AN INTRODUCTION TO 3-MANIFOLDS

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Introduction

In these lecture notes we will give a quick introduction to 3–manifolds, with a special emphasis on their fundamental groups. The lectures were held at the summer school 'groups and manifolds' held in Münster July 18 to 21 2011.

In the first section we will show that given $k \geq 4$ any finitely presented group is the fundamental group of a closed k–dimensional manifold. This is not the case for 3–manifolds, we will for example see that $\mathbb{Z}, \mathbb{Z}/n, \mathbb{Z} \oplus \mathbb{Z}/2$ and $\mathbb{Z}^3$ are the only abelian groups which arise as fundamental groups of closed 3–manifolds. In the second section we recall the classification of surfaces via their geometry and outline the proofs for several basic properties of surface groups. We will furthermore summarize the Thurston classification of diffeomorphisms of surfaces.

We will then shift our attention to 3–manifolds. In the third section we will first introduce various examples of 3–manifolds, e.g. lens spaces, Seifert fibered spaces, fibered 3–manifolds and exteriors of knots and links, we will furthermore see that new examples can be constructed by connected sum and by gluing along tori. The goal in the remainder of the lecture notes will then be to bring some order into the world of 3–manifolds. The prime decomposition theorem of Kneser and Thurston stated in Section 4.1 will allow us to restrict ourselves to prime 3–manifolds. In Section 4.2 we will state Dehn's lemma and the sphere theorem, the combination of these two theorems shows that most prime 3–manifolds are aspherical and that most of their topology is controlled by the fundamental group.

In Section 2 we had seen that 'most' surfaces are hyperbolic, in Section 5 we will therefore study properties hyperbolic 3–manifolds. The justification for studying hyperbolic 3–manifolds comes from the Geometrization Theorem conjectured by Thurston and proved by Perelman. The theorem says that any prime manifold can be constructed by gluing Seifert fibered spaces and hyperbolic manifolds along incompressible 3–manifolds.

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Caveat. These are lecture notes and they will still contain inaccuracies, for precise statements we refer to the references.

1. Finitely presented groups and high dimensional manifolds

1.1. Finitely presented groups.

Definition. Let $x_1, \dots, x_n$ be symbols, then we denote by

$$\langle x_1, \dots, x_n \rangle$$

the free group with generators $x_1, \dots, x_n$. If $r_1, \dots, r_m$ are words in $x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}$, then we denote by

$$\langle x_1, \dots, x_n \,|\, r_1, \dots, r_m \rangle$$

the quotient of $\langle x_1, \dots, x_n \rangle$ by the normal closure of $r_1, \dots, r_m$, i.e. the quotient by the smallest normal subgroup which contains $r_1, \dots, r_m$. We call $x_1, \dots, x_n$ *generators* and $r_1, \dots, r_m$ *relators*.

Definition. If G is isomorphic to a group of the form $\langle x_1, \dots, x_n \,|\, r_1, \dots, r_m \rangle$, then we say that G is *finitely presented*, and we call

$$\langle x_1, \dots, x_n \,|\, r_1, \dots, r_m \rangle$$

a *presentation of* G. We call $n - m$ the deficiency of the presentation, and we define the *deficiency of* G to be the maximal deficiency of any presentation of G.

Example.

(1) The free abelian group $\mathbb{Z}^3$ is isomorphic to

$$\langle x_1, x_2, x_3 \,|\, [x_1, x_2], [x_1, x_3], [x_2, x_3] \rangle,$$

the deficiency of this presentation is zero, and one can show that the deficiency of $\mathbb{Z}^3$ is indeed zero.

(2) The free abelian group $\mathbb{Z}^4$ is isomorphic to

$$\langle x_1, x_2, x_3, x_4 \,|\, [x_1, x_2], [x_1, x_3], [x_1, x_4], [x_2, x_3], [x_2, x_4], [x_3, x_4] \rangle,$$

the deficiency of this presentation is -2, and one can show that the deficiency of $\mathbb{Z}^4$ is indeed -2. Similarly, the deficiency of any free abelian group of rank greater than three is negative.

1.2. **Fundamental groups of high dimensional manifolds.** Let M be a manifold. (Here, and throughout these lectures, manifold will always mean a smooth, compact, connected, orientable manifold, we will not assume though that manifolds are closed.) Any manifold has a CW structure with one 0–cell and finitely many 1–cells and 2–cells. This decomposition gives rise to a presentation for $\pi = \pi_1(M)$, where the generators correspond to the 1–cells and the relators correspond to the 2–cells. We thus see that $\pi_1(M)$ is finitely presented. The following question now naturally arises:

Question 1.1. *Which finitely presented groups can arise as fundamental groups of manifolds?*

Already by looking at dimensions 1 and 2 it is clear that the answer depends on the dimension. It turns out that the question has a simple answer once we go to manifolds of dimension greater than three.

Theorem 1.2. *Let G be a finitely presented group and let $k \geq 4$. Then there exists a closed k–manifold M with $\pi_1(M) = G$.*

Proof. We pick a finite presentation

$$G = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_m \rangle.$$

We consider the connected sum of n copies of $S^1 \times S^{k-1}$; its fundamental group is canonically isomorphic to $\langle x_1, \dots, x_n \rangle$. We now represent $r_1, \dots, r_m$ by disjoint embedded closed curves $c_1, \dots, c_m$. We consider the inclusion map

$$\begin{array}{l} X := (S^1 \times S^{k-1} \# \dots \# S^1 \times S^{k-1}) \setminus (c_1 \times D^{k-1} \cup \dots \cup c_m \times D^{k-1}) \\ \qquad\quad \downarrow \iota \\ Y := S^1 \times S^{k-1} \# \dots \# S^1 \times S^{k-1}. \end{array}$$

This map induces an epimorphism of fundamental groups, since by general position any closed curve can be pushed off the curves $c_1, \dots, c_m$. But this map also induces a monomorphism. Indeed, if a curve $c \subset X$ bounds a disk $D \subset Y$ in $S^1 \times S^{k-1} \# \dots \# S^1 \times S^{k-1}$, then by a general position argument we can push the disk off the curves $c_1, \dots, c_m$ (here we used that $n \geq 4 > 2 + 1$). The curve c thus already bounds a disk in X, i.e. it is null homotopic in X.

We now consider the closed manifold

$$(S^1 \times S^{k-1} \# \dots \# S^1 \times S^{k-1}) \setminus \bigcup_{i=1}^{m} c_i \times D^{k-1} \ \cup \bigcup_{i=1}^{m} D^2 \times S^{k-2},$$

where we glue a disk to each curve c_i. It follows from the van Kampen theorem, that this closed manifold has the desired fundamental group. $\square$

It is well–known that the isomorphism problem for finitely presented groups is not solvable (see [Mi92]), we thus obtain the following immediate corollary to the previous theorem:

Corollary 1.3. *Let $k \geq 4$. Then there is no algorithm which can decide whether two k–manifolds are diffeomorphic or not.*

We will now show that the statement of the theorem does not hold in dimension 3:

Proposition 1.4. *Let N be a closed 3–manifold, then $\pi_1(N)$ admits a presentation of deficiency zero.*

Note that 'most' abelian groups have negative deficiency, and one can thus use the proposition to show if $\pi_1(N)$ is abelian, then $\pi_1(N)$ is isomorphic to $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, \mathbb{Z}/n$ and $\mathbb{Z} \oplus \mathbb{Z}/n$.

one can use the proposition to show that the only abelian groups which appear as fundamental groups of closed 3–manifolds are [1]

$$\mathbb{Z} = \pi_1(S^1 \times S^2), \mathbb{Z}/n = \pi_1(\text{lens space}) \text{ and } \mathbb{Z}^3 = \pi_1(3 - \text{torus}).$$

Proof. We pick a triangulation of N and we denote by H a tubular neighborhood of the 1–skeleton. Note that H and $\overline{N \setminus H}$ are handlebodies of the same genus, say g. This decomposition gives rise to a presentation for $\pi_1(N)$ with g generators and g relators. □

2. Surface groups

2.1. **The classification of surfaces.** We now study fundamental groups of surfaces. Surface groups are well understood and many have nice properties, which will be guiding us later in the study of 3–manifold groups. Surface groups will also play a key rôle in the study of 3–manifolds. For the most part we will in the following also include the case of non–orientable surfaces.

