ON A CONJECTURE OF B. BERNDT AND B. KIM

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Abstract. We prove a recent conjecture of B. Berndt and B. Kim regarding the positivity of the coefficients in the asymptotic expansion of a class of partial theta functions. This generalizes results found in Ramanujan's second notebook, and recent work of Galway and Stanley.

1. Introduction

In his second notebook [1], page 324, Ramanujan claims an asymptotic expansion for the partial theta function

$$2\sum_{n=0}^{\infty}(-1)^n q^{n^2+n} \sim 1+t+t^2+2t^3+5t^4+\dots \tag{1.1}$$

with $q=\frac{1-t}{1+t}$ as $t\to 0^+$. It is not clear, just given the left-hand side of (1.1), that the coefficients of its asymptotic expansion (in t) are always positive integers, nor that one should expect this. Galway [3] proved this curious fact to be true. Stanley [5], answering a question of Galway, then gave a nice combinatorial interpretation of the coefficients in the asymptotic expansion (1.1) as the number of fixed-point-free alternating involutions in the symmetric group S_{2n}, providing a second proof that the coefficients are positive integers.

Berndt and Kim [2] study more general partial theta functions.

$$f_b(t) := 2\sum_{n=0}^{\infty}(-1)^n q^{n^2+bn}, \tag{1.2}$$

where b is real. They show that, similarly to (1.1), $f_b(t)$ admits an asymptotic expansion of the form

$$f_b(t) \sim \sum_{n=0}^{\infty} a_n t^n, \tag{1.3}$$

where the a_n are given explicitely in term of Euler numbers and Hermite polynomials [2, Theorem 1.1]. For the purposes of this paper, we do not require the explicit shape of the a_n, so we do not state it here. (We point out a small typo in [2] equation (1.3): the exponent of $(1-t)/(1+t)$ should read $(1-2b)/4$ rather than $(2b-1)/4$, as the authors correctly state in [2] equation (2.9).)

In analogy to the results established by Galway and Stanley pertaining to the coefficients of (1.1), Berndt and Kim prove that the coefficients a_n of the generalized partial theta functions defined in (1.2) are integers if $b\in\mathbb{N}$ [2, Theorem 2.5], and make the following conjecture regarding their positivity.

Conjecture (Berndt-Kim [2])**.** *For any positive integer b, for sufficiently large n, the coefficients a_n in the asymptotic expansion* (1.3)*, have the same sign.*

The purpose of this note is to prove this conjecture of Berndt and Kim.

Theorem 1. *The Berndt-Kim conjecture is true. More precisely, if $b \equiv 1, 2 \pmod 4$, then $a_n > 0$ for $n \gg 0$, and if $b \equiv 0, 3 \pmod 4$, then $a_n < 0$ for $n \gg 0$.*

2. Proof of Theorem 1

To prove the theorem, we first observe that for integers $b \geq 2$,

$$f_{b+1}(t) = -q^{-b}\Big(f_{b-1}(t) - 2\Big). \tag{2.1}$$

We proceed by induction on b to prove Theorem 1. The case $b = 1$ follows from [3] as mentioned in §1. To prove the case $b = 2$ we employ the fact that the q-series in this case is essentially a modular form. To be more precise, we have that

$$f_2(t) = -q^{-1}\left(g\left(\frac{i\theta}{2\pi}\right) - 1\right),$$

where we have adopted the notation $\theta = \log\left(\frac{1+t}{1-t}\right)$ from [2]. The function $g(\tau)$, where $\tau \in \mathbb{H}$, is the modular form defined by

$$g(\tau) := \sum_{n\in\mathbb{Z}} (-1)^n e^{2\pi i n^2 \tau} = \frac{\eta(\tau)^2}{\eta(2\tau)},$$

where $\eta(\tau) := e^{\frac{2\pi i \tau}{24}} \prod_{n\geq 1}(1 - e^{2\pi i n\tau})$ is Dedekind's eta-function, a well known modular form of weight 1/2. Employing the modular transformation of η (see [4] e.g.)

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)$$

we obtain that

$$g\left(\frac{i\theta}{2\pi}\right) \to 0$$

as $\theta \to 0^+$, and thus as $t \to 0^+$. Therefore

$$f_2(t) \sim q^{-1} = \frac{1+t}{1-t},$$

which clearly has positive coefficients in its t-expansion.

We now assume Theorem 1 holds for some $b - 1 \geq 1$, and prove that it also holds for $b + 1$. (The first two inductive cases $b - 1 = 1, 2$ are proven above.) We use (2.1) and split

$$q^{-b} = (1+t)^b (1-t)^{-b}.$$

Since the t-expansion of $(1+t)^b$ is finite and contains only positive coefficients, it suffices to prove that the t-coefficents in the asymptotic expansion of

$$-(1-t)^{-b}\left(f_{b-1}(t) - 2\right)$$

eventually all have the same sign. We will address the fact that the sign is dependent on the residue class of $b \pmod 4$ as stated in Theorem 1 later.

It is easy to show that

$$(1-t)^{-b} = \sum_{j=0}^{\infty} \binom{b-1+j}{b-1} t^j.$$

By induction, we may assume without loss of generality that

$$f_{b-1}(t) - 2 \sim \sum_{n \geq 0} \alpha_n t^n,$$

as $t \to 0^+$, where $\alpha_n > 0$ for $n \geq L$, for some $L \geq 0$. Set

$$\beta_n := \binom{b-1+n}{b-1}.$$

It suffices to show that

$$\sum_{0 \leq n \leq m} \alpha_n \beta_{m-n} \tag{2.2}$$

is positive for $m \gg 0$. We break the sum (2.2) into two parts

$$\Sigma_1 := \sum_{0 \leq n \leq L-1} \alpha_n \beta_{m-n}, \qquad \Sigma_2 := \sum_{L \leq n \leq m} \alpha_n \beta_{m-n},$$

where we assume $m \gg 0$ is sufficiently large to ensure $m \geq L$.

Since L is independent of m and the β_j's are monotonically decreasing (in j, for fixed b), we may bound

$$\Sigma_1 \ll \beta_m \ll m^{b-1},$$

as $m \to \infty$. Similarly, Σ_2 may be estimated from below by

$$\Sigma_2 \gg \sum_{L \leq n \leq m} \beta_{m-n} = \sum_{0 \leq n \leq m-L} \beta_n \gg \sum_{1 \leq n \leq m} n^{b-1} \gg \int_0^m x^{b-1} dx \gg m^b,$$

as $m \to \infty$. Since clearly $m^b \gg m^{b-1}$, the positivity of Σ_2 dominaties, and we have that the coefficients in the asymptotic t-expansion of $f_{b+1}(t)$ have for sufficiently large m the same sign as claimed.

The more precise claim that the coefficients in the asymptotic t-expansion for $f_b(t)$ are eventually positive for $b \equiv 1, 2 \pmod 4$ and eventually negative for $b \equiv 0, 3 \pmod 4$ follows easily by induction using (2.1), and the previously established facts that the coefficients in the asymptotic t-expansion of $f_1(t)$ and $f_2(t)$ are eventually positive.

References

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