A TRIPLE PRODUCT CALCULATION FOR $\mathrm{GL}_2(\mathbb{R})$

MICHAEL WOODBURY

Let F be a number field and $\mathbb{A} = \mathbb{A}_F$ the ring of adeles. Let T be the subgroup of GL_2 consisting of diagonal matrices with $Z \subseteq T$ the center. Let $N \subseteq \mathrm{GL}_2$ be the subgroup of upper triangle unipotent matrices so that $P = TN$ the standard Borel.

Given automorphic representations π_1, π_2, π_3 of GL_2 over F such that the product of the central characters is trivial, one can consider the so-called triple product L-function $L(s, \Pi)$ attached to $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$, or the completed L-function $\Lambda(s, \Pi)$. This L-function is closely related to periods of the form

$$I(\varphi) = \int_{[\mathrm{GL}_2]} \varphi_1(g)\varphi_2(g)\varphi(g)dg$$

where $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$ with $\varphi_i \in \pi_i$, and $[\mathrm{GL}_2] = \mathbb{A}^\times \mathrm{GL}_2(F)\backslash GL_2(\mathbb{A})$.

One example of this relationship arises in the case that π_1 and π_2 are cupsidal and π_3 is an Eisenstein series. Then $L(s, \Pi)$ is the Rankin-Selberg L-function $L(s, \pi_1 \times \pi_2)$, and for appropriately chosen φ_3, the period I gives an integral representation. Another example occurs when all three representations are cuspidal. In this case, formulas for $L(s, \Pi)$ have been given by Garrett[4], Gross-Kudla[7], Harris-Kudla[8], Watson[15] and Ichino[9].

Let us write $\pi_i = \times_v \pi_{i,v}$ as a (restricted) tensor product over the places v of F, with each $\pi_{i,v}$ an admissible representation of $\mathrm{GL}_2(F_v)$. Let $\langle \cdot, \cdot \rangle_v$ be a (Hermitian) form on π_i. Then, assuming that $\varphi_i = \otimes \varphi_{i,v}$ is factorizable[1], for each v we can consider the matrix coefficient

$$I'(\varphi_v) = \int_{\mathrm{PGL}_2(F_v)} \langle \pi_v(g_v)\varphi_{1,v}, \varphi_{1,v}\rangle_v \langle \pi_v(g_v)\varphi_{2,v}, \varphi_{2,v}\rangle_v \langle \pi_v(g_v)\varphi_{3,v}, \varphi_{3,v}\rangle_v dg_v,$$

and the normalized matrix coefficient

$$I_v(\varphi_v) = \zeta_{F_v}(2)^{-2} \frac{L_v(1, \Pi_v, \mathrm{Ad})}{L_v(1/2, \Pi_v)} I'_v(\varphi_v). \tag{1}$$

When each of the representations π_i is cuspidal, Ichino proved in [9] that there is a constant C (depending only on the choice of measures) such that

$$\frac{|I(\varphi)|^2}{\prod_{j=1}^3 \int_{[\mathrm{GL}_2]} |\varphi_j(g)|^2 \, dg} = \frac{C}{2^3} \cdot \zeta_F(2)^2 \cdot \frac{\Lambda(1/2, \Pi)}{\Lambda(1, \Pi, \mathrm{Ad})} \prod_v \frac{I_v(\varphi_v)}{\langle \varphi_v, \varphi_v \rangle_v} \tag{2}$$

whenever the denominators are nonzero. We remark that, due to the choice of normalizations, the product on the right hand side of (2) is in fact a finite product over some number of "bad" places.

While Ichino's formula is extremely general, for number theoretic applications it is often important to understand well the bad factors. For example, subconvexity for the triple

Date: July 9, 2013.

[1]As a restricted tensor product, we have chosen vectors $\varphi^0_{i,v} \in \pi_v$ for almost all places v. We require that the local inner forms must satisfy $\langle \varphi^0_{i,v}, \varphi^0_{i,v} \rangle_v = 1$ for almost all such v.

product L-function as proved by Bernstein-Reznikov in [1] and Venkatesh [14] used, in the former case, Watson's formula from [15] or, in the latter, the present author's paper [16].

