings also return to their original values.*

Proof. The new parameters shall be some subtraction-free rational functions in the old parameters. However, for every specific assignment of positive real values to them, injectivity in Theorem 7.5 implies that there exists at most one factorization into Chevalley generators along a prescribed reduced word. Since the set $\mathbb{R}^k_{>0}$ is Zariski closed in $\mathbb{R}^k$, the statement follows. □

Now we give a different prove of the same fact. Consider the graph G_w, whos vertices are the commutation classes of reduced words for w. Two vertices of G_w are connected if the two reduced words are different from each other by a single braid move. The process above can be described as follows: we walk along the edges of G_w and eventually come back to the original vertex. Consider the following two types of cycles in G_w.

- 4-**cycle**

$$i, i+1, i, \dots, j, j+1, j \to i+1, i, i+1, \dots, j, j+1, , \to i+1, i, i+1, \dots, j+1, j, j+1$$

$$\to i, i+1, i, \dots, j+1, j, j+1 \to i, i+1, i, \dots, j, j+1, j;$$

- 8-**cycle**

$$i, i+1, i+2, i, i+1, i \to i, i+1, i+2, i+1, i, i+1 \to i, i+2, i+1, i+2, i, i+1 \to$$

$$i+2, i+1, i, i+1, i+2, i+1 \to i+2, i+1, i, i+2, i+1, i+2 \to i+1, i+2, i+1, i, i+1, i+2 \to$$

$$i+1, i, i+2, i+1, i+2, i \to i+1, i, i+1, i+2, i+1, i \to i, i+1, i+2, i, i+1, i.$$

**** **figure**

One can verify by a direct computation that as we go around such cycles, the parameters at crossings come back to the original values.The following lemma would imply the statement of the Corollary 7.14.

Lemma 7.15. *Every cycle in G_w can be "filled" with 4-cycles and 8-cycles.*

Proof. Define the *rank* of a reduced word by
$$r(s_{i_1}\dots s_{i_k}) = i_1 + \dots + i_k.$$
The rank is well-defined for commutation classes, the braid moves change it by one. Let γ be a cycle in G_w, denote $r(\gamma)$ the maximal of the ranks of the vertices in γ. We prove the statement by induction on $r(\gamma)$. The base is clear, since the only cycle that contains only vertices of fixed rank is the trivial cycle.

Assume now the statement holds for ranks smaller than r but fails for r. Furthermore, among all cycles γ where it fails choose the one with smallest number of vertices with rank r. Let $v \in \gamma$ be a vertex with $r(v) = r$. Consider the two edges in γ adjacent to v: vv' and vv''. Each corresponds to a single braid move applied to a triple $\dots s_i s_j s_i \dots$, $|i-j|$ in v.

There are three cases to consider. First, the two triples can coinside. In that case we can consider cycle γ' obtained from γ by ommiting the steps $v' \to v \to v'' = v'$. This cycle has rank at most r and strictly less vertices of rank r - contradiction. Second, the two triples can be disjoint. In that case we can complete v', v, v'' to a 4-cycle $v' \to v \to v'' \to v''' \to v'$, and consider a new cycle γ' where the path taken from v' to v'' is through v''', not v. Since $r(v''') = r(v) - 2$, we conclude that γ' has strictly less vertices of rank r - contradiction.

Finally, we need to consider the case when the two triples used in braid moves overlap but do not coinside. They cannot possibly share two elements. Indeed, if $\dots s_i s_j s_i \dots s_j \dots$ is a reduced word such that the last three shown factors $s_j s_i \dots s_j$ can be brought together via commutation moves, then it is easy to see all four factors can be grouped together: $\dots s_i s_j s_i s_j \dots$, which cannot happen in a reduced word. Thus, the two triples share exactly one common element. There are two possibilities: $\dots s_i s_j s_i \dots s_j \dots s_i \dots$ and $\dots s_i s_j s_i \dots s_k \dots s_i \dots$. Here $k = 2i - j$ and one is able to group the last three shown factors together using only the commutation moves. The second case is impossible since in that case one of the v' and v'' would have larger rank than v. Thus we must have $v = \dots s_{i+1} s_i s_{i+1} \dots s_i \dots s_{i+1} \dots$. Whatever is inbetween the last s_i and s_{i+1} one should be able to commute through s_{i+1} in order to perform the braid move, thus we can assume there is nothing there. In the interval $s_{i+1} \dots s_i$ there may be terms one can commute through s_{i+1} but not through s_i, all other terms we can get rid of by pushing them to the right. Such terms can only be s_{i-1}-s. Furthermore, there can be at most one such factor, and one must have at least one since $s_{i+1} s_i s_{i+1} s_i s_{i+1}$ is not reduced. Thus the location under consideration must look like $\dots s_{i+1} s_i s_{i+1} s_{i-1} s_i s_{i+1} \dots$, which is one of the vertices of the 8-cycle. Once we know that, we can take the path from v' to v'' along the other part of the 8-cycle. This reduces the number of rank r vartices and thus leads to a contradiction. $\square$

