# ASYMPTOTICS FOR $d$-FOLD PARTITION DIAMONDS AND RELATED INFINITE PRODUCTS

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Abstract. We prove an asymptotic formula for the number of $d$-fold partition diamonds of $n$ and their Schmidt-type counterparts. In order to do so, we study the asymptotic behavior of certain infinite products. We also remark on interesting potential connections with mathematical physics and Bloch groups.

## 1. Introduction and statement of results

A *partition* of a non-negative integer $n$ is a finite sequence $\lambda = (a_0, a_1, \cdots, a_k)$ of positive integers such that $|\lambda| := a_0 + a_1 + \cdots + a_k = n$. The theory of partitions has a long and rich history in combinatorics and number theory, which is overviewed in Andrews' book [3]. In this paper, we are primarily concerned with the asymptotic properties of partitions. The modern viewpoint on this study began with the famous paper of Hardy and Ramanujan [17], in which they studied the function $p(n)$ which counts the number of partitions of $n$ and proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \quad (\text{as } n \to \infty).$$

They showed this theorem by developing the Circle Method, which has since spawned a huge number of variations with applications across all of analytic number theory.

In 2001, Andrews, Paule, and Riese [8] reinitiated the study of partition analysis, an algebraic framework designed by MacMahon for the deduction of generating functions for different kinds of plane partitions. This began a long series of papers from Andrews and collaborators on this topic, including papers on hypergeometric multisums [4], magic squares [10] and recently partitions with $n$ copies of $n$ [6]. In particular, this new research spawned a great interest in plane partition diamonds. A *plane partition diamond* (or just a *partition diamond*) as defined in [7] is a pair of sequences of integers $\{a_j\}_{j\geq 0}$, $\{b_j\}_{j\geq 0}$ such that for every $j \in \mathbb{N}_0$ we have $a_j \geq \max\{b_{2j}, b_{2j+1}\} \geq a_{j+1}$. The naming convention of partition diamonds comes from the fact that these can be

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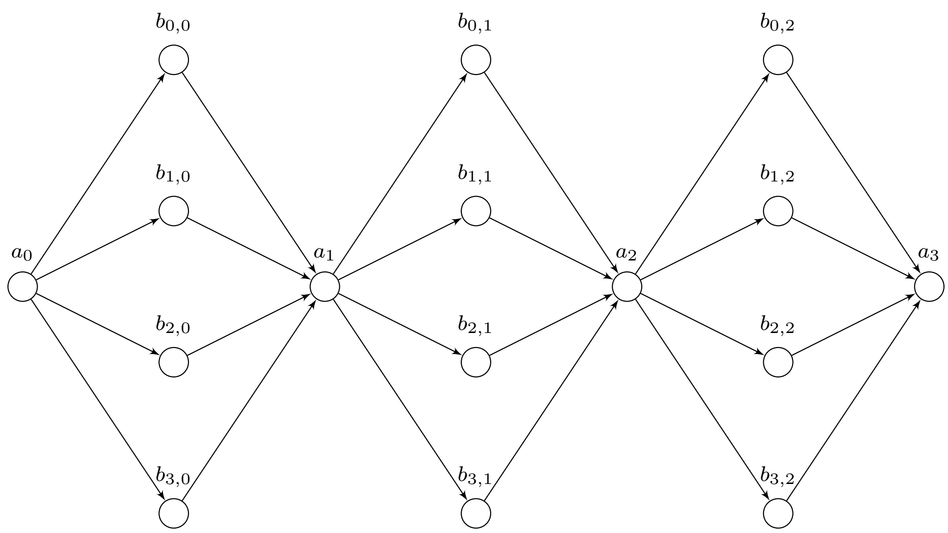
represented graphically as a kind of directed graph of diamonds, with the direction of edges denoting the inequalities imposed. Partition diamonds have been the subject of many very interesting studies; for example, they give examples of modular forms [5, 9], and their generalizations exhibit interesting congruence properties [7, 13, 23, 24].

In this paper, we consider a recent generalization of plane partition diamonds. In [16], a *$d$-fold partition diamond* is defined as a collection of non-negative integer sequences $\{a_k\}_{k\geq 0}, \{b_{j,k}\}_{k\geq 0,\, 0\leq j\leq d-1}$ such that for every $k\in\mathbb{N}_0$, we have the inequalities $a_k \geq \max\limits_{0\leq j\leq d-1} b_{j,k} \geq a_{k+1}$. Observe that standard integer partitions can be viewed as 1-fold partition diamonds, and the previously defined plane partition diamonds can be viewed as 2-fold partition diamonds. The inequalities exhibited by a $d$-fold partition diamond can also be represented cleanly with a directed graph; we exhibit how this works for 4-fold diamond partitions below.




In line with recent work on Schmidt-type partitions of $n$ [6], we define the *Schmidt size* of a $d$-fold partition diamond $\{a_k\}_{k\geq 0}, \{b_{j,k}\}_{k\geq 0,\, 0\leq j\leq d-1}$ as the size of the subpartition $\{a_k\}_{k\geq 0}$; in terms of the directed graph above, the Schmidt size of a $d$-fold partition diamond is the sum of the central nodes. Questions related to Schmidt-style modified size functions on partitions have been popular recently in the theory of partitions [2, 15, 19, 20].

In this paper, we consider the functions that count $d$-fold partition diamonds of size $n$ and Schmidt size $n$ and compute their asymptotic expansions. In line with [16], we define by $r_d(n)$ the number of $d$-fold partition diamonds of $n$ and we let $s_d(n)$ be the number of $d$-fold partition diamonds with Schmidt size $n$. In order to state these asymptotic formulas, we need to define certain constants. Let

$$C_d := \int_0^\infty \log\left(A_d\left(e^{-x}\right)\right) dx,$$

with $A_d(x)$ the Eulerian polynomials defined in (3.2). Then we have the following.

**Theorem 1.1.** *As $n \to \infty$ we have that*

$$s_d(n) \sim \frac{\left(C_d + \frac{\pi^2(d+1)}{6}\right)^{\frac{d}{4}+\frac{1}{2}}}{\sqrt{2}(2\pi)^{\frac{d}{2}+1}\sqrt{d!}n^{\frac{d}{4}+1}} e^{2\sqrt{\left(C_d+\frac{\pi^2(d+1)}{6}\right)n}}.$$

Our second main result is the following theorem.

