

ASYMPTOTICS OF COMMUTING ℓ -TUPLES IN SYMMETRIC GROUPS AND LOG-CONCAVITY

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ABSTRACT. Denote by $N_\ell(n)$ the number of ℓ -tuples of elements in the symmetric group S_n with commuting components, normalized by the order of S_n . In this paper, we prove asymptotic formulas for $N_\ell(n)$. In addition, general criteria for log-concavity are shown, which can be applied to $N_\ell(n)$ among other examples. Moreover, we obtain a Bessenrodt–Ono type theorem which gives an inequality of the form $c(a)c(b) > c(a+b)$ for certain families of sequences $c(n)$.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper we consider asymptotics for commuting ℓ -tuples in S_n , where S_n denotes the symmetric group for $n, \ell \in \mathbb{N}$. To be more precise, let $|M|$ be the cardinality of a set M and define

$$C_{\ell,n} := \left\{ (\pi_1, \dots, \pi_\ell) \in S_n^\ell : \pi_j \pi_k = \pi_k \pi_j \text{ for } 1 \leq j, k \leq \ell \right\}. \quad (1.1)$$

The numbers $|C_{\ell,n}|$ are divisible by $|S_n|$ and appear as the specialization of the ℓ -th orbifold characteristic to the n -th symmetric product of a manifold of ordinary Euler characteristic 1 (see [1] for a combinatorial approach and Theorem 2.1 of [10]). In this paper, we prove asymptotics and log-concavity of

$$N_\ell(n) := \frac{|C_{\ell,n}|}{|S_n|}. \quad (1.2)$$

Bryan and Fulman [10]¹ proved the following.²

Theorem 1.1 (Bryan and Fulman, see also [1], p. 3). *For $\ell \in \mathbb{N}$, we have*

$$\sum_{n=0}^{\infty} |C_{\ell,n}| \frac{q^n}{n!} = \prod_{n=1}^{\infty} (1 - q^n)^{-g_{\ell-1}(n)} = \exp \left(\sum_{n=1}^{\infty} g_\ell(n) \frac{q^n}{n} \right),$$

where $g_\ell(n)$ denote the number of subgroups of \mathbb{Z}^ℓ of index n (we refer to the work of Lubotzky and Segal [19] for more background on the group theoretic interpretation).

Since $g_2(n) = \sigma_1(n)$, where for $m \in \mathbb{N}$, $\sigma_m(n) := \sum_{d|n} d^m$, we obtain that $N_2(n) = p(n)$, where $p(n)$ denotes the number of partitions of n (for more background, see [3, 23]). Thus, $N_2(n)$ equals to the number of conjugacy classes in S_n [13]. Note that $\ell = 3$ reveals an interesting connection to topology as highlighted by Britnell (see introduction of [9]) due to

2020 *Mathematics Subject Classification*. Primary 05A17, 11P82; Secondary 05A20.

Key words and phrases. Generating functions, log-concavity, partition numbers, symmetric group.

¹The result of Bryan–Fulman had been strongly influenced by Stanley ([10], acknowledgments).

²Further proofs have been obtained by [1] and [28]. The work of Bryan and Fulman can be considered as a generalization of a combinatorial formulae due to Macdonald [20] and Hirzebruch and Höfer [17].

work of Liskovets and Medynkh [18]. Namely $N_3(n)$ counts the number of non-equivalent n -sheeted coverings of a torus.

Recently, Neuhauser and one of the authors [2] proved that $N_\ell(n)$ is log-concave for $n \geq 20$ for almost all ℓ if and only if $n \equiv 0 \pmod{3}$. Moreover it was shown by Nicolas ([22], Proposition 1) and reproved by DeSalvo and Pak ([12], Theorem 1.1) that $N_2(n)$ is log-concave for $n \geq 26$. In [2] it was conjectured that $N_3(n)$ is also log-concave for $n \geq 22$ (numerically verified there for $n \leq 10^5$). Based on numerical experiments they also speculated that for fixed ℓ , $N_\ell(n)$ is log-concave for almost all n .

In this paper, using results by Bridges, Brindle, and two of the authors in [7], we prove asymptotic formulas for $N_\ell(n)$ for arbitrary ℓ , and partially answer the question (5) posed in Section 6 there. Note that we are in the situation of multiple poles which is not covered by the classical result of Meinardus [21].

Theorem 1.2. *For $\ell \geq 6$ we have,³ as $n \rightarrow \infty$,*

$$N_\ell(n) \sim \frac{(\ell-1)!^{\frac{1}{2\ell}} \sqrt{Z_\ell}}{\sqrt{2\pi\ell} n^{\frac{\ell+1}{2\ell}}} \exp\left(\frac{\ell\Gamma(\ell)^{\frac{1}{\ell}} Z_\ell}{\ell-1} n^{\frac{\ell-1}{\ell}} + \sum_{k=2}^{\ell} A_{\ell,k} n^{\frac{\ell-k}{\ell}}\right) \left(1 + \sum_{j=1}^{\infty} \frac{B_{\ell,j}}{n^{\frac{j}{\ell}}}\right),$$

where $Z_\ell := (\zeta(2) \cdot \zeta(3) \cdots \zeta(\ell))^{\frac{1}{\ell}}$ for certain $A_{\ell,k}$ and $B_{\ell,j}$.

Remarks.

- (1) *We give a more explicit description for the constants $A_{\ell,k}$ in (3.21). Theoretically, the values $B_{\ell,j}$ can also be calculated explicitly. They result from the calculation methods in [7] by a rather complicated procedure.*
- (2) *The cases $\ell \in \{3, 4, 5\}$ are simpler and of special interest. They are treated separately in Theorem 3.2 and Theorem 3.3.*

It turns out that asymptotics like the one in Theorem 1.2 are sufficient to prove log-concavity⁴ of sequences. Recall that a sequence a_n is called *log-concave* if $a_n^2 \geq a_{n+1}a_{n-1}$. We prove the following general result.

Theorem 1.3. *Assume that $c(n)$ is a sequence with*

$$c(n) \sim \frac{C}{n^\kappa} \exp\left(\sum_{\lambda \in \mathcal{S}} A_\lambda n^\lambda\right) \sum_{\mu \in \mathcal{T}} \frac{\beta_\mu}{n^\mu} \quad (n \rightarrow \infty).$$

Here $\kappa \in \mathbb{R}$, $\mathcal{S} \subset \mathbb{Q}^+ \cap (0, 1)$ is finite, $\mathcal{T} \subset \mathbb{Q}_0^+$, $C, A_\lambda, \beta_\mu \in \mathbb{R}$ with $\beta_0 = 1$. Let $\lambda^* := \max\{\lambda \in \mathcal{S} : A_\lambda \neq 0\}$ and assume that $A_{\lambda^*} > 0$. Then, for n sufficiently large, $c(n)$ is log-concave.⁵

In particular, from Theorem 1.3 and Theorem 1.2, we conclude the following.

Corollary 1.4. *Let⁶ $\ell \geq 2$. For n sufficiently large $N_\ell(n)$ is log-concave.*

³Here and in the following such series are meant as asymptotic expansions.

⁴For the definition, more background and applications in combinatorics see Stanley [27] and Section 5 of Brändén [6].

⁵The result can probably also be extended to real exponents, but this case is not required for any of our applications and thus we do not allow this case here.

⁶Note that the case $\ell = 1$ is trivially true.

Remark. In Corollary 4.2 we give as further examples partitions into k -gonal numbers, as well as n -dimensional representation numbers of the groups $\mathfrak{su}(3)$ and $\mathfrak{so}(5)$.

In 2016, Bessenrodt and Ono ([5], Theorem 2.1) proved that if $a, b \in \mathbb{N}$ satisfy $a, b > 1$ and $a + b > 8$, then $p(a)p(b) \geq p(a + b)$ with equality if and only if $\{a, b\} = \{2, 7\}$. We also show a Bessenrodt–Ono type theorem for general sequences, which implies Bessenrodt’s and Ono’s result on $p(n)$ for a, b sufficiently large.

Theorem 1.5. *Let $c(n)$ be a sequence satisfying*

$$c(n) \sim \frac{C}{n^\kappa} \exp\left(\sum_{\lambda \in \mathcal{S}} A_\lambda n^\lambda\right)$$

with $\kappa \in \mathbb{R}$, $\mathcal{S} \in \mathbb{R} \cap (0, 1)$, $C, A_\lambda \in \mathbb{R}$. Let $\lambda^* := \max\{\lambda \in \mathcal{S} : A_\lambda \neq 0\}$ with $A_{\lambda^*} > 0$. If $a, b \gg 1$, then

$$c(a)c(b) > c(a + b).$$

Again, we can apply this to the sequences $N_\ell(n)$.

