

IDENTITIES FOR GENERALIZED APPELL FUNCTIONS AND THE BLOW-UP FORMULA

KATHRIN BRINGMANN¹, JAN MANSCHOT², AND LARRY ROLEN³

ABSTRACT. In this paper, we prove identities for a class of generalized Appell functions which are based on the A_2 root lattice. The identities are reminiscent of periodicity relations for the classical Appell function, and are proven using only analytic properties of the functions. Moreover they are a consequence of the blow-up formula for generating functions of invariants of moduli spaces of semi-stable sheaves of rank 3 on rational surfaces. Our proof confirms that in the latter context, different routes to compute the generating function (using the blow-up formula and wall-crossing) do arrive at identical q -series. The proof also gives a clear procedure for how to prove analogous identities for generalized Appell functions appearing in generating functions for sheaves with rank $r > 3$.

1. INTRODUCTION

Paul Émile Appell's study of doubly periodic functions led to the definition of the Appell function [1]

$$(1.1) \quad A(u, v) = A(u, v; \tau) := e^{\pi i u} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\frac{n(n+1)}{2}} e^{2\pi i n v}}{1 - e^{2\pi i u} q^n},$$

with $u \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$, $v \in \mathbb{C}$, $q := e^{2\pi i \tau}$, $\tau \in \mathbb{H}$. Among the intriguing properties of this function are its behavior under elliptic transformations $(u, v) \mapsto (u + n_1\tau + m_1, v + n_2\tau + m_2)$, $n_1, n_2, m_1, m_2 \in \mathbb{Z}$ and under modular transformations $(u, v, \tau) \mapsto (\frac{u}{c\tau+d}, \frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d})$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. About one century ago, Srinivasa Ramanujan closely studied related q -hypergeometric series, and his discoveries led him to coin the term “mock theta function” for such objects. The precise transformation properties of (1.1) were only clarified in 2002 by Sander Zwegers [21]. In particular, he showed that addition of a non-holomorphic integral to $A(u, v)$ gives a function which transforms as a multi-variable Jacobi form.

Over the years various generalizations of the Appell function have arisen, for example higher level Appell functions and multi-variable Appell functions $A_Q(u, \mathbf{v})$ with Q an n -dimensional positive definite quadratic form and $\mathbf{v} \in \mathbb{C}^n$ [22]. The $A_Q(u, \mathbf{v})$ differ from

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$A(u, v)$ by having a more general quadratic form Q in the numerator. Such functions occurred as characters of Lie superalgebras [11, 17] and also as certain generating functions in the context of rank 2 sheaves on rational surfaces [3, 18, 19]. More recently it was established that besides the possibly higher dimensional quadratic form, generalizations of (1.1) with multiple terms in the denominator also occur as characters of Lie superalgebras [11] and for rank > 2 sheaves [14]. These new functions can be expressed as specializations of the following general shape (cf. [11, equation (0.13)] and [14, equation (4.2)]):

$$(1.2) \quad A_{Q, \{\mathbf{m}_j\}}(\mathbf{u}, \mathbf{v}) = A_{Q, \{\mathbf{m}_j\}}(\mathbf{u}, \mathbf{v}; \tau) := \sum_{\mathbf{k} \in \Lambda} \frac{q^{\frac{1}{2}Q(\mathbf{k})} e^{2\pi i \mathbf{v} \cdot \mathbf{k}}}{\prod_{\ell=1}^{n_-} (1 - q^{\mathbf{k} \cdot \mathbf{m}_\ell} e^{2\pi i u_\ell})},$$

where Q is the quadratic form for the n_+ -dimensional positive definite lattice Λ , $\mathbf{m}_j \in \Lambda^*$ with Λ^* the lattice dual to Λ and $j = 1, \dots, n_-$, and where \cdot denotes the (Euclidean) dot product. Furthermore, the argument \mathbf{u} is a vector $\mathbf{u} = (u_1, \dots, u_{n_-})$ with $u_j \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$, and $\mathbf{v} \in \Lambda^* \otimes \mathbb{C} \simeq \mathbb{C}^{n_+}$. In the context of Lie superalgebras, $\{\mathbf{m}_j\}$ is a set of pairwise orthogonal vectors [11], but for sheaves this is typically not the case. One can expand the terms in the denominator as geometric series, which leads to an $(n_+ + n_-)$ -dimensional quadratic form of signature (n_+, n_-) . Therefore we naturally refer to (1.2) as an Appell function with signature (n_+, n_-) .

The study of Appell functions with more general signatures than the mock modular objects in Zwegers' thesis is very natural. One expects that there is a more complicated notion of "completion" for such objects than is necessary for mock objects. Besides the two contexts mentioned above, these objects have arisen in a number of places in the recent literature on q -series; for example, the reader is referred to [4] and [15] where it is explained how certain special cases of general type Appell sums can be reduced to products of mock modular objects, and what the relation of these Appell sums are to the theory of H -harmonic Maass-Jacobi forms and to the theory of Gromov-Witten potentials. Further examples of Appell-like sums with two factors in the denominator were studied recently by Ismail and Zhang (see Corollary 4.4 of [9]) in the context of Rogers-Ramanujan type identities.

However, the general structure of these objects is still unknown (although results have been obtained in unpublished work of Raum and of Zagier-Zwegers). In this paper, we take a first step towards understanding such objects, which was also the first step in the proof of the modularity results of [4]. Namely, one first considers the error to satisfying the elliptic transformation satisfied by Jacobi forms, and writes this obstruction to transforming as a Jacobi forms in terms of simpler mock-type objects. Here we establish such formulas for our special cases. A similar analysis could also be performed on many other examples of higher-index Appell sums using the methods used here.

These special cases are instances of (1.2) which occur as building blocks of generating functions of topological invariants of moduli spaces of sheaves with rank r on a ruled or rational surface S [14]. Expressing such generating functions in terms of generalized Appell functions is a major simplification compared to previously determined generating functions for $r = 3$, and also allows one to write completely explicit generating functions for $r > 3$ [14]. For the surface \mathbb{P}^2 and rank r sheaves, the quadratic form Q in (1.2) corresponds to the one of the A_{r-1} root lattice, which is defined by [5]

$$A_{r-1} := \{(k_1, \dots, k_r) \in \mathbb{Z}^r : k_1 + \dots + k_r = 0\}.$$

Moreover, the generalized Appell functions are naturally divided by classical theta functions of the A_{r-1} root lattice, such that we arrive at an A_{r-1} generalization of the Appell-Lerch sum $\mu(u, v) := \frac{A(u, v)}{\vartheta(v)}$ [21], where $\vartheta(z)$ is the Jacobi theta function defined in (2.2).¹