Surfaces have been completely classified, more precisely the following theorem was already proved in the 19th century:

Theorem 2.1. *Two surfaces are diffeomorphic if and only if they have the same Euler characteristic, the same number of components and the same orientability.*

[1] If we allow non–orientable manifolds, then we have to add

$$\mathbb{Z} \times \mathbb{Z}/2 = \pi_1(S^1 \times \mathbb{R}P^2).$$

In the study of surfaces it is helpful to take a geometric point of view. In particular, note that if a closed surface Σ admits a Riemannian metric of area A and constant curvature K, then it follows from the Gauss–Bonnet theorem, that

$$K \cdot A = 2\pi\chi(\Sigma),$$

in particular the Euler characteristic gives an obstruction to what type of constant curvature metric a surface can possibly admit.

The uniformization theorem says, that a constant curvature metric which is allowed by the Gauss–Bonnet theorem, will actually occur. More precisely, we get the following table:

$\chi(\Sigma)$	>0	$=0$	<0
type of surface	S^2 or $\mathbb{R}\,\mathrm{P}^2$	torus or Klein bottle	everything else
Σ admits metric of constant curvature	$\equiv 1$	$\equiv 0$	$\equiv -1$
universal cover	S^2	$(\mathbb{R}^2,$ Euclidean metric$)$	$\mathbb{H}^2$

Here we think of S^2 and $\mathbb{R}^2$ as equipped with the usual metrics of constant curvature +1 respectively 0, and we denote by

$$\mathbb{H}^2 = \{(x,y) \mid y > 0\}$$

the upper half plane together with unique complete the metric of curvature -1, namely

$$\frac{1}{y} \cdot \text{standard metric on } \mathbb{R}^2.$$

The action of $\pi_1(\Sigma)$ on the universal cover $\widetilde{\Sigma}$ shows that $\pi_1(\Sigma)$ is a discrete subgroup of $\mathrm{Isom}(\widetilde{\Sigma})$ which acts on $\widetilde{\Sigma}$ cocompactly and without fixed points. For orientable surfaces we thus obtain the following table:

$\chi(\Sigma)$	>0	$=0$	<0
$\mathrm{Isom}(\widetilde{\Sigma})$	$O(3)$	$O(2) \ltimes \mathbb{R}^2$	$PS^*L(2,\mathbb{R})$
$\pi_1(\Sigma)$	0 or $\mathbb{Z}/2$	$\mathbb{Z}^2$ or $\langle a, b \mid abab^{-1}\rangle$	torsion-free Fuchsian group

Here

$$\begin{aligned} S^*L(2,\mathbb{R}) &= \{A \in \mathrm{GL}(2,\mathbb{R}) \mid \det(A) = \pm 1\}, \\ PS^*L(2,\mathbb{R}) &= \{A \in \mathrm{GL}(2,\mathbb{R}) \mid \det(A) = \pm 1\}/\pm \mathrm{id} \end{aligned}$$

acts on $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}.$$

Remark.
1. The fact that every surface supports a complete metric of constant curvature is often referred to as the 'uniformization theorem'.
2. The hyperbolic structure on a surface is not necessarily unique, in fact the space of hyperbolic structures on a closed surface of genus g is $(6g-6)$–dimensional. [2]
3. The fundamental group of an orientable hyperbolic surface is a discrete subgroup of
$$PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm\,\mathrm{id}\}.$$
4. Surfaces with boundary can be classified in a very similar fashion.

We obtain the following corollary to the uniformization theorem:

Lemma 2.2. *Let $\Sigma \neq S^2, \mathbb{R}P^2$ be a surface.*
1. *Σ is aspherical, in particular Σ is an Eilenberg–Maclane space for π.*
2. *$\pi_1(\Sigma)$ is torsion-free.*

Proof.
1. We denote by $\widetilde{\Sigma}$ the universal cover of Σ. For any $k \geq 2$ we have $\pi_k(\Sigma) \cong \pi_k(\widetilde{\Sigma})$, but the latter groups are zero since $\widetilde{\Sigma} = \mathbb{R}^2$ or $\widetilde{\Sigma} = \mathbb{H}^2$ by the uniformization theorem.
2. This will follow from (1) and the following more general claim:

   Claim. Let π be a group which admits a finite dimensional $K(\pi, 1)$, then π is torsion free.

   Let X be a finite dimensional Eilenberg–Maclane space for π. Let $G \subset \pi$ be a cyclic subgroup. We have to show that G is infinite cyclic. We denote by $\hat{X}$ the cover corresponding $G \subset X$. Then $\hat{X}$ is an Eilenberg–Maclane space for G. Since $\hat{X}$ is finite dimensional it follows that $H_i(G;\mathbb{Z}) = 0$ for all but finitely many dimensions. Since finite cyclic groups have non–trivial homology in all odd dimensions it follows that G is infinite cyclic.

□

[2] Given a fixed surface we can associate to each hyperbolic structure a vector in $\mathbb{R}^{6g-6}$ by taking the lengths of certain fixed $6g-6$ curves. This defines a homeomorphism. We refer to
`www.math.sunysb.edu/~jabehr/GeomandTeich.ps`
for details.

2.2. **Fundamental groups of surfaces.** Given a space X with fundamental group π, we want to answer the following questions:

(1) Is π *is linear over the ring* R, i.e. does there exist a monomorphism $\pi \to \mathrm{GL}(n, R)$ for a sufficiently large n?
(2) Does π have 'many finite index quotients', i.e. does X have many finite covers?
(3) Does π admit finite index subgroups with large homology?

Positive answers are useful for various reasons:

(1) linear groups are reasonably well understood and have many good properties, e.g. they are residually finite (see Proposition 2.4) and they satisfy the Tits alternative [3] (see [Ti72]),
(2) the existence of 'many finite covers' allows us to study X through its finite covers, for example if N is a smooth 4–manifold, then the Seiberg–Witten invariants of its finite covers will in general contain more information then the Seiberg–Witten invariants of N alone,
(3) 'large homology groups' means that a space has 'lots of interesting submanifolds'. For example, if N is a n–manifold, then $H_{n-1}(N;\mathbb{Z}) = H^1(N;\mathbb{Z}) \neq 0$ implies that N admits codimension one submanifolds along which we can decompose N into hopefully easier piece.

We will now see that surface groups have, perhaps not surprisingly, very good properties, in particular we will get 'best possible' answers to the above questions.