In this appendix we calculate I_v in the case that $v \mid \infty$ is a real place, $\pi_{1,v} = \pi^k_{\mathrm{dis}}$ is the discrete series representation of (even) weight k, and $\pi_{v,2} = \pi_{it_2}$ and $\pi_{3,v} = \pi_{it_3}$ are principal series representations where $\pi_{it} = \mathrm{Ind}_P^G(|\cdot|^{it} \otimes |\cdot|^{-it})$ is obtained as the normalized induction of the character

$$|\cdot|^{it} \otimes |\cdot|^{-it} : T(\mathbb{R}) \to \mathbb{C}.$$

Recall that if $f \in \pi_{it}$ then

$$f(\left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right)\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) g) = |y|^{\frac{1}{2}+it} f(g)$$

for all $g \in \mathrm{GL}_2(\mathbb{R})$.

Remark. If π_{it} corresponds to the archimedean component of the automorphic representation associated to a Maass form f of eigenvalue λ under the Laplacian, then $\lambda = \frac{1}{4} + t^2$.

Let

$$K = O(2) \supseteq \mathrm{SO}(2) = \left\{ \kappa_\theta = \left(\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array} \right) \middle| \theta \in \mathbb{R} \right\}.$$

Recall that a function $f_i \in \pi_i$ is said to have *weight* m if $f_i(g\kappa_\theta) = f_i(g)e^{im\theta}$ for all $g \in \mathrm{GL}_2(\mathbb{R})$. As is well known, for each $m \in \mathbb{Z}$ the subspace of π_i consisting of functions of weight m is at most 1-dimensional.

Theorem 1. *Let $f_1 \in \pi^k_{\mathrm{dis}}$ be the vector of weight k, let $f_2 \in \pi_{it_2}$ be the vector of weight zero, and let $f_3 \in \pi_{it_3}$ be the vector of weight $-k$ (each normalized[2] so that $f_i(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)) = 1$.) Then*

$$(3) \quad I'_v(f_1 \otimes f_2 \otimes f_3) = \frac{4\pi}{(k-1)!(\frac{1}{2}+it_3)_{\frac{k}{2}}(\frac{1}{2}-it_3)_{\frac{k}{2}}} \times \frac{\Gamma(\frac{k}{2}+it_2+it_3)\Gamma(\frac{k}{2}+it_2-it_3)\Gamma(\frac{k}{2}-it_2-it_3)\Gamma(\frac{k}{2}-it_2+it_3)}{\Gamma(\frac{1}{2}+it_2)\Gamma(\frac{1}{2}-it_2)\Gamma(\frac{1}{2}+it_3)\Gamma(\frac{1}{2}-it_3)}$$

and

$$(4) \qquad I_v(f_1 \otimes f_2 \otimes f_3) = \frac{2^{k-1}\pi^k}{(\frac{1}{2}+it_3)_{\frac{k}{2}}(\frac{1}{2}-it_3)_{\frac{k}{2}}}.$$

where $(z)_m = z(z-1)\cdots(z-m+1)$.

1. Real local factors

For the remainder of this note, we work locally over a real place. Since the place v is assumed fixed, we remove subscripts from the associated L-functions. We trust that no confusion will arise between these and the global L-function considered above. (For example, $L(s,\Pi)$, to be defined below, represents the local L-factor $L_v(s,\Pi)$ appearing in equation (1).)

We will assume, however, that the discrete series π_{it} is unitary. (This is automatically true if π_{it} is the local component of an automorphic representation.) This implies that t is real or that t purely imaginary of absolute value less than 1/2. This requirement will be used implicitly to guarantee that certain integrals converge and that certain functions are real valued. We will use this facts without further mention.

