ASYMPTOTICS FOR RANK AND CRANK MOMENTS

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Abstract. Moments of the partition rank and crank statistics have been studied for their connections to combinatorial objects such as Durfee symbols, as well as for their connections to harmonic Maass forms. This paper proves a conjecture due to two of the authors that refined a conjecture of Garvan. Garvan's original conjecture states that the moments of the crank function are always larger than the moments of the rank function, even though the moments have the same main asymptotic term. The refined version provides precise asymptotic estimates for both the moments and their differences. Our proof uses the Hardy-Ramanujan circle method, multiple sums of Bernoulli polynomials, and the theory of quasimock theta functions.

1. Introduction and statement of results

The theory of partitions has long motivated the study of hypergeometric series and automorphic forms. A foundational example for the interplay between these fields is Euler's partition function $p(n)$, which has the generating function

$$P(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{1.1}$$

If $q := e^{2\pi i\tau}$ is the standard uniformizer at infinity, then this coincides with $q^{\frac{1}{24}}/\eta(\tau)$ where $\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's weight 1/2 modular form.

On the other hand, many partition generating functions do not naturally appear in the theory of modular forms but rather in more general automorphic contexts, such as the theory of harmonic Maass forms. Ramanujan's famous *mock theta functions* are prime examples of this phenomenon [27]. As a result, the study of generating functions from partition theory has inspired a number of important results about mock theta functions and harmonic Maass forms [9, 12, 14, 15]. Harmonic Maass forms are real analytic generalizations of modular forms in that they satisfy the same linear fractional transformation laws and (weak) growth conditions at cusps, but they are not holomorphic functions of the complex upper half plane, and instead are only required to be annihilated by the weight k hyperbolic Laplacian.

Many of these modern results originated in Ramanujan's original results on the arithmetic of the partition function [25, 26]. Most famously, the three "Ramanujan congruences" state that for all n,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned} \tag{1.2}$$

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In an effort to provide a combinatorial explanation of Ramanujan's congruences, Dyson introduced [18] the *rank* of a partition, which is defined as

$$\operatorname{rank}(\lambda) := \text{largest part of } \lambda - \text{number of parts of } \lambda.$$

He conjectured that the partitions of $5n+4$ (resp. $7n+5$) form 5 (resp. 7) groups of equal size when sorted by their rank modulo 5 (resp. 7). Building on Dyson's observations, Atkin and Swinnerton-Dyer later proved Dyson's rank conjectures [8].

Dyson further conjectured the existence of an analogous statistic, the *crank*, that would explain all three congruences simultaneously. Garvan finally found the crank while studying q-series of the sort seen in Ramanujan's "Lost Notebook" [20], and together with Andrews presented the following definition [5]. Let $o(\lambda)$ denote the number of ones in a partition, and define $\mu(\lambda)$ as the number of parts strictly larger than $o(\lambda)$. Then

$$\operatorname{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if } o(\lambda) = 0, \\ \mu(\lambda) - o(\lambda) & \text{if } o(\lambda) > 0. \end{cases} \tag{1.3}$$

Works of the first two authors and Ken Ono show that both the rank and crank also play key roles in understanding the infinitely many other congruences for $p(n)$ [10, 14, 22].

It is more useful here to work with generating functions than combinatorial definitions. If $\mathcal{M}(m,n)$ and $\mathcal{N}(m,n)$ are the number of partitions of n with crank and rank m, respectively, then, aside from the anomalous case of $\mathcal{M}(m,n)$ when $n=1$, the two-parameter generating functions may be written as [5, 8]

$$C(x;q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} \mathcal{M}(m,n)x^m q^n = \prod_{n\geq 1} \frac{1-q^n}{(1-xq^n)(1-x^{-1}q^n)} = \frac{1-x}{(q)_\infty}\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n(n+1)/2}}{1-xq^n},$$

$$R(x;q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} \mathcal{N}(m,n)x^m q^n = \sum_{n\geq 0} \frac{q^{n^2}}{(xq;q)_n(x^{-1}q;q)_n} = \frac{1-x}{(q)_\infty}\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n(3n+1)/2}}{1-xq^n}.$$

Although the final expressions for $C(x;q)$ and $R(x;q)$ appear quite similar, their analytic behaviors are markedly different. For example, if $x \neq 1$ is a fixed root of unity, then $C(x;q)$ is essentially a meromorphic modular form [22], whereas $R(x;q)$ corresponds to the holomorphic part of a harmonic Maass form [14, 29].

In addition to the importance of the rank and crank statistics in the study of Ramanujan's congruences, a number of recent works highlight the importance of the weighted moments of the crank and rank statistics. For example, Andrews obtained an elegant description of the smallest parts partition function $\operatorname{spt}(n)$, the number of smallest parts among the integer partitions of n, in terms of $p(n)$ and $N_2(n) := \sum_m m^2 N(m,n)$. Andrews proved [3] that

$$\operatorname{spt}(n) = np(n) - \frac{1}{2}N_2(n).$$

Additionally, Andrews introduced *marked Durfee symbols* in [4], which are combinatorial generalizations of partitions, and are intrinsically connected to crank and rank moments. Atkin and Garvan's original study of rank and crank moments also had important applications to partition congruences [7].