First note, that it follows easily from the uniformization theorem that surface groups are linear over $\mathbb{R}$. But in fact a stronger statement holds:

Proposition 2.3. *Let Σ be a surface, then $\pi := \pi_1(\Sigma)$ is linear over $\mathbb{Z}$.*

Proof. A general principle says that a 'generic' pair of matrices $A, B \in \mathrm{GL}(n,\mathbb{Z})$ will generate a free group on two generators. For example, one can use the 'ping-pong lemma' to show that

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

[3]The Tits alternative says that a finitely generated linear group either contains a non–abelian free group or it admits a finite index subgroup which is solvable

generate a free group. [4] In particular $\mathrm{SL}(2,\mathbb{Z})$ contains a free group on two generators, and it thus contains any free group, in particular it contains the fundamental group of any surface with boundary.

If Σ is closed, then Newman [Ne85] has shown that there exists an embedding $\pi_1(\Sigma) \to \mathrm{SL}(8,\mathbb{Z})$. [5] $\square$

Definition. Let P be a property of groups. We say that a group π is *residually* P if given any non–trivial $g \in \pi$ there exists a homomorphism $\alpha\colon \pi \to G$ to a group G which has property π. [6]

Remark.
1. Any finitely generated abelian group is residually finite.
2. The group $(\mathbb{Q},+)$ is not residually finite, in fact it has no finite quotients at all.
3. Let p be a prime. Any finitely generated free abelian group is residually a p–group, i.e. residually a group of p–power order.
4. If a finitely presented group is residually finite, then it has solvable word problem, i.e. it can be decided whether a given word in the generators represents the trivial element or not. We refer to [Moa66] for details. [7]

Proposition 2.4. *Let Σ be a surface, then $\pi := \pi_1(\Sigma)$ is residually finite.*

In fact a stronger statement holds: for any prime p the group π is residually p.

Proof. We pick a monomorphism $\alpha\colon \pi \to \mathrm{GL}(n,\mathbb{Z})$. Let $g \in \pi$ be non–trivial. Pick $k \in \mathbb{N}$ such that not all entries of $\alpha(g)$ are divisible by k. Then the image of g under the map

$$\alpha\colon \pi \to \mathrm{GL}(n,\mathbb{Z}) \to \mathrm{GL}(n,\mathbb{Z}/k)$$

is non–trivial. $\square$

Definition. We say that a group π is *subgroup separable* if for any finitely generated subgroup A and any $g \in \pi \setminus A$ there exists a homomorphism $\alpha\colon \pi \to G$ to a finite group such that $\alpha(g) \notin \alpha(A)$. [8]

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[4] For a proof see:
`http://en.wikipedia.org/wiki/Ping-pong_lemma`

[5] I do not know whether '8' is optimal, presumably not, but this is the only reference I am aware of.

[6] Put differently, a group π is residually P if we can detect any non trivial element in a P–quotient.

[7] See also
`www.math.umbc.edu/~ campbell/CombGpThy/RF_Thesis/1_Decision_Problems.html`

[8] Put differently, a group π is subgroup separable if given any finitely generated group A and $g \notin A$ we can tell that $g \notin A$ by going to a finite quotient.

Remark.

1. A subgroup separable group is in particular residually finite, this follows immediately from applying the definition to $A = \{e\}$.
2. Any finitely generated abelian group is subgroup separable, indeed, given $A \subset \pi$ the group π/A is again finitely generated, in particular residually finite.
3. If a finitely presented group is subgroup separable, then the extended word problem is solvable, i.e. it can be decided whether a given finitely generated subgroup contains a given element or not. We refer again to [Moa66] for details.

The following theorem was proved by Scott [Sc78] in 1978:

Theorem 2.5. (Scott's theorem) *Let Σ be a surface, then $\pi_1(\Sigma)$ is subgroup separable.*

Given a space X and a ring R we write

$$vb_1(X;R) := \sup\{b_1(X';R)\,|\,X' \to X \text{ finite covering }\} \in \mathbb{N}\cup\{\infty\}.$$

Put differently, $vb_1(X;R) = \infty$ if X admits finite covers with arbitrarily large first R–Betti numbers.

Lemma 2.6. *Let Σ be a hyperbolic surface, then $vb_1(X;R) = \infty$ for any ring.*

Proof. We consider the case that Σ is closed, the bounded case is proved the same way. Let Σ' be an n–fold cover of Σ. Then it follows from the multiplicativity of the Euler characteristic under finite covers that

$$b_1(\Sigma') - 2 = -\chi(\Sigma') = -n\chi(\Sigma) \geq n.$$

□

2.3. **The mapping class group and Dehn twists.** Let Σ be an orientable surface. We want to study

$$\mathrm{Diff}(\Sigma) := \{\text{orientation preserving diffeomorphisms of } \Sigma\}$$

and the mapping class group

$$\mathrm{MCG}(\Sigma) := \mathrm{Diff}(\Sigma)/\text{homotopy} = \pi_0(\mathrm{Diff}(\Sigma)).$$

Definition. Let $c \subset \Sigma$ be an oriented simple closed curve. The *Dehn twist along c* is defined to be the diffeomorphism

$$\begin{array}{rcl} \Sigma & \to & \Sigma \\ x & \mapsto & \begin{cases} x, & \text{if } x \in \Sigma \setminus c \times [0,1] \\ (e^{2\pi i t}z, t), & \text{if } x = (z,t) \in c \times [0,1]. \end{cases} \end{array}$$

The following theorem was proved by Lickorish [Li62]:

Theorem 2.7. (Lickorish's theorem) *Let Σ be a surface, then any element in* $\mathrm{MCG}(\Sigma)$ *is the composition of finitely many Dehn twists.*

In fact the mapping class group is generated by Dehn twists on certain $3g+1$–curves and one can give a finite presentation for the mapping class group.

2.4. **Classification of diffeomorphisms.** Up to diffeotopy S^2 and $\mathbb{R}P^2$ admit no orientation preserving diffeomorphisms, i.e.

$$\mathrm{MCG}(S^2) = 0.$$

So let us turn to the torus $T = \mathbb{R}^2/\mathbb{Z}^2$. Any diffeomorphism of T lifts to an diffeomorphism of $\mathbb{R}^2$ which preserves $\mathbb{Z}^2$ as a set. One can show that the diffeotopy class of the diffeomorphism is determined by the restriction to $\mathbb{Z}^2$, i.e.

$$\mathrm{MCG}(T) = \mathrm{SL}(2, \mathbb{Z}).$$

Given $A \in \mathrm{SL}(2, \mathbb{Z})$ there are three cases we have to distinguish:

(1) A is not diagonalizable, then 1 or 1 is an eigenvalue, i.e. A fixes a line or reverses a line,
(2) A is diagonalizable with complex eigenvalues, in fact the only possible complex eigenvalues are $\pm i$ and $\pm e^{\pi i/3}$, i.e. A has finite order,
(3) A has two real eigenvalues λ and λ^{-1}.

In terms of diffeomorphisms for tori this means that given any $\varphi \in \mathrm{MCG}(T)$ one of the following happens:

(1) φ is reducible, i.e. φ fixes an essential curve as a set, [9] this happens for example if φ is the Dehn twist along one curve,
(2) φ is periodic, i.e. φ has finite order,
(3) φ is *Anosov*, i.e. there exists a transverse pair of geodesic curves c and d on T and a $\lambda > 1$ such that c gets 'stretched' by $\lambda > 1$ and d gets 'compressed' by the factor $\lambda^{-1} < 1$.