The identities proved here are implied by the blow-up formula. This is a well-known formula in algebraic geometry which relates the generating function of Poincaré polynomials of semi-stable rank r sheaves on a complex algebraic surface S and the generating function for the blow-up $\phi : \tilde{S} \rightarrow S$ with exceptional divisor C_e [8, 12, 20]. Here, we briefly recall the blow-up formula (see [13] for more details and references). Let $H_{r, c_1}(z, \tau; S, J) = H_{r, c_1}(z; S, J)$ be the generating function of stack invariants of semi-stable sheaves on the algebraic surface S . The generating function sums over the second Chern class of the sheaves and keeps the rank r and first Chern class c_1 fixed. The sheaves are μ -semi-stable with respect to the polarization J . Then the blow-up formula states:

$$(1.3) \quad H_{r, \tilde{c}_1}(z; \tilde{S}, \phi^* J) = \frac{b_{r, k}(z)}{\eta^r} H_{r, c_1}(z; S, J)$$

with $\tilde{c}_1 := \phi^* c_1 - kC_e$, and where $b_{r, k}(z)$ and η are defined by (2.5) and (2.1) respectively. Besides giving an intriguing relation between generating functions for S and \tilde{S} , the blow-up formula also gives interesting relations for $H_{r, \tilde{c}_1}(z; \tilde{S}, J)$, since for different c_1 and k one can arrive at the same $\tilde{c}_1 = \phi^* c_1 - kC_e$. Since determining $H_{r, \tilde{c}_1}(z; \tilde{S}, \phi^* J)$ gives rather different expressions depending on \tilde{c}_1 [13], these relations imply surprising and intriguing identities for the generalized Appell functions. For $r = 2$ and $S = \mathbb{P}^2$, these identities reduce to known periodicity relations of $A(u, v)$ [21].

The identities for $r = 3$ and $S = \mathbb{P}^2$ can be found in [14, Section 4.3]. We prove those identities using only analytic properties of the generating functions. This result is desirable from the point of view of q -series, since the intricate identities are proven without referring to an underlying meaning of the coefficients as topological invariants of moduli spaces, and further improves the understanding of these complicated objects. The result is also of interest for algebraic geometry, since it shows that the different “routes” for calculating $H_{3, c_1}(z; \mathbb{P}^2)$ do lead to identical generating functions [13]. It therefore confirms consistency of all ingredients (i.e., suitable polarization, wall-crossing and blow-up formula) going into the calculations. The proof also indicates how to prove similar identities for the generalized Appell functions for A_{r-1} lattices following from equation (1.3) for $r > 3$.

Statement of results. Let ϑ as before be defined in (2.2). We denote by $b_{r, k}$ certain theta functions summing over the root lattice A_{r-1} (see (2.5) for the definition). We further let $w := e^{2\pi iz}$, $z \in \mathbb{C}$.

Our main results are the identities given in the following theorems. Our first identity expresses the difference of two generalized Appell functions with signature $(2, 2)$ in terms of higher level Appell functions, the eta function and Jacobi theta function.

¹We often suppress the τ -dependence of a function whenever we wish to emphasize the dependence on an elliptic variable and the modular variable τ is clear from context.

Theorem 1.1. *As meromorphic functions, we have*

$$\begin{aligned} & \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-k_1-2k_2+2} q^{k_1^2+k_2^2+k_1k_2+k_1+k_2}}{(1-w^2q^{2k_1+k_2})(1-w^2q^{k_2-k_1})} - \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-k_1-2k_2+1} q^{k_1^2+k_2^2+k_1k_2+2k_1+2k_2+1}}{(1-w^2q^{2k_1+k_2+1})(1-w^2q^{k_2-k_1})} \\ & + \frac{i\eta^3}{\vartheta(2z)} \left(\sum_{k \in \mathbb{Z}} \frac{w^{-3k+2} q^{3k^2+2k}}{1-w^3q^{3k}} + \sum_{k \in \mathbb{Z}} \frac{w^{-3k+1} q^{3k^2+k}}{1-w^3q^{3k}} - \sum_{k \in \mathbb{Z}} \frac{w^{-3k+1} q^{3k^2+4k+1}}{1-w^3q^{3k+1}} - \sum_{k \in \mathbb{Z}} \frac{w^{-3k+2} q^{3k^2-k}}{1-w^3q^{3k-1}} \right) \\ & = \frac{\eta^6 \vartheta(z)}{\vartheta(2z)^2 \vartheta(3z)}. \end{aligned}$$

Remark. We note that taking into account the τ variable, one can also determine the modularity properties of the difference of the first two terms of Theorem 1.1 using the μ -function of Zwegers [22].

Our second identity is similar to the first, but differs in that the various terms are divided by cubic theta functions $b_{3,k}(z)$ and Jacobi theta functions.

Theorem 1.2. *We have*

$$\begin{aligned} & \frac{1}{b_{3,0}(z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-k_1-2k_2+3} q^{k_1^2+k_2^2+k_1k_2-\frac{1}{3}}}{(1-w^2q^{2k_1+k_2-1})(1-w^2q^{k_2-k_1})} - \frac{1}{b_{3,1}(z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-k_1-2k_2+1} q^{k_1^2+k_2^2+k_1k_2+2k_1+2k_2+1}}{(1-w^2q^{2k_1+k_2+1})(1-w^2q^{k_2-k_1})} \\ & + \frac{i\eta^3}{\vartheta(2z)b_{3,0}(z)} \left(\sum_{k \in \mathbb{Z}} \frac{w^{-3k+3} q^{3k^2-\frac{1}{3}}}{1-w^3q^{3k-1}} + \sum_{k \in \mathbb{Z}} \frac{w^{-3k} q^{3k^2+3k+\frac{2}{3}}}{1-w^3q^{3k+1}} \right) \\ & - \frac{i\eta^3}{\vartheta(2z)b_{3,1}(z)} \left(\sum_{k \in \mathbb{Z}} \frac{w^{-3k+1} q^{3k^2+4k+1}}{1-w^3q^{3k+1}} + \sum_{k \in \mathbb{Z}} \frac{w^{-3k+2} q^{3k^2-k}}{1-w^3q^{3k-1}} \right) = 0. \end{aligned}$$

Our third identity is similar to the previous first two identities.

Theorem 1.3. *We have*

$$\begin{aligned} & \frac{1}{b_{3,0}(z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-k_1-2k_2} q^{k_1^2+k_2^2+k_1k_2}}{(1-w^2q^{2k_1+k_2})(1-w^2q^{k_2-k_1})} - \frac{1}{b_{3,1}(z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-k_1-2k_2-1} q^{k_1^2+k_2^2+k_1k_2+k_1+k_2+\frac{1}{3}}}{(1-w^2q^{2k_1+k_2+1})(1-w^2q^{k_2-k_1})} \\ & + \frac{2i\eta^3}{\vartheta(2z)b_{3,0}(z)} \sum_{k \in \mathbb{Z}} \frac{w^{-3k} q^{3k^2}}{1-w^3q^{3k}} - \frac{i\eta^3}{\vartheta(2z)b_{3,1}(z)} \left(\sum_{k \in \mathbb{Z}} \frac{w^{-3k+1} q^{3k^2-2k+\frac{1}{3}}}{1-w^3q^{3k-1}} + \sum_{k \in \mathbb{Z}} \frac{w^{-3k-1} q^{3k^2+2k+\frac{1}{3}}}{1-w^3q^{3k+1}} \right) \\ & = \frac{\eta^6 \vartheta(z)}{\vartheta(2z)^2 \vartheta(3z) b_{3,0}(z)}. \end{aligned}$$

The paper is organized as follows. In Section 2, we review the definitions and a few properties of the Dedekind eta function and the theta functions which occur in the statement of the main results. We give the proofs of these results in Section 3.