[2] This normalization ensures that $\langle f_i, f_i\rangle = 1$.

We record the relevant local factors for representations of $\mathrm{GL}_2(\mathbb{R})$. Let

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2), \qquad \text{and} \qquad \Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s)$$

where $\Gamma(s) = \int_0^\infty y^s e^{-y} d^\times y$ when $\mathrm{Re}(s) > 0$ and is extended by analytic continuation elsewhere. Note that

$$\Gamma_{\mathbb{R}}(1) = 1, \qquad \Gamma_{\mathbb{R}}(2) = \frac{1}{\pi}, \qquad \text{and} \qquad \Gamma_{\mathbb{C}}(m) = \frac{(m-1)!}{2^{m-1}\pi^m}. \tag{5}$$

We recall basic facts about the local Langlands correspondence for $\mathrm{GL}_2(\mathbb{R})$ as found in Knapp [11]. The Weil group $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^2$ where $j^2 = -1$ and $jzj^{-1} = \bar{z}$ for $z \in \mathbb{C}^\times$. The irreducible representations of $W_{\mathbb{R}}$ are all either 1-dimensional or 2-dimensional. The 1-dimensional representations are parametrized by $\delta \in \{0, 1\}$ and $t \in \mathbb{C}$:

$$\rho_1(\delta, t) : \begin{array}{c} z \mapsto |z|^t \\ j \mapsto (-1)^\delta. \end{array}$$

The irreducible 2-dimensional representations are parametrized by positive integers m and $t \in \mathbb{C}$:

$$\rho_2(m,t) : \begin{array}{l} re^{i\theta} \mapsto \begin{pmatrix} r^{2t}e^{im\theta} & 0 \\ 0 & r^{2t}e^{-im\theta} \end{pmatrix} \\ \\ j \mapsto \begin{pmatrix} 0 & (-1)^m \\ 1 & 0 \end{pmatrix} \end{array}$$

Defining $\rho_2(0,t) = \rho_1(0,t) \oplus \rho_1(1,t)$ and $\rho_2(m,t) = \rho_2(|m|, t)$, the following is an elementary exercise.

Lemma 2. *Every (semisimple) finite dimensional representation of $W_{\mathbb{R}}$ is a direct sum of irreducibles each of dimension one or two. Under the operations of direct sum and tensor product, the following is a complete set of relations.*

$$\begin{gathered} \rho_2(m,t) \simeq \rho_2(-m,t) \\ \rho_2(0,t) \simeq \rho_1(0,t) \oplus \rho_1(1,t) \\ \rho_1(\delta_1,t_1) \otimes \rho_1(\delta_2,t_2) \simeq \rho_1(\delta, t_1+t_2) \\ \rho_1(\delta,t_1) \otimes \rho_2(m,t_2) \simeq \rho_2(m, t_1+t_2) \\ \rho_2(m_1,t_1) \otimes \rho_2(m_2,t_2) \simeq \rho_2(m_1+m_2, t_1+t_2) \oplus \rho_2(m_1-m_2, t_1+t_2) \end{gathered}$$

In the third line, $\delta = \delta_1 + \delta_2 \pmod 2$. Moreover, if $\widetilde{\rho}$ denotes the contragradient of ρ then

$$\widetilde{\rho_1(\delta,t)} \simeq \rho_1(\delta,-t), \quad \textit{and} \quad \widetilde{\rho_2(m,t)} \simeq \rho_1(m,-t).$$

Attached to each irreducible representation ρ of $W_{\mathbb{R}}$ is an L-factor

$$L(s, \rho_1(\delta,t)) = \Gamma_{\mathbb{R}}(s+t+\delta), \qquad \text{and} \qquad L(s,\rho_2(m,t)) = \Gamma_{\mathbb{C}}(s+t+\frac{m}{2}).$$

Writing a general representation ρ as a direct sum of irreducibles $\rho_1 \oplus \cdots \oplus \rho_r$, we define

$$L(s,\rho) = \prod_{i=1}^r L(s,\rho_i).$$

In particular, given ρ, the adjoint representation is

$$\mathrm{Ad}(\rho) \simeq \rho \otimes \widetilde{\rho} \ominus \rho_1(0,0)$$

since $\rho_1(0,0)$ is the trivial representation.