The 'generic' diffeomorphism of a torus is of that type.

Thurston in the late 1970's showed that a complete analogue holds for hyperbolic surfaces: Let Σ be a hyperbolic surface and $\varphi \in \mathrm{MCG}(\Sigma)$, then

(1) φ is reducible, or
(2) φ is periodic, or

[9]We call a simple closed curve *essential* if it does not bound a disk and if it is not boundary parallel

(3) φ is *Pseudo–Anosov*, i.e. there exists a transverse pair of measured foliations c and d and a $\lambda > 1$ such that the foliations are preserved by φ and their transverse measures are multiplied by λ and λ^{-1}.

For the precise meaning of (3) we refer to [Th88] and [CB88]. The third case is again the generic case.

3. Examples and constructions of 3–manifolds

For the remainder of this lecture course we will study 3–manifolds. In dimension three we do not have to distinguish between the categories of topological, smooth and PL manifolds: by Moise's theorem any topological 3–manifold also admits a unique PL and a unique smooth structure. [10] We will throughout restrict ourselves to 3–manifolds which are either closed or which have toroidal boundary.

3.1. Examples of 3–manifolds.

3.1.1. *Lens spaces.* The most basic example of a 3–manifold is of course the 3–sphere:

$$S^3 = \{x \in \mathbb{R}^4 \,|\, ||x|| = 1\} = \{(w, z) \in \mathbb{C}^2 \,|\, |w|^2 + |z|^2 = 1\}.$$

The quotients of S^3 by cyclic groups form already an interesting class of manifolds. More precisely, let $p, q \in \mathbb{N}$ and let $\xi = e^{2\pi i/p}$. We then consider the lens space

$$L(p, q) := S^3/\sim \text{ where } (z, w) \sim (\xi z, \xi^q w),$$

We can thus view $L(p, q)$ as the quotient of S^3 by a free action of $\mathbb{Z}/p$. It follows that

$$\pi_1(L(p, q)) = \mathbb{Z}/p.$$

Lens spaces are completely classified:

Theorem 3.1. (Reidemeister 1937) *Let $L(p, q)$ and $L(p, q')$ be two lens spaces.*

(1) *$L(p, q)$ and $L(p, q')$ are homotopy equivalent if and only if*
$$q' \equiv \pm a^2 q \bmod p \text{ for some } a.$$

(2) *$L(p, q)$ and $L(p, q')$ are diffeomorphic if and only if*
$$q' \equiv \pm q^{\pm 1} \bmod p.$$

[10] The analogous statement does of course not hold in dimension four, for example it follows from the work of Freedman and Donaldson that $\mathbb{R}^4$ admits uncountably many smooth structures.

Proof. The 'if' directions can be shown directly, the first 'only if' direction follows from studying the linking forms on the first homology, the second 'only if' direction follows from considering Reidemeister torsion. □

We see in particular that the fundamental group of a 3–manifold does not determine the homotopy type, and the homotopy type does not determine the diffeomorphism type of a lens space.

3.1.2. *Complements of knots and links.* Given any knot or link L in S^3 we can consider its exterior

$$X(L) := S^3 \setminus \nu L.$$

Important examples are given by

(1) the unknot, the trefoil knot and the figure 8 knot,
(2) torus knots and links, i.e. knots and links which lie on the standard torus in S^3. Note that the trefoil knot is a torus knot, whereas the figure 8 knot is not.

3.1.3. *Mapping tori.* Let Σ be a surface and $\varphi \in \operatorname{Diff}(\Sigma)$, then we can consider the mapping torus

$$([0,1] \times \Sigma) \,/\, (0,x) \sim (1,\varphi(x)).$$

If N can be written that way, then we say that N *fibers over* S^1. For example the exteriors of torus knots and of the figure 8 knot fiber over S^1, but 'most' knot exteriors do not fiber over S^1.

3.1.4. *Seifert fibered spaces.*

Definition. A *Seifert fibered 3-manifold* is a 3-manifold N together with a decomposition into disjoint simple closed curves (called *fibers*) such that each fiber has a tubular neighborhood that forms a standard fibered torus. A *standard fibered torus* corresponding to a pair of coprime integers (a, b) with $a > 0$ is the surface bundle of the diffeomorphism of a disk given by rotation by an angle of $2\pi b/a$, with the natural fibering by circles. If $a > 1$, then the middle fiber is called *singular*.

There are various different equivalent ways to think about Seifert fibered manifolds:

(1) Seifert fibered manifolds are S^1–bundles over a surface with isolated 'singular fibers',
(2) Seifert fibered manifolds are S^1–bundles over 2–dimensional orbifolds,

(3) A 3–manifold is Seifert fibered if and only if it is finitely covered by an S^1–bundle over a surface.

Example. The following are Seifert fibered spaces:

(1) $S^1 \times \Sigma$, in particular the three torus,
(2) S^3 with the Hopf fibration,
(3) lens spaces,
(4) exteriors of torus links,
(5) mapping tori of periodic surface diffeomorphisms.

Seifert fibered spaces are well understood and completely classified. We refer to [Or72, He76, Ja80, Br93] for further information and for the classification of Seifert fibered 3-manifolds. For future reference we record the following lemma:

Lemma 3.2. *Let N be a Seifert fibered 3–manifold with infinite fundamental group. Then a regular fiber of the Seifert fibration generates an infinite cyclic normal subgroup of $\pi_1(N)$.*

Since Seifert fibered spaces are finitely covered by circle bundles over surfaces it is not difficult to show that fundamental groups of Seifert fibered spaces share many of the nice properties of surface groups, more precisely we have the following proposition:

Proposition 3.3. *Let N be a Seifert fibered space. Then the following hold:*

(1) *$\pi_1(N)$ is linear over $\mathbb{Z}$,*
(2) *$\pi_1(N)$ is residually finite,*
(3) *$\pi_1(N)$ is subgroup separable.*

3.2. **Operations on 3–manifolds.** Given 2–oriented 3–manifolds N_1 and N_2 we can consider the connected sum

$$N_1 \# N_2 = (N_1 \setminus \text{3-ball}) \cup (N_2 \setminus \text{3-ball}),$$

where we identify the two boundary spheres using an orientation reversing homeomorphism. Note that the diffeomorphism type of the connected sum depends in general on the orientation, i.e. in general

$$N_1 \# N_2 \not\cong N_1 \# - N_2.$$

Similarly, given two 3–manifolds N_1 and N_2 with toroidal boundary, we can create a new manifold by gluing N_1 to N_2 along a boundary torus.

4. 3–manifolds up to 1973

4.1. **The prime decomposition theorem.** A 3-manifold N is called *prime* if N can not be written as a non-trivial connected sum of two manifolds, i.e. if $N = N_1 \# N_2$, then $N_1 = S^3$ or $N_2 = S^3$. Furthermore N is called *irreducible* if every embedded S^2 bounds a 3-ball. Note that an irreducible 3-manifold is prime, conversely if N is a prime 3-manifold, then either N is irreducible or $N = S^1 \times S^2$. We now have the following theorem:

Theorem 4.1. (Prime decomposition theorem) *Let N be an oriented 3–manifold.*

1. *There exists a decomposition $N \cong N_1 \# \dots \# N_r$, where the 3-manifolds $N_1, \dots, N_r$ are oriented prime 3-manifolds.*
2. *If $N \cong N_1 \# \dots \# N_r$ and $N \cong N_1' \# \dots \# N_s'$ where the 3-manifolds N_i and N_i' are oriented prime 3-manifolds, then $r = s$ and (possibly after reordering) there exists an orientation preserving diffeomorphism $N_i \to N_i'$.*

Remark.

