NOTES ON THE PAIRING BETWEEN A REPRESENTATION AND ITS CONTRAGRADIENT

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These notes discuss the pairing between an admissible representation of $\mathrm{GL}_2(F)$ with F a p-adic field and its contragradient based on a discussion with Steve Kudla on August 14, 2008 and (per his suggestion) the paper [?] of Bernstein and Zelevinsky.

1. Right and left Haar measures

We take G to be a topological l-group as in [?], X to be an l-space, and $S(X)$ to be the set of compactly supported, locally constant functions on X. For our purposes it is fine to think of G as the F points of an algebraic group over F, and X as a space on which G acts (for example, G itself.) A linear functional on $S(X)$ is called a distribution. The space of all such functionals is denoted $S^*(X)$. The pairing $S(X) \times S*(X) \to \mathbb{C}$ is denoted by

$$(f, T) \mapsto T(f) = \int_X f(x) dT(x) = \int_X f \; dT.$$

We let $S^*_c(X) \subset S^*(X)$ be the space of compactly supported distributions, and $C^\infty(X)$ the set of locally constant functions. The pairing above extends to one on $C^\infty(X) \times S^*_c(X)$.

We consider the actions of G on itself given by left and right translation. Which are defined via

$$\lambda(g)g_0 = gg_0 \quad \text{and} \quad \rho(g)g_0 = g_0 g^{-1}$$

respectively. There is a unique up to scalar left-invariant *Haar measure* μ_G on G. This means that $\mu_G \in S^*(G)$ satisfies

$$\int_G f(g_0 g) d\mu_G(g) = \int_G f(g) d\mu_G(g_0^{-1} g) = \int_G f(g) d\mu_G(g)$$

for all $g \in G$ and $f \in S(X)$. (Similarly, there is a right-invariant Haar measure.) We can take μ_G such that $\mu_G(f) > 0$ for every nonzero nonnegative function $f \in S(X)$.

A *character* on G is a continuous (i.e. locally constant) homomorphism from G to $\mathbb{C}^\times$. By the uniqueness of the Haar measure there is a function $\Delta_G : G \to \mathbb{C}$ satisfying

$$\rho(g)\mu_G = \Delta_G(g)\mu_G.$$

This means that since

$$\rho(g_0)\mu_G(f) = \int_G f(g) d\mu_G(g g_0^{-1}) = \int_G f(g g_0) d\mu_G(g)$$

is still left invariant, $\rho(g_0)\mu_G$ must be a multiple of μ_G. $\Delta_G(g_0)$ is defined to be that number. It turns out that Δ_G is a character. We call it the *modulus character* of G. It satisfies the following additional properties.

- The restriction of Δ_G to any compact subgroup is trivial[1].
- The distribution $\Delta_G^{-1}\mu_G$ is right invariant.

If $\Delta_G \equiv 1$, we say that G is *unimodular*. Note that the first assertion above implies that if G is generated by compact subgroups it is unimodular.

2. Haar measure on a quotient space

Now we consider $B \subset G$ a closed subgroup. Let $\Delta = \Delta_G/\Delta_B$, and define

$$S(G,B,\Delta) = \left\{ f \in C^\infty(G) \;\middle|\; \begin{array}{l} f(hg) = \Delta(h)f(g) \text{ for all } h \in H, g \in G \\ f \text{ is finite modulo } B \end{array} \right\}$$

The second condition means that there is a compact set Ω such that supp$f \subset HK$. If $\Delta = 1$, $S(G,H,\Delta) \simeq S(H\backslash G)$.

Proposition 1. *There exists a unique up to scalar functional $\Lambda \in S^*(G,B,\Delta)$ that is right G-invariant.*

The idea is to start with the functional on $S(G)$

$$\phi \mapsto \int_G \phi(g)d\mu_G(g).$$

This is clearly G-invariant. So, using a projection $p : S(G) \to S(G,H,\Delta)$, we can use this to define the functional on $f \in S(G,H,\Delta)$ applying the above to $\phi \in p^{-1}(f)$. One then argues existence and uniqueness using the uniqueness of the Haar measure on G.

Of course, this method requires one to come up with the linear map p, and show that it is surjective. Moreover, one needs to show that the value $\Lambda(f)$ does not depend on the choice of $\phi \in p^{-1}(f)$. We accomplish this in a series of lemmas.

Lemma 2. *If $f \in S(G)$, the element pf defined by*

$$(pf)(g) = \int_B f(bg)\Delta_G^{-1}(b)d\mu_B(b)$$

belongs to $S(G,B,\Delta)$.

Proof. This is a straightforward computation.

$$\begin{aligned} pf(b_0g) &= \int_B f(bb_0g)\Delta_G^{-1}(b)d\mu_B(b) \\ &= \int_B f(bg)\Delta_G^{-1}(bb_0^{-1})d\mu_B(bb_0^{-1}) \\ &= \int_B f(bg)\Delta_G(b)\Delta_G(b_0)\Delta_B^{-1}(b_0)d\mu_B(b) \\ &= \Delta(b_0)pf(g). \end{aligned}$$

□

Note that p clearly commutes with the action of G on the right.

Lemma 3. *The map $p : S(G) \to S(G,B,\Delta)$ is surjective.*

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[1]This fact follows from considering the pairing $\mu_G(1_K)$ where K is the compact group in question, and 1_K its characteristic function. Then by positivity $\Delta_G(k) \in \mathbb{R}^+$. Therefore, since it's a character $\Delta_G(K)$ is a subgroup of $\mathbb{C}^\times \int \mathbb{R}^+ = \{1\}$.