Corollary 1.6. *Let $\ell \geq 2$. We have, for $a, b \gg 1$*

$$N_\ell(a)N_\ell(b) > N_\ell(a + b).$$

The paper is organized as follows. In Section 2, we recall known results. In Section 3 we prove Theorem 1.2. In Section 4, we show our main results concerning log-concavity and give some examples. In Section 5 we provide a proof of Theorem 1.5. In Section 6, we state some open questions.

ACKNOWLEDGMENTS

The first author received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 101001179).

2. PRELIMINARIES

2.1. Results from [7]. We require some results from [7]. Let $f : \mathbb{N} \rightarrow \mathbb{N}_0$, set $\Lambda := \mathbb{N} \setminus f^{-1}(\{0\})$, and for $q = e^{-z}$ ($z \in \mathbb{C}$ with $\operatorname{Re}(z) > 0$), define

$$G_f(z) := \sum_{n \geq 0} p_f(n)q^n := \prod_{n \geq 1} \frac{1}{(1 - q^n)^{f(n)}}, \quad L_f(s) := \sum_{n \geq 1} \frac{f(n)}{n^s}.$$

We let \mathcal{P} be the set of poles of $L_f^*(s) := \Gamma(s)\zeta(s+1)L_f(s)$, and for $R > 0$ we denote by \mathcal{P}_R the union of the poles of L_f^* greater than $-R$ with $\{0\}$. We require the following key properties of these objects:

- (P1) All poles of L_f are real. Let $\alpha > 0$ be the largest pole of L_f . There exists $L \in \mathbb{N}$, such that for all primes p , we have $|\Lambda \setminus (p\mathbb{N} \cap \Lambda)| \geq L > \frac{\alpha}{2}$.
- (P2) Condition (P2) is attached to $R \in \mathbb{R}^+$. The series $L_f(s)$ converges for some $s \in \mathbb{C}$, has a meromorphic continuation to $\{s \in \mathbb{C} : \operatorname{Re}(s) \geq -R\}$, and is holomorphic on the line $\{s \in \mathbb{C} : \operatorname{Re}(s) = -R\}$. The function $L_f^*(s) = \Gamma(s)\zeta(s+1)L_f(s)$ has only real poles $0 < \alpha := \gamma_1 > \gamma_2 > \dots$ that are all simple, except the possible pole at $s = 0$, that may be double.

(P3) For some $a < \frac{\pi}{2}$, in every strip $\sigma_1 \leq \sigma \leq \sigma_2$ in the domain of holomorphicity of $L_f(s)$, we uniformly have, for $s = \sigma + it$,

$$L_f(s) = O_{\sigma_1, \sigma_2} \left(e^{a|t|} \right), \quad |t| \rightarrow \infty.$$

We have the following asymptotic behavior of $p_f(n)$.

Theorem 2.1 ([7], Theorem 1.4). *Assume (P1) for $L \in \mathbb{N}$, (P2) for $R > 0$, and (P3). Then, for some $M, N \in \mathbb{N}$, we have*

$$p_f(n) = \frac{C}{n^b} \exp \left(A_1 n^{\frac{\alpha}{\alpha+1}} + \sum_{j=2}^M A_j n^{\alpha_j} \right) \left(1 + \sum_{j=2}^N \frac{B_j}{n^{\beta_j}} + O_{L,R} \left(n^{-\min\{\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\}} \right) \right).$$

Here, $0 \leq \alpha_M < \alpha_{M-1} < \dots < \alpha_2 < \alpha_1 = \frac{\alpha}{\alpha+1}$ are given by \mathcal{L}

$$\mathcal{L} := \frac{1}{\alpha+1} \mathcal{P}_R + \sum_{\mu \in \mathcal{P}_R} \left(\frac{\mu+1}{\alpha+1} - 1 \right) \mathbb{N}_0.$$

The exponents $0 < \beta_2 < \beta_3 < \dots$ are given by $\mathcal{M} + \mathcal{N}$, where \mathcal{M} and \mathcal{N} are defined by

$$\mathcal{M} := \left\{ \theta = \frac{\alpha}{\alpha+1} n + \sum_{\mu \in \mathcal{P}_R} \left(1 - \frac{\mu+1}{\alpha+1} \right) n_\mu : n, n_\mu \in \mathbb{N}_0, \theta \in \left[0, \frac{R+\alpha}{\alpha+1} \right) \right\},$$

$$\mathcal{N} := \left\{ \sum_{j=1}^K b_j \theta_j : b_j, K \in \mathbb{N}_0, \theta_j \in (-\mathcal{L}) \cap \left(0, \frac{R}{\alpha+1} \right) \right\}.$$

The coefficients A_j and B_j can be calculated explicitly; A_1 , C , and b are given by

$$A_1 := \left(1 + \frac{1}{\alpha} \right) (\omega_\alpha \Gamma(\alpha+1) \zeta(\alpha+1))^{\frac{1}{\alpha+1}}, \quad C := \frac{e^{L'_f(0)} (\omega_\alpha \Gamma(\alpha+1) \zeta(\alpha+1))^{\frac{\frac{1}{2}-L_f(0)}{\alpha+1}}}{\sqrt{2\pi(\alpha+1)}},$$

$$b := \frac{1 - L_f(0) + \frac{\alpha}{2}}{\alpha+1},$$

where $\omega_\nu := \text{Res}_{s=\nu} L_f(s)$. Moreover, if α is the only positive pole of L_f , then we have $M = 1$.

The situation of exactly two positive poles was worked out explicitly in Theorem 4.4 of [7].

Theorem 2.2. *Assume that $f : \mathbb{N} \rightarrow \mathbb{N}_0$ satisfies the conditions of Theorem 2.1 and that L_f has exactly two positive poles $\alpha > \beta$, such that $\frac{\lambda+1}{\lambda} \beta < \alpha \leq \frac{\lambda}{\lambda-1} \beta$ for some $\lambda \in \mathbb{N}$. Then*

$$p_f(n) = \frac{C}{n^b} \exp \left(A_1 n^{\frac{\alpha}{\alpha+1}} + A_2 n^{\frac{\beta}{\alpha+1}} + \sum_{k=3}^{\lambda+1} A_k n^{\frac{(k-1)\beta}{\alpha+1} + \frac{k-2}{\alpha+1} + 2-k} \right)$$

$$\times \left(1 + \sum_{j=2}^N \frac{B_j}{n^{\nu_j}} + O_{L,R} \left(n^{-\min\{\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\}} \right) \right), \quad (n \rightarrow \infty),$$

with

$$A_1 := (\omega_\alpha \Gamma(\alpha+1) \zeta(\alpha+1))^{\frac{1}{\alpha+1}} \left(1 + \frac{1}{\alpha} \right), \quad A_2 := \frac{\omega_\beta \Gamma(\beta) \zeta(\beta+1)}{(\omega_\alpha \Gamma(\alpha+1) \zeta(\alpha+1))^{\frac{\beta}{\alpha+1}}},$$

and for $k \geq 3$

$$\begin{aligned}
A_k := & K_k + \frac{c_1^{\frac{1}{\alpha+1}}}{\alpha} \sum_{m=1}^{\lambda} \binom{-\alpha}{m} \sum_{\substack{0 \leq j_1, \dots, j_\lambda \leq m \\ j_1 + \dots + j_\lambda = m \\ j_1 + 2j_2 + \dots + \lambda j_\lambda = k-1}} \binom{m}{j_1, j_2, \dots, j_\lambda} \frac{K_2^{j_1} \cdots K_{\lambda+1}^{j_\lambda}}{c_1^{\frac{m}{\alpha+1}}} \\
& + \frac{c_2}{\beta c_1^{\frac{\beta}{\alpha+1}}} \sum_{m=1}^{\lambda} \binom{-\beta}{m} \sum_{\substack{0 \leq j_1, \dots, j_\lambda \leq m \\ j_1 + \dots + j_\lambda = m \\ j_1 + 2j_2 + \dots + \lambda j_\lambda = k-2}} \binom{m}{j_1, j_2, \dots, j_\lambda} \frac{K_2^{j_1} \cdots K_{\lambda+1}^{j_\lambda}}{c_1^{\frac{m}{\alpha+1}}}.
\end{aligned}$$

Here, $\binom{m}{m_1, m_2, \dots, m_k} := \frac{m!}{m_1! m_2! \cdots m_k!}$ with $\sum_{j=1}^k m_j = m$ denotes the multinomial coefficient. The ν_j run through $\mathcal{M} + \mathcal{N}$, the K_j are given in Lemma 4.3 of [7], and c_1 , c_2 , and c_3 are defined by

$$c_1 := \omega_\alpha \Gamma(\alpha + 1) \zeta(\alpha + 1), \quad c_2 := \omega_\beta \Gamma(\beta + 1) \zeta(\beta + 1), \quad c_3 := L_f(0).$$