1. The first part of the theorem was proved by Kneser [Kn29]. The difficulty of course lies in showing that the process of decomposing a 3–manifold will end after finitely many steps. The second statement was proved by Milnor [Mi62].
2. The theorem says in particular, that any 3–manifold group can be written as the free product of fundamental groups of prime 3–manifolds. In fact the converse holds: if $\pi_1(N^3)$ is isomorphic to a free product $A * B$, then there exist 3–manifolds X and Y with $\pi_1(X) = A$, $\pi_1(Y) = B$ and $N = A \# B$. This statement is referred to as the 'Kneser conjecture' and was first proved by Stallings [St59].
3. Schubert proved that a similar theorem holds for knots: a knot can be uniquely written as the connect sum of finitely many prime knots.

We will henceforth restrict ourselves to irreducible 3–manifolds.

4.2. **Dehn's lemma and the sphere theorem.** The following theorem was first formulated by Dehn in 1910. In 1927 Kneser showed that Dehn's proof had a gap, and a correct proof was finally given by Papakyriakopoulos [Pa57] [11] in 1957:

[11] I now quote from the wikipedia article about Papakyriakopoulos: 'The following limerick was composed by John Milnor, shortly after learning of several graduate students' frustration at completing a project where the work of every Princeton mathematics faculty member was to be summarized in a limerick:

Theorem 4.2. **(Dehn's lemma)** *Let N be a 3–manifold, let T be a boundary component, and suppose that*

$$K := \operatorname{Ker}\{\pi_1(T) \to \pi_1(N)\}$$

is non–trivial. Then there exists a properly embedded disk $D \subset N$ such that ∂D represents a non–trivial element in K.

Remark. This formulation of Dehn's lemma is also known as the loop theorem.

We obtain immediately the following corollary:

Corollary 4.3. *Let $K \subset S^3$ be a knot. Then K is trivial if and only if $\pi_1(S^3 \setminus \nu K) = \mathbb{Z}$.*

Proof. If K is trivial, then $\pi_1(S^3 \setminus \nu K) = \pi_1(S^1 \times D^2) = \mathbb{Z}$. Conversely, if $\pi_1(S^3 \setminus K) = \mathbb{Z}$, then the longitude of K represents an element in

$$\operatorname{Ker}\{\pi_1(\partial(S^3 \setminus \nu K)) \to \pi_1(S^3 \setminus \nu K)\}.$$

But by the loop theorem the longitude now bounds a disk, i.e. the knot is trivial. □

Papakyriakopoulos [Pa57] also proved the following deep theorem:

Theorem 4.4. **(Sphere theorem)** *Let N be a 3–manifold such that $\pi_2(N) \neq 0$. Then there exists an embedded essential* [12] *sphere in N, in particular N is reducible.*

Remark.
(1) We can summarize Dehn's lemma and the sphere theorem slightly sloppily as follows: 'Dehn's lemma says, that if there exists a singular disk, then it can be replaced by an embedded disk', and the sphere theorem says, morally speaking, 'a singular sphere can be replaced by an embedded sphere'.
(2) If N is a manifold of dimension $n \geq 5$, then it follows from a general position argument, that any class in $\pi_2(N)$ can be represented by an embedded sphere. The analogous statement

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The perfidious lemma of Dehn
Was every topologist's bane
'Til Christos Papa-
kyriakopou-
los proved it without any strain.

The phrase 'without any strain' is not meant to indicate that Papa did not expend much energy in his efforts. Rather, it refers to Papa's 'tower construction', which quite nicely circumvents much of the difficulty in the cut-and-paste efforts that preceded Papa's proof.'

[12] An embedded sphere is called essential if it does not bound a 3–ball

does not hold in dimension four. This can be viewed as the source of all the troubles (or exciting phenomena, depending on your point of view) in dimension four.

Corollary 4.5. *Let N be an irreducible 3–manifold with infinite fundamental group. Then N is aspherical.*

Proof. It follows from the sphere theorem, that $\pi_2(N) = 0$. We now denote by $\widetilde{N}$ the universal cover of N. It follows from Hurewicz that

$$\pi_3(N) = \pi_3(\widetilde{N}) = H_3(\widetilde{N}).$$

Our assumption that $\pi_1(N)$ is infinite implies that $\widetilde{N}$ is not compact, i.e. that $H_3(\widetilde{N}) = 0$. By induction we can now show that in fact $\pi_k(N) = 0$ for all $k \geq 3$. □

Note that this corollary applies in particular to knot complements. By the argument of Lemma 2.2 we now obtain the following corollary:

Corollary 4.6. *Let N be an irreducible 3–manifold with infinite fundamental group. Then $\pi_1(N)$ is torsion–free.*

We can now give a complete answer to the question, which abelian groups can arise as fundamental groups of 3–manifolds:

Proposition 4.7. *The only abelian groups which appear as fundamental groups of closed 3–manifolds are* [13]

$$\mathbb{Z} = \pi_1(S^1 \times S^2), \mathbb{Z}/n = \pi_1(\textit{lens space}) \textit{ and } \mathbb{Z}^3 = \pi_1(3 - \textit{torus}).$$

Proof. Let N be a 3–manifold with abelian fundamental group. By Proposition 1.4 we already know that $\pi_1(N)$ is isomorphic to $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, \mathbb{Z}/n$ or $\mathbb{Z} \oplus \mathbb{Z}/n$. If N is not prime, then $\pi_1(N)$ is a free product, and we can thus assume that N is in fact prime. If N has infinite, non–cyclic fundamental group, then it follows from Corollary 4.5 that N is aspherical, i.e. $N = K(\pi_1, 1)$. In Corollary 4.6 we showed that this implies that $\pi_1(N)$ is torsion–free, we can thus exclude the possibility that $\pi_1(N) = \mathbb{Z} \oplus \mathbb{Z}/n$. Finally note that $H_3(\mathbb{Z}^2) \neq H_3(N) = \mathbb{Z}$, so we can also exclude the case that $\pi_1(N) = \mathbb{Z}^2$. □

[13] If we allow non–orientable manifolds, then we have to add

$$\mathbb{Z} \times \mathbb{Z}/2 = \pi_1(S^1 \times \mathbb{R}P^2).$$

4.3. Haken manifolds.

Definition. (1) A surface $\Sigma \subset N$ is called *incompressible*, if $\pi_1(\Sigma) \to \pi_1(N)$ is injective.
(2) A 3-manifold N is called *Haken* if N is irreducible and if N admits a non-simply connected embedded incompressible surface $\Sigma \subset N$.

The following lemma shows in particular that the exteriors of knots are Haken:

Lemma 4.8. *Let $N \neq S^1 \times D^2$ be any irreducible 3–manifold (for once with no restrictions on the boundary) with $b_1(N) \geq 1$, then N is Haken.*

Proof. Let $\Sigma \subset N$ be a properly embedded surface of 'minimal complexity' [14] representing a given non–trivial element in $H^1(N;\mathbb{Z}) = H_2(N, \partial N; \mathbb{Z})$.

Claim. The surface Σ is incompressible.

If Σ is not incompressible, then applying Dehn's lemma to $N \setminus \nu\Sigma$ we can find an embedded essential curve c on Σ which bounds an embedded disk D in N. We can thus do surgery on Σ along c and we obtain a surface of smaller complexity which represents the same homology class. This concludes the proof of the claim.