Under the Langlands correspondence, admissible representations π of $\mathrm{GL}_2(\mathbb{R})$ correspond to 2-dimensional representations $\rho = \rho(\pi)$ of $W_{\mathbb{R}}$. For example, $\rho(\pi_{it}) = \rho_1(0,it) \oplus \rho_1(0,-it)$ and $\rho(\pi_{\mathrm{dis}}^k) = \rho_2(0,k-1)$. Thus the local factors for the discrete series and principal series representations are

$$L(s,\pi_{\mathrm{dis}}^k) = \Gamma_{\mathbb{C}}(s+(k-1)/2), \quad \text{and} \quad L(s,\pi_{it}) = \Gamma_{\mathbb{R}}(s+it)\Gamma_{\mathbb{R}}(s-it).$$

We define

$$L(s,\Pi) = L(s, \rho(\pi_{\mathrm{dis}}^k) \otimes \rho(\pi_{it_2}) \otimes \rho(\pi_{it_3}))$$

and

$$L(s,\Pi,\mathrm{Ad}) = L(s, \mathrm{Ad}\,\rho(\pi_{\mathrm{dis}}^k) \oplus \mathrm{Ad}\,\rho(\pi_{it_2}) \oplus \mathrm{Ad}\,\rho(\pi_{it_3})).$$

Lemma 3. *Let $\Pi = \pi_{\mathrm{dis}}^k \otimes \pi_{it_2} \otimes \pi_{it_3}$. The normalizing factor relating I_v and I_v' in (1) at a real place v is*

$$\frac{L(1,\Pi_v,\mathrm{Ad})}{\Gamma_{\mathbb{R}}(2)^2 L(1/2,\Pi_v)} =$$

$$2^{k-3}\pi^{k-1}(k-1)! \frac{\Gamma(\frac{1}{2}+it_2)\Gamma(\frac{1}{2}-it_2)\Gamma(\frac{1}{2}+it_3)\Gamma(\frac{1}{2}-it_3)}{\Gamma(\frac{k}{2}+it_2+it_3)\Gamma(\frac{k}{2}-it_2+it_3)\Gamma(\frac{k}{2}+it_2-it_3)\Gamma(\frac{k}{2}-it_2-it_3)}.$$

Proof. Using Lemma 2, one can easily show that

$$L(1/2,\Pi) = \prod_{\varepsilon,\varepsilon' \in \{\pm 1\}} \Gamma_{\mathbb{C}}\left(\varepsilon i t_2 + \varepsilon' i t_3 + \frac{k}{2}\right) = 2^4(2\pi)^{-2k} \prod_{\varepsilon,\varepsilon' \in \{\pm 1\}} \Gamma\left(\frac{k}{2} + \varepsilon i t_2 + \varepsilon' i t_3\right)$$

and, applying (5), $L(1,\Pi,\mathrm{Ad})$ is equal to

$$\begin{aligned}&(\Gamma_{\mathbb{C}}(k)\Gamma_{\mathbb{R}}(2)) \cdot (\Gamma_{\mathbb{R}}(1+2it_2)\Gamma_{\mathbb{R}}(1-2it_2)\Gamma_{\mathbb{R}}(1)) \cdot (\Gamma_{\mathbb{R}}(1+2it_3)\Gamma_{\mathbb{R}}(1-2it_3)\Gamma_{\mathbb{R}}(1)) \\ &= \frac{(k-1)!}{2^{k-1}\pi^{k+3}} \Gamma\left(\frac{1}{2}+it_2\right)\Gamma\left(\frac{1}{2}-it_2\right)\Gamma\left(\frac{1}{2}+it_3\right)\Gamma\left(\frac{1}{2}-it_3\right).\end{aligned}$$