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2. PRELIMINARIES

This section defines the eta and theta functions which appear in the theorems and the proofs, and recalls the elliptic transformations of the theta functions. The Dedekind eta and Jacobi theta functions are defined by

$$(2.1) \quad \eta = \eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

$$(2.2) \quad \vartheta(z) = \vartheta(z; \tau) := -iw^{-\frac{1}{2}} q^{\frac{1}{8}} (q)_{\infty} (w^{-1}q)_{\infty} (w)_{\infty} = i \sum_{r \in \frac{1}{2} + \mathbb{Z}} (-1)^{r-\frac{1}{2}} w^r q^{\frac{r^2}{2}}.$$

Here $(a)_n = (a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j)$, and the last equality is the content of the famous Jacobi triple product identity. The Dedekind eta function is well-known to be a modular form of weight $1/2$, whereas the Jacobi theta function is a Jacobi form of weight and index both $1/2$.

In particular, we need the elliptic transformations of ϑ ($n \in \mathbb{Z}$):

$$(2.3) \quad \vartheta(z + n\tau) = (-1)^n q^{-\frac{n^2}{2}} w^{-n} \vartheta(z).$$

Moreover, it is easily verified using the infinite product expression for ϑ in (2.2) that

$$(2.4) \quad \vartheta'(0) = -2\pi\eta^3.$$

We next define the theta functions $b_{r,k}$ which appeared in equation (1.3). They sum over the A_{r-1} root lattices and are defined by

$$(2.5) \quad b_{r,k}(z) = b_{r,k}(z; \tau) := \sum_{\substack{\sum_{j=1}^r a_j = 0 \\ a_j \in \frac{k}{r} + \mathbb{Z}}} q^{-\sum_{j < \ell} a_j a_{\ell}} w^{\frac{1}{2} \sum_{j < \ell} (a_j - a_{\ell})}.$$

We are mainly interested in the case $r = 3$, the cubic theta functions, in which the theta functions are explicitly given by

$$b_{3,0}(z) = \sum_{m,n \in \mathbb{Z}} q^{m^2 + n^2 + mn} w^{m+2n},$$

$$b_{3,1}(z) = \sum_{m,n \in \mathbb{Z}} q^{m^2 + n^2 + mn + m + n + \frac{1}{3}} w^{m+2n+1}.$$

These satisfy the following transformation formula

$$(2.6) \quad b_{3,k}(z + \tau) = q^{-1} w^{-2} b_{3,k}(z).$$

These ‘‘cubic’’ theta functions have been of great interest in the literature, starting with their introduction by Borwein and Borwein [2]. In particular, the general theory of theta functions of shapes such as those in $b_{3,k}$ in a general context analogous to the usual Jacobian elliptic functions has been worked out in great detail by Schultz [16].

3. PROOFS OF THEOREMS

Proof of Theorem 1.1: The first key step is to show that the difference of the left- and right-hand sides satisfies the elliptic transformation law of a Jacobi form of negative index.

By a function φ satisfying this *elliptic transformation law of index m* , we mean a function which transforms as

$$(3.1) \quad \varphi(z + \lambda\tau + \mu; \tau) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)} \varphi(z; \tau) \quad \text{for all } \lambda, \mu \in \mathbb{Z}.$$

For a meromorphic function φ satisfying this equation, the number of zeros minus the number of poles is equal to $2m$ (cf. Theorem 1.2 of [6]). Thus, if $m < 0$, as in our case, there must either be poles in z or the function is identically zero. In the second step of our proof, we compare the location of poles (all of which happen to be simple) and residues at these poles for the two sides of our identity and show that they all match. Thus, the difference is holomorphic and transforms with negative index, and is hence zero.

For the first step, we begin by denoting the right-hand and the left-hand sides of the claimed equation by r and ℓ , respectively. Then by (2.3), we find that

$$r(z + \tau) = q^8 w^{16} r(z).$$

We now aim to show that ℓ satisfies the same transformation law. Write $\ell = \ell_1 + \ell_2$ with ℓ_1 the first and ℓ_2 the second line of the equation defining ℓ .

We first consider ℓ_1 . We call the first summand (resp. negative of the second) summand in ℓ_1 , ℓ_{11} (resp. ℓ_{12}). We get

$$\ell_{11}(z + \tau) - q^8 w^{16} \ell_{11}(z) = \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-k_1 - 2k_2 + 6} q^{k_1^2 + k_2^2 + k_1 k_2 - 2k_1 - 5k_2 + 8}}{(1 - w^2 q^{2k_1 + k_2})(1 - w^2 q^{k_2 - k_1})} (1 - w^{12} q^{3k_1 + 6k_2}),$$

where for the first summand, we shift $k_2 \mapsto k_2 - 2$. We then substitute the identity

$$(3.2) \quad 1 - w^{12} q^{3k_1 + 6k_2} = (1 - w^2 q^{k_2 - k_1}) (1 + w^4 q^{k_1 + 2k_2} + w^8 q^{2k_1 + 4k_2}) \\ + (1 - w^2 q^{2k_1 + k_2}) (w^2 q^{k_2 - k_1} + w^6 q^{3k_2} + w^{10} q^{k_1 + 5k_2}).$$

The contribution from the first term, which we call G_1 , gives

$$G_1(z) = \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1 - 2k_2 + 6} q^{3k_1^2 + k_2^2 + 3k_1 k_2 - 8k_1 - 5k_2 + 8}}{1 - w^2 q^{k_2}} (1 + w^4 q^{3k_1 + 2k_2} + w^8 q^{6k_1 + 4k_2}),$$

where we let $k_2 \mapsto k_2 - 2k_1$ and then $k_1 \mapsto -k_1$. Similarly the contribution from the second term in (3.2), denoted G_2 , is

$$G_2(z) = \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1 - 2k_2 + 8} q^{3k_1^2 + k_2^2 + 3k_1 k_2 - 7k_1 - 4k_2 + 8}}{1 - w^2 q^{k_2}} (1 + w^4 q^{3k_1 + 2k_2} + w^8 q^{6k_1 + 4k_2}),$$

where we sent $k_2 \mapsto k_2 + k_1$.