Our assumption that $N \neq S^1 \times D^2$ and that N is irreducible implies that $\Sigma \neq D^2$ and $\Sigma \neq S^2$. We thus found an incompressible non–simply connected surface in N. □

The basic idea for the study of Haken manifolds is very simple: given an incompressible surface $\Sigma \subset N$ we can cut N along Σ, we obtain a (possibly disconnected) 3–manifold such that each component has positive first Betti number, i.e. each component is Haken again. We can thus iterate the process and reduce the 'complexity' of the 3–manifold along the way till we obtain 3–balls. We refer to [Ha62] for details.

Waldhausen [Wa68b, Corollary 6.5] used this approach to prove the following theorem:

Theorem 4.9. *Let N and N' be two closed Haken manifolds with $\pi_1(N) \cong \pi_1(N')$, then N and N' are homeomorphic.*

[14] The complexity of a surface Σ with connected components $\Sigma_1, \dots, \Sigma_k$ is defined as

$$\chi_-(\Sigma) = \sum_{i=1}^{k} \max(0, -\chi_i(\Sigma)).$$

This turns out to be the correct generalization of the concept of 'minimal genus' to disconnected surfaces. We refer to [Th86a] for details.

Remark. If N and N' are Haken manifolds with non–trivial toroidal boundary, then it is in general not true, that their diffeomorphism type is determined by their fundamental group. In fact if K denotes the trefoil knot, then

$$S^3 \setminus \nu(K\#K) \text{ and } S^3 \setminus \nu(K\# -K)$$

are not diffeomorphic, even though the fundamental groups are isomorphic. This can be dealt with as follows: Let N be a 3-manifold with incompressible boundary. Then we refer to the fundamental group of $\pi_1(N)$ together with the conjugacy classes of subgroups determined by the boundary components as the *peripheral structure of* N. Waldhausen showed that Haken 3–manifolds with non–spherical boundary are determined by their 'peripheral structure'. We refer to [Wa68b] for details.

5. Interlude: Hyperbolic 3–manifolds

We say that a 3–manifold N, possibly with toroidal boundary, is *hyperbolic* if the interior of N admits a complete metric of constant curvature equal to -1. If N is hyperbolic, then its universal cover is given by

$$\mathbb{H}^3 = \{(x, y, z) \,|\, z > 0\}$$

equipped with the metric

$$\frac{1}{z} \cdot \text{standard metric on } \mathbb{R}^3.$$

Note that $\text{Isom}(\mathbb{H}^3) = \text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/ \pm \text{id}$.

Proposition 5.1. *A 3–manifold N is hyperbolic if and only if there exists a discrete and faithful representation $\alpha\colon \pi_1(N) \to \text{SL}(2, \mathbb{C})$ such that $\alpha(\pi_1(N)) \subset \text{SL}(2, \mathbb{C})$ is torsion–free and has finite covolume.*

Proof. If N is a hyperbolic 3–manifold, then there exists a discrete and faithful representation $\alpha\colon \pi_1(N) \to \text{PSL}(2, \mathbb{C})$, such that the image is torsion–free and has finite covolume. Thurston showed that this representation lifts to a representation to $\text{SL}(2, \mathbb{C})$ (see [Sh02, Section 1.6]). The converse follows immediately from the observation that $\mathbb{H}/\alpha(\pi_1(N))$ is a hyperbolic 3–manifold, diffeomorphic to N. □

Riley [Ri74] used this approach in 1974 to show that the figure 8 knot [15] is hyperbolic. But in most cases it is very difficult to directly show that a 3–manifold is hyperbolic.

[15] In this case one can write down the representation explicitly: The fundamental group of the figure 8 knot complement has the following presentation

$$\langle x, y \,|\, xyx^{-1}y^{-1}x = yxy^{-1}x^{-1}y \rangle$$

In the following we say that a 3-manifold N is *atoroidal* if any incompressible torus is boundary parallel, i.e. can be isotoped into the boundary. The following lemma gives an obstruction to a 3–manifold being hyperbolic:

Lemma 5.2. *Let N be a hyperbolic 3–manifold and let $A \subset N$ be a subgroup isomorphic to $\mathbb{Z}^2$. Then there exists a boundary torus T such that $A = \pi_1(T)$ as subgroups in $\pi_1(N)$.*

It follows directly from the lemma that a closed hyperbolic 3–manifold does not contain any incompressible tori.

Proof. We can view $\pi_1(N)$ as a discrete subgroup of $\mathrm{SL}(2,\mathbb{C})$. A basic argument shows that any discrete subgroups of $\mathrm{SL}(2,\mathbb{C})$ isomorphic to $\mathbb{Z}^2$ is of the form

$$\begin{pmatrix} \varepsilon & \mathbb{Z}+\lambda\mathbb{Z} \\ 0 & \varepsilon \end{pmatrix}$$

where $\varepsilon = \pm 1$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. One can now show that any such subgroup corresponds to a boundary torus ([Bo02, Theorem 2.9]). □

In the case of surfaces we noted that any surface Σ with $\chi(\Sigma) < 0$ is hyperbolic, but the hyperbolic structure is by no means unique. Amazingly the situation is completely different in dimension three (see [Mob68] and [Pr73]):

Theorem 5.3. (Mostow–Prasad rigidity) *Let N be a hyperbolic 3–manifold, then the hyperbolic structure of N is unique up to isometry.*

Put differently, if two hyperbolic 3–manifolds are homeomorphic, then they are already isometric. This implies that all geometric invariants of hyperbolic 3–manifolds (e.g. volume) are in fact topological invariants. It follows also, that a hyperbolic 3–manifold admits a unique (up to conjugation) discrete and faithful representation $\alpha\colon \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$.

Remark.
1. Rigidity also holds in fact for closed hyperbolic manifolds of any dimension greater than two.
2. One can use rigidity to show that the the discrete and faithful representation $\alpha\colon \pi_1(N) \to \mathrm{SL}(2,\mathbb{C})$ is conjugate to a representation $\pi_1(N) \to \mathrm{SL}(2,\overline{\mathbb{Q}})$ over the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ (see [MR03, Corollary 3.2.4]).

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and a discrete and faithful representation is given by

$$x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } y \mapsto \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$$

where $z = e^{\pm \pi i/3}$.

We conclude this section with the following variation on Proposition 2.4.

Proposition 5.4. *Let N be a hyperbolic 3–manifold, then $\pi_1(N)$ is residually finite.*

Proof. We can view $\pi_1(N)$ as a subgroup of $\mathrm{SL}(2,\mathbb{C})$. It thus suffices to prove the following claim:

Claim. Any finitely generated subgroup of $\mathrm{GL}(n,\mathbb{C})$ is residually finite.

