The Gaussian Free Field, Random Interlacements, and Isomorphism Theorems

Exercise sheet 7

Random interlacement point process

Throughout this exercise sheet, we call \((W^*, \mathcal{W}^*)\) the space of doubly infinite trajectories in \(\mathbb{Z}^d\) modulo time-shift, endowed with its canonical filtration, and \((\mathcal{W}_+, \mathcal{W}^+)\) the space of forward trajectories in \(\mathbb{Z}^d\). We also call \(\Omega\) the space of locally finite point measures on \(W^* \times \mathbb{R}_+\) and \(\mathcal{A}\) the sigma-algebra generated by the evaluation maps of \(W^* \otimes \mathcal{B}(\mathbb{R}_+)\). Moreover, let \(\mathbb{P}\) be the probability on \((\Omega, \mathcal{A})\) such that the random element \(\omega\) of \((\Omega, \mathcal{A}, \mathbb{P})\) is the random interlacement point process. For all \(u > 0\), and \(\omega = \sum_{n \geq 0} \delta_{(w_n^*, u_n)} \in \Omega\), we define the random interlacement set at level \(u\) by

\[
\mathcal{I}^u = \bigcup_{u_n \leq u} \text{range } (w_n^*),
\]

where \text{range}(w^*) is the set of vertices visited by \(w^* \in W^*\). Let \(\nu\) be the measure on \((W^*, \mathcal{W}^*)\) such that \(\omega\) is a Poisson point process with intensity measure \(\nu \otimes \lambda\), where \(\lambda\) is the Lebesgue measure on \(\mathbb{R}_+\). For all \(K \subset \subset \mathbb{Z}^d\) and \(w^* \in W^*\) which hit \(K\), we call \(s_K(w^*)\) the only element of \((\pi^*)^{-1}(w^*)\) which hit \(K\) for the first time at time 0, and \(s_{K, +}(w^*) \in W_+\) the forward trajectory of \(s_K(w^*)\). For each \(x \in \mathbb{Z}^d\), let \(\mathbb{P}_x\) be the probability on \((W_+, \mathcal{W}_+)\) such that the canonical coordinate process \((X_n)_{n \geq 0}\) is a simple random walk beginning in \(x\) under \(\mathbb{P}_x\). Finally, for all \(K \subset \subset \mathbb{Z}^d\), we call

\[
\partial_m K = \{x \in K : \exists y \notin K \text{ such that } y \sim x\}
\]

and, for all \(L \geq 1\), we call \(B(L) = \{x \in \mathbb{Z}^d : |x|_\infty \leq L\}\).

Exercise 7.1 (7 points)

Let us fix some \(u > 0\) and \(K \subset \subset \mathbb{Z}^d\), and let \(\emptyset = K_0 \subset K = K_1 \subset K_2 \subset K_3 \subset \ldots\) be an infinite sequence of finite subsets of \(\mathbb{Z}^d\) such that \(\bigcup_{i=0}^{\infty} K_i = \mathbb{Z}^d\). Under a probability \(\mathbb{Q}\), for each \(i \in \mathbb{N}\), we define independently \((w^*_{i,j})_{j \in \mathbb{N}}\) an i.i.d. sequence of random variables in \((W^*, \mathcal{W}^*)\) such that for all \(A \in \mathcal{W}^*\),

\[
\mathbb{Q}(w^*_{i,j} \in A) = \frac{\nu(A \cap (W^*_{K_i} \setminus W^*_{K_{i-1}}))}{\text{cap}(K_i \setminus K_{i-1})}, \quad \text{for all } j \in \mathbb{N},
\]

\((u_{i,j})_{j \in \mathbb{N}}\) an i.i.d. sequence of random variables uniformly distributed on \([u \times (i - 1), u \times i]\) and \(N_i\) a random variable with law Poisson\((u \text{cap}(K_i \setminus K_{i-1}))\). Show that