**Theorem 1.2.** *As $n \to \infty$ we have that*

$$r_d(n) \sim \frac{\left(\frac{C_d}{d+1} + \frac{\pi^2}{6}\right)^{\frac{1}{2}} e^{\frac{d-1}{2(d+1)}} d!}{2\sqrt{2}\pi d!^{\frac{d}{2(d+1)}} n} e^{2\sqrt{\left(\frac{C_d}{d+1}+\frac{\pi^2}{6}\right)n}}.$$

The remainder of our paper is laid out as follows. In Section 2 we outline the main asymptotic techniques we apply in our analysis. In Section 3, we explain certain preliminary facts about Eulerian polynomials and certain two-variable deformations of Eulerian polynomials given in [16], and give evaluations of certain integrals which emerge in the process of proving the main theorems. In Sections 4 and 5, we prove Theorems 1.1 and 1.2, respectively, as well as very broad generalizations of these results. Finally, in Section 6 we discuss some final remarks, including possible applications to physics and connections of certain constants in our formulas with an open question about Bloch groups.

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## 2. Asymptotic techniques

2.1. **A variation of Euler–Maclaurin summation.** We say that a function $f$ is of *sufficient decay* in an (unbounded) domain $D \subset \mathbb{C}$ if there exists some $\varepsilon > 0$ such that $f(w) \ll w^{-1-\varepsilon}$ as $|w| \to \infty$ in $D$. We need to use a version of Euler–Maclaurin summation which has been popularized by Zagier [26]. We quote Theorem 1.2 of [12] which follows from the Euler–Maclaurin summation formula.

**Proposition 2.1.** *Suppose that $0 \leq \theta < \frac{\pi}{2}$ and let $D_\theta := \{re^{i\alpha} : r \geq 0 \text{ and } |\alpha| \leq \theta\}$. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic in a domain containing $D_\theta$, so that in particular $f$ is holomorphic at the origin, and assume that $f$ and all of its derivatives are of sufficient decay. Then for $a \in \mathbb{R}$ and $N \in \mathbb{N}_0$,*

$$\sum_{m\geq 0} f((m+a)w) = \frac{I_f}{w} - \sum_{n=0}^{N-1} \frac{B_{n+1}(a) f^{(n)}(0)}{(n+1)!} w^n + O_N\left(w^N\right),$$

*uniformly, as $w \to 0$ in $D_\theta$. Here $I_f := \int_0^\infty f(x)dx$.*

We also require the more general result, which is given in the proof of Theorem 1.2 of [12] (see equation (5.8) there).

**Proposition 2.2.** *Assume the conditions from Proposition 2.1 are satisfied. We have for any $N \in \mathbb{N}$ that*

$$\sum_{m\geq 0} f((m+a)w) = \frac{I_f}{w} - \sum_{n=0}^{N-1} \frac{B_{n+1}(a) f^{(n)}(0)}{(n+1)!} w^n - \sum_{k\geq N} \frac{f^{(k)}(0) a^{k+1}}{(k+1)!} w^k$$
$$- \frac{w^N}{2\pi i} \sum_{n=0}^{N-1} \frac{B_{n+1}(0) a^{N-n}}{(n+1)!} \int_{C_R(0)} \frac{f^{(n)}(z)}{z^{N-n}(z-aw)} dz - (-w)^{N-1} \int_{aw}^{w\infty} \frac{f^{(N)}(z) \widetilde{B}_N\left(\frac{z}{w} - a\right)}{N!} dz,$$

*where $\widetilde{B}_n(x) := B_n(x - \lfloor x \rfloor)$ and $C_R(0)$ denotes the circle of radius $R$ centred at the origin, where $R$ is such that $f$ is holomorphic in $C_R(0)$.*

2.2. **Ingham's Tauberian theorem.** In order to compute the asymptotic behavior of the coefficients $s_d(n)$ and $r_d(n)$ as $n \to \infty$, we make use of a Tauberian Theorem variant proved by Jennings-Shaffer, Mahlburg, and the first author, following work of Ingham. In essence, for generating functions carrying certain analytic properties[1] this gives an easy-to-use method of obtaining the main-term asymptotic of its Fourier coefficients. We quote the special case $\alpha = 0$ of Theorem 1.1 of [12], which follows from Ingham's Theorem [18].

**Proposition 2.3.** *Let $B(q) = \sum_{n\geq 0} b(n) q^n$ be a power series with non-negative real coefficients and radius of convergence at least one and that the $b(n)$ are weakly increasing. Assume that $\lambda$, $\beta$, $\gamma \in \mathbb{R}$ with $\gamma > 0$ exist such that*

$$B\left(e^{-t}\right) \sim \lambda t^\beta e^{\frac{\gamma}{t}} \quad \text{as } t \to 0^+, \qquad B\left(e^{-z}\right) \ll |z|^\beta e^{\frac{\gamma}{|z|}} \quad \text{as } z \to 0, \tag{2.1}$$

[1]The second condition is often dropped in (2.1) which makes the proposition unfortunately incorrect (see [12]).

*with* $z = x + iy$ *(*$x, y \in \mathbb{R}, x > 0$*) in each region of the form* $|y| \leq \Delta x$ *for* $\Delta > 0$. *Then*

$$b(n) \sim \frac{\lambda \gamma^{\frac{\beta}{2}+\frac{1}{4}}}{2\sqrt{\pi} n^{\frac{\beta}{2}+\frac{3}{4}}} e^{2\sqrt{\gamma n}} \qquad \text{as } n \to \infty.$$

To use Proposition 2.3 to study the asymptotic growth of $r_d(n)$ and $s_d(n)$, we need to verify that these are weakly increasing. We quickly prove that these properties hold.