7.4. **Type B and G relations.** The transformation of parameters takes different form in not simply laced types.

Theorem 7.16. [BZ, Theorem 3.1]

- *Assume nodes i and j are connected in the Dynkin diagram by an edge of multiplicity 2. The corresponding Chevalley generators satisfy the following braid relation:*
$$e_i(t_1)e_j(t_2)e_i(t_3)e_j(t_4) = e_j(p_1)e_i(p_2)e_j(p_3)e_i(p_4),$$
where
$$p_1 = \frac{t_2 t_3^2 t_4}{\pi_2},\ p_2 = \frac{\pi_2}{\pi_1},\ p_3 = \frac{\pi_1^2}{\pi_2},\ p_4 = \frac{t_1 t_2 t_3}{\pi_1},$$

where

$$\pi_1 = t_1t_2 + (t_1+t_3)t_4, \quad \pi_2 = t_1^2t_2 + (t_1+t_3)^2t_4.$$

- *Assume nodes i and j are connected in the Dynkin diagram by an edge of multiplicity 3. The corresponding Chevalley generators satisfy the following braid relation:*

$$e_i(t_1)e_j(t_2)e_i(t_3)e_j(t_4)e_i(t_5)e_j(t_6) = e_j(p_1)e_i(p_2)e_j(p_3)e_i(p_4)e_j(p_5)e_i(p_6),$$

where

$$p_1 = \frac{t_2t_3^3t_4^2t_5^3t_6}{\pi_3},\ p_2 = \frac{\pi_3}{\pi_2},\ p_3 = \frac{\pi_2^3}{\pi_3\pi_4},\ p_4 = \frac{\pi_4}{\pi_1\pi_2},\ p_5 = \frac{\pi_1^3}{\pi_4},\ p_6 = \frac{t_1t_2t_3^2t_4t_5}{\pi_1},$$

where

$$\begin{aligned}
\pi_1 &= t_1t_2t_3^2t_4 + t_1t_2(t_3+t_5)^2t_6 + (t_1+t_3)t_4t_5^2t_6,\\
\pi_2 &= t_1^2t_2^2t_3^3t_4 + t_1^2t_2^2(t_3+t_5)^3t_6 + (t_1+t_3)^2t_4^2t_5^3t_6 + t_1t_2t_4t_5^2t_6(3t_1t_3 + 2t_3^2 + 2t_3t_5 + 2t_1t_5),\\
\pi_3 &= t_1^3t_2^2t_3^3t_4 + t_1^3t_2^2(t_3+t_5)^3t_6 + (t_1+t_3)^3t_4^2t_5^3t_6 + t_1^2t_2t_4t_5^2t_6(3t_1t_3 + 3t_3^2 + 3t_3t_5 + 2t_1t_5),\\
\pi_4 &= t_1^2t_2^2t_3^3t_4(t_1t_2t_3^3t_4 + 2t_1t_2(t_3+t_5)^3t_6 + (3t_1t_3 + 3t_3^2 + 3t_3t_5 + 2t_1t_5)t_4t_5^2t_6) +\\
&\quad t_6^2(t_1t_2(t_3+t_5)^2 + (t_1+t_3)t_4t_5^2)^3.
\end{aligned}$$

Proof. We borrow the proof from [BZ]. The Lie group can be realized inside complition of universal enveloping algebra. We show the calculation in types A_2 and B_2, and leave it as an exercise to extend to G_2.