Combining these, we arrive at the desired formula. □

2. Whittaker models

As a matter of notation, set

$$a(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}, \quad z(u) = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \quad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Let π be an infinite dimensional representation of G with central character ω and $\psi : \mathbb{R} \to \mathbb{C}^\times$ a nontrivial additive character. Then there is a unique space of functions $\mathcal{W}(\pi,\psi)$ isomorphic to π such that

$$W(z(u)n(x)g) = \omega(u)\psi(x)W(g) \tag{6}$$

for all $g \in G$. Recall that the inner product on $\mathcal{W}(\pi,\psi)$ is given by

$$\langle W, W' \rangle = \int_{\mathbb{R}^\times} W(a(y))\overline{W'(a(y))}d^\times y.$$

We fix $\psi : \mathbb{R} \to \mathbb{C}^\times$ once and for all to be the character $\psi(x) = e^{2\pi i x}$.

If the central character of π is trivial, and $W \in \mathcal{W}(\pi, \psi)$ has weight k, (6) becomes

$$W(z(u)n(x)a(y)\kappa_\theta) = e^{2\pi i x} W(a(y)) e^{im\theta}. \tag{7}$$

This, by the Iwasawa decomposition, determines W completely provided we can describe $w(y) = W(a(y))$. This can be accomplished for the weight k vector $W_k^k \in \mathcal{W}(\pi_{\text{dis}}^k, \psi)$ by utilizing the fact that W_k^k is annihilated by the lowering operator $X^- \in \mathrm{Lie}(\mathrm{GL}_2(\mathbb{R}))$. Applying X^- to (7), one finds that $w(y)$ satisfies a certain differential equation whose solution is easily obtained. The unique solution with moderate growth is, up to a constant,

$$W_k^k(a(y)) = \begin{cases} y^{k/2} e^{-2\pi y} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0. \end{cases} \tag{8}$$

We calculate directly that

$$\int_0^\infty W_k^k(a(y)) W_{k'}^{k'}(a(y)) y^{s-1} d^\times s = \int_0^\infty y^{(k+k')/2} e^{-4\pi y} d^\times y = \frac{\Gamma(s-1+(k+k')/2)}{(4\pi)^{s-1+(k+k')/2}} \tag{9}$$

By letting $s = 1$ and $k = k'$, this implies that

$$\langle W_k^k, W_k^k \rangle = \frac{(k-1)!}{(4\pi)^k}. \tag{10}$$

Analogously, if $W_m^\lambda \in \mathcal{W}(\pi_{it}, \psi)$ is a weight m-vector which is an eigenvector for the action of the Laplace operator Δ of eigenvalue λ, one can apply Δ to (6) to see that $w(y) = W_m^\lambda(a(y))$ satisfies the confluent geometric differential equation

$$w'' + \left[-\frac{1}{4} + \frac{m}{2y} + \frac{\lambda}{y^2} \right] w = 0. \tag{11}$$

Therefore, $W_m^\lambda(a(y)) = W_{\frac{m}{2}, it}(|y|)$ is the unique solution of (11) with exponential decay as $|y| \to \infty$ and $\lambda = \frac{1}{2} + t^2$. (See ...) The weight zero vector W_0^λ can be expressed in terms of the incomplete Bessel function:

$$W_0^\lambda(a(y)) = W_{0,it}(y) = 2\pi^{-1/2} |y|^{1/2} K_{it}(2\pi |y|). \tag{12}$$

By formula (6.8.48) of [2], it follows that

$$\begin{aligned} \int_0^\infty W_{0,it_1}(a(y)) W_{0,it_2}(a(y)) y^{s-1} d^\times y &= \frac{4}{\pi} \int_0^\infty K_{it_1}(2\pi y) K_{it_2}(2\pi y) y^s d^\times y \\ &= \frac{1}{2\pi^{s+1}} \frac{\Gamma(\frac{s+it_1+it_2}{2})\Gamma(\frac{s-it_1+it_2}{2})\Gamma(\frac{s+it_1-it_2}{2})\Gamma(\frac{s-it_1-it_2}{2})}{\Gamma(s)}. \end{aligned} \tag{13}$$