**Lemma 2.4.** *For* $d \in \mathbb{N}$, *the sequences* $s_d(n)$ *and* $r_d(n)$ *are weakly increasing.*

*Proof.* Let $\mathcal{R}_d(n)$ and $\mathcal{S}_d(n)$ be the collections of $d$-fold partition diamonds of size and Schmidt-size $n$, respectively, so that $r_d(n) = |\mathcal{R}_d(n)|, s_d(n) = |\mathcal{S}_d(n)|$. It is enough to construct injections $\mathcal{R}_d(n) \hookrightarrow \mathcal{R}_d(n+1)$ and $\mathcal{S}_d(n) \hookrightarrow \mathcal{S}_d(n+1)$. Such a map is immediately furnished in both cases by that function which takes a $d$-fold diamond partition $(\{a_k\}_{k\geq 0}, \{b_{j,k}\}_{j,k})$ and adds 1 to $a_0$ and leaves all other part sizes fixed. $\square$

## 3. Preliminaries

### 3.1. **Asymptotics of the $q$-Pochhammer symbol.**

We recall the famous asymptotic formula for the inverse of $q$-Pochhammer symbol, which follows from the modularity of the Dedekind $\eta$-function, and is given by

$$\frac{1}{(e^{-z}; e^{-z})_\infty} \sim \sqrt{\frac{z}{2\pi}} e^{\frac{\pi^2}{6z}} \qquad \text{as } z \to 0. \tag{3.1}$$

### 3.2. **Eulerian polynomials.**

We consider here the *Eulerian polynomials*, which we denote by $A_d(x)$, and some of their basic properties. For more properties and proofs, see [22, 26.14]. These polynomials can be defined by the power series identity

$$\sum_{j\geq 0} (j+1)^d x^j = \frac{A_d(x)}{(1-x)^{d+1}}. \tag{3.2}$$

Based on this property, these polynomials can also be defined recursively by $A_0(x) = 1$ and for each $d \in \mathbb{N}$,

$$A_d(x) = (1 + (d-1)x) A_{d-1}(x) + x(1-x)A'_{d-1}(x). \tag{3.3}$$

The first few Eulerian polynomials are

$$A_1(x) = 1, \quad A_2(x) = 1 + x, \quad A_3(x) = 1 + 4x + x^2.$$

We require a few special values of these polynomials. In particular, by induction on $d$ it is not hard to prove that

$$A_d(1) = d!. \tag{3.4}$$

We also obtain by differentiating (3.3) and induction on $d$ that[2]

$$A_d'(1) = \frac{(d-1)\cdot d!}{2}. \tag{3.5}$$

We need to use an important and well-known symmetry property of the Eulerian polynomials: for $d \in \mathbb{N}$, we have

$$A_d(x) = x^{d-1} A_d\left(\frac{1}{x}\right). \tag{3.6}$$

We make use of the following lemma, which follows directly from (3.2).

**Lemma 3.1.** *For $d \in \mathbb{N}$, $A_d(x)$ has no zeros in $[0,1]$.*

3.3. **Deformed Eulerian polynomials.** In order to analyze $d$-fold partition diamonds, we need to consider certain polynomials $F_d(x,y)$ which were introduced in [16]. These polynomials are defined recursively by

$$F_1(x,y) = 1, \quad F_d(x,y) = \frac{\left(1-xy^d\right) F_{d-1}(x,y) - y\,(1-x)\, F_{d-1}(xy,y)}{1-y}.$$

The first cases are

$$F_2(x,y) = 1 + xy, \quad F_3(x,y) = 1 + 2xy + 2xy^2 + x^2y^3.$$

For later convenience, we set

$$H_d(x,y) := \left(1-xy^d\right) F_{d-1}(x,y) - y\,(1-x)\, F_{d-1}(xy,y),$$

so that $F_d(x,y) = \frac{H_d(x,y)}{1-y}$. We refer to $F_d(x,y)$ as a deformation of the Eulerian polynomials because of the following lemma.

**Lemma 3.2.** *We have for $d \in \mathbb{N}$ that $F_d(x,1) = A_d(x)$.*

*Proof.* For this proof and for later convenience, we observe that by simple differentiation rules that

$$\begin{aligned} -H_d^{(0,1)}(x,y) = dxy^{d-1}F_{d-1}(x,y) - \left(1-xy^d\right) F_{d-1}^{(0,1)}(x,y) + (1-x)F_{d-1}(xy,y) \\ + xy(1-x)F_{d-1}^{(1,0)}(xy,y) + y(1-x)F_{d-1}^{(0,1)}(xy,y). \end{aligned} \tag{3.7}$$

Now, $F_d(x,1) = \lim_{y\to 1} \frac{H_d(x,y)}{1-y}$, and we can apply L'Hopitals rule to obtain

$$F_d(x,1) = -\lim_{y\to 1} H_d^{(0,1)}(x,y) = (1+(d-1)x)\,F_{d-1}(x,1) + x\,(1-x)\,F_{d-1}^{(1,0)}(x,1).$$

Observing that this recurrence matches (3.3) and that $F_1(x,1) = A_1(x) = 1$, the claim follows. □

[2]We note that $A_d'(1)$ are known as the Lah numbers (OEIS A001286).

We need a brief lemma which specifies that $F_d(x,y)$ does not have zeros of a certain type. This lemma follows by combining the fact that $F_d(x,y)$ is continuous with Lemma 3.1 and Lemma 3.2.

**Lemma 3.3.** *For $d \in \mathbb{N}$, there exists a neighborhood $\mathcal{N}_d$ of $y=1$ such that $F_d(x,y) \neq 0$ for all $x \in [0,1]$ and $y \in \mathcal{N}_d$.*

We also need a certain differential equation satisfied by $F_d(x,y)$, which is centrally important for the evaluation of the asymptotic expansion of $r_d(n)$.

**Lemma 3.4.** *We have for $d \in \mathbb{N}$ that*

$$F_d^{(0,1)}(x,1) = \frac{dx}{2} F_d^{(1,0)}(x,1).$$

*Proof.* We prove this claim by induction on $d$. The identity is clear for $d=1$. We next assume that for fixed $d \geq 2$, the claim holds. We next reduce the claim to an expression in terms of $H_d(x,y)$. By using L'Hopital's rule, it is not hard to see that

$$\begin{aligned} F_d^{(1,0)}(x,1) &= \lim_{y \to 1} \frac{H_d^{(1,0)}(x,y)}{1-y} = -H_d^{(1,1)}(x,1), \\ F_d^{(0,1)}(x,1) &= \lim_{y \to 1} \frac{(1-y)H_d^{(0,1)}(x,y) + H_d(x,y)}{(1-y)^2} = -\frac{1}{2} H_d^{(0,2)}(x,1). \end{aligned}$$

Therefore, in order to prove the lemma we only need to prove that

$$-H_d^{(0,2)}(x,1) = -dxH_d^{(1,1)}(x,1). \tag{3.8}$$

We use (3.7) as a stepping stone for proving (3.8). By taking the derivative of (3.7) with respect to $x$ and evaluating subsequently at $y=1$, it is not hard to see that

$$\begin{aligned} &- dxH_d^{(1,1)}(x,1) \\ &\quad = d(d-1)xF_{d-1}(x,1) + dx\left(2+(d-3)x\right) F_{d-1}^{(1,0)}(x,1) + dx^2\left(1-x\right) F_{d-1}^{(2,0)}(x,1). \end{aligned}$$