Recall that $e_i(t) = exp(t\epsilon_i)$, where ϵ_i is the Chevalley generator in the Lie algebra. Thus we have

$$(1 + t_1\epsilon_1 + t_1^2\epsilon_1^2/2 + \ldots)(1 + t_2\epsilon_2 + t_2^2\epsilon_2^2/2 + \ldots)(1 + t_3\epsilon_1 + t_3^2\epsilon_1^2/2 + \ldots) =$$
$$(1 + p_1\epsilon_2 + p_1^2\epsilon_2^2/2 + \ldots)(1 + p_2\epsilon_1 + p_2^2\epsilon_1^2/2 + \ldots)(1 + p_3\epsilon_2 + p_3^2\epsilon_2^2/2 + \ldots).$$

Now, the only important thing to know about the Serre relations is that they are of degree three, thus the coefficients of all terms of degree at most two must coinside. This gives us $t_1 + t_3 = p_2$, $t_2 = p_1 + p_3$, $t_1t_2 = p_2p_3$, $t_2t_3 = p_1p_2$. This is enough to solve uniquely for p-s in terms of t-s.

Similarly, in type B we have

$$(1+t_1\epsilon_1+t_1^2\epsilon_1^2/2+\ldots)(1+t_2\epsilon_2+t_2^2\epsilon_2^2/2+\ldots)(1+t_3\epsilon_1+t_3^2\epsilon_1^2/2+\ldots)(1+t_4\epsilon_2+t_4^2\epsilon_2^2/2+\ldots) =$$
$$(1+p_1\epsilon_2+p_1^2\epsilon_2^2/2+\ldots)(1+p_2\epsilon_1+p_2^2\epsilon_1^2/2+\ldots)(1+p_3\epsilon_2+p_3^2\epsilon_2^2/2+\ldots)(1+p_4\epsilon_1+p_4^2\epsilon_1^2/2+\ldots).$$

Since the type B Serre relation is of degree four, this gives the equations $t_1 + t_3 = p_2 + p_4$, $t_2 + t_4 = p_1 + p_3$, $t_1t_2 + t_1t_4 + t_3t_4 = p_2p_3$, $t_1^2t_2 + t_1^2t_4 + t_3^2t_4 + 2t_1t_3t_4 = p_2^2p_3$, which allow to uniquely solve for p-s.

There is also a different way to deduce the type B relation by realizing the hyperoctahedral group inside the symmetric group. Recall that one can think of the generators s_i of the hyperoctahedral group as follows: for $i > 0$, s_i swaps i-th entry with $i+1$-st, and simultaniously $-i$-th with $-(i+1)$-st. The s_0 swaps 1-st entry with -1-st. This means that if we write the type B braid relation above in terms of the type A Chevalley generators, it would look as follows:

$$e_0(t_1)e_1(t_2)e_{-1}(t_2)e_0(t_3)e_1(t_4)e_{-1}(t_4) = e_1(p_1)e_{-1}(p_1)e_0(p_2)e_1(p_3)e_{-1}(p_3)e_0(p_4).$$

****** computation** □

7.5. M**-variables.** Let us consider a wiring diagram of the commutation class of reduced decomposition $w = s_{i_1} \dots s_{i_l}$. We shall number the wires by the index of their beginning. Let the parameter at the crossing of i-th and j-th wires be t_{ij}. The wires cut the plane into regions we call *chambers*. ****** figure** Label the chambers by the set of wires that pass below. Assign to a chamber $I \subseteq [n]$ variable M_I as follows:

$$M_I = \left(\prod_{i<j, i \notin I, j \in I} t_{ij} \right)^{-1}.$$

Use notation $Ii = I \cup i$, $Iij = I \cup i \cup j$, etc.

Lemma 7.17. *If I, Ii, Ij and Iij are the four chambers surrounding parameter t_{ij}, we have*

$$t_{ij} = \frac{M_I M_{Iij}}{M_{Ii} M_{Ij}}.$$

Proof. For each t_{kl} count how many times it occurs in the numerator and how many times in the denominator. □

Theorem 7.18. *Assume we do a braid move involving i-th, j-th and k-th wire, $i < j < k$. Then while the parameters t change according to*

$$t'_{ij} = t_{ik} t_{ij} / (t_{ij} + t_{jk}), \quad t'_{ik} = t_{ij} + t_{jk} \quad t'_{jk} = t_{jk} t_{ik} / (t_{ij} + t_{jk}),$$

the M-variables M_{Iik} and M_{Ij} get exchanged for each other in the picture according to

$$M_{Ij} M_{Iik} = M_{Ik} M_{Iij} + M_{Ii} M_{Ijk},$$

while the rest of the M-s stay fixed.