Remark. By Lemma 4.3 of [7], the first values of K_j are given by

$$\begin{aligned}
K_1 &= c_1^{\frac{1}{\alpha+1}}, \quad K_2 = \frac{c_2}{(\alpha + 1) c_1^{\frac{\beta}{\alpha+1}}}, \quad K_3 = \frac{c_2^2 (\alpha - 2\beta)}{2(\alpha + 1)^2 c_1^{\frac{2\beta+1}{\alpha+1}}}, \\
K_4 &= \frac{c_2^3 (2\alpha^2 - 9\alpha\beta - 2\alpha + 9\beta^2 + 3\beta)}{6(\alpha + 1)^3 c_1^{\frac{3\beta+2}{\alpha+1}}}, \\
K_5 &= \frac{c_2^4 (6\alpha^3 - 44\alpha^2\beta - 15\alpha^2 + 96\alpha\beta^2 + 56\alpha\beta + 6\alpha - 64\beta^3 - 48\beta^2 - 8\beta)}{24(\alpha + 1)^4 c_1^{\frac{4\beta+3}{\alpha+1}}}.
\end{aligned}$$

Remark. As the number of positive poles increases, the situation quickly becomes much more complicated. The focus of this paper is, in particular, the case of three poles.

We also require the behavior of a certain saddle point function. We adopt the notation for the coefficients of asymptotic expansions from [7] and write for a sequence $g(n)$

$$g(n) = \sum_{j=1}^N \frac{a_{g,j}}{n^{\nu_j}} + O_R(n^{-R}), \quad (\nu_1 < \nu_2 < \cdots < \nu_R < R).$$

Proposition 2.3 (Corollary 3.4 of [7]). *Let $\Phi_f := \text{Log}(G_f)$ and assume that $f: \mathbb{N} \rightarrow \mathbb{N}_0$ satisfies the conditions of Theorem 2.1. Let $\varrho_n > 0$ solve⁷*

$$-\Phi'_f(\varrho) = n.$$

Then

$$\varrho_n = \sum_{1 \leq j \leq N_e} \frac{a_{\varrho,j}}{n^{\nu_{\varrho,j}}} + O\left(n^{-\frac{R}{\alpha+1}-1}\right)$$

⁷We go back and forth between functions and sequences in our notations here. For more details see Subsection 2.2 in [7].

with $a_{\varrho,1} = a_{-\Phi_f,1}^{\frac{1}{\alpha+1}} = (\omega_\alpha \Gamma(\alpha+1) \zeta(\alpha+1))^{\frac{1}{\alpha+1}}$ and we have

$$\{\nu_{\varrho,j} : 1 \leq j \leq N_\varrho\} = \left(\frac{1}{\alpha+1} - \sum_{\mu \in \mathcal{P}_R} \left(\frac{\mu+1}{\alpha+1} - 1 \right) \mathbb{N}_0 \right) \cap \left[\frac{1}{\alpha+1}, \frac{R}{\alpha+1} + 1 \right).$$

The following is required for a more explicit investigation of the involved constants.

Lemma 2.4 (Lemma 4.1 of [7]). *Let $f : \mathbb{N} \rightarrow \mathbb{N}_0$ satisfy the conditions of Theorem 2.1. Then*

$$p_f(n) = \frac{e^{n\varrho_n} G_f(\varrho_n)}{\sqrt{2\pi}} \left(\sum_{j=1}^M \frac{d_j}{n^{\nu_j}} + O_{L,R} \left(n^{-\min\left\{\frac{L+1}{\alpha+1}, \frac{R+\alpha}{\alpha+1} + \frac{\alpha+2}{2(\alpha+1)}\right\}} \right) \right)$$

for some $M \in \mathbb{N}$, $d_1 = \frac{1}{\sqrt{\alpha+1}} (\omega_\alpha \Gamma(\alpha+1) \zeta(\alpha+1))^{\frac{1}{2(\alpha+1)}}$, and ν_j runs through

$$\frac{\alpha+2}{2(\alpha+1)} + \frac{\alpha}{\alpha+1} \mathbb{N}_0 + \left(- \sum_{\mu \in \mathcal{P}_R} \left(\frac{\mu+1}{\alpha+1} - 1 \right) \mathbb{N}_0 \right) \cap \left[0, \frac{R+\alpha}{\alpha+1} \right).$$

In particular, we have $\nu_1 = \frac{\alpha+2}{2(\alpha+1)}$.

2.2. Results from complex analysis. We require the Lagrange inversion formula.

Lemma 2.5 (Corollary 11.2 of [11]). *Let $\phi : B_r(0) \rightarrow D$, where for $r \in \mathbb{R}^+$ as usual $B_r(0) := \{z \in \mathbb{C} : |z| < r\}$, be a holomorphic function such that $\phi(0) = 0$ and $\phi'(0) \neq 0$, with $\phi(z) =: \sum_{n \geq 1} a_n z^n$. Then ϕ is locally biholomorphic and its local inverse has a power series expansion $\phi^{-1}(w) =: \sum_{k \geq 1} b_k w^k$, with*

$$b_k = \frac{1}{ka_1^k} \sum_{\substack{\ell_1, \ell_2, \ell_3, \dots \geq 0 \\ \ell_1 + 2\ell_2 + \dots = k-1}} (-1)^{\ell_1 + \ell_2 + \ell_3 + \dots} \frac{k \cdots (k-1 + \ell_1 + \ell_2 + \dots)}{\ell_1! \ell_2! \ell_3! \cdots} \left(\frac{a_2}{a_1} \right)^{\ell_1} \left(\frac{a_3}{a_1} \right)^{\ell_2} \cdots$$

To study the asymptotics of $N_\ell(n)$ we first have to investigate the analytic properties of the Dirichlet series attached to the sequence $g_{\ell-1}(n)$ from Theorem 1.1 with $f_\ell := g_{\ell-1}$. Solomon [26] (see also [19], Theorem 15.1, p. 296) proved that, for $\ell \geq 2$,

$$L_{f_\ell}(s) = \prod_{k=0}^{\ell-2} \zeta(s-k). \quad (2.1)$$

For the following investigations the following result is essential; its proof follows from standard properties of the Riemann zeta function.

Proposition 2.6. *The function L_{f_ℓ} has one pole at $s = 1$ for $\ell = 2$, two poles $s \in \{1, 2\}$ for $\ell = 3$, and three poles $s \in \{\ell-3, \ell-2, \ell-1\}$ for $\ell \geq 4$. If $\ell \geq 4$, then we have for $\nu \in \{\ell-3, \ell-2, \ell-1\}$*

$$\text{Res}_{s=\nu} L_{f_\ell}^*(s) = (\nu-1)! \zeta(\nu+1) \prod_{\substack{0 \leq k \leq \ell-2 \\ k \neq \nu-1}} \zeta(\nu-k)$$

and $L_{f_\ell}(0) = 0$. Additionally, for $\ell \geq 6$ the function $L_{f_\ell}^*$ only has simple poles in $s \in \{\ell - 3, \ell - 2, \ell - 1\}$. For $\ell \in \{4, 5\}$, $L_{f_\ell}^*$ has an additional simple pole in $s = 0$, with residue

$$\operatorname{Res}_{s=0} L_{f_4}^*(s) = \frac{\zeta'(-2)}{24}, \quad \operatorname{Res}_{s=0} L_{f_5}^*(s) = \frac{\zeta'(-2)}{2880}.$$

3. ASYMPTOTIC EXPANSIONS FOR $N_\ell(n)$

3.1. Exponent sets. For our calculations, we need the following lemma which follows by a direct calculation.

Lemma 3.1. *For $\ell \geq 3$ and $f = f_\ell$, we have, as $R \rightarrow \infty$,*

$$\mathcal{L} = \frac{\ell - 1}{\ell} - \frac{1}{\ell} \mathbb{N}_0, \quad \mathcal{M} = \frac{1}{\ell} \mathbb{N}_0, \quad \mathcal{N} = \frac{1}{\ell} \mathbb{N}_0.$$

3.2. The case $\ell = 2$. This case is the partition function and is classical. In the setting of the present paper it is

$$N_2(n) = p(n) = \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3n}} \left(1 + \sum_{j=1}^N \frac{B_j}{n^{\frac{j}{2}}} + O_N \left(n^{-\frac{N+1}{2}} \right) \right)$$

for certain B_j and was treated in [7].