Evaluating this at $s = 1$ in the case that $t_1 = t_2 = t$, we have that

$$\langle W_0^\lambda, W_0^\lambda \rangle = \frac{\Gamma(\frac{1}{2} + it)\Gamma(\frac{1}{2} - it)}{\pi}. \tag{14}$$

Note that we have used that $W_0^\lambda(a(y))$ is an even function and $\Gamma(1/2) = \sqrt{\pi}$.

Remark. An explicit intertwining map $\pi \to \mathcal{W}(\pi, \psi)$ is given, when the integral is convergent, by

$$f \mapsto W_f \qquad W_f(g) = \pi^{-1/2} \int_{\mathbb{R}} f(wn(x)g)\psi(x) dx \tag{15}$$

where $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and this can be extended by analytic continuation elsewhere.

As an alternative to the strategy above, one can deduce equations (9) and (13) by working directly from (15). (See [3].) The normalization in (12) coincides with this choice of intertwiner.

3. Proof of Theorem 1

We are now in a position to prove Theorem 1. Having laid the groundwork above, it is a simple consequence of the following result due to Michel-Venkatesh [12].

Lemma 4 (Michel-Venkatesh)**.** *Let π_1, π_2, π_3 be tempered representations of $\mathrm{GL}_2(\mathbb{R})$ with π_3 a principal series. Fixing an isometry $\pi_i \to \mathcal{W}(\pi_i, \psi)$ for $i = 1, 2$ we may associate for $f_i \in \pi_i$ vectors W_i in the Whittaker model. Then the form $\ell_{\mathrm{RS}} : \pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbb{C}$ given by*

$$\ell_{\mathrm{RS}}(f_1 \otimes f_2 \otimes f_3) = \int_K \int_{\mathbb{R}^\times} W_1(a(y)\kappa) W_2(a(y)\kappa) f_3(a(y)\kappa) \left|y\right|^{-1} d^\times y d\kappa \tag{16}$$

satisfies $\left|\ell_{\mathrm{RS}}\right|^2 = I'(f_1 \otimes f_2 \otimes f_3)$

For $i = 1, 2$ we have $\lambda_i = \frac{1}{4} + t_i^2$. Recall our choice of test functions: $W_1 = W_k^k$, $W_2 = W_0^{\lambda_2}$, and $f_3 \in \pi_{it_3}$ of weight $-k$. Since the sum of the weights of these is zero, the integral over K in (16) is trivial, and

$$\begin{aligned}
\ell_{\mathrm{RS}}(W_1 \otimes W_2 \otimes f_3) &= \int_0^\infty W_1(a(y)) W_2(a(y)) f_3(a(y)) \left|y\right|^{-1} d^\times y \\
&= \int_0^\infty e^{-2\pi y} y^{k/2} 2\pi^{-1/2} y^{1/2} K_{it_2}(2\pi y) y^{1/2+it_3} y^{-1} d^\times y \\
&= 2\pi^{-1/2} \int_0^\infty e^{-2\pi y} K_{it_2}(2\pi y) y^{k/2+it_3} d^\times y \\
&= \frac{2}{(4\pi)^{k/2+it_3}} \frac{\Gamma(\frac{k}{2} + it_2 + it_3)\Gamma(\frac{k}{2} - it_2 + it_3)}{\Gamma(\frac{1}{2} + \frac{k}{2} + it_3)}
\end{aligned}$$

In the final line we have used equation (6.8.28) from [2]. This simplifies further by using the identity $\Gamma(z + m) = \Gamma(z)(z)_m$.