Next

$$\ell_{12}(z + \tau) - q^8 w^{16} \ell_{12}(z) = \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-k_1 - 2k_2 + 5} q^{k_1^2 + k_2^2 + k_1 k_2 - k_1 - 4k_2 + 6}}{(1 - w^2 q^{2k_1 + k_2 + 1})(1 - w^2 q^{k_2 - k_1})} (1 - w^{12} q^{3k_1 + 6k_2 + 3}),$$

where in the first term, we shift $k_2 \mapsto k_2 - 2$. Similarly as before, we decompose

$$1 - w^{12} q^{3k_1 + 6k_2 + 3} = (1 - w^2 q^{k_2 - k_1}) (1 + w^4 q^{k_1 + 2k_2 + 1} + w^8 q^{2k_1 + 4k_2 + 2}) \\ + (1 - w^2 q^{2k_1 + k_2 + 1}) (w^2 q^{k_2 - k_1} + w^6 q^{3k_2 + 1} + w^{10} q^{k_1 + 5k_2 + 2}).$$

We call the contribution from the first term G_3 . After changing variables to $k_2 \mapsto k_2 - 2k_1 - 1$ and then $k_1 \mapsto -k_1 - 1$, we get

$$(3.3) \quad G_3(z) = \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1 - 2k_2 + 4} q^{3k_1^2 + k_2^2 + 3k_1 k_2 - 4k_1 - 3k_2 + 4}}{1 - w^2 q^{k_2}} (1 + w^4 q^{3k_1 + 2k_2 + 2} + w^8 q^{6k_1 + 4k_2 + 4}).$$

The contribution of the second summand G_4 is, shifting $k_2 \mapsto k_2 + k_1$,

$$(3.4) \quad G_4(z) = \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1 - 2k_2 + 7} q^{3k_1^2 + k_2^2 + 3k_1 k_2 - 5k_1 - 3k_2 + 6}}{1 - w^2 q^{k_2}} (1 + w^4 q^{3k_1 + 2k_2 + 1} + w^8 q^{6k_1 + 4k_2 + 2}).$$

Substituting the expressions for G_j , we find that

$$\begin{aligned} G_1(z) + G_2(z) - G_3(z) - G_4(z) &= \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1 - 2k_2} q^{3k_1^2 + k_2^2 + 3k_1 k_2 + 8}}{1 - w^2 q^{k_2}} \left(w^6 q^{-8k_1 - 5k_2} \right. \\ &+ w^{10} q^{-5k_1 - 3k_2} + w^{14} q^{-2k_1 - k_2} + w^8 q^{-7k_1 - 4k_2} + w^{12} q^{-4k_1 - 2k_2} + w^{16} q^{-k_1} - w^4 q^{-4k_1 - 3k_2 - 4} \\ &\left. - w^8 q^{-k_1 - k_2 - 2} - w^{12} q^{2k_1 + k_2} - w^7 q^{-5k_1 - 3k_2 - 2} - w^{11} q^{-2k_1 - k_2 - 1} - w^{15} q^{k_1 + k_2} \right). \end{aligned}$$

We label the twelve terms in the brackets by $j = 1, 2, \dots, 12$. For $j = 1, 2, 4$ and 10 , we shift $k_1 \mapsto k_1 + 1$ and for $j = 9$ we shift $k_1 \mapsto k_1 - 1$. This gives

$$\begin{aligned} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1 - 2k_2} q^{3k_1^2 + k_2^2 + 3k_1 k_2 + 8}}{1 - w^2 q^{k_2}} &\left(w^3 q^{-2k_1 - 2k_2 - 5} + w^7 q^{k_1 - 2} + w^{14} q^{-2k_1 - k_2} + w^5 q^{-k_1 - k_2 - 4} \right. \\ &+ w^{12} q^{-4k_1 - 2k_2} + w^{16} q^{-k_1} - w^4 q^{-4k_1 - 3k_2 - 4} - w^8 q^{-k_1 - k_2 - 2} - w^{15} q^{-4k_1 - 2k_2 + 1} \\ &\left. - w^4 q^{k_1 - 4} - w^{11} q^{-2k_1 - k_2 - 1} - w^{15} q^{k_1 + k_2} \right). \end{aligned}$$

We denote the terms in the brackets of the last expression by \mathcal{D}_j . Substituting now

$$\begin{aligned} \mathcal{D}_1 &= w^3 q^{-2k_1 - 2k_2 - 5} (1 - w^2 q^{k_2} + w^2 q^{k_2}), & \mathcal{D}_6 &= w^{14} q^{-k_1 - k_2} (w^2 q^{k_2} - 1 + 1), \\ \mathcal{D}_7 &= -w^4 q^{-4k_1 - 3k_2 - 4} (1 - w^2 q^{k_2} + w^2 q^{k_2}), & \mathcal{D}_{12} &= -w^{13} q^{k_1} (w^2 q^{k_2} - 1 + 1) \end{aligned}$$

and dividing out in these terms $1 - w^2 q^{k_2}$, we obtain

$$(3.5) \quad \begin{aligned} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1 - 2k_2} q^{3k_1^2 + k_2^2 + 3k_1 k_2 + 8}}{1 - w^2 q^{k_2}} &\left(w^5 q^{-2k_1 - k_2 - 5} + w^7 q^{k_1 - 2} + w^{14} q^{-2k_1 - k_2} + w^5 q^{-k_1 - k_2 - 4} \right. \\ &+ w^{12} q^{-4k_1 - 2k_2} + w^{14} q^{-k_1 - k_2} - w^6 q^{-4k_1 - 2k_2 - 4} - w^8 q^{-k_1 - k_2 - 2} \\ &\left. - w^{15} q^{-4k_1 - 2k_2 + 1} - w^4 q^{k_1 - 4} - w^{11} q^{-2k_1 - k_2 - 1} - w^{13} q^{k_1} \right) \\ &+ \sum_{k_1, k_2 \in \mathbb{Z}} w^{-3k_1 - 2k_2} q^{3k_1^2 + k_2^2 + 3k_1 k_2 + 8} \left(w^3 q^{-2k_1 - 2k_2 - 5} - w^{14} q^{-k_1 - k_2} - w^4 q^{-4k_1 - 3k_2 - 4} + w^{13} q^{k_1} \right). \end{aligned}$$

The last line is a sum of four theta functions T_j ($j = 1, \dots, 4$), which we show vanishes. To this end, substitute in T_1 and T_2 , $k_1 \mapsto -k_1$ and then $k_2 \mapsto k_2 + 3k_1$. Next replacing in T_1 , $k_1 \mapsto k_1 + 1$, and $k_2 \mapsto k_2 - 2$ shows that $T_1 + T_3 = 0$, and replacing in T_2 , $k_1 \mapsto k_1 + 1$ and

$k_2 \mapsto k_2 - 1$, shows that $T_2 + T_4 = 0$. Grouping the terms of the other lines of (3.5) gives our final expression for

$$\ell_1(z + \tau) - q^8 w^{16} \ell_1(z) = G_1(z) + G_2(z) - G_3(z) - G_4(z),$$

namely

$$(3.6) \quad \begin{aligned} & (1 - w^3 q^2 + w^9 q^4) \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1 - 2k_2} q^{3k_1^2 + k_2^2 + 3k_1 k_2 + 8}}{1 - w^2 q^{k_2}} (w^5 q^{-k_1 - k_2 - 4} - w^4 q^{k_1 - 4}) \\ & + (1 - w^6 q^4 + w^9 q^5) \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1 - 2k_2} q^{3k_1^2 + k_2^2 + 3k_1 k_2 + 8}}{1 - w^2 q^{k_2}} (w^5 q^{-2k_1 - k_2 - 5} - w^6 q^{-4k_1 - 2k_2 - 4}). \end{aligned}$$