3.3. The case $\ell = 3$. The following theorem gives the asymptotic behavior of $N_3(n)$.

Theorem 3.2. *We have, as $n \rightarrow \infty$,*

$$N_3(n) \sim \frac{e^{-\frac{\zeta'(-1)}{2} - \frac{\pi^2}{288\zeta(3)} \zeta(3)^{\frac{11}{72}}}}{2^{\frac{11}{24}} \cdot 3^{\frac{47}{72}} \cdot \pi^{\frac{11}{72}} \cdot n^{\frac{47}{72}}} \exp \left(\frac{(3\pi)^{\frac{2}{3}} \zeta(3)^{\frac{1}{3}}}{2} n^{\frac{2}{3}} - \frac{\pi^{\frac{4}{3}}}{4 \cdot 3^{\frac{2}{3}} \cdot \zeta(3)^{\frac{1}{3}}} n^{\frac{1}{3}} \right) \left(1 + \sum_{j=1}^{\infty} \frac{B_{3,j}}{n^{\frac{j}{3}}} \right)$$

for certain numbers $B_{3,j}$.

Proof. We have by (2.1) (see [4], p. 231)

$$L_{f_3}(s) = \zeta(s)\zeta(s-1) = \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n^s}. \quad (3.1)$$

Thus the only positive poles of $L_{f_3}(s)$ are $\alpha = 2$ and $\beta = 1$. Moreover, as $f_3(n) = \sigma_1(n) \neq 0$ for $n \in \mathbb{N}$, we have $f_3^{-1}(\{0\}) = \emptyset$, and hence we can choose L arbitrarily large. We see that (P1) is satisfied. We have $\mathcal{P} = \{1, 2\}$ and these poles are simple, so (P2) is satisfied. Note that we may choose R arbitrarily large, as $\zeta(s)\zeta(s-1)$ has a meromorphic continuation to the entire complex plane. By properties of the Riemann zeta function, $L_{f_3}(\sigma + it) \ll e^{a|t|}$ on vertical strips with finite width for arbitrary $a > 0$, so in particular (P3) is satisfied. Thus we can use Theorem 2.2.

First, we note that $\lambda = 2$ satisfies $\frac{\lambda+1}{\lambda}\beta < \alpha \leq \frac{\lambda}{\lambda-1}\beta$. Using (3.1), we compute

$$\omega_2 = \frac{\pi^2}{6}, \quad \omega_1 = -\frac{1}{2}, \quad L_{f_3}(0) = \frac{1}{24}, \quad L'_{f_3}(0) = \frac{\log(2\pi)}{24} - \frac{\zeta'(-1)}{2}.$$

Thus

$$C = \frac{e^{L'_{g_3}(0)} \zeta(3)^{\frac{11}{72}}}{\sqrt{2} \cdot 3^{\frac{47}{72}} \cdot \pi^{\frac{7}{36}}}, \quad b = \frac{47}{72}, \quad A_1 = \frac{(3\pi)^{\frac{2}{3}} \zeta(3)^{\frac{1}{3}}}{2}, \quad A_2 = -\frac{\pi^{\frac{4}{3}}}{4 \cdot 3^{\frac{2}{3}} \cdot \zeta(3)^{\frac{1}{3}}}. \quad (3.2)$$

Using that $c_2 = -\frac{\pi^2}{12}$ and $c_1 = \frac{\pi^2\zeta(3)}{3}$ we next compute

$$A_3 = -\frac{\pi^2}{288\zeta(3)}.$$

Finally, we find with Theorem 2.2 and (3.2)

$$N_3(n) \sim \frac{e^{-\frac{\zeta'(-1)}{2} - \frac{\pi^2}{288\zeta(3)}\zeta(3)\frac{11}{72}}}{2^{\frac{11}{24}} \cdot 3^{\frac{47}{72}} \cdot \pi^{\frac{11}{72}} \cdot n^{\frac{47}{72}}} \exp\left(\frac{(3\pi)^{\frac{2}{3}}\zeta(3)^{\frac{1}{3}}}{2}n^{\frac{2}{3}} - \frac{\pi^{\frac{4}{3}}}{4 \cdot 3^{\frac{2}{3}} \cdot \zeta(3)^{\frac{1}{3}}}n^{\frac{1}{3}}\right).$$

With Lemma 3.1 and Theorem 2.1 we conclude that the exponents in the polynomial terms in the expansions of $N_3(n)$ are given by $\frac{1}{3}\mathbb{N}_0$, as $(\mathcal{M} + \mathcal{N}) \cap [0, \infty) = \frac{1}{3}\mathbb{N}_0$. \square

3.4. The cases $\ell \in \{4, 5\}$. We are now ready to determine the asymptotic behavior of $N_4(n)$ and $N_5(n)$.

Theorem 3.3. *We have, as $n \rightarrow \infty$,*

$$N_4(n) \sim \frac{e^{\frac{\zeta'(-2)}{24}\pi^{\frac{1}{4}}\zeta(3)^{\frac{1}{8}}}}{2^{\frac{13}{8}} \cdot 3^{\frac{1}{4}} \cdot 5^{\frac{1}{8}} \cdot n^{\frac{5}{8}}} \exp\left(\frac{2^{\frac{7}{4}} \cdot \pi^{\frac{3}{2}} \cdot \zeta(3)^{\frac{1}{4}}}{3^{\frac{3}{2}} \cdot 5^{\frac{1}{4}}}n^{\frac{3}{4}} + A_{4,2}n^{\frac{1}{2}} + A_{4,3}n^{\frac{1}{4}} + A_{4,4}\right) \left(1 + \sum_{j=1}^{\infty} \frac{B_{4,j}}{n^{\frac{j}{4}}}\right),$$

$$N_5(n) \sim \frac{e^{\frac{\zeta'(-2)}{2880}(\pi\zeta(3)\zeta(5))^{\frac{1}{10}}}}{2^{\frac{2}{5}} \cdot 3^{\frac{1}{5}} \cdot 5^{\frac{3}{5}} \cdot n^{\frac{3}{5}}} \times \exp\left(\frac{5^{\frac{4}{5}} \cdot \pi^{\frac{6}{5}} \cdot (\zeta(3)\zeta(5))^{\frac{1}{5}}}{2^{\frac{9}{5}} \cdot 3^{\frac{2}{5}}}n^{\frac{4}{5}} + A_{5,2}n^{\frac{3}{5}} + A_{5,3}n^{\frac{2}{5}} + A_{5,4}n^{\frac{1}{5}} + A_{5,5}\right) \left(1 + \sum_{j=1}^{\infty} \frac{B_{5,j}}{n^{\frac{j}{5}}}\right),$$

with computable constants $A_{4,j}$ ($2 \leq j \leq 4$) and $A_{5,j}$ ($2 \leq j \leq 5$) and certain $B_{4,j}$ and $B_{5,j}$.

Proof. By Theorem 2.1 we have, for $\ell \in \{4, 5\}$, as $\alpha = \ell - 1$, $L_{f_\ell}(0) = 0$, and $\omega_{\ell-1} = \zeta(2)\zeta(3) \cdots \zeta(\ell - 1)$ (all by Proposition 2.6)

$$C = \frac{e^{L'_{f_\ell}(0)}(\ell - 1)!^{\frac{1}{2\ell}}\sqrt{Z_\ell}}{\sqrt{2\pi\ell}}, \quad b = \frac{\ell + 1}{2\ell}.$$

We find

$$L'_{f_4}(0) = \frac{\zeta'(-2)}{24}, \quad L'_{g_5}(0) = \frac{\zeta'(-2)}{2880}.$$

The constants $A_{4,1}$ and $A_{5,1}$ can now be computed by straightforward calculations. With Lemma 3.1 and Theorem 2.1 we conclude that the exponents in the polynomial terms in the expansions of $N_\ell(n)$ are given by $\frac{1}{\ell}\mathbb{N}_0$ (for $\ell \in \{4, 5\}$), as $(\mathcal{M} + \mathcal{N}) \cap [0, \infty) = \frac{1}{\ell}\mathbb{N}_0$. \square