Similarly, by taking the derivative of (3.7) with respect to $y$ and substituting $y=1$, we obtain

$$\begin{aligned} - H_d^{(0,2)}(x,1) &= d(d-1)xF_{d-1}(x,1) + 2\left(\left(d-1\right)x+1\right) F_{d-1}^{(0,1)}(x,1) \\ &\quad + 2x\left(1-x\right) F_{d-1}^{(1,0)}(x,1) + 2x\left(1-x\right) F_{d-1}^{(1,1)}(x,1) + x^2\left(1-x\right) F_{d-1}^{(2,0)}(x,1). \end{aligned} \tag{3.9}$$

Now, using the induction hypothesis, we can show that

$$F_{d-1}^{(1,1)}(x,1) = \frac{\partial}{\partial x} F_{d-1}^{(0,1)}(x,1) = \frac{\partial}{\partial x} \frac{(d-1)x}{2} F_{d-1}^{(1,0)}(x,1)$$
$$= \frac{d-1}{2} F_{d-1}^{(1,0)}(x,1) + \frac{(d-1)x}{2} F_{d-1}^{(2,0)}(x,1).$$

Substituting this into (3.9) and comparing the formula to that for $-dxH_d^{(1,1)}(x,1)$, we obtain (3.8) and therefore the lemma is proven. ☐

3.4. **Generating functions for $s_d(n)$ and $r_d(n)$.** Here we recall the generating functions for $s_d(n)$ and $r_d(n)$, each of which were proven in [16]. Firstly, for $s_d(n)$ Theorem 1.2 of [16] gives that

$$\sum_{n\geq 0} s_d(n) q^n = \prod_{n\geq 1} \frac{A_d\left(q^n\right)}{\left(1-q^n\right)^{d+1}}. \tag{3.10}$$

We also require the generating function for $r_d(n)$, proven in Theorem 1.1 of [16],

$$\sum_{n\geq 0} r_d(n) q^n = \prod_{n\geq 1} \frac{F_d\left(q^{(d+1)(n-1)+1}, q\right)}{1-q^n}. \tag{3.11}$$

3.5. **Evaluating integrals.** In the process of evaluating the constants in our main theorems, the evaluation of certain integrals is paramount. In order to evaluate these integrals, we need the *dilogarithm function* defined for $|z| \leq 1$ by

$$\mathrm{Li}_2\left(z\right) := \sum_{n\geq 1} \frac{z^n}{n^2}$$

and on $\mathbb{C} \setminus [1,\infty)$ by the analytic continuation (see [27, page 5])

$$\mathrm{Li}_2\left(z\right) := -\int_0^z \log(1-u) \frac{du}{u}.$$

For many interesting properties of this function, see [27]. We now prove a proposition to evaluate certain integrals.

**Proposition 3.5.** *Let $P(x) = \prod_{j=1}^d (x-\alpha_j) \in \mathbb{R}[x]$ be a monic polynomial of degree $d \in \mathbb{N}$ such that $P(0) = 1$ and such that $P(x)$ has no zeros on the interval $[0,1]$. Define the integrals*

$$\mathcal{I}_P := \int_0^\infty \mathrm{Log}\left(P(e^{-x})\right) dx.$$

*Then we have*

$$\mathcal{I}_P = -\sum_{j=1}^{d} \mathrm{Li}_2\left(\frac{1}{\alpha_j}\right).$$

*Proof.* Observe firstly that the assumptions that $P$ is monic, that $P(0) = 1$ and that $P(x)$ has no zeros in the interval $[0, 1]$ imply that $\mathcal{I}_P$ converges. Using integration by parts, we obtain

$$\mathcal{I}_P = \int_0^\infty \frac{xe^{-x}P'(e^{-x})}{P(e^{-x})}dx.$$

By further substituting $u = e^{-x}$, we obtain

$$\mathcal{I}_P = -\int_0^1 \log(u)\frac{P'(u)}{P(u)}du.$$

Since $P$ is a monic polynomial, we have

$$\frac{P'(u)}{P(u)} = \sum_{j=1}^{d} \frac{1}{u-\alpha_j},$$

and therefore

$$\mathcal{I}_P = -\sum_{j=1}^{d} \int_0^1 \frac{\log(u)}{u-\alpha_j}du.$$

We now consider for $a \notin [0, 1]$ the integrals

$$I(a) := \int_0^1 \frac{\log(u)}{u-a}du.$$

We claim that $I(a) = \mathrm{Li}_2(\frac{1}{a})$. Because $\mathrm{Li}_2(z)$ is analytic in $\mathbb{C} \setminus [1, \infty)$ (see [27]), both sides of this formula are analytic functions of $a$ away from $[0, 1]$, and therefore to prove our claim we only need to prove its truth in the region $a > 1$. Here, the identity $\frac{d}{du}\mathrm{Li}_2(u) = -\frac{1}{u}\log(1-u)$ is valid for $|u| < 1$ because of the series expansion of $\mathrm{Li}_2(u)$, and so it is straightforward to show that

$$\frac{d}{du}\left(\mathrm{Li}_2\left(\frac{u}{a}\right) + \log(u)\log\left(1-\frac{u}{a}\right)\right) = \frac{\log(u)}{u-a}$$

for $a > 1$. Since $\mathrm{Li}_2(0) = 0$ and $\log(u)\log(1-\frac{u}{a}) \to 0$ as $u \to 0^+$, we therefore obtain for $a > 1$ that

$$\int_0^1 \frac{\log(u)}{u-a}du = \mathrm{Li}_2\left(\frac{1}{a}\right),$$

and by analytic continuation the identity holds for $a \in \mathbb{C}\backslash[0,1]$. Since the polynomial $P$ has no zeros in the interval $[0,1]$, the claim follows. □

## 4. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Recall the generating function for $s_d(n)$ in (3.10). To ease notation, we define

$$F_d(q) := \prod_{n\geq 1} A_d\left(q^n\right).$$

We begin with a preparatory lemma on the asymptotic of $F_d(q)$.