Next we consider the second line, ℓ_2 , and set

$$\lambda_2(z) := \frac{-i\vartheta(2z)}{\eta^3} \ell_2(z).$$

Using (2.3) we thus aim to compute the combination

$$(3.7) \quad L_2(z) := \lambda_2(z + \tau) - q^6 w^{12} \lambda_2(z).$$

We do this termwise and only carry out the details for the first term. This yields the following contribution to (3.7)

$$\begin{aligned} & \sum_{k \in \mathbb{Z}} \frac{w^{-3k+2} q^{3k^2 - k + 2}}{1 - w^3 q^{3k+3}} - \sum_{k \in \mathbb{Z}} \frac{w^{-3k+14} q^{3k^2 + 2k + 6}}{1 - w^3 q^{3k}} = \sum_{k \in \mathbb{Z}} \frac{w^{-3k+5} q^{3k^2 - 7k + 6}}{1 - w^3 q^{3k}} (1 - w^9 q^{9k}), \\ & = \sum_{k \in \mathbb{Z}} w^{-3k+2} q^{3k^2 - k + 2} + \sum_{k \in \mathbb{Z}} w^{-3k+8} q^{3k^2 - 4k + 6} + \sum_{k \in \mathbb{Z}} w^{-3k+11} q^{3k^2 - k + 6}, \end{aligned}$$

where we shift $k \mapsto k - 1$ in the first summand on the first line. To obtain the second line we use

$$(1 - w^9 q^{9k}) = (1 - w^3 q^{3k}) (1 + w^3 q^{3k} + w^6 q^{6k}),$$

and map $k \mapsto k + 1$ in the first summand. Treating the other terms similarly, we obtain that

$$(3.8) \quad \begin{aligned} L_2(z) &= (1 - w^3 q^2 + w^9 q^4) \sum_{k \in \mathbb{Z}} \left(w^{-3k+2} q^{3k^2 - k + 2} - w^{-3k+4} q^{3k^2 - 5k + 4} \right) \\ &+ (1 - w^6 q^4 + w^9 q^5) \sum_{k \in \mathbb{Z}} \left(w^{-3k+1} q^{3k^2 - 2k + 1} - w^{-3k+2} q^{3k^2 - 4k + 2} \right). \end{aligned}$$

Summarizing, we have to show the following identity:

$$\ell_1(z + \tau) - q^8 w^{16} \ell_1(z) + q^2 w^4 \frac{i\eta^3}{\vartheta(2z)} L_2(z) = 0.$$

Since both $\ell_1(z + \tau) - q^8 w^{16} \ell_1(z)$ (see (3.6)) and L_2 (see (3.8)) contain a term with $(1 - w^3 q^2 + w^9 q^4)$ and $(1 - w^6 q^4 + w^9 q^5)$, the identity in particular holds if the following two

identities hold:

$$(3.9) \quad \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1-2k_2+4} q^{3k_1^2+k_2^2+3k_1k_2+4}}{1-w^2q^{k_2}} (wq^{-k_1-k_2} - q^{k_1}) \\ = \frac{i\eta^3}{\vartheta(2z)} \sum_{k \in \mathbb{Z}} \left(w^{-3k+8} q^{3k^2-5k+6} - w^{-3k+6} q^{3k^2-k+4} \right),$$

$$(3.10) \quad \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1-2k_2+5} q^{3k_1^2+k_2^2+3k_1k_2+3}}{1-w^2q^{k_2}} (q^{-2k_1-k_2} - wq^{-4k_1-2k_2+1}) \\ = \frac{i\eta^3}{\vartheta(2z)} \sum_{k \in \mathbb{Z}} \left(w^{-3k+6} q^{3k^2-4k+4} - w^{-3k+5} q^{3k^2-2k+3} \right).$$

Denote the left-hand side of (3.9) by \mathcal{L}_1 and the right-hand side by \mathcal{R}_1 , and similarly define \mathcal{L}_2 and \mathcal{R}_2 for (3.10). We again use elliptic transformations and residue computations to prove these identities. We obtain for the shift $z \mapsto z + \tau$ of the \mathcal{R}_j 's, using (2.3) and the shift $k \mapsto k + 1$ in the first summand,

$$(3.11) \quad \mathcal{R}_1(z + \tau) + w^4 q^8 \mathcal{R}_2(z) = 0, \quad \mathcal{R}_2(z + \tau) + wq^4 \mathcal{R}_1(z) = 0.$$

Thus, we want to prove the following identities, which show the desired elliptic transformation properties of the left and right hand sides:

$$(3.12) \quad \mathcal{L}_1(z + \tau) + w^4 q^8 \mathcal{L}_2(z) = 0, \quad \mathcal{L}_2(z + \tau) + wq^4 \mathcal{L}_1(z) = 0.$$

In $\mathcal{L}_1(z + \tau)$ we change $k_1 \mapsto k_1 + 1$ and $k_2 \mapsto k_2 - 2$. After adding $w^4 q^8 \mathcal{L}_2(z)$, we can divide out by $1 - w^2 q^{k_2}$, to obtain

$$\mathcal{L}_1(z + \tau) + w^4 q^8 \mathcal{L}_2(z) \\ = \sum_{k_1, k_2 \in \mathbb{Z}} w^{-3k_1-2k_2+1} q^{3k_1^2+k_2^2+3k_1k_2+5} \left(w^5 q^{-4k_1-4k_2+7} (1 + w^2 q^{k_2}) - w^4 q^{-2k_1-3k_2+6} (1 + w^2 q^{k_2}) \right).$$

This is a sum of four theta functions S_j . A direct calculation shows that $S_1 = -S_3$ and $S_2 = -S_4$, which proves the first identity of (3.12). The second identity in (3.12) is proven similarly.

One may verify from the quasi-periodicity (3.11) of \mathcal{R}_j , that $w^{-11} \mathcal{R}_1(2z)$ and $w^{-8} \mathcal{R}_2(2z)$ satisfy the elliptic transformation law (3.1) with index $m = -5$. Since we have thus negative index Jacobi forms, we finish the proofs of (3.9) and (3.10) by showing that both sides have the same poles and the same residues.