Proof. Let L be a compact open subgroup of G and $g \in G$. Define

$$S(G)_g^L = \{f \in S(G) \mid \mathrm{supp} f \subset BgL, \text{ and } f(gl) = f(g) \text{ for all } l \in L\}.$$

We similarly define $S(G, B, \Delta)_g^L$. Since p is linear, and such functions generate $S(G)$ it suffices to prove the lemma for these subspaces.

So let $f \in S(G, B, \Delta)_g^L$. (Notice that f is determined by its value on g.) If we multiply f by c_{gL}, the characteristic function of gL, the result, call it ϕ', is clearly an element of $S(G)_g^L$. It is easy to see that $p(\phi')$ has support in BgL. So it is determined by its value on g:

$$\begin{aligned} p(\phi')(g) &= \int_B c_{gL}(bg) f(bg) \Delta_G^{-1}(b) d\mu_B(b) \\ &= f(g) \cdot \int_B c_{gL}(bg) \Delta(b) \Delta_G^{-1}(b) d\mu_B(b) \\ &= f(g) \int_B c_{gL}(bg) \Delta_B^{-1}(b) d\mu_B(b) \end{aligned}$$

which differs from $f(g)$ by a constant c. Taking $\phi = c^{-1}\phi'$ gives that $p\phi = f$. □

Lemma 4. *If $f \in S(G)$ and $pf = 0$ then $\int_G f(g) d\mu_G(g) = 0$.*

Proof. Again, it suffices to consider the set $S(G)_g^L$ as above. If $f \in S(G)_g^L$, then f is zero outside of BgL and constant on right L cosets of such. So $S(G)_g^L$ can be identified with the set of finite functions on the set BgL/L.

By uniqueness of Haar measure, all functionals on $S(G)_g^L$ that translate according to Δ_G^{-1} on the left must be proportional. Both $f \mapsto pf(g)$ and $\int_G f(g)\Delta()\mu_G(g)$ satisfy this translation property, and so the result follows. □

Proof of Proposition 1. As suggested, for $f \in S(G, B, \Delta)$ let $\phi \in p^{-1}(f)$. By Lemma 3 such a ϕ exists. Define

$$\Lambda(f) = \int_G \phi(g) d\mu_G(g).$$

Clearly, Λ is G-invariant, and by Lemma 4 it is well defined.

To see uniqueness note that if Λ' is another functional, the pull back to $S^*(G)$, it must be a multiple of the Haar measure on G. Pushing this forward, one sees that Λ' is a multiple of Λ. □

3. Application to $G = \mathrm{GL}_2(F)$

Let $G = \mathrm{GL}_2(F)$ and $B = \{\left(\begin{smallmatrix} a_1 & b \\ & a_2 \end{smallmatrix}\right) \in G\}$. Then $\Delta_G = 1$ and $\Delta_B(\left(\begin{smallmatrix} a_1 & b \\ & a_2 \end{smallmatrix}\right)) = abs\frac{a_1}{a_2}$. We now let dg denote the left invariant Haar measure on G. Therefore, Proposition 1 gives a functional on the set of functions that are right invariant by some compact open set, and transform according to

$$f(\left(\begin{smallmatrix} a_1 & b \\ & a_2 \end{smallmatrix}\right) g) = abs\frac{a_1}{a_2} f(g). \tag{1}$$

Such functions naturally occur when considering induced representations. Let χ_1, χ_2 be quasicharacters of $F^\times$. Then define the induced representation

$$I(\chi_1, \chi_2) = \mathrm{Ind}_B^G(\chi_1 |\cdot|^{1/2} \otimes \chi_2 |\cdot|^{-1/2}).$$

This is a space of functions $f : G \to \mathbb{C}$ satisfying:

- f is fixed by some $L \subset G$ compact, and

- $f(\left(\begin{smallmatrix} a_1 & b \\ & a_2 \end{smallmatrix}\right) g) = \chi_1(a_1)\chi_2(a_2) \left|\frac{a_1}{a_2}\right|^{1/2} f(g)$ for all $g \in G$.

Although the functions in $I(\chi_1, \chi_2)$ don't transform according to (1), if we take the product of $f \in I(\chi_1, \chi_2)$ and $f' \in I(\chi_1^{-1}, \chi_2^{-1})$, then it does. So the above theory tells that there exists a pairing

$$I(\chi_1, \chi_2) \times I(\chi_1^{-1}, \chi_2^{-1}) \longrightarrow \mathbb{C}.$$

In the remainder of this section, we prove the following.

Proposition 5. *Let* $\pi = I(\chi_1, \chi_2)$ *and* $\widetilde{\pi} = I(\chi_1^{-1}, \chi_2^{-1})$. *The pairing on* $\langle \cdot, \cdot \rangle : \pi \times \widetilde{\pi} \to \mathbb{C}$ *given by*

$$\langle f, f' \rangle = \int_K f f'(k) dk$$

is G-invariant and unique up to a constant.

Let $\mathcal{O}_F$ be the ring of integers of F, and $K = \mathrm{GL}_2(\mathcal{O}_F)$ a maximal compact subgroup of G. Let $B_K = B \cap K$. The Iwasawa decomposition says that $G = BK$. Therefore, functions in $I(\chi_1, \chi_2)$ are determined by their restriction to K. So we have an isomorphism

$$S(B_K \backslash K) \simeq S(K, B_K, 1) \to S(G, B, \Delta).$$

(Notice that since K is compact the modulus characters Δ_K and Δ_{B_K} are trivial.)

This gives the following picture:

$$\begin{array}{ccccc} & & S(K) & \subset & S(G) \\ S(B_K \backslash K) & \simeq & S(K, B_K, 1) & \simeq S(G, B, \Delta) \to & \end{array}$$

[Add discussion of why the diagram is commutative, and hence why the pairing is as described.]

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