3.5. Proof of Theorem 1.2. To prove Theorem 1.2, we do some preliminary considerations. By Lemma 3.2 of [7] we have that, as $z \rightarrow 0$ in a cone in the right half plane,

$$\Phi_{f_\ell}(z) = \sum_{\nu \in -\mathcal{P}_R \setminus \{0\}} \text{Res}_{s=-\nu} L_{f_\ell}^*(s) z^\nu - L_{f_\ell}(0) \text{Log}(z) + L'_{f_\ell}(0) + O_R(|z|^R) \quad (3.3)$$

and for $k \in \mathbb{N}$

$$\Phi_{f_\ell}^{(k)}(z) = \sum_{\nu \in -\mathcal{P}_R \setminus \{0\}} (\nu)_k \text{Res}_{s=-\nu} L_{f_\ell}^*(s) z^{\nu-k} + \frac{(-1)^k (k-1)! L_{f_\ell}(0)}{z^k} + O_{R,k}(|z|^{R-k}), \quad (3.4)$$

where $(s)_k := s(s-1)\cdots(s-k+1)$. By Proposition 2.6, L_{f_ℓ} has exactly three positive poles $\{\ell-3, \ell-2, \ell-1\}$. As $L_{f_\ell}(0) = 0$ by Proposition 2.6 the above becomes

$$-\Phi'_{f_\ell}(z) = \frac{C_1}{z^\ell} + \frac{C_2}{z^{\ell-1}} + \frac{C_3}{z^{\ell-2}} + O_R(|z|^{R-1}), \quad (3.5)$$

where we have, according to Proposition 2.6 and $k = 1$ in (3.4),

$$\begin{aligned} C_1 &= (\ell-1)!\zeta(\ell) \prod_{\substack{0 \leq k \leq \ell-2 \\ k \neq \ell-2}} \zeta(\ell-1-k) > 0, & C_2 &= (\ell-2)!\zeta(\ell-1) \prod_{\substack{0 \leq k \leq \ell-2 \\ k \neq \ell-3}} \zeta(\ell-2-k) \neq 0, \\ C_3 &= (\ell-3)!\zeta(\ell-2) \prod_{\substack{0 \leq k \leq \ell-2 \\ k \neq \ell-4}} \zeta(\ell-3-k) \neq 0. \end{aligned} \quad (3.6)$$

To calculate the exponent sets \mathcal{L}, \mathcal{M} , and \mathcal{N} , we apply Proposition 2.3. Let $\alpha := \ell-1$. By Proposition 2.3 the exponents of the saddle point function $\varrho_{\ell,n}$ belonging to f_ℓ lie in $\frac{1}{\ell}\mathbb{N}$. Our next goal is to prove the following version of Lemma 4.3 of [7].

Lemma 3.4. *For $\ell \geq 4$, let $f := f_\ell$. Then, as $n \rightarrow \infty$,*

$$\varrho_{\ell,n} \sim \sum_{j=1}^{\infty} \frac{K_{\ell,j}}{n^{\frac{j}{\ell}}}$$

for some constants $K_{\ell,j}$ independent of n that can be calculated explicitly using $a_k(x)$, $b_k(x)$, and e_h defined below in (3.10), (3.13), and (3.14), respectively. We have

$$K_{\ell,1} = C_1^{\frac{1}{\ell}}. \quad (3.7)$$

Proof. Note that it suffices to treat all of the following asymptotic expansions formally, as in Section 3 of [7] it was shown that the analytical considerations can be made rigorous under the assumptions of Theorem 2.1.

By Proposition 2.3, the exponents of $\varrho_{\ell,n}$ lie in $\frac{1}{\ell}\mathbb{N}$. Hence, by Proposition 3.3 of [7] there exist $K_{\ell,j} \in \mathbb{R}$, such that

$$\varrho_{\ell,n} \sim \sum_{j=1}^{\infty} \frac{K_{\ell,j}}{n^{\frac{j}{\ell}}} \quad (n \rightarrow \infty).$$

With (3.5) we obtain $a_{-\Phi'_{f_\ell},1} = C_1$. We also have $a_{\varrho_{\ell},1} = a_{-\Phi'_{f_\ell},1}^{\frac{1}{\ell}}$ by Proposition 2.3, since $\alpha+1 = \ell$. We conclude that $K_{\ell,1} = a_{\varrho_{\ell},1} = C_1^{\frac{1}{\ell}}$.

For the exact values of the coefficients $K_{\ell,j}$ we need to resolve

$$\frac{C_1}{\varrho_{\ell,n}^\ell} + \frac{C_2}{\varrho_{\ell,n}^{\ell-1}} + \frac{C_3}{\varrho_{\ell,n}^{\ell-2}} - n = 0 \quad (3.8)$$

by (3.5) and Proposition 2.3. In order to do so, we divide (3.8) by n to obtain

$$C_1 \left(\frac{n^{-\frac{1}{\ell}}}{\varrho_{\ell,n}} \right)^\ell + C_2 n^{-\frac{1}{\ell}} \left(\frac{n^{-\frac{1}{\ell}}}{\varrho_{\ell,n}} \right)^{\ell-1} + C_3 n^{-\frac{2}{\ell}} \left(\frac{n^{-\frac{1}{\ell}}}{\varrho_{\ell,n}} \right)^{\ell-2} - 1 = 0.$$

Thus we are interested in points $(x, z) = (x, z(x))$ on the curve

$$C_1 z^\ell + C_2 x z^{\ell-1} + C_3 x^2 z^{\ell-2} - 1 = 0$$

for small values of x , where $z = \frac{n^{-\frac{1}{\ell}}}{\varrho_{\ell,n}}$. Making the change of variables

$$z =: w + C_1^{-\frac{1}{\ell}} \quad (3.9)$$

we obtain

$$C_1 \sum_{k=0}^{\ell} \binom{\ell}{k} w^k C_1^{-\frac{\ell-k}{\ell}} + C_2 x \sum_{k=0}^{\ell-1} \binom{\ell-1}{k} w^k C_1^{-\frac{\ell-1-k}{\ell}} + C_3 x^2 \sum_{k=0}^{\ell-2} \binom{\ell-2}{k} w^k C_1^{-\frac{\ell-2-k}{\ell}} - 1 = 0,$$

using the Binomial Theorem. We rewrite this in the shape:

$$\sum_{k=0}^{\ell} a_k(x) w^k = 0,$$

where

$$a_k(x) := \binom{\ell}{k} C_1^{\frac{k}{\ell}} + C_2 \binom{\ell-1}{k} C_1^{\frac{k+1}{\ell}-1} x + C_3 \binom{\ell-2}{k} C_1^{\frac{k+2}{\ell}-1} x^2 - \delta_{k=0}, \quad (3.10)$$

where $\delta_S = 1$ if a statement S holds and $\delta_S = 0$ otherwise. In other words, each $a_k(x)$ is a polynomial in x of degree at most 2. As

$$a_0(x) = C_2 C_1^{\frac{1}{\ell}-1} x + C_3 C_1^{\frac{2}{\ell}-1} x^2,$$

we obtain the identity

$$\varphi_x(w) := \sum_{k=1}^{\ell} a_k(x) w^k = -a_0(x) = -C_2 C_1^{\frac{1}{\ell}-1} x - C_3 C_1^{\frac{2}{\ell}-1} x^2. \quad (3.11)$$

We now apply Lemma 2.5 to (3.11) to Taylor approximate its root close to $w = 0$ as $x \rightarrow 0$. As $C_1 \neq 0$ (we even show $C_1 > 0$ in (3.6)), we have, by definition

$$a_1(x) = \ell C_1^{\frac{1}{\ell}} + (\ell-1) C_2 C_1^{\frac{2}{\ell}-1} x + (\ell-2) C_3 C_1^{\frac{3}{\ell}-1} x^2 \neq 0. \quad (3.12)$$

for all x sufficiently small (i.e., all n sufficiently large). Thus $\varphi_x(0) = 0$ and $\varphi'_x(0) = a_1(x) \neq 0$. Thus we may apply Lemma 2.5 and obtain that the coefficients of the inverse function are

$$b_k(x) = \frac{1}{k a_1(x)^k} \sum_{\substack{\ell_1, \ell_2, \ell_3, \dots \geq 0 \\ \ell_1 + 2\ell_2 + \dots = k-1}} (-1)^{\ell_1 + \ell_2 + \ell_3 + \dots} \frac{k \cdots (k-1 + \ell_1 + \ell_2 + \dots)}{\ell_1! \ell_2! \ell_3! \cdots} \times \left(\frac{a_2(x)}{a_1(x)} \right)^{\ell_1} \left(\frac{a_3(x)}{a_1(x)} \right)^{\ell_2} \cdots. \quad (3.13)$$

As a result, for x sufficiently small, we can solve (3.11) with

$$w(x) = \sum_{k=1}^{\infty} b_k(x) \left(-C_2 C_1^{\frac{1}{\ell}-1} x - C_3 C_1^{\frac{2}{\ell}-1} x^2 \right)^k =: \sum_{h=0}^{\infty} e_h x^h \quad (3.14)$$

with $e_h \in \mathbb{R}$. We now have an expansion for $w = z - C_1^{-\frac{1}{\ell}}$ in powers of $n^{-\frac{1}{\ell}}$. Resolving (3.9) (recall $z = \frac{n^{-\frac{1}{\ell}}}{\varrho_{\ell,n}}$), i.e.,

$$\frac{n^{-\frac{1}{\ell}}}{\varrho_{\ell,n}} = C_1^{-\frac{1}{\ell}} + w \left(n^{-\frac{1}{\ell}} \right)$$

leads to the closed expression

$$\varrho_{\ell,n} = \frac{n^{-\frac{1}{\ell}}}{C_1^{-\frac{1}{\ell}} + w\left(n^{-\frac{1}{\ell}}\right)} = C_1^{\frac{1}{\ell}} n^{-\frac{1}{\ell}} \sum_{m=0}^{\infty} C_1^{\frac{m}{\ell}} w\left(n^{-\frac{1}{\ell}}\right)^m.$$