We start with \mathcal{R}_1 which has at most simple poles for $z \in \frac{1}{2}\mathbb{Z}\tau + \frac{1}{2}\mathbb{Z}$. Given the elliptic transformations (3.12) we only need to match residues modulo $\mathbb{Z}\tau + \mathbb{Z}$. Firstly consider the pole at $z = 0$. We compute the residue to be

$$-\frac{i}{4\pi} \sum_{k \in \mathbb{Z}} \left(q^{3k^2-5k+6} - q^{3k^2-k+4} \right) = 0,$$

where we shift in the first summand $k \mapsto -k + 1$ to find that the residue vanishes and also use (2.4). At $z = 1/2$, using again (2.4) and the equation $\vartheta(z+1) = -\vartheta(z)$ (as follows easily

from the first identity in (2.2), we obtain the residue

$$\frac{i}{4\pi} \sum_{k \in \mathbb{Z}} \left((-1)^k q^{3k^2-5k+6} - (-1)^k q^{3k^2-k+4} \right) = -\frac{i}{2\pi} \sum_{k \in \mathbb{Z}} (-1)^k q^{3k^2-k+4},$$

where we let $k \mapsto -k + 1$ in the first summand. At $z = \tau/2$, the residue is

$$\frac{iq^{\frac{1}{2}}}{4\pi} \sum_{k \in \mathbb{Z}} \left(q^{-\frac{3k}{2}+4} q^{3k^2-5k+6} - q^{-\frac{3k}{2}+3} q^{3k^2-k+4} \right) = \frac{i}{4\pi} \sum_{k \in \mathbb{Z}} \left(q^{3k^2+\frac{k}{2}+7} - q^{3k^2-\frac{5k}{2}+\frac{15}{2}} \right),$$

shifting again $k \mapsto -k + 1$ in the first summand. Moreover, we use (2.3) to compute that

$$\vartheta'(\tau) = -q^{-\frac{1}{2}} \frac{\partial}{\partial z} [w^{-1}\vartheta(z)]_{z=0} = -q^{-\frac{1}{2}} \vartheta'(0)$$

since ϑ is odd as a function of z . Finally at $z = \tau/2 + 1/2$, our residue becomes

$$\begin{aligned} -\frac{iq^{\frac{1}{2}}}{4\pi} \sum_{k \in \mathbb{Z}} \left((-1)^k q^{-\frac{3k}{2}+4} q^{3k^2-5k+6} - (-1)^k q^{-\frac{3k}{2}+3} q^{3k^2-k+4} \right) \\ = \frac{i}{4\pi} \sum_{k \in \mathbb{Z}} \left((-1)^k q^{3k^2+\frac{k}{2}+7} + (-1)^k q^{3k^2-\frac{5k}{2}+\frac{15}{2}} \right), \end{aligned}$$

making again the shift $k \mapsto -k + 1$ in the first summand.

We next turn to \mathcal{L}_1 . We see that we again have at most simple poles at $\frac{1}{2}\mathbb{Z} + \tau\frac{1}{2}\mathbb{Z}$. For $z = 0$ a pole can only come from the term $k_2 = 0$ yielding the contribution

$$\frac{i}{4\pi} \sum_{k_1 \in \mathbb{Z}} q^{3k_1^2+4} (q^{-k_1} - q^{k_1}) = 0$$

by changing $k_1 \mapsto -k_1$ in the second term. At $z = 1/2$ again the pole comes from $k_2 = 0$ yielding

$$\frac{i}{4\pi} \sum_{k_1 \in \mathbb{Z}} (-1)^{k_1} q^{3k_1^2+4} (-q^{-k_1} - q^{k_1}) = -\frac{i}{2\pi} \sum_{k_1 \in \mathbb{Z}} (-1)^{k_1} q^{3k_1^2-k_1+4}$$

which matches the residue of \mathcal{R}_1 . At $z = \tau/2$, the pole comes from $k_2 = -1$ yielding

$$\frac{i}{4\pi} \sum_{k_1 \in \mathbb{Z}} q^{3k_1^2-\frac{9k_1}{2}+8} \left(q^{-k_1+\frac{3}{2}} - q^{k_1} \right) = \frac{i}{4\pi} \left(\sum_{k_1 \in \mathbb{Z}} q^{3k_1^2+\frac{k_1}{2}+7} - \sum_{k_1 \in \mathbb{Z}} q^{3k_1^2-\frac{5k_1}{2}+\frac{15}{2}} \right),$$

where for the first summand we change $k_1 \mapsto k_1+1$, and in the second summand $k_1 \mapsto -k_1+1$.

Finally at $z = \tau/2 + 1/2$ the pole comes from $k_2 = -1$ yielding

$$\begin{aligned} -\frac{i}{4\pi} \sum_{k_1 \in \mathbb{Z}} (-1)^{k_1} q^{3k_1^2-\frac{9k_1}{2}+8} \left(q^{-k_1+\frac{3}{2}} + q^{k_1} \right) \\ = \frac{i}{4\pi} \left(\sum_{k_1 \in \mathbb{Z}} (-1)^{k_1} q^{3k_1^2+\frac{k_1}{2}+7} + \sum_{k_1 \in \mathbb{Z}} (-1)^{k_1} q^{3k_1^2-\frac{5k_1}{2}+\frac{15}{2}} \right), \end{aligned}$$

where for the first summand we shift again $k_1 \mapsto k_1 + 1$, and in the second summand $k_1 \mapsto -k_1 + 1$. So we see that all residues of \mathcal{L}_1 and \mathcal{R}_1 match. The calculation of the residues of \mathcal{L}_2 and \mathcal{R}_2 is similar to those \mathcal{L}_1 and \mathcal{R}_1 , so we omit the calculation.

At this point, we have proven that the left and right hand sides of Theorem 1.1 transform identically under the elliptic transformation $z \mapsto z + \tau$. It is straightforward to check that they also transform the same under $z \mapsto z + 1$. To finish the proof, one needs again to verify that the poles and residues of both sides match. To this end, we multiply both sides by $\frac{\vartheta(2z)^2}{\eta^6}$. We see that both sides still satisfy the elliptic transformation law of negative index Jacobi forms with simple poles at $\frac{m}{3}\tau + \frac{n}{3}$, with $(m, n) = (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1)$ and $(2, 2)$. Note that at $(0, 0)$ we have a removable singularity. We omit the details of the calculations and only list the residues. For $(m, n) = (0, 1)$, the residue equals $\frac{\vartheta(\frac{1}{3})}{6\pi\eta^3}$; for $(m, n) = (0, 2)$, $-\frac{\vartheta(\frac{1}{3})}{6\pi\eta^3}$; for $(m, n) = (1, 0)$, $\frac{q^{\frac{1}{2}}\vartheta(\frac{\tau}{3})}{6\pi\eta^3}$; for $(m, n) = (1, 1)$, $-\frac{q^{\frac{1}{2}}\vartheta(\frac{\tau+1}{3})}{6\pi\eta^3}$; for $(m, n) = (1, 2)$, $\frac{q^{\frac{1}{2}}\vartheta(\frac{\tau+2}{3})}{6\pi\eta^3}$; for $(m, n) = (2, 0)$, $-\frac{q^2\vartheta(\frac{2\tau}{3})}{6\pi\eta^3}$; for $(m, n) = (2, 1)$, $\frac{q^2\vartheta(\frac{2\tau+1}{3})}{6\pi\eta^3}$; and for $(m, n) = (2, 2)$, $-\frac{q^2\vartheta(\frac{2\tau+2}{3})}{6\pi\eta^3}$. \square