Let $\pi \subset \mathrm{GL}(n,\mathbb{C})$ be a subgroup which is generated by $g_1,\dots,g_k$. We denote by R the ring which is generated by the entries of $\alpha(g_1),\dots,\alpha(g_k)$. Note that R is finitely generated over R. Since R is finitely generated over R we can find for any $r \in R$ a maximal ideal $m \subset R$ such that r is non–trivial in R/m. The field R/m has a prime characteristic and is finitely generated, hence R/m is in fact a finite field. Since m is maximal the quotient R/m is Given a non–trivial matrix $A \subset \pi$ can now find a maximal ideal $m \subset R$ such that A represents a non–trivial element in the finite group $\mathrm{GL}(n,R/m)$. □

We thus see that hyperbolic 3–manifolds have interesting properties, but in contrast to surfaces it is very difficult to construct examples of hyperbolic 3–manifolds 'by hand'. Historically the first example of a closed hyperbolic 3–manifold is the Seifert–Weber manifold constructed in 1933. We refer to [SW33] for details. [16] In [BRT09] it is shown that the Seifert–Weber manifold is not Haken. The figure 8 knot complement, the Seifert–Weber manifold and their finite covers were some of the few examples of hyperbolic 3–manifolds known till the mid 1970's.

6. The JSJ and the geometric decomposition

6.1. **The statement of the theorems.** In Section 4.1 we saw that 3–manifolds have a unique prime decomposition, i.e. they have a unique decomposition along spheres. In this section we will see that 3–manifolds also have a canonical decomposition along incompressible tori. After this second stage of decompositions we will finally end up with pieces which are either hyperbolic or Seifert fibered.

The following theorem was proved independently by Jaco and Shalen [JS79] and Johannson [Jo79].

Theorem 6.1. (JSJ Decomposition Theorem) *Let N be an irreducible 3-manifold. Then there exists a collection of disjointly embedded*

[16] Or alternatively see:
http://en.wikipedia.org/wiki/Seifert-Weber_space

incompressible tori $T_1, \dots, T_k$ such that each component of N cut along $T_1 \cup \dots \cup T_k$ is atoroidal or Seifert fibered. Furthermore any such collection with a minimal number of components is unique up to isotopy.

In the following we will refer to the tori $T_1, \dots, T_k$ as the *JSJ tori* and we will refer to the components of N cut along $\cup_{i=1}^k T_i$ as the *JSJ components.*

The goal now is to determine which 3–manifolds are atoroidal. In the following we say that a closed 3-manifold is *spherical* if it admits a complete metric of curvature $+1$. Note that the universal cover of a spherical 3–manifold is necessarily S^3. It is clear that spherical 3–manifolds have finite fundamental groups, in particular they are atoroidal. By the discussion of the previous section we also know that hyperbolic 3–manifolds are atoroidal. Thurston [Th82] conjectured that these are all examples of atoroidal 3–manifolds.

This conjecture was proved by Thurston for Haken manifolds. The proof for the general case was first given by Perelman in his seminal papers [Pe02, Pe03a, Pe03b], we refer to [MT07] for full details. More precisely, Perelman proved the following theorem:

Theorem 6.2. (Perelman) *Let N be an irreducible atoroidal 3–manifold. Then either N is spherical or N is hyperbolic.*

It is well-known that S^3 equipped with the canonical metric is the only spherical simply connected 3-manifold. We thus obtain the following theorem:

Theorem 6.3. (Poincaré conjecture) *The 3–sphere S^3 is the only simply connected, closed 3-manifold.*

Remark. It also follows that a 3-manifold N is spherical if and only if it is the quotient of S^3 by a finite group, which acts freely and isometrically, in particular we can view $\pi_1(N)$ as a finite subgroup of SO(4) which acts freely on S^3. We refer to [Or72, Chapter 1, Theorem 1] and [Or72, Chapter 2, Theorem 2] for details and for the complete list of 3-manifolds with finite fundamental groups.

Note that spherical 3-manifolds are in fact Seifert fibered (see [Bo02, Theorem 2.8]). We thus obtain the following theorem:

Theorem 6.4. (Geometrization Theorem) *Let N be an irreducible 3-manifold with empty or toroidal boundary. Then there exists a collection of disjointly embedded incompressible tori $T_1, \dots, T_k$ such that each component of N cut along $T_1 \cup \dots \cup T_k$ is hyperbolic or Seifert fibered. Furthermore any such collection with a minimal number of components is unique up to isotopy.*

6.2. **Geometric structures on 3–manifolds.** We had seen that any surface admits a metric of constant curvature. For 3–manifolds the situation is more complicated: we first have to decompose a given manifold along embedded spheres and incompressible tori. The resulting pieces are then either hyperbolic or Seifert fibered. The Seifert fibered pieces with finite fundamental groups are spherical, and the Seifert fibered pieces with infinite abelian fundamental groups are Euclidean, i.e. they have a metric of constant curve zero. Seifert fibered pieces with infinite, non–abelian fundamental groups do not admit a metric of constant curvature. But they do carry a unique geometry if we expand our definition of 'geometry of a manifold'. The five extra geometries needed are referred to as

$$\mathrm{Sol}, \widetilde{\mathrm{SL}(2,\mathbb{R})}, \mathrm{Nil}, S^2 \times \mathbb{R}, H^2 \times \mathbb{R}.$$

Even though the geometric point of view is very pretty, in practice these geometries are studied very little, since they all correspond to Seifert fibered spaces or torus bundles, which are well understood anyway. The only geometric 3–manifolds which are not well understood are the most important manifolds: hyperbolic 3–manifolds.

For completeness' sake we also describe how the geometry of a Seifert fibered space N can be determined from the topology of N: As we mentioned earlier, any Seifert fibered space is finitely covered by an S^1–bundle over a surface Σ. We denote by e the Euler class of the S^1–bundle. Then $\chi(\Sigma)$ and e determine the geometry of N:

$$\begin{array}{r|ccc} & \chi > 0 & \chi = 0 & \chi < 0 \\ \hline e = 0 & S^2 \times \mathbb{R} & \mathbb{E}^3 & \mathbb{H}^2 \times \mathbb{R} \\ e \neq 0 & S^3 & \mathrm{Nil} & \widetilde{\mathrm{SL}(2,\mathbb{R})}. \end{array}$$

The Sol–geometry appears only for torus bundles which are not Seifert fibered spaces. Their JSJ decomposition is given by cutting the torus bundle along a fiber. We refer to [Th82], [Sc83] and [Bo02] for more information on the eight geometries.

The above discussion shows that one can reformulate the geometrization theorem in such a way, that the name 'geometrization theorem' is fully justified:

Theorem 6.5. (Geometrization Theorem) *Let N be an irreducible 3-manifold with empty or toroidal boundary. Then there exists a collection of disjointly embedded incompressible tori $T_1, \dots, T_k$ such that each component of N cut along $T_1 \cup \dots \cup T_k$ is geometric. Furthermore* *any such collection with a minimal number of components is unique up to isotopy.* [17]

Finally, Thurston's eight geometries also inspired fashion designer Issey Miyake:
`http://www.youtube.com/watch?v=lMneAQsAZUA`

6.3. **Examples of the decompositions.** We now want to see what the geometrization theorem does for several 'real life' examples.

Let N_1 and N_2 be two hyperbolic 3–manifolds such that $T_i = \partial N_i, i = 1, 2$ consists of one torus each. Using Dehn's lemma one can easily show that the boundary tori are incompressible. We then glue N_1 and N_2 along their boundary tori using any diffeomorphism. The resulting 3–manifold N contains an incompressible torus. By the discussion of Section 5 the manifold N can therefore not be hyperbolic. One can also show that N is not Seifert fibered. [18] We thus need at least one torus to cut N into a union of Seifert fibered spaces and hyperbolic pieces. The torus T of course does the trick.

In general, if we glue two 3–manifolds N_1 and N_2 along a torus boundary component, then in almost all cases [19] the JSJ decomposition of the resulting manifold N is given by the JSJ tori of N_1, together with the JSJ tori of N_2 and the gluing torus.