The lemma now follows as the previous equation gives the desired expansion. \square

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We identify the part of the asymptotic in

$$e^{n\varrho_{\ell,n}} G_{f_{\ell}}(\varrho_{\ell,n}) = e^{n\varrho_{\ell,n} + \Phi_{f_{\ell}}(\varrho_{\ell,n})}$$

with non-negative exponents. In other words, we ask for the term

$$e^{n\varrho_{\ell,n} + \Phi_{f_{\ell}}(\varrho_{\ell,n})} = \exp\left([n\varrho_{\ell,n} + \Phi_{f_{\ell}}(\varrho_{\ell,n})]_*\right) (1 + o(1)), \quad (3.15)$$

where the notation $[]_*$ means the part of an asymptotic expansion $\sum_{j=1}^{\infty} a_j n^{\beta_j}$ with $\beta_j \geq 0$. First we get with Lemma 3.4, as $n \rightarrow \infty$,

$$n\varrho_{\ell,n} = \sum_{j=1}^{\ell} K_{\ell,j} n^{\frac{\ell-j}{\ell}} + O\left(n^{-\frac{1}{\ell}}\right). \quad (3.16)$$

Moreover, with (3.3) and integrating (3.5) we get as $z \rightarrow 0^+$ (in a cone)

$$\Phi_{f_{\ell}}(z) = \frac{C_1}{(\ell-1)z^{\ell-1}} + \frac{C_2}{(\ell-2)z^{\ell-2}} + \frac{C_3}{(\ell-3)z^{\ell-3}} + L'_{g_{\ell}}(0) + o(1),$$

since $L_{f_{\ell}}(0) = 0$. Thus in particular, as $n \rightarrow \infty$,

$$\Phi_{f_{\ell}}(\varrho_{\ell,n}) = \frac{C_1}{(\ell-1)\varrho_{\ell,n}^{\ell-1}} + \frac{C_2}{(\ell-2)\varrho_{\ell,n}^{\ell-2}} + \frac{C_3}{(\ell-3)\varrho_{\ell,n}^{\ell-3}} + L'_{g_{\ell}}(0) + o(1). \quad (3.17)$$

We next use (3.7) and rewrite

$$\frac{1}{\varrho_{\ell,n}} \sim \frac{1}{C_1^{\frac{1}{\ell}} n^{-\frac{1}{\ell}} + \sum_{j=2}^{\infty} K_{\ell,j} n^{-\frac{j}{\ell}}} = \frac{n^{\frac{1}{\ell}}}{C_1^{\frac{1}{\ell}}} \left(1 + \frac{1}{C_1^{\frac{1}{\ell}}} \sum_{j=1}^{\infty} \frac{K_{\ell,j+1}}{n^{\frac{j}{\ell}}}\right)^{-1}.$$

We can work with the following identity regarding formal power series:

$$\frac{1}{1 + \sum_{k=1}^{\infty} b_k z^k} = \sum_{m=0}^{\infty} (-1)^m \left(\sum_{k=1}^{\infty} b_k z^k\right)^m = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

with

$$c_n = \sum_{\substack{j_1, j_2, \dots, j_n \geq 0 \\ j_1 + 2j_2 + 3j_3 + \dots = n}} (-1)^{j_1 + j_2 + \dots} \binom{j_1 + j_2 + j_3 + \dots}{j_1, j_2, j_3, \dots} b_1^{j_1} b_2^{j_2} b_3^{j_3} \dots.$$

We thus obtain

$$\frac{1}{\varrho_{\ell,n}} \sim n^{\frac{1}{\ell}} \sum_{m=0}^{\infty} \frac{D_m}{n^{\frac{m}{\ell}}},$$

where

$$D_m := C_1^{-\frac{1}{\ell}} \sum_{\substack{j_1, j_2, \dots \geq 0 \\ j_1 + 2j_2 + 3j_3 + \dots = m}} (-1)^{j_1 + j_2 + \dots} \binom{j_1 + j_2 + j_3 + \dots}{j_1, j_2, j_3, \dots} C_1^{-\frac{m}{\ell}} K_{\ell,2}^{j_1} K_{\ell,3}^{j_2} K_{\ell,4}^{j_3} \dots.$$

Using the Multinomial Theorem again, we find

$$\begin{aligned}
& \frac{C_1}{(\ell-1)\varrho_{\ell,n}^{\ell-1}} + \frac{C_2}{(\ell-2)\varrho_{\ell,n}^{\ell-2}} + \frac{C_3}{(\ell-3)\varrho_{\ell,n}^{\ell-3}} \\
&= \frac{C_1}{\ell-1} n^{\frac{\ell-1}{\ell}} \sum_{k=0}^{\infty} n^{-\frac{k}{\ell}} \sum_{\substack{m_1, m_2, \dots \geq 0 \\ m_1+m_2+\dots=\ell-1 \\ m_2+2m_3+\dots=k}} \binom{\ell-1}{m_1, m_2, \dots} D_0^{m_1} D_1^{m_2} \dots \\
&+ \frac{C_2}{\ell-2} n^{\frac{\ell-2}{\ell}} \sum_{k=0}^{\infty} n^{-\frac{k}{\ell}} \sum_{\substack{m_1, m_2, \dots \geq 0 \\ m_1+m_2+\dots=\ell-2 \\ m_2+2m_3+\dots=k}} \binom{\ell-2}{m_1, m_2, \dots} D_0^{m_1} D_1^{m_2} \dots \\
&+ \frac{C_3}{\ell-3} n^{\frac{\ell-3}{\ell}} \sum_{k=0}^{\infty} n^{-\frac{k}{\ell}} \sum_{\substack{m_1, m_2, \dots \geq 0 \\ m_1+m_2+\dots=\ell-3 \\ m_2+2m_3+\dots=k}} \binom{\ell-3}{m_1, m_2, \dots} D_0^{m_1} D_1^{m_2} \dots.
\end{aligned} \tag{3.18}$$

We next determine closed formulas for the exponent coefficients $A_{\ell,k}$. We first find with Theorem 2.1

$$A_{\ell,1} = \left(1 + \frac{1}{\ell-1}\right) (\zeta(2) \cdots \zeta(\ell-1) (\ell-1)! \zeta(\ell))^{\frac{1}{\ell}}. \tag{3.19}$$

Also note that, combing Theorem 2.1 and Lemma 2.4, we obtain

$$\begin{aligned}
& \frac{e^{n\varrho_{\ell,n}} G_{f_{\ell}}(\varrho_{\ell,n})}{\sqrt{2\pi}} \left(\sum_{j=1}^N \frac{d_j}{n^{\nu_j}} + O_{L,R} \left(n^{-\min\left\{\frac{L+1}{\alpha+1}, \frac{R+\alpha}{\alpha+1} + \frac{\alpha+2}{2(\alpha+1)}\right\}} \right) \right) \\
&= \frac{C}{n^b} \exp \left(A_1 n^{\frac{\alpha}{\alpha+1}} + \sum_{j=2}^M A_j n^{\alpha_j} \right) \left(1 + \sum_{j=2}^N \frac{B_j}{n^{\beta_j}} + O_{L,R} \left(n^{-\min\left\{\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\right\}} \right) \right),
\end{aligned}$$

for some numbers $0 \leq \alpha_M < \cdots < \alpha_2 < \frac{\alpha}{\alpha+1}$. Consequently with (3.15), comparing the exponential terms with powers in $\mathcal{L} \cap [0, \infty)$ (note that the term $L'_{f_{\ell}}(0)$ is treated separately, so we have to subtract it),

$$[n\varrho_{\ell,n} + \Phi_{f_{\ell}}(\varrho_{\ell,n}) - L'_{g_{\ell}}(0)]_* = A_1 n^{\frac{\alpha}{\alpha+1}} + \sum_{j=2}^M A_j n^{\alpha_j} =: \sum_{k=1}^{\ell} A_{\ell,k} n^{\frac{\ell-k}{\ell}}. \tag{3.20}$$