Recall that we have chosen f_i such that $\langle f_i, f_i\rangle = 1$ for each i. Therefore, in order to apply Lemma 4, we must normalize ℓ_{RS}:

$$\begin{aligned}
I'(f_1 \otimes f_2 \otimes f_3) = \frac{\left|\ell_{\mathrm{RS}}(W_1 \otimes W_2 \otimes f_3)\right|^2}{\langle W_1, W_2\rangle\langle W_2, W_2\rangle} &= \frac{4\pi}{(k-1)!(\frac{1}{2} - it_3)_{\frac{k}{2}}(\frac{1}{2} + it_3)_{\frac{k}{2}}} \times \\
&\times \frac{\Gamma(\frac{k}{2} + it_2 + it_3)\Gamma(\frac{k}{2} + it_2 - it_3)\Gamma(\frac{k}{2} - it_2 - it_3)\Gamma(\frac{k}{2} - it_2 + it_3)}{\Gamma(\frac{1}{2} + it_2)\Gamma(\frac{1}{2} - it_2)\Gamma(\frac{1}{2} + it_3)\Gamma(\frac{1}{2} - it_3)}
\end{aligned}$$

To complete the proof, we multiply by the normalizing factor of Lemma 3.

Remark. If one or more of the representations π_{it_j} is a complementary series (i.e. if $\lambda_j < \frac{1}{4}$) then the result of Theorem 1 still holds, but the explicit calculation is somewhat different. In this case, it is no longer true that for $r \in \mathbb{R}$

$$\left|\Gamma(r + it_j)\right|^2 = \Gamma(r + it_j)\Gamma(r - it_j),$$

nor is it true that $\langle f_j, f_j \rangle = 1$. Taking into account these differences, however, the final answer ends up agreeing with what has been calculated above.

References

[1] Joseph Bernstein and Andre Reznikov. Periods, subconvexity of L-functions and representation theory. *J. Differential Geom.*, 70(1):129–141, 2005.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Tables of integral transforms. Vol. I.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. Based, in part, on notes left by Harry Bateman.

[3] Paul B. Garrett. Standard archimedean integrals for $GL(2)$. available at http://www.math.umn.edu/∼garrett/.

[4] Paul B. Garrett. Decomposition of Eisenstein series: Rankin triple products. *Ann. of Math. (2)*, 125(2):209–235, 1987.

[5] Roger Godement and Hervé Jacquet. *Zeta functions of simple algebras.* Lecture Notes in Mathematics, Vol. 260. Springer-Verlag, Berlin, 1972.

[6] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, 4th ed., Academic Press, New York, 1965.

[7] Benedict H. Gross and Stephen S. Kudla. Heights and the central critical values of triple product L-functions. *Compositio Math.*, 81(2):143–209, 1992.

[8] M. Harris, S. Kudla, *The Central Critical Value of a Triple Product L-Function*, Ann. of Math. (2) V. 133, 605-672.

[9] A. Ichino, *Trilinear forms and the central values of triple product L-functions*, Duke Math. J. Volume 145, Number 2 (2008), 281-307.

[10] A. Ichino, T. Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture.* GAFA, V.19, 2010, 1378-1425.

[11] A. W. Knapp. Local Langlands correspondence: the Archimedean case. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 393–410. Amer. Math. Soc., Providence, RI, 1994.

[12] P. Michel and A. Venkatesh. The subconvexity problem for $gl(2)$. *Publications Math IHES*, 2009.

[13] I. Piatetski-Shapiro and S. Rallis, *Rankin Triple L-functions*, Compositio Math. 1987, 31-115.

[14] Akshay Venkatesh. Sparse equidistribution problems, period bounds and subconvexity. *Ann. of Math. (2)*, 172(2):989–1094, 2010.

[15] T. Watson, *Central Value of Rankin Triple L-function for Unramified Maass Cusp Forms,* Princeton thesis, 2004.

[16] Michael Woodbury. Trilinear forms and subconvexity of the triple product L-function. available at http://www.math.columbia.edu/∼woodbury/research/.