Proof of Theorem 1.2: We follow the same strategy as in the proof for Theorem 1.1. Since many steps are almost identical to those in the proof of Theorem 1.1, we only give the main intermediate results. We denote the first line of Theorem 1.2 by $\ell_1^*(z)$. First we consider the periodicity of the generalized Appell functions in $\ell_1^*(z)$ under $z \mapsto z + \tau$ without the $b_{3,k}$. We define ℓ_{11}^* as $b_{3,0}$ times the first summand in ℓ_1^* . Sending $w \mapsto wq$ in ℓ_{11}^* and adding $-q^8w^{16}$ times the original function, we have

$$(3.13) \quad \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-k_1-2k_2+7} q^{k_1^2+k_2^2+k_1k_2-3k_1-6k_2+\frac{32}{3}}}{(1-w^2q^{2k_1+k_2-1})(1-w^2q^{k_2-k_1})} (1-w^{12}q^{3k_1+6k_2-3}).$$

Similar to the proof of Theorem 1.1, we substitute the equality

$$(3.14) \quad \begin{aligned} 1-w^{12}q^{3k_1+6k_2-3} &= (1-w^2q^{k_2-k_1}) (1+w^4q^{k_1+2k_2-1} + w^8q^{2k_1+4k_2-2}) \\ &\quad + (1-w^2q^{2k_1+k_2-1}) (w^2q^{k_2-k_1} + w^6q^{3k_2-1} + w^{10}q^{k_1+5k_2-2}), \end{aligned}$$

and write (3.13) as $G_1^* + G_2^*$, where G_1^* and G_2^* come from the two terms in (3.14). They are given by

$$\begin{aligned} G_1^*(z) &= \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1-2k_2+5} q^{3k_1^2+k_2^2+3k_1k_2-6k_1-4k_2+\frac{17}{3}}}{1-w^2q^{k_2}} (1+w^4q^{3k_1+2k_2+1} + w^8q^{6k_1+4k_2+2}), \\ G_2^*(z) &= \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1-2k_2+6} q^{3k_1^2+k_2^2+3k_1k_2-3k_1-2k_2+\frac{14}{3}}}{1-w^2q^{k_2}} (1+w^4q^{3k_1+2k_2+2} + w^8q^{6k_1+4k_2+4}), \end{aligned}$$

where for G_1^* we shifted $k_2 \mapsto k_2 - 2k_1 + 1$ followed by $k_1 \mapsto -k_1$, and for G_2^* we shifted $k_2 \mapsto k_2 + k_1$ followed by $k_1 \mapsto k_1 + 1$. After a few more elementary manipulations, we see

that we can write $G_1^* + G_2^*$ as

$$\begin{aligned}
(3.15) \quad & (1 + w^6 q^4 + w^9 q^5) \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1 - 2k_2 + 4} q^{3k_1^2 + k_2^2 + 3k_1 k_2 + \frac{8}{3}}}{1 - w^2 q^{k_2}} \\
& + (1 + w^3 q^2 + w^9 q^4) \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1 - 2k_2 + 8} q^{3k_1^2 + k_2^2 + 3k_1 k_2 - 3k_1 - k_2 + \frac{14}{3}}}{1 - w^2 q^{k_2}} \\
& + w q^{\frac{5}{3}} (1 + w^3 q^2 + w^6 q^4) b_{3,0}(z).
\end{aligned}$$

We define ℓ_{12}^* as $-b_{3,1}$ times the second summand in ℓ_1^* . Comparing with Theorem 1.1, we see that ℓ_{12}^* is identical to ℓ_{12} . Therefore, $\ell_{12}^*(z + \tau) - q^8 w^{16} \ell_{12}^*(z)$ equals $G_3 + G_4$ with G_3 and G_4 given respectively in (3.3) and (3.4). With a few manipulations, we bring $G_3 + G_4$ in the following form:

$$\begin{aligned}
(3.16) \quad & (1 + w^9 q^4) \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1 - 2k_2 + 9} q^{3k_1^2 + k_2^2 + 3k_1 k_2 - 5k_1 - 2k_2 + 6}}{1 - w^2 q^{k_2}} \\
& + (1 + w^9 q^5) \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1 - 2k_2 + 6} q^{3k_1^2 + k_2^2 + 3k_1 k_2 - 4k_1 - 2k_2 + 4}}{1 - w^2 q^{k_2}} \\
& + w^6 q^4 \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1 - 2k_2 + 5} q^{3k_1^2 + k_2^2 + 3k_1 k_2 - 2k_1 - k_2 + 3}}{1 - w^2 q^{k_2}} \\
& + w^3 q^2 \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1 - 2k_2 + 7} q^{3k_1^2 + k_2^2 + 3k_1 k_2 - k_1 + 4}}{1 - w^2 q^{k_2}} + w q^{\frac{5}{3}} (1 + w^3 q^2 + w^6 q^4) b_{3,1}(z),
\end{aligned}$$

where we pull out the factors $w^6 q^4$ and $w^3 q^2$ for later convenience.

We continue with the second line of Theorem 1.2, which we denote by ℓ_2^* . Defining

$$\lambda_2^*(z) := \frac{-i\vartheta(2z)b_{3,0}(z)}{\eta^3} \ell_2^*(z),$$

we determine for the shift $z \mapsto z + \tau$ that

$$\begin{aligned}
(3.17) \quad & \lambda_2^*(z + \tau) - w^{12} q^6 \lambda_2^*(z) \\
& = (1 + w^3 q^2 + w^9 q^4) \sum_{k \in \mathbb{Z}} w^{-3k+3} q^{3k^2 - 3k + \frac{8}{3}} + (1 + w^6 q^4 + w^9 q^5) \sum_{k \in \mathbb{Z}} w^{-3k} q^{3k^2 + \frac{2}{3}}.
\end{aligned}$$

Similarly, we denote the third line by ℓ_3^* and define $\lambda_3^*(z) := \frac{i\vartheta(2z)b_{3,1}(z)}{\eta^3} \ell_3^*(z)$. We obtain

$$\begin{aligned}
(3.18) \quad & \lambda_3^*(z + \tau) - w^{12} q^6 \lambda_3^*(z) = (1 + w^9 q^4) \sum_{k \in \mathbb{Z}} w^{-3k+1} q^{3k^2 + k + 2} + w^6 q^4 \sum_{k \in \mathbb{Z}} w^{-3k+1} q^{3k^2 - 2k + 1} \\
& + (1 + w^9 q^5) \sum_{k \in \mathbb{Z}} w^{-3k+2} q^{3k^2 - 4k + 2} + w^3 q^2 \sum_{k \in \mathbb{Z}} w^{-3k+2} q^{3k^2 - k + 2}.
\end{aligned}$$