Using Lemma 3.1 employing (3.16), (3.17), and (3.18), to obtain for $k \geq 2$,

$$\begin{aligned}
A_{\ell,k} &= K_{\ell,k} + \frac{C_1}{\ell-1} \sum_{\substack{m_1, m_2, \dots \geq 0 \\ m_1+m_2+\dots=\ell-1 \\ m_2+2m_3+\dots=k-1}} \binom{\ell-1}{m_1, m_2, \dots} D_0^{m_1} D_1^{m_2} \dots \\
&+ \frac{C_2}{\ell-2} \sum_{\substack{m_1, m_2, \dots \geq 0 \\ m_1+m_2+\dots=\ell-2 \\ m_2+2m_3+\dots=k-2}} \binom{\ell-2}{m_1, m_2, \dots} D_0^{m_1} D_1^{m_2} \dots
\end{aligned}$$

$$+ \frac{C_3}{\ell-3} \sum_{\substack{m_1, m_2, \dots \geq 0 \\ m_1 + m_2 + \dots = \ell-3 \\ m_2 + 2m_3 + \dots = k-3}} \binom{\ell-3}{m_1, m_2, \dots} D_0^{m_1} D_1^{m_2} \dots \quad (3.21)$$

If $\ell \geq 6$, then we have $L'_{f_\ell}(0) = 0$, since L_{f_ℓ} has a zero in $s = 0$ of order at least 2. This simplifies the formula to the one in Theorem 1.2. Theorem 2.1, Proposition 2.6, and

$$C = \frac{(\ell-1)!^{\frac{1}{2\ell}} Z_\ell^{\frac{1}{2}}}{\sqrt{2\pi\ell}}$$

as well as (3.19) gives the value of $A_{\ell,1}$. With Lemma 3.1 and Theorem 2.1 we conclude that the exponents in the polynomial terms in the expansions of $N_\ell(n)$ are given by $\frac{1}{\ell}\mathbb{N}_0$, as $(\mathcal{M} + \mathcal{N}) \cap [0, \infty) = \frac{1}{\ell}\mathbb{N}_0$. \square

4. LOG-CONCAVITY AND THE PROOF OF THEOREM 1.3

4.1. Proof of Theorem 1.3. In this subsection, we prove our general result on log-concavity.

Proof of Theorem 1.3. We have

$$c(n)^2 - c(n+1)c(n-1) \sim C^2 \left(\frac{\exp(2 \sum_{\lambda \in \mathcal{S}} A_\lambda n^\lambda)}{n^{2\kappa}} \left(\sum_{\mu \in \mathcal{T}} \frac{\beta_\mu}{n^\mu} \right)^2 - \frac{\exp(\sum_{\lambda \in \mathcal{S}} A_\lambda ((n+1)^\lambda + (n-1)^\lambda))}{(n+1)^\kappa (n-1)^\kappa} \sum_{\mu \in \mathcal{T}} \frac{\beta_\mu}{(n+1)^\mu} \sum_{\nu \in \mathcal{T}} \frac{\beta_\nu}{(n-1)^\nu} \right). \quad (4.1)$$

We now claim that for all $\lambda \in \mathcal{S}$, we have

$$\exp\left(A_\lambda \left((n+1)^\lambda + (n-1)^\lambda - 2n^\lambda\right)\right) = 1 + \frac{\gamma_{\lambda,1}}{n^{2-\lambda}} + o\left(n^{-2+\lambda}\right) \quad (4.2)$$

for $\gamma_{\lambda,1} \in \mathbb{R}$. To see this, write $\lambda = \frac{a}{b}$ with $\gcd(a, b) = 1$ and set $x := n^{-\frac{1}{b}}$. Then

$$\begin{aligned} (n+1)^{\frac{a}{b}} + (n-1)^{\frac{a}{b}} - 2n^{\frac{a}{b}} &= \left(\frac{1}{x^b} + 1\right)^{\frac{a}{b}} + \left(\frac{1}{x^b} - 1\right)^{\frac{a}{b}} - 2x^{-a} \\ &= \frac{1}{x^a} \left(\left(1 + x^b\right)^{\frac{a}{b}} + \left(1 - x^b\right)^{\frac{a}{b}} - 2 \right) =: f_{a,b}(x). \end{aligned}$$

Now, as $y \rightarrow 0$,

$$g_\lambda(y) := (1+y)^\lambda + (1-y)^\lambda - 2 = O(y^2).$$

Thus $f_{a,b}$ has a Taylor expansion of the shape

$$f_{a,b}(x) = \sum_{j \geq 1} \alpha_{\lambda,j} x^{2bj-a}.$$

Thus

$$f_{a,b}\left(n^{-\frac{1}{b}}\right) = \sum_{j \geq 1} \frac{\alpha_{\lambda,j}}{n^{\frac{2bj-a}{b}}} = \frac{\alpha_{\lambda,1}}{n^{2-\lambda}} + o\left(n^{-2+\lambda}\right).$$

Plugging in the expansion of the exponential function gives the claim, using that $0 < \lambda < 1$.

Using (4.2), we obtain

$$\begin{aligned} \exp\left(\sum_{\lambda \in \mathcal{S}} A_\lambda \left((n+1)^\lambda + (n-1)^\lambda - 2n^\lambda\right)\right) &= \prod_{\lambda \in \mathcal{S}} \left(1 + \frac{\gamma_{\lambda,1}}{n^{2-\lambda}} + o\left(n^{-2+\lambda}\right)\right) \\ &= 1 + \frac{\gamma_{\lambda^*,1}}{n^{2-\lambda^*}} + o\left(n^{-2+\lambda^*}\right). \end{aligned}$$

Note that $\gamma_{\lambda^*,1} = A_{\lambda^*} \alpha_{\lambda^*,1}$ and that $\alpha_{\lambda,1} = \lambda(\lambda-1) < 0$.

Next we claim that (for $n > 1$)

$$\frac{1}{(n+1)^\kappa (n-1)^\kappa} = \frac{1}{(n^2-1)^\kappa} = n^{-2\kappa} \left(1 + \sum_{j \geq 1} \frac{\delta_j}{n^{2j}}\right) \quad (4.3)$$

for certain δ_j . To see this, set $x := n^{-2}$. Then we want

$$\frac{x^{-\kappa}}{\left(\frac{1}{x}-1\right)^\kappa} = 1 + \sum_{j \geq 1} \delta_j x^j.$$

The left-hand side is $\frac{1}{(1-x)^\kappa}$ and the claim follows.

Thus, by (4.1),

$$\begin{aligned} &c(n)^2 - c(n+1)c(n-1) \\ &\sim C^2 \frac{\exp\left(2 \sum_{\lambda \in \mathcal{S}} A_\lambda n^\lambda\right)}{n^{2\kappa}} \left(\left(\sum_{\mu \in \mathcal{T}} \frac{\beta_\mu}{n^\mu} \right)^2 \right. \\ &\quad \left. - \left(1 + \frac{\gamma_{\lambda^*,1}}{n^{2-\lambda^*}} + o\left(n^{-2+\lambda^*}\right)\right) (1 + O(n^{-2})) \sum_{\mu \in \mathcal{T}} \frac{\beta_\mu}{(n+1)^\mu} \sum_{\nu \in \mathcal{T}} \frac{\beta_\nu}{(n-1)^\nu} \right). \end{aligned}$$

The sign of this is dictated by

$$\sum_{\mu, \nu \in \mathcal{T}} \frac{\beta_\mu \beta_\nu}{n^{\mu+\nu}} - \left(1 + \frac{\gamma_{\lambda^*,1}}{n^{2-\lambda^*}} + o\left(n^{-2+\lambda^*}\right)\right) (1 + O(n^{-2})) \sum_{\mu \in \mathcal{T}} \frac{\beta_\mu}{(n+1)^\mu} \sum_{\nu \in \mathcal{T}} \frac{\beta_\nu}{(n-1)^\nu}.$$

As $\beta_0 = 1$, the above equals mod $(o(n^{-2+\lambda^*}))$,

$$\begin{aligned} &\sum_{\substack{\mu, \nu \in \mathcal{T} \\ 0 \leq \mu + \nu \leq 2 - \lambda^*}} \frac{\beta_\mu \beta_\nu}{n^{\mu+\nu}} - \sum_{\substack{\mu, \nu \in \mathcal{T} \\ 0 \leq \mu + \nu \leq 2 - \lambda^*}} \frac{\beta_\mu \beta_\nu}{(n+1)^\mu (n-1)^\nu} - \frac{\gamma_{\lambda^*,1}}{n^{2-\lambda^*}} \\ &= 2 \sum_{\substack{\mu \in \mathcal{T} \\ 0 \leq \mu \leq 1 - \frac{\lambda^*}{2}}} \beta_\mu^2 \left(\frac{1}{n^{2\mu}} - \frac{1}{(n+1)^\mu (n-1)^\mu} \right) \\ &\quad + \sum_{\substack{\mu \in \mathcal{T} \\ 0 \leq \mu < \nu \\ 1 \leq \mu + \nu \leq 2 - \lambda^*}} \beta_\mu \beta_\nu \left(\frac{2}{n^{\mu+\nu}} - \frac{1}{(n+1)^\mu (n-1)^\nu} - \frac{1}{(n+1)^\nu (n+1)^\mu} \right) - \frac{\gamma_{\lambda^*,1}}{n^{2-\lambda^*}}. \end{aligned} \quad (4.4)$$