Proof. One checks directly that the M-s that stay in the picture are preserved, for example M_{jk} stays the same since $t'_{ij} t'_{ik} = t_{ij} t_{ik}$. The relation between M_{Iik} and M_{Ij} follows from

$$(t_{jk} t'_{ij})^{-1} = (t_{ik} t_{jk})^{-1} + (t_{ij} t_{ik})^{-1},$$

which is easily verified. □

Proposition 7.19. *The value of M_J depends only on J and not on the wiring diagram.*

Proof. We argue that any two wiring diagrams with chamber set J can be connected by braid moves while keeping chamber J present at all times. That would clearly imply the statement.

Let $\bar{w}$ be a reduced word for w such that the corresponding wiring diagram has chamber labelled by J. Cut this J-chamber by a vertical line, thus obtaining a decomposition $\bar{w} = \bar{w}_1 \bar{w}_2$. The $\bar{w}_1$ can be changed into a reduced word where we first sort elements of J and its complement between each other, and then apply some permutation within eahc set. Similar change can be done to $\bar{w}_2$ except for it we first sort the elements within J and its complement, and then sort them with each other. Finally, any two wiring diagrams of such form clearly can be connected by braid moves without losing chamber set J, which completes the proof. □

****** example**

7.6. **Twist map.** For $X \in GL_n(\mathbb{R})$ let $[X]_+$ be the upper triangular factor in the LDU decomposition of X. For upper unitriangular $X \in B^-_{w_0}$ in the top Bruhat cells call the map $X \mapsto Y = w_0[Xw_0]^T_+ w_0$ the *twist map*. Recall that the LDU factorization is given by

$$l_{i,j} = \frac{|X_{[j-1]\cup i,[j]}|}{|X_{[j],[j]}|}; \quad u_{i,j} = \frac{|X_{[i],[i-1]\cup j}|}{|X_{[i],[i]}|}; \quad d_{i,i} = \frac{|X_{[i],[i]}|}{|X_{[i-1],[i-1]}|}.$$

Example 7.2. The twist of

$$X = \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{is} \quad Y = \begin{pmatrix} 1 & \frac{x_{23}}{x_{12}x_{23}-x_{13}} & \frac{1}{x_{13}} \\ 0 & 1 & \frac{x_{12}}{x_{13}} \\ 0 & 0 & 1 \end{pmatrix}.$$

Lemma 7.20. *The inverse of the twist map is given by* $Y \mapsto X = [w_0 Y^t]_+$.

Theorem 7.21. *The flag minors* $|Y_J| = |Y_{[|J|],J}|$ *coinside with* M_J.

Example 7.3. Take the product $e_1(a)e_2(b)e_1(c)$ and the corresponding wiring diagram. Number wires on the left from bottom to top, and mark as chamber sets the sets of wires that pass *above* the chamber. Then the product and its twist are

$$X = \begin{pmatrix} 1 & a+c & ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & \frac{1}{c} & \frac{1}{ab} \\ 0 & 1 & \frac{a+c}{ab} \\ 0 & 0 & 1 \end{pmatrix}.$$

It remains to compare $M_2 = 1/c$, $M_3 = 1/ab$, $M_{23} = 1/bc$, etc.

We shall deduce the theorem from the following two lemmas.

Lemma 7.22. *We have*

$$|Y_{[p,q]}| = \frac{|X_{[q-p+1]\cup[n-p+2,n]}|}{|X_{[n-q+1]}|}, \quad 1 \le a \le b \le n.$$

Proof. We have

$$|Y_{[p,q]}| = Y^T_{[p,q],[n+p-q,n]} = Y^T_{[p-1]\cup[n+p-q,n]} = \frac{|(Xw_0)_{[p-1]\cup[n+p+-q,n]}|}{|(Xw_0)_{[q]}|} = \frac{|X_{[q-p+1]\cup[n-p+2,n]}|}{|X_{[n-q+1]}|}.$$

□

Lemma 7.23. *We have*

$$M_{[p,q]} = \frac{|X_{[q-p+1]\cup[n-p+2,n]}|}{|X_{[n-q+1]}|}, \quad 1 \le a \le b \le n.$$

Proof. Proof by example. Take $n = 4$, $[p,q] = [2,3]$ and reduced word $e_1e_2e_3e_1e_2e_1$. □

Now we can prove the theorem.