**Lemma 4.1.** *As $w \to 0$ in $D_\theta$, we have*

$$F_d\left(e^{-w}\right) = \frac{e^{\frac{C_d}{w}+\frac{d-1}{24}w}}{\sqrt{d!}}\left(1+O\left(w^N\right)\right)$$

*for any $N \in \mathbb{N}$.*

*Proof.* Let

$$\mathcal{F}_d(q) := \mathrm{Log}(F_d(q)) = \sum_{n\geq 1} \mathrm{Log}\left(A_d\left(q^n\right)\right),$$

where throughout we use the principal branch of the logarithm. Then

$$\mathcal{F}_d\left(e^{-w}\right) = \sum_{n\geq 1} f_d(nw),$$

where

$$f_d(z) := \mathrm{Log}\left(A_d\left(e^{-z}\right)\right).$$

Note that by (3.4) we have $A_d(1) = d! > 0$, that by (3.2) we have $A_d(0) = 1$, and that $A_d(e^{-w})$ is holomorphic in $w$. Therefore, in the limit $w \to 0$ (i.e., for $|w|$ suitably small) we have that $A_d(e^{-nw})$ is arbitrarily close to $d!$, and avoids the branch of the complex logarithm on the cut $(-\infty, 0]$.

Recall that $A_d(x)$ has no roots on the interval $[0,1]$ by Lemma 3.1. Applying Proposition 2.1 gives that for $w \to 0$ in $D_\theta$ we have that

$$\mathcal{F}_d\left(e^{-w}\right) \sim \frac{I_{f_d}}{w} - \sum_{n\geq 0} \frac{B_{n+1}(1)}{(n+1)!} f_d^{(n)}(0) w^n. \tag{4.1}$$

We adopt the usual convention that $f(z) \sim \sum_{n\geq -1} a_n z^n$ means that for each $N \geq -1$, we have $f(z) = \sum_{n=-1}^{N} a_n z^n + O(z^{N+1})$.

To determine the term $n = 0$, we compute, using (3.4),

$$f_d(0) = \log(A_d(1)) = \log(d!).$$

Thus the term $n=0$ in (4.1) equals $-\frac{\log(d!)}{2}$. We next determine the term $n=1$. By definition

$$f_d'(0)=\left[\frac{\partial}{\partial z}\log\left(A_d\left(e^{-z}\right)\right)\right]_{z=0}=-\frac{A_d'(1)}{A_d(1)}.$$

Using (3.4) and (3.5), we obtain $\frac{d-1}{24}$ for the term $n=1$. Plugging into (4.1) we therefore obtain

$$\mathcal{F}_d\left(e^{-w}\right)\sim\frac{I_{f_d}}{w}-\frac{\log(d!)}{2}+\frac{d-1}{24}w+O\left(w^2\right).$$

We are left to show that the asymptotic expansion has no further terms than the three given on the right-hand side. Using (6.2), it is not difficult to show that

$$f_d^*(z):=f_d(z)+\frac{d-1}{2}z=\log\left(A_d\left(e^{-z}\right)\right)+\frac{d-1}{2}z$$

is an even function. Therefore, in (4.1) only the term $n=1$ and $n$ even terms survive. However, for $n\geq 2$ even it is well-known that $B_{n+1}(1)=0$, and thus only the terms $n=0$ and $n=1$ in the sum of (4.1) contribute to the asymptotic. Combining these observations gives the claim. □

We are now in a position to prove Theorem 1.1.

*Proof of Theorem 1.1.* Recall the generating function in (3.10). Using (3.1) and Lemma 4.1, we have as $w\to 0$ in $D_\theta$ that

$$\sum_{n\geq 0}s_d(n)e^{-nw}\sim\left(\sqrt{\frac{w}{2\pi}}e^{\frac{\pi^2}{6w}}\right)^{d+1}\frac{e^{\frac{C_d}{w}+\frac{d-1}{24}w}}{\sqrt{d!}}\sim\frac{1}{(2\pi)^{\frac{d+1}{2}}\sqrt{d!}}w^{\frac{d+1}{2}}e^{\left(C_d+\frac{\pi^2(d+1)}{6}\right)\frac{1}{w}}.$$

Plugging into Proposition 2.3, with $\lambda=\frac{1}{(2\pi)^{\frac{d+1}{2}}\sqrt{d!}}$, $\beta=\frac{d+1}{2}$, and $\gamma=C_d+\frac{\pi^2(d+1)}{6}$ then gives the claimed asymptotic for $s_d(n)$. □

Using similar techniques, it is not hard to prove the following theorem for a general class of polynomials. Note that in general we do not obtain a terminating asymptotic expansion.

**Theorem 4.2.** *Let $P(x)\in\mathbb{R}[x]$ be a monic polynomial with $P(0)=1$, $P(1)>0$, and assume that $P$ has no zeros in $[0,1]$. Let $H(q):=\prod_{n\geq 1}P(q^n)$. Denote the Fourier coefficients of $H(q)$ by $c(n)$. Suppose that $c(n)$ are non-negative and weakly increasing for $n\gg 0$. Define*

$$\mathcal{C}_P:=\int_0^\infty\log\left(P(e^{-x})\right)dx.$$

*Then as $n \to \infty$ we have that*

$$c(n) \sim \frac{\mathcal{C}_P^{\frac{1}{4}}}{2\sqrt{\pi P(1)} n^{\frac{3}{4}}} e^{2\sqrt{\mathcal{C}_P n}}.$$

**Remarks.**

(1) One may use Proposition 3.5 to obtain that

$$\mathcal{C}_P = -\sum_{j=1}^{d} \mathrm{Li}_2\left(\frac{1}{\alpha_j}\right),$$

where the sum runs over all roots $\alpha_j$ of $P$ counted with multiplicity.

(2) The results can be extended immediately to products of rational functions, provided the numerator and denominator satisfy the hypotheses. This is done in more generality in Theorem 5.3. One could also avoid the need for monotonicity of the coefficients if stronger asymptotic properties away from $q \to 1$ are derived.

(3) For certain choices of polynomial $P$, it is not hard to see that the sum of dilogarithms defining $\mathcal{C}_P$ simplifies considerably. For example, let $\ell$ be a fixed prime. If $P$ is chosen to be the $\ell$-th cyclotomic polynomial $\Phi_\ell$, the roots are precisely all of the primitive $\ell$-th roots of unity. Then using the distribution property for dilogarithms (see e.g. [27, page 9]), we recover the $\ell$-regular partition asymptotic. This agrees with the asymptotic arising from the $\ell$-regular partition generating function

$$\prod_{n\geq 1} \Phi_\ell(q^n) = \prod_{n\geq 1} \frac{1-q^{\ell n}}{1-q^n}$$

where the asymptotic for the coefficients of the right-hand side can be evaluated using standard techniques - see e.g. [14]. We discuss the possibility of finding simpler expressions for $\mathcal{C}_P$ in more generality in Section 6.