Combining these results, we can again find identities which imply that the required periodicity holds as in Theorem 1.1. To determine the identities, first note that the last terms of (3.15) and (3.16) cancel each other in the computation of $\ell_1^*(z + \tau) - q^9 w^{18} \ell_1^*(z)$ (note

that we use (2.6)). The first identity then follows by combining the sums $\sum_{k_1, k_2 \in \mathbb{Z}}$ in (3.15), (3.16), (3.17), and (3.18) which are multiplied $(1 + w^9 q^4)$ on the left:

$$(3.19) \quad \begin{aligned} & \frac{1}{b_{3,0}(z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1-2k_2+8} q^{3k_1^2+k_2^2+3k_1k_2-3k_1-k_2+\frac{14}{3}}}{1-w^2q^{k_2}} \\ & - \frac{1}{b_{3,1}(z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1-2k_2+9} q^{3k_1^2+k_2^2+3k_1k_2-5k_1-2k_2+6}}{1-w^2q^{k_2}} \\ & + w^4 q^2 \frac{i\eta^3}{\vartheta(2z)} \left(\frac{1}{b_{3,0}(z)} \sum_{k \in \mathbb{Z}} w^{-3k+3} q^{3k^2-3k+\frac{8}{3}} - \frac{1}{b_{3,1}(z)} \sum_{k \in \mathbb{Z}} w^{-3k+1} q^{3k^2+k+2} \right) = 0. \end{aligned}$$

The second identity follows by combining the sums $\sum_{k_1, k_2 \in \mathbb{Z}}$ multiplied by $(1 + w^9 q^5)$

$$\begin{aligned} & \frac{1}{b_{3,0}(z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1-2k_2+4} q^{3k_1^2+k_2^2+3k_1k_2+\frac{8}{3}}}{1-w^2q^{k_2}} \\ & - \frac{1}{b_{3,1}(z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1-2k_2+6} q^{3k_1^2+k_2^2+3k_1k_2-4k_1-2k_2+4}}{1-w^2q^{k_2}} \\ & + w^4 q^2 \frac{i\eta^3}{\vartheta(2z)} \left(\frac{1}{b_{3,0}(z)} \sum_{k \in \mathbb{Z}} w^{-3k} q^{3k^2+\frac{2}{3}} - \frac{1}{b_{3,1}(z)} \sum_{k \in \mathbb{Z}} w^{-3k+2} q^{3k^2-4k+2} \right) = 0. \end{aligned}$$

The third identity follows by combining the sums $\sum_{k_1, k_2 \in \mathbb{Z}}$ multiplied by $w^6 q^4$

$$\begin{aligned} & \frac{1}{b_{3,0}(z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1-2k_2+4} q^{3k_1^2+k_2^2+3k_1k_2+\frac{8}{3}}}{1-w^2q^{k_2}} \\ & - \frac{1}{b_{3,1}(z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1-2k_2+5} q^{3k_1^2+k_2^2+3k_1k_2-2k_1-k_2+3}}{1-w^2q^{k_2}} \\ & + w^4 q^2 \frac{i\eta^3}{\vartheta(2z)} \left(\frac{1}{b_{3,0}(z)} \sum_{k \in \mathbb{Z}} w^{-3k} q^{3k^2+\frac{2}{3}} - \frac{1}{b_{3,1}(z)} \sum_{k \in \mathbb{Z}} w^{-3k+1} q^{3k^2-2k+1} \right) = 0. \end{aligned}$$

The fourth identity follows by combining the sums $\sum_{k_1, k_2 \in \mathbb{Z}}$ multiplied by $w^3 q^2$

$$\begin{aligned} & \frac{1}{b_{3,0}(z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1-2k_2+8} q^{3k_1^2+k_2^2+3k_1k_2-3k_1-k_2+\frac{14}{3}}}{1-w^2q^{k_2}} \\ & - \frac{1}{b_{3,1}(z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-3k_1-2k_2+7} q^{3k_1^2+k_2^2+3k_1k_2-k_1+4}}{1-w^2q^{k_2}} \\ & + w^4 q^2 \frac{i\eta^3}{\vartheta(2z)} \left(\frac{1}{b_{3,0}(z)} \sum_{k \in \mathbb{Z}} w^{-3k+3} q^{3k^2-3k+\frac{8}{3}} - \frac{1}{b_{3,1}(z)} \sum_{k \in \mathbb{Z}} w^{-3k+2} q^{3k^2-k+2} \right) = 0. \end{aligned}$$

We prove the first identity (3.19) and leave the proofs of the other identities as an exercise for the reader. First we argue as in the proof of Theorem 1.1 that (a power of w times) the function satisfies the elliptic transformation law of a negative index Jacobi form. To this

end, consider the change of variables $z \mapsto z + 2\tau$. For the third term of the left-hand side of (3.19) we get after letting $z \mapsto z + 2\tau$ and using (2.3) and (2.6) that

$$\frac{w^4 q^{10}}{b_{3,0}(z + 2\tau)} \frac{i\eta^3}{\vartheta(2z + 4\tau)} \sum_{k \in \mathbb{Z}} w^{-3k+3} q^{3k^2-9k+\frac{26}{3}} \stackrel{k \mapsto k+1}{=} \frac{w^{16} q^{22}}{b_{3,0}(z)} \frac{i\eta^3}{\vartheta(2z)} \sum_{k \in \mathbb{Z}} w^{-3k} q^{3k^2-3k+\frac{5}{3}},$$

which is $w^9 q^{20}$ times the original third term. One can show similarly that the first term plus the second term, and the fourth term, are multiplied by $w^9 q^{20}$ under $z \mapsto z + 2\tau$. It is also easy to check that the two sides have the same behavior under $z \mapsto z + 1$. Therefore, the left-hand side satisfies the elliptic transformation law of a Jacobi form of negative index. If we multiply (3.19) by $b_{3,0}b_{3,1}$, we still have a function which satisfies the elliptic transformation law of a Jacobi form of negative index, since $b_{3,k}(z + 2\tau) = w^{-4} q^{-4} b_{3,k}(z)$ (by (2.6)). Under the transformation $z \mapsto z + \tau$, (3.19) becomes the third identity multiplied by q^5 . Thus, we can work as before with a vector-valued identity, and consider the periodicity $z \mapsto z + \tau$.

Since the index is negative, what remains is to show that the residues within the parallelogram spanned by 1 and τ of the left-hand side of (3.19) vanish. This is a tedious, but straightforward proof, which we omit. The proof of the Identities 2, 3, 4, and, consequently, Theorem 1.2 follow similarly. \square

Proof of Theorem 1.3: The proof of Theorem 1.3 is almost identical to the proof of Theorem 1.2, and so we omit the details. \square

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¹ MATHEMATICAL INSTITUTE, UNIVERSITY OF COLOGNE, WEYERTAL 86-90, 50931 COLOGNE, GERMANY

E-mail address: kbringma@math.uni-koeln.de

² SCHOOL OF MATHEMATICS, TRINITY COLLEGE DUBLIN, DUBLIN 2, IRELAND

E-mail address: manschot@maths.tcd.ie

³ DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802

E-mail address: larryrolen@psu.edu