Proof. Clearly its enough to prove it for one reduced word, since we can get to any other by braid moves. Along the way we would conclude that all new chamber sets must coinside with corresponding minors of Y since they satisfy the same exchange relation. It remains to note that for a particular reduced word the chamber minors are exactly the solid flag minors $|Y_{[p,q]}|$. □

Remark 7.1. In the case $X \in B^-_w$ of arbitrary bruhat cell one should twist by $X \mapsto Y = w^{-1}[Xw^{-1}]^T_+ w$. All the reuslts extend verbatim, in particular one still has $|Y_J| = M_J$.

7.7. **Criteria for total positivity.**

Theorem 7.24. *Let $X \in U$, and let $\bar{w}$ be any reduced word for w_0. The following are equivalent:*

- *X is upper totally positive;*
- *all flag minors $|X_J|$, as J runs over chamber sets in $\bar{w}$, are positive.*

Proof. We can get from any reduced word to any other reduced word by braid move. If we know positivity of all chamber minors before a braid move, it implies their positivity after the move. It remains to note that for a special reduced word we have exactly the soild minors $X_{[i,j]}$ of Fekete's criterion. Equivalently, we can use the fact that any subset $J \subseteq [n]$ can be a chamber minor for some reduced word. □

In fact, the following stronger statement holds.

Theorem 7.25. *Let $X \in B_w^+$, and let $\bar{w}$ be any reduced word for w. The following are equivalent:*

- *X is totally nonnegative;*
- *all flag minors $|X_J|$, as J runs over chamber sets in $\bar{w}$, are positive.*

Exercise 7.26. *Prove this theorem.*

****** example**

7.8. **Double wiring diagrams.**

7.9. **Strong and weak separation.** We wish to understand families of minors that provide a criteria for total positivity. For simplicity let us look at $U_{\geq 0}$ and flag minors X_I, $I \subseteq [n]$. Let us call two subsets I and J of $[n]$ *strongly separated* if either $I - J \prec J - I$ or $J - I \prec I - J$. Here $I - J$ denotes the elements of I that are not in J, and $I \prec J$ denotes the order in which every element of I is smaller than every element of J. Let us call a collection of subsets of $[n]$ strongly separated if every two subsets in the collection are.

Example 7.4. $(1, 4)$ and $(2, 3)$ are not strongly separated, but $(1, 4)$ and $(2, 4, 5)$ are.

Exercise 7.27. *Show that a collection of subsets is strongly separated if and only if it is part of set of chamber minors of a wiring diagram of w_0.*

Example 7.5. The collection $\emptyset, (1), (4), (1, 2), (1, 3), (1, 4), (3, 4), (1, 2, 3), (1, 3, 4), (2, 3, 4), (1, 2, 3, 4)$ is strongly separated. Those are exactly the chamber minors for the reduced word $s_2s_3s_2s_1s_2s_3$.

Consider the simplicial comlex on the set of subset of $[n]$. A collection of subsets is a face if and only if it is strongly separated. The following statement is clear from the exercise.

Corollary 7.28. *The complex of strong separation is pure of dimension $n+1+\binom{n}{2}$. Each facet of the strong separation complex provides a total positivity criterion in $U_{\geq 0}$.*

Example 7.6. The collection in the previous example is one of the facets for $n = 4$.

It is natural to ask if there are any not strongly separated sets that provide criteria for total positivity. The answer is affirmative. For example, because of the relation

$$X_{(1,3,4)}X_{(2,3)} = X_{(1,2,3)}X_{(3,4)}X_{(2,3,4)}X_{(1,3)},$$

we can exchange $(1,3,4)$ for $(2,3)$ in the collection above and still have a total positivity criteria. The resulting collection is not strongly separated however since for example sets $(2,3)$ and $(1,4)$ are not.