## 5. Proof of Theorem 1.2

In this section we prove Theorem 1.2. Recall the generating function for $r_d(n)$ given in (3.11). To ease notation, we let

$$G_d(q) \coloneqq \prod_{n\geq 0} F_d\left(q^{(d+1)n+1}, q\right),$$

where $F_d(x,y)$ is defined in Subsection 2.3.

We again being with a preparatory lemma on the asymptotic of $G_d(q)$.

**Lemma 5.1.** *As $w \to 0$ in $D_\theta$, we have that*

$$G_d\left(e^{-w}\right) \sim \frac{e^{\frac{C_d}{(d+1)w}+\left(\frac{1}{2}-\frac{1}{d+1}\right)d!}}{(d!)^{\frac{d}{2(d+1)}}}.$$

*Proof.* We have

$$\mathcal{G}_d(q) := \operatorname{Log}(G_d(q)) = \sum_{n\geq 0} \operatorname{Log}\left(F_d\left(q^{(d+1)n+1}, q\right)\right).$$

Write

$$\mathcal{G}_d\left(e^{-w}\right) = \sum_{n\geq 0} g_{d,w}\left(\left(n+\frac{1}{d+1}\right)(d+1)w\right),$$

where

$$g_{d,w}(x) := \operatorname{Log}\left(F_d\left(e^{-z}, e^{-w}\right)\right).$$

Observe that $F_d(0,0) = 1$ and that by Lemma 3.3, $F_d\left(e^{-tw}, e^{-w}\right)$ does not vanish for $|w|$ small and $t \in \mathbb{R}_0^+$. Note that by Lemma 3.2 and (3.4) we have that $F_d(1,1) = A_d(1) = d! > 0$ and that $F_d(e^{-tw}, e^{-w})$ is holomorphic in $w$ for $t \in \mathbb{R}_0^+$. Therefore, in the limit $w \to 0$ (i.e., for $|w|$ suitably small) we again avoid the branch of the complex logarithm on the cut $(-\infty, 0]$.

Then applying Proposition 2.2 with $N = 2$ gives that

$$\begin{aligned}\mathcal{G}_d\left(e^{-w}\right) = {} & \frac{I_{g_{d,w}}}{(d+1)w} - \sum_{n=0}^{1} \frac{B_{n+1}\left(\frac{1}{d+1}\right) g_{d,w}^{(n)}(0)}{(n+1)!}(d+1)^n w^n \\ & - \frac{1}{d+1}\sum_{k\geq 2}\frac{g_{d,w}^{(k)}(0)}{(k+1)!}w^k - \frac{w^2}{2\pi i}\sum_{n=0}^{1}\frac{(d+1)^n B_{n+1}(0)}{(n+1)!}\int_{C_R(0)}\frac{g_{d,w}^{(n)}(z)}{z^{2-n}(z-w)}dz \\ & - \frac{(d+1)w}{2}\int_w^{w\infty} g''_{d,w}(z)\widetilde{B}_2\left(\frac{z}{(d+1)w}-\frac{1}{d+1}\right)dz. \end{aligned} \tag{5.1}$$

The main asymptotic contribution comes from the term

$$\frac{I_{g_{d,0}}}{(d+1)w} = \frac{C_d}{(d+1)w}$$

by Lemma 3.2.

The constant term in (5.1) is

$$\frac{I_{\left[\frac{\partial}{\partial w} g_{d,w}\right]_{w=0}}}{d+1} - B_1\left(\frac{1}{d+1}\right) g_{d,0}(0).$$

We have by Lemma 3.2 and (3.4) that

$$g_{d,0}(0) = A_d(1) = d!.$$

We then compute

$$\left[\frac{\partial}{\partial w} g_{d,w}(x)\right]_{w=0} = \left[\frac{\partial}{\partial w} \log\left(F_d\left(e^{-x}, e^{-w}\right)\right)\right]_{w=0} = -\frac{F_d^{(0,1)}\left(e^{-x},1\right)}{F_d\left(e^{-x},1\right)}.$$

Using Lemma 3.4 we therefore obtain that

$$\begin{aligned} I_{\left[\frac{\partial}{\partial w} g_{d,w}\right]_{w=0}} dx &= -\int_0^\infty \frac{F_d^{(0,1)}\left(e^{-x},1\right)}{F_d\left(e^{-x},1\right)} dx = \frac{d}{2}\int_0^\infty \frac{\frac{\partial}{\partial x} F_d(e^{-x},1)}{F_d\left(e^{-x},1\right)} dx \\ &= \frac{d}{2}\int_0^\infty \frac{\partial}{\partial x} \log\left(F_d\left(e^{-x},1\right)\right) dx = \frac{d}{2}(\log(F_d(0,1)) - \log(F_d(1,1))). \end{aligned}$$

We now claim that $F_d(0,1) = 1$. By Lemma 3.2, we have $F_d(0,1) = A_d(0)$. We plug into (3.3) and obtain

$$A_d(0) = A_{d-1}(0) = 1,$$

as $A_1(0) = 1$. Moreover, again by Lemma 3.2 and (3.4), we have

$$F_d(1,1) = A_d(1) = d!.$$

Thus,

$$I_{\left[\frac{\partial}{\partial w} g_{d,w}\right]_{w=0}} = -\frac{d}{2}\log(d!).$$

So the constant term in (5.1) is equal to

$$-\frac{d\log(d!)}{2(d+1)} + \left(\frac{1}{2} - \frac{1}{d+1}\right) d!.$$

Exponentiating gives the claim, noting that the remaining terms go into the error. $\square$

We are now in a position to prove Theorem 1.2.