If you have any questions, please do not hesitate to contact me at aprevost@math.uni-koeln.de
a) the random point measure
\[ \sum_{i=1}^{\infty} \sum_{j=1}^{N_i} \delta_{(w_{i,j}^*, u_{i,j})} \]
has the same law as the random interlacement point process,
(hint: use the colouring lemma and infinite divisibility) (2 p.)
b) for all \( B \in \mathcal{W}_+ \) such that \( w(0) \in \partial_{int} K \) for all \( w \in B \),
\[ \nu((s_{K,+})^{-1}(B)) = \text{cap}(K) \mathbb{P}_{\bar{e}_K}(B), \]
(1 p.)
c) the random process
\[ \sum_{j=1}^{N_1} \delta_{s_{K,+}(w_{1,j}^*)} \]
is a Poisson point process with intensity measure \( u \text{cap}(K) \mathbb{P}_{\bar{e}_K} \),
(hint: use exercise 5.3) (2 p.)
d) for all \( n \in \mathbb{N} \),
\[ \mathbb{P}(K \subset \mathcal{I}^u \mid N_1 = n) \geq 1 - |K| \left(1 - \frac{\text{cap}(0)}{\text{cap}(K)}\right)^n. \]
(2 p.)

Exercise 7.2 (3 points)
Let us fix some \( u > 0 \), we say that an infinite connected set \( K \subset \mathbb{Z}^d \) is an infinite maximal connected component of the random interlacement set \( \mathcal{I}^u \) if
- \( K \subset \mathcal{I}^u \),
- \( |K| = \infty \),
- for all \( x, y \in K \), there exist \( n \in \mathbb{N} \) and a path \((x_0 = x, x_1, \ldots, x_{n-1}, x_n = y) \in K^{n+1}\)
of length \( n + 1 \) such that \( x_i \sim x_{i-1} \) for all \( i \in \{1, \ldots, n\} \),
- for all \( x \in \partial_{int}(K^c) \), \( x \not\in \mathcal{I}^u \).
We call \( N^u \in \mathbb{N} \cup \{\infty\} \) the number of infinite maximal connected components of \( \mathcal{I}^u \).
Show that
a) for all \( k \in \mathbb{N} \cup \{\infty\} \),
\[ \mathbb{P}(N^u = k) \in \{0, 1\}, \]
(hint: use ergodicity) (1 p.)
b) there exists a constant $K^u \in \mathbb{N} \cup \{\infty\}$ such that $\mathbb{P}(N^u = K^u) = 1$ and for all $i \in (\mathbb{N} \cup \{\infty\}) \setminus \{K^u\}$, $\mathbb{P}(N^u = i) = 0$. (1 p.)

c) $K^u \neq 0$. (1 p.)

**Exercise 7.3** (5 points)

Show that

a) for all $L \in \mathbb{N}$, $K \subset B(L)$ and $y \in K$,

$$\text{cap}(B(2L)) \inf_{x \in \partial_{int}B(2L)} \mathbb{P}_x(H_K < \infty, X_{H_K} = y) \leq e_K(y)$$

and

$$e_K(y) \leq \text{cap}(B(2L)) \sup_{x \in \partial_{int}B(2L)} \mathbb{P}_x(H_K < \infty, X_{H_K} = y),$$

(2 p.)

b) for all $L \geq 1$, $K \subset B(L)$, $z \in \partial_{int}B(2L)$ and $y \in K$, the function

$$x \mapsto \mathbb{P}_x(H_K < \infty, X_{H_K} = y)$$

is harmonic on $B(z, \frac{L}{2})$. (1 p.)

Using the Harnack inequality as in Exercise 1.4, one can show that there exists a constant $C_H = C_H(d)$ such that for all $L \geq 1$, $K \subset B(L)$ and $y \in K$,

$$\sup_{x \in \partial_{int}B(2L)} \mathbb{P}_x(H_K < \infty, X_{H_K} = y) \leq C_H \inf_{x \in \partial_{int}B(2L)} \mathbb{P}_x(H_K < \infty, X_{H_K} = y),$$

and you don’t have to prove this result again. Show that

c) there exists a constant $C = C(d)$ and $c = c(d)$ such that for all $L \geq 1$, $K \subset B(L)$, $y \in K$ and $x \in \partial_{int}B(2L)$

$$c L^{2-d} e_K(y) \leq \mathbb{P}_x(H_K < \infty, X_{H_K} = y) \leq C L^{2-d} e_K(y).$$

(2 p.)

**Total: 15 points**

In order to be allowed to take the exam you must obtain at least 50% of the homework points.