Remark. It follows from the above arguments that a knot $K \subset S^3$ is either a torus knot, a hyperbolic knot, or a satellite knot, i.e. a knot which is given by wrapping a non–trivial knot in a solid torus around another non–trivial knot.

Exercise 6.6. *Let K_1 and K_2 be two non–trivial hyperbolic knots. What is the JSJ decomposition of $S^3 \setminus (K_1 \cup K_2)$?*

In Section 4.3 we obtained new 3–manifolds by connect sum operation and by gluing along tori. The prime decomposition theorem and the geometrization theorem now unravel these two operations: the

[17] The two decompositions in the two formulations of the geometrization theorem are identical with one exception: if N is a torus bundle over S^1 with Sol geometry, then in the second formulation we do not need any tori, whereas in the first formulation the decomposition will consist of one fiber, which then cuts N into two manifolds of the form $T \times [0, 1]$, which are both Seifert fibered spaces.

[18] For example incompressible tori in Seifert fibered spaces are well-understood, in particular the complement of an incompressible torus in a Seifert fibered space has to be Seifert fibered again.

[19] The only exception is the case that the JSJ components abutting the gluing torus are both Seifert fibered, and if the Seifert fibrations match up.

prime decomposition theorem detects the prime components and the geometrization theorem detects 'almost all' the gluing tori.

As a final example we discuss fibered 3–manifolds. The following theorem was proved by Thurston [Th86a, Th86b] in the late 1970's:

Theorem 6.7. . *Let Σ be a surface with $\chi(\Sigma) < 0$ and let $\varphi \in \operatorname{MCG}(\Sigma)$. We denote by N the mapping torus of φ.*

(1) *If φ is reducible, then N admits an incompressible torus,*
(2) *if φ is periodic, then N is Seifert fibered,*
(3) *if φ is pseudo–Anosov, then N is hyperbolic.*

6.4. **Applications.** We will first prove the following lemma:

Lemma 6.8. *Let N be a 3–manifold which contains a subgroup isomorphic to $\mathbb{Z}^2$, then N contains an incompressible torus.*

Proof. If N is hyperbolic, then we saw at the end of Section 5 that any subgroup isomorphic to $\mathbb{Z}^2$ comes from a boundary torus. On the other hand, if N is Seifert fibered, then the statement follows easily from the classification of Seifert fibered spaces. Now suppose that N is neither hyperbolic nor Seifert fibered. It then follows from geometrization that N has a non–trivial JSJ decomposition, in particular it contains an incompressible torus. □

Theorem 6.9. *Let N and N' be two closed, prime 3-manifolds with $\pi_1(N) \cong \pi_1(N')$. Then either N and N' are homeomorphic, or N and N' are both lens spaces.*

Recall that a 3–manifold N is aspherical if and only if N is irreducible and if $\pi_1(N)$ is infinite. The theorem thus in particular proves the Borel conjecture for 3–manifolds.

Proof. Let N and N' be two closed, prime 3-manifolds with $\pi_1(N) \cong \pi_1(N')$. We first assume that N contains an incompressible torus. It follows from Lemma 6.8 that N' also contains an incompressible torus. In particular N and N' are both Haken, and the statement follows from Theorem 4.9.

We now consider the case that neither N nor N' contain an incompressible torus. It follows from geometrization that N and N' are either Seifert fibered or hyperbolic. First suppose that N is Seifert fibered. It follows from Lemma 3.2 that $\pi_1(N)$ contains an infinite cyclic normal subgroup. An argument similar to the proof of Lemma 5.2 shows that fundamental groups of hyperbolic 3–manifolds do not contain infinite cyclic normal subgroups. We thus see that N' is also Seifert fibered. It now follows from the classification of Seifert fibered spaces (see [Sc83, Theorem 3.1] and [Or72, p. 113]) that N and N' are diffeomorphic.

Finally consider the case that N and N' are hyperbolic. We identify $\pi_1(N)$ with $\pi_1(N')$. Let $\alpha\colon \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$ and $\alpha'\colon \pi_1(N') \to \mathrm{PSL}(2,\mathbb{C})$ be the discrete and faithful representations. Note that

$$N = \mathbb{H}/\alpha(\pi_1(N)) \text{ and } N' = \mathbb{H}/\alpha'(\pi_1(N')). \tag{6.1}$$

By Mostow–Prasad rigidity all discrete and faithful representations $\pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$ with cocompact image are conjugate. It follows that α and α' are conjugate, but this implies by (6.1) that N and N' are diffeomorphic. $\square$

If N is an irreducible 3–manifold we can now write $\pi_1(N)$ as an iterated HNN extension and amalgam of fundamental groups of Seifert fibered spaces and hyperbolic 3–manifolds along torus groups. We already saw that fundamental groups of hyperbolic 3–manifolds and Seifert fibered spaces are residually finite. Hempel [He87], building on ideas of Thurston [Th82], and assuming/using the Geometrization Theorem showed the following theorem:

Theorem 6.10. *Let N be any 3–manifold, then $\pi_1(N)$ is residually finite.*

7. Hyperbolic 3–manifolds

The previous section shows that the key to understanding hyperbolic 3–manifolds lies in understanding hyperbolic 3–manifolds. Unfortunately progress in our understanding of hyperbolic 3–manifolds has till recently been rather slow.

Before we state some of the most important questions regarding hyperbolic 3–manifolds we introduce one more definition: Let P be a property of 3–manifolds, then we say that a 3–manifold N is *virtually P* if N admits a finite cover which has Property P.

We now list some questions:

1. Is every hyperbolic 3–manifold virtually Haken?
2. Does every hyperbolic 3–manifold admit a finite cover with positive first Betti number?
3. Is every hyperbolic 3–manifold virtually fibered?

Note that we already saw that not every hyperbolic 3–manifold is Haken, but it was recently shown by Kahn and Markovic [KM09] that the fundamental group of every closed hyperbolic 3–manifold contains the fundamental group of a closed surface. This is a very important step towards proving (1).

Question (3) is the most ambitious question, and at first glance it sounds rather dubious. Fibered 3–manifolds are clearly very special and fairly simply to describe. Why then should any hyperbolic 3–manifold admit a finite cover which fibers over S^1? This question was first asked by Thurston in [Th82] and except for studying a few very specific examples progress has been very slow till a few years ago.

But recently there has been some progress towards proving (3). To state the results we need the following definition:

Definition. Let Γ be a finite graph with vertex set V, then it gives rise to a group presentation as follows:

$$\langle \{g_v\}_{v\in V} \,|\, [g_u, g_v] = 1 \text{ if } u \text{ and } v \text{ are connected by an edge}\rangle.$$

Any group which is isomorphic to such a group is called a *right angled Artin group (RAAG)*. RAAGs are also often referred to as graph groups or free partially commutative groups.

Agol [Ag08] proved the following theorem:

Theorem 7.1. *Let N be a hyperbolic 3–manifold such that $\pi_1(N)$ is a subgroup of a RAAG, then N virtually fibers over S^1.*

This raises the question, which hyperbolic 3–manifolds have the property that their fundamental groups are subgroups of a RAAG. Wise [Wi09, Wi11] announced a proof for the statement that a Haken 3–manifold is either virtually fibered, or its fundamental group is a subgroup of a RAAG. Wise's results together with Agol's theorem would thus imply that a hyperbolic Haken 3–manifold is virtually fibered.

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