*Proof of Theorem 1.2.* Recall the generating function for $r_d(n)$ in (3.11). We then use Lemma 5.1 and (3.1) to obtain that

$$\sum_{n\geq 0} r_d(n)e^{-nw} \sim \sqrt{\frac{w}{2\pi}} e^{\frac{\pi^2}{6w}} \frac{e^{\frac{C_d}{(d+1)w} + \left(\frac{1}{2}-\frac{1}{d+1}\right)d!}}{(d!)^{\frac{d}{2(d+1)}}} = \frac{e^{\frac{d-1}{2(d-1)}d!}}{\sqrt{2\pi}(d!)^{\frac{d}{2(d+1)}}}\sqrt{w}e^{\left(\frac{C_d}{d+1}+\frac{\pi^2}{6}\right)\frac{1}{w}}$$

as $w \to 0$ in $D_\theta$. Applying Proposition 2.3 with $\lambda = (2\pi)^{-\frac{1}{2}} e^{\frac{d-1}{2(d+1)}d!} d!^{-\frac{d}{2(d+1)}}$, $\beta = \frac{1}{2}$, and $\gamma = \frac{C_d}{d+1} + \frac{\pi^2}{6}$ gives the claim. $\square$

We note here that asymptotic "tricks" used to prove Theorem 1.2 can be generalized quite broadly. The main point is that Lemma 5.1 can be greatly generalized to many products of the form $\prod_{n\geq 0} P\left(q^n, q\right)$ where $P(x,y) \in \mathbb{R}[x,y]$. The basic idea is to take a logarithm in order to reduce the question to asymptotics for $\operatorname{Log}\left(P\left(e^{-nz}, e^{-z}\right)\right)$. The main idea of our method is to use the Euler–Maclaurin formula as stated in Proposition 2.2 to give an exact formula for $\operatorname{Log}\left(P\left(e^{-z}, e^{-w}\right)\right)$ for $w$ fixed. Then, suitable holomorphic properties of this expression permit the substitution $w = z$ as $z \to 0$, and we can then compute suitable asymptotics as $z \to 0$. We execute these objectives in the following two results.

**Lemma 5.2.** *Let $P \in \mathbb{R}[x,y]$ be a polynomial such that $P(1,1) > 0$, $P(0,1) = 1$, and such that $P(x,1)$ has no zeros for $0 \leq x \leq 1$. Let $a$, $b \in \mathbb{N}$ with $0 \leq a < b$ and*

$$G_P(q) := \prod_{n\geq 0} P\left(q^{bn+a}, q\right).$$

*Then in each region $D_\theta$ with $0 < \theta < \frac{\pi}{2}$, we have as $w \to 0$ in $D_\theta$ that*

$$G_P\left(e^{-w}\right) \sim P(1,1)^{\frac{1}{2}-\frac{a}{b}} e^{\mathcal{D}_P} \cdot e^{\frac{\mathcal{C}_{\mathcal{P}}}{bw}}$$

*where we define $\mathcal{P}(x) := P(x,1)$, $\mathcal{C}_P$ as in Theorem 4.2 and*

$$\mathcal{D}_P := I_{\left[\frac{\partial}{\partial w} g_{P,w}\right]_{w=0}} = -\int_0^\infty \frac{P^{(0,1)}\left(e^{-x},1\right)}{P\left(e^{-x},1\right)} dx.$$

*Proof.* We consider $\mathcal{G}_P(q) := \operatorname{Log}\left(G_P(q)\right)$. Then we have

$$\mathcal{G}_P\left(e^{-w}\right) = \sum_{n\geq 0} g_{P,w}\left(\left(n + \frac{a}{b}\right) bw\right),$$

where for any fixed $w$ we define

$$g_{P,w}\left(z\right) := \operatorname{Log}\left(P\left(e^{-z}, e^{-w}\right)\right).$$

As in previous results, the conditions we assume for $P(x,y)$ ensure that $g_{P,w}(z)$ satisfies the analytic conditions necessary for convergence of $\mathcal{G}_P\left(e^{-w}\right)$ and the application of Proposition 2.2. By applying Proposition 2.2 in this setting, we see that

$$\mathcal{G}_P\left(e^{-w}\right) = \frac{I_{g_{P,w}}}{bw} - \sum_{n=0}^{1} \frac{B_{n+1}\left(\frac{a}{b}\right) g_{P,w}^{(n)}(0)}{(n+1)!}(bw)^n$$
$$- \frac{1}{b}\sum_{k\geq 2} \frac{g_{P,w}^{(k)}(0)a^{k+1}}{(k+1)!}w^k - \frac{(bw)^2}{2\pi i}\sum_{n=0}^{1} \frac{\left(\frac{a}{b}\right)^{2-n} B_{n+1}(0)}{(n+1)!}\int_{C_R(0)} \frac{g_{P,w}^{(n)}(z)}{z^{2-n}(z-aw)}dz$$
$$+ \frac{bw}{2}\int_{aw}^{w\infty} g''_{P,w}(z)\widetilde{B}_2\left(\frac{z}{bw}-\frac{a}{b}\right)dz.$$

In order to apply Proposition 2.3, we need the terms up through the constant term in $w$ of this expansion; we see that

$$\mathcal{G}_P\left(e^{-w}\right) = \frac{I_{g_{P,w}}}{bw} - B_1\left(\frac{a}{b}\right)g_{P,w}(0) + O(w)$$
$$= \frac{\mathcal{C}_{\mathcal{P}}}{bw} + \mathcal{D}_P + \left(\frac{1}{2}-\frac{a}{b}\right)\operatorname{Log}\left(P(1,1)\right) + O(w).$$

This completes the proof. □

On the basis of this lemma, we can prove asymptotic formulas for the coefficients of these very general rational products.

**Theorem 5.3.** *Let $P, Q \in \mathbb{R}[x,y]$ be polynomials such that $P(1,1), Q(1,1) > 0$, $P(0,1) = Q(0,1) = 1$, and such that $P(x,1)$ and $Q(x,1)$ have no zeros for $0 \leq x \leq 1$. Let*

$$H(q) := \sum_{n\geq 0} c(n)q^n := \prod_{n\geq 0} \frac{P\left(q^{An+a},q\right)}{Q\left(q^{Bn+b},q\right)},$$

*and suppose that for $n \gg 0$ the $c(n)$ are increasing functions. Then as long as $\frac{\mathcal{C}_{\mathcal{P}}}{A} > \frac{\mathcal{C}_{\mathcal{Q}}}{B}$, we have, as $n\to\infty$,*

$$c(n) \sim \frac{\lambda_{P,Q,a,A,b,B}\cdot \mathcal{C}_{P,Q,A,B}^{\frac{1}{4}}}{2\sqrt{\pi}n^{\frac{3}{4}}}e^{2\sqrt{\mathcal{C}_{P,Q,A,B}n}},$$