Let us call two sets I and J *weakly separated* if at least one of the following two conditions holds:

- $|I| \geq |J|$ and $J - I$ can be partitioned into disjoint union $J' \cup J''$ so that $J' \prec I - J \prec J''$;
- $|I| \leq |J|$ and $I - J$ can be partitioned into disjoint union $I' \cup I''$ so that $I' \prec J - I \prec I''$.

Example 7.7. The pairs $(1,4)$ and $(2,3)$, $(1,5)$ and $(2,3,4)$ are weakly separated, the pairs $(2,5)$ and (3), $(1,3)$ and $(2,5)$ are not.

Let us call a collection of subsets of $[n]$ weakly separated if every two subsets in the collection are. We can also define the complex of weak separation where facets are the pairwise weakly separated collections. Clearly, every strongly separated collection is also weakly separated, but not the other way around.

Example 7.8. The collection $\emptyset, (1), (4), (1,2), (1,3), (1,4), (3,4), (1,2,3), (2,3), (2,3,4), (1,2,3,4)$ is weakly separated.

Theorem 7.29 (Leclerc-Zelevinsky)**.** *The maximal possible size of a collection of weakly separated subsets is* $n + 1 + \binom{n}{2}$.

Proof. Plot every subset $I \subseteq [n]$ as a path from lattice point $(0,0)$ to a lattice point (a,b) with $a+b=n$. For that, if $i \in I$, let the i-th step be up, otherwise let it be to the right. ****** picture example** . Let us start with the origin $(0,0)$ and add lattice points to it one after another, so that at any given moment the points form a south-west ideal. For example first we add $(1,0)$, then $(0,1)$, then maybe $(1,1)$ and $(2,0)$, etc. At the end we shall have exactly all the points (a,b) in the region $a \geq 0$, $b \geq 0$ and $a+b \leq n$.

Plot all elements of a collection of weakly separated sets as paths in the lattice as above. At any intermidiate step we consider the initial parts of the paths that lie inside already included part of the lattice. Note that such initial parts for some of the paths may coinside, in which case we do not distinguish them. At each intermidiate step we count how many *different* initial segments of paths are there.

Claim: with each extra lattice node added, this number increases by at most one. Indeed, if there are two distinct paths that both branch at a given node, one can easily find a not weakly separated pair of paths. Thus, counting the nodes at which the number can go up by one, we arrive to the statement of the theorem. □

One has a similar theorem in all cells of all partial flag varieties. For example, in the case of the Grassmannian $Gr(k,n)$, we have the following.

Theorem 7.30. *The maximal possible size of a collection of weakly separated subsets of* $[n]$ *of size* k *each is* $k(n-k)+1$.

Proof. The same as above, only inside the region $0 \leq a \leq n-k, 0 \leq b \leq k$. □

Conjecture 7.31 (Leclerc-Zelevinsky)**.** *The complex of weak separation is pure and is exactly the part of the cluster complex consisting of Plücker coordinates. In particular, each facet is a TP criterion. All facets can be connected with each other via sequences of flips*

$$\{Ii, Ij, Ik, Iij, Ijk\} \longleftrightarrow \{Ii, Ik, Iij, Ijk, Iik\},$$

where $i < j < k$ are disjoint from I.

This conjecture was proved for $Gr(3, n)$ Grassmannians by Scott, for complete flag varieties by Danilov, Karzanov and Koshevoy, and for arbitrarily Grassmannians by Oh, Postnikov and Speyer.

Exercise 7.32. *Consider the complex of weakly separated subsets of $[n]$ of size three. Show that it is pure.*

References

[BB] A. Bjorner and F. Brenti: Combinatorics of Coxeter groups.

[BFZ] A. Berenstein, S. Fomin and A. Zelevinsky: Parametrizations of canonical bases and totally positive matrices.

[BZ] A. Berenstein and A. Zelevinsky: Total positivity in Schubert varieties.

[Ker] S. Kerov: Asymptotic representation theory of the symmetric group and its applications in analysis.

[LP] T. Lam and P. Pylyavskyy: Total positivity in loop groups I: whirls and curls.

[Mac] I. G. Macdonald: Notes on Schubert polynomials.