Next, by (4.3) we obtain that (for $n > 1$)

$$\frac{1}{(n+1)^\mu (n-1)^\mu} = n^{-2\mu} + O\left(n^{-2\mu-2}\right).$$

Similarly, we see that, for certain ρ_r ,

$$\frac{1}{(n+1)^\mu(n-1)^\nu} + \frac{1}{(n+1)^\nu(n-1)^\mu} = 2n^{-\mu-\nu} \left(1 + \sum_{r \geq 1} \frac{\rho_r}{n^{2r}} \right).$$

Thus (4.4) becomes

$$O(n^{-2}) - \frac{\gamma_{\lambda^*,1}}{n^{2-\lambda^*}}.$$

To conclude the claim, we note that we have

$$\operatorname{sgn}(-\gamma_{\lambda^*,1}) = \operatorname{sgn}(A_{\lambda^*}) = 1. \quad \square$$

4.2. Examples. We can show the following result.

Corollary 4.1. *Let $f: \mathbb{N} \rightarrow \mathbb{N}_0$ satisfy all conditions of Theorem 2.1, and assume that we can choose L in (P1) arbitrarily large. Assume furthermore that $L_f(s)$ has a meromorphic continuation to \mathbb{C} with only rational poles. Then $p_f(n)$ is log-concave for n sufficiently large.*

Proof. Theorem 2.1 provides an expansion for $p_f(n)$ as in Theorem 1.3, since the rational poles of $L_f(s)$ guarantee that all exponents occurring in the expansion of $p_f(n)$ are again rational. Note that $\lambda^* = \frac{\alpha}{\alpha+1}$ and⁸ $A_1 > 0$ again by Theorem 2.1. This gives then the claim. \square

There are several further applications. Examples are, besides the classical partitions and plane partitions, partitions into k -gonal numbers $p_k(n)$, and the number of n -dimensional representations for the groups $\mathfrak{su}(3)$ and $\mathfrak{so}(5)$, denoted by $r_{\mathfrak{su}(3)}(n)$ and $r_{\mathfrak{so}(5)}(n)$, respectively. The asymptotic behavior of the numbers $r_{\mathfrak{su}(3)}(n)$ was first studied by Romik [25] and later refined by two of the authors in [8]. Asymptotic expressions for $r_{\mathfrak{so}(5)}(n)$ and $p_k(n)$ were given in [7]. We directly obtain the following:

Corollary 4.2. *Let $k \geq 3$. For n sufficiently large, the sequences $p_k(n)$, $r_{\mathfrak{su}(3)}(n)$, and $r_{\mathfrak{so}(5)}(n)$ are log-concave.*

Finally, we give another example in a slightly different direction. For $d \in \mathbb{N}_0$, let⁹

$$\sum_{n \geq 0} p_d(n) q^n := \prod_{n \geq 1} (1 - q^n)^{-n^d}. \quad (4.5)$$

Note that we have $p_0(n) = p(n)$ and $p_1(n) = \operatorname{pp}(n)$ is the plane partition function. Note that the corresponding L -series is given by

$$\sum_{n \geq 1} \frac{n^d}{n^s} = \zeta(s - d).$$

It is not hard to check that the conditions (P1), (P2), and (P3) are satisfied, and we have

$$\mathcal{P}_R \subseteq \{d+1\} \cup \{-1, -2, -3, \dots\}$$

for the set of poles $s \neq 0$ of $L_d^*(s) := \Gamma(s)\zeta(s+1)\zeta(s-d)$. Note that $\alpha = d+1$ and

$$b = \frac{1}{2(d+2)} - \frac{\zeta(-d)}{d+2} + \frac{1}{2}, \quad A_1 = \left(1 + \frac{1}{d+1}\right) ((d+1)! \zeta(d+2))^{\frac{1}{d+2}},$$

⁸Note the abuse of notation.

⁹see [15], where a generalization of generating function for the partition and plane partition function was studied

$$C = \frac{e^{\zeta'(-d)}((d+1)!\zeta(d+2))^{\frac{\frac{1}{2}-\zeta(-d)}{d+2}}}{\sqrt{2\pi(d+2)}}.$$

As a consequence, we have by Theorem 2.1

$$p_d(n) \sim \frac{C}{n^b} e^{A_1 n^{\frac{d+1}{d+2}}} \left(1 + \sum_{j \geq 1} \frac{E_{d,j}}{n^{\frac{j}{d+2}}} \right)$$

with certain $E_{d,j}$. Note that, by the same arguments used for the other examples above, $p_d(n)$ is log-concave for n sufficiently large. In [16] it had been proven that in the case of plane partitions, $p_1(n)$ is log-concave for almost all n and conjectured that this is already valid for $n \geq 12$. This conjecture had been proven by Ono, Pujahari, and Rolin [24].

5. PROOF OF THEOREM 1.5

The idea of the following proof is similar to that of Theorem 1.3.

Proof of Theorem 1.5. Assume that $a, b \gg 1$. Then

$$\begin{aligned} c(a)c(b) - c(a+b) &\sim \frac{C^2}{a^\kappa b^\kappa} \exp\left(\sum_{\lambda \in \mathcal{S}} A_\lambda (a^\lambda + b^\lambda)\right) - \frac{C}{(a+b)^\kappa} \exp\left(\sum_{\lambda \in \mathcal{S}} A_\lambda (a+b)^\lambda\right) \\ &= \frac{C^2}{a^\kappa b^\kappa} \exp\left(\sum_{\lambda \in \mathcal{S}} A_\lambda (a^\lambda + b^\lambda)\right) \left(1 - \frac{a^\kappa b^\kappa}{C(a+b)^\kappa} \exp\left(\sum_{\lambda \in \mathcal{S}} A_\lambda ((a+b)^\lambda - a^\lambda - b^\lambda)\right)\right). \end{aligned}$$

Without loss of generality we may assume that $a \geq b$ and write $a = xb$, $x \geq 1$. Then

$$\begin{aligned} \frac{a^\kappa b^\kappa}{C(a+b)^\kappa} \exp\left(\sum_{\lambda \in \mathcal{S}} A_\lambda ((a+b)^\lambda - a^\lambda - b^\lambda)\right) \\ = \frac{x^\kappa b^\kappa}{C(x+1)^\kappa} \exp\left(\sum_{\lambda \in \mathcal{S}} A_\lambda b^\lambda ((1+x)^\lambda - x^\lambda - 1)\right). \end{aligned}$$

Let $f_\lambda(x) := (1+x)^\lambda - x^\lambda - 1$. It is not hard to see that $f'_\lambda(x) < 0$. As $f_\lambda(0) = 0$, we have $f_\lambda(x) < 0$ for $x \geq 1$. Moreover, using l'Hospital,

$$\lim_{x \rightarrow \infty} \left((x+1)^\lambda - x^\lambda - 1 \right) = -1.$$

We conclude $-1 \leq f_\lambda(x) < 0$ for all $x \geq 1$. This gives the claim. \square

6. OPEN QUESTIONS

We leave some open questions to the interested reader.

- (1) What can be said about Turán type inequalities for the sequences studied? Note that the case $\ell = 2$ of the partition function was treated in ([14], Theorems 1 and 2) by showing that certain classes of Jensen polynomials have real roots. The more general case $\ell \geq 3$ will be much harder.

- (2) It would be interesting to make the results of this paper (in particular log-concavity) explicit. The hard part would be to make [7] explicit. For example, the case $\ell = 2$ of the partition function was treated in ([12], Theorem 1.1). Note that this turns out to be much easier than the higher cases of ℓ as in the partition case one has an exact formula, which yields inequalities for the partition function.
- (3) According to a conjecture by Chen (see Conjecture 1.2 in [12]), the sequence $|C_{2,n}| = n!p(n)$ is log-convex for all $n > 1$, which was shown by DeSalvo and Pak 2015 ([12], Theorem 4.1). The question now is whether this is also true for sufficiently large n for the $|C_{\ell,n}|$ for all $\ell \geq 3$, and to what extent this could be related to the results of this work.

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