*where we define*

$$\mathcal{C}_{\mathcal{P},\mathcal{Q},A,B} := \frac{\mathcal{C}_{\mathcal{P}}}{A} - \frac{\mathcal{C}_{\mathcal{Q}}}{B} > 0, \quad \mathcal{D}_{P,Q} := \mathcal{D}_P - \mathcal{D}_Q,$$
$$\lambda_{P,Q,a,A,b,B} := P(1,1)^{\frac{1}{2}-\frac{a}{A}}Q(1,1)^{\frac{b}{B}-\frac{1}{2}}e^{\mathcal{D}_{P,Q}}.$$

*Proof.* Because we assume that $c(n)$ is increasing for $n \gg 0$, we need only calculate suitable asymptotics as $z \to 0$ in regions $D_\theta$ for certain $0 < \theta < \frac{\pi}{2}$. From Lemma 5.2 we obtain

$$\prod_{n \geq 0} \frac{P\left(q^{An+a}, q\right)}{Q\left(q^{Bn+b}, q\right)} \sim P(1,1)^{\frac{1}{2}-\frac{a}{A}} Q(1,1)^{\frac{b}{B}-\frac{1}{2}} \cdot e^{\left(\frac{\mathcal{C}_P}{A}-\frac{\mathcal{C}_Q}{B}\right)\frac{1}{w}+\mathcal{D}_{P,Q}}.$$

This completes the proof by Proposition 2.3 with $\lambda = \lambda_{P,Q,a,A,b,B}$, $\beta = 0$, and $\gamma = \mathcal{C}_{P,Q,A,B} > 0$. $\square$

## 6. Final Remarks

6.1. **Applications in mathematical physics.** There are potential applications of our method to the computation of asymptotic formulas for coefficients of thermal partition functions in super Yang–Mills theory [1, 11, 21]. For example the partition functions in [1, equation (5.10)], [11, equation (2.6)], and [21, equation (7.3)], can all be treated with this approach. These partition functions often take the form of infinite products over rational functions evaluated at $q^n$. Asymptotic formulas for these partition functions can be used to derive information about the entropy of the relevant system. Although the physics literature does contain some elementary methods for computing asymptotics for these coefficients, our method is capable of vast generalization. In particular, since our asymptotic method is based upon the exact formula given in Proposition 2.2, one could compute asymptotics to much higher degrees of precision, and therefore obtain more accurate entropies.

6.2. **Dilogarithms and Bloch groups.** Let $P$ be a polynomial of degree $d$ with integral coefficients with $P(0) = 1$ and no roots in the interval $[0,1]$. If $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{C} \backslash [0,1]$ are the zeros of $P$, then by Proposition 3.5 we have

$$\mathcal{C}_P = -\sum_{j=1}^{d} \mathrm{Li}_2\left(\frac{1}{\alpha_j}\right).$$

It is natural to ask whether there is a simpler representation for $\mathcal{C}_P$. Since the values $\frac{1}{\alpha_j}$ are the zeros of the reciprocal polynomial $z^d P(\frac{1}{z})$, we reframe this question in the following slightly more general way.

**Question.** *Let $P(z) \in \mathbb{Z}[z]$ be a monic polynomial. Then under what circumstances does the value*

$$\sum_{P(\alpha)=0} \mathrm{Li}_2(\alpha)$$

*simplify in some sense?*

It has been pointed out to the authors by Zagier that this somewhat vague question is closely connected to the so-called Bloch group. This is defined in terms of the *Bloch–Wigner dilogarithm function*

$$D(z) := \mathrm{Im}\left(\mathrm{Li}_2(z)\right) + \mathrm{Arg}\,(1-z)\log|z|,$$

which is real-analytic for $z \in \mathbb{C}\backslash\{0,1\}$ and satisfies

$$D(z) = -D(1-z) = -D\left(\frac{1}{z}\right)$$

along with a five-term relation [27, p. 11]

$$D\left(x\right) + D\left(y\right) + D\left(\frac{1-x}{1-xy}\right) + D\left(1-xy\right) + D\left(\frac{1-y}{1-xy}\right) = 0.$$

Motivated by connections to the volumes of hyperbolic 3-manifolds (as explained in [27]), Bloch defined a group structure based on this functional equation. To be more precise, the *Bloch group* of a field $L \subseteq \overline{\mathbb{Q}}$, denoted $\mathcal{B}_L$, is defined as all formal linear combinations of symbols $[\alpha], \alpha \in L^\times \setminus \{1\}$ subject to the relations

$$[x] + \left[\frac{1}{x}\right] = [x] + [1-x] = [x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1-xy] + \left[\frac{1-y}{1-xy}\right] = 0.$$

Now, $\mathcal{B}_L$ is certainly abelian and countable, its rank is the number of pairs of complex embeddings of $L$ into $\mathbb{C}$, and nontrivial elements are easy to produce [27, p. 15–16]. Torsion elements have been well-studied, and the basic result is that for $\xi \in L$, we have $[\xi] \in \mathcal{B}_L$ is torsion if and only if $D\left(\xi^\sigma\right) = 0$ for all complex embeddings $\sigma \in \mathrm{Gal}\left(L/\mathbb{Q}\right)$ (see Section B on page 36 of [27]). If we compare this fact with the properties of the so-called Rogers dilogarithm [27, p. 23], then we see that this is equivalent to $\mathrm{Li}_2\left(\xi\right) \in \log(\overline{\mathbb{Q}})^{\otimes 2}$, that is, that $\mathrm{Li}_2\left(\xi\right)$ can be expressed as linear combinations of forms $\log\left(\alpha\right)\log\left(\beta\right)$ for $\alpha, \beta \in \overline{\mathbb{Q}}$. Therefore, the previous question about simplified values of $\mathcal{C}_P$ motivates the following question:

**Question.** *Let $P \in \mathbb{Z}[x]$ be a monic polynomial with splitting field $L$. Then under what circumstances is*

$$\sum_{P(\alpha)=0} [\alpha] \in \mathcal{B}_L$$

*a torsion element in the Bloch group $\mathcal{B}_L$, and if it is torsion, what is its order?*

For the Eulerian polynomials $A_d(x)$ in particular, by using and the dilogarithm identity $\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{1}{x}\right) = -\zeta(2) - \frac{\log(-x)^2}{2}$, it can be shown that $\mathcal{C}_{A_d}$ is, up to an explicit multiple of $\zeta(2)$, an element of $\log\left(\overline{\mathbb{Q}}\right)^{\otimes 2}$. It would be quite interesting to understand this question more deeply using the tools of Bloch groups and the dilogarithm identities.

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