Furthermore, rank moments have particularly interesting automorphic properties. In joint work with Garvan, the first two authors showed that the generating function is (essentially) a *quasimock theta function*, defined as the holomorphic part of sums of weak Maass forms and their derivatives (just as Atkin and Garvan showed that the crank moments are related to quasimodular forms) [7, 11, 12]. These types of q-series arise quite frequently in mathematical physics in the study of topological Yangs-Mills from string theory [2, 23].

To be precise, for a nonnegative integer k, define the k-th *crank* (resp. *rank*) *moment* as

$$M_k(n) := \sum_{m\in\mathbb{Z}} m^k \mathcal{M}(m,n), \tag{1.4}$$
$$N_k(n) := \sum_{m\in\mathbb{Z}} m^k \mathcal{N}(m,n).$$

Both the crank and rank moments vanish when k is odd due to the symmetries of the statistics [7]. Thus we need only consider even moments, and for $k \in \mathbb{N}$, we therefore define

$$C_{2k}(q) := \sum_n M_{2k}(n) q^n$$
$$R_{2k}(q) := \sum_n N_{2k}(n) q^n$$

be the generating functions for $M_{2k}(n)$ and $N_{2k}(n)$. As alluded to earlier, $C_{2k}(q)$ is essentially a quasimodular form, and $R_{2k}(q)$ is (up to a fractional power of q) a *quasimock theta function* [12].

In this paper we focus on conjectural observations of Garvan regarding the relative size of the crank and rank moments [20].

Conjecture (Garvan)**.** *Let $k \geq 1$ be an integer.*

1. *As $n \to \infty$, we have $M_{2k}(n) \sim N_{2k}(n)$.*
2. *For all $n \geq 2$, we have $M_{2k}(n) > N_{2k}(n)$.*

Remark. Garvan's conjecture can be interpreted as stating that the distribution of the crank statistic is slightly "wider" than that of the rank, but not enough to affect the main asymptotic behavior. This is unexpected, as there is little about the combinatorial definitions of the crank and rank that suggest any close relations. Also, part (1) of his conjecture does not appear in [20], but was mentioned in private communication to the first two authors.

A refined conjecture was given by the first two authors in [13].

Refined Conjecture. *Suppose that $k \geq 1$.*

1. *As $n \to \infty$,*
$$M_{2k}(n) \sim N_{2k}(n) \sim \alpha_{2k} \cdot n^k\, p(n),$$
where $\alpha_k \in \mathbb{Q}$ is non-negative.
2. *Garvan's inequality holds for all n, and as $n \to \infty$,*
$$D_{2k}(n) := M_{2k}(n) - N_{2k}(n) \sim \beta_{2k}\, n^{k-\frac{1}{2}} \cdot p(n),$$
where $\beta_{2k} \in \frac{\sqrt{6}}{\pi}\mathbb{Q}$ is positive

Remark. Garvan also personally provided numerical data to the authors that suggested the shape of part (1) of this conjecture. Furthermore, we emphasize out that neither part of this conjecture is a strict refinement of Garvan's conjectures, as they requires sufficiently large values of n.

The first two authors proved this conjecture in the cases $k = 2$ and 4. However, this proof did not suggest the general form of α_{2k} or β_{2k}, and the techniques rapidly became too unwieldy for use on higher k. In this paper, we amplify and improve our use of the circle method by using more precise modular transformations; this was inspired by Ramanujan's original treatment of asymptotics for the partition function. We prove the refined conjecture in all cases by explicitly computing the first two terms of the asymptotic expansions for the rank and crank moments.

In the following result, we use the standard notations where $B_n(x)$ denotes the Bernoulli polynomials, and $I_\alpha(x)$ denotes the modified Bessel functions.

Theorem 1.1. *We have*

$$M_{2k}(n) = \pi\xi_{2k}(24n-1)^{k-3/4}I_{3/2}(y_n) + \widetilde{\xi}_{2k}(24n-1)^{k-5/4}I_{1/2}(y_n) + O\left(n^{k-7/4}\cdot n^{-1/4}\exp(y_n)\right),$$

where

$$y_n := \frac{\pi}{6}\sqrt{24n-1}.$$

The constants are given by

$$\xi_{2k} := (-1)^k 2B_{2k}(1/2) \qquad \text{and} \qquad \widetilde{\xi}_{2k} := -3(2k)(2k-3)\xi_{2k} + \xi'_{2k}, h$$

where $\xi'_0 := 0$, *and for* $k > 0$,

$$\xi'_{2k} := -\frac{1}{4}(2k)(2k-1)\xi_{2k-2}.$$

Remark. At $x = 1/2$, the Bernoulli polynomials evaluate to $B_{2k}(1/2) = \left(2^{1-2k}-1\right)B_{2k}$, where B_{2k} is the usual Bernoulli number. In particular, we have that $\xi_{2k} = 2(-1)^k B_{2k}(1/2) > 0$.

Remark. Regardless of index, each of the modified Bessel functions has the main asymptotic term $I_a(y) \sim \dfrac{e^y}{y}$ (see [6]).

The proof of Theorem 1.1 uses a recursive relation for the crank moment generating functions given in [7]. We then relate these to the rank moments with Atkin-Garvan's "Rank-Crank PDE" [7], which we state precisely in Section 4. This PDE is a recursive formula for the rank moment generating functions that involves triple products of crank moment generating functions, and arises from a heat operator relation for Jacobi Maass forms, as shown by the first author and Zwegers in [16]. Using evaluations of multiple sums of Bernoulli polynomials to obtain an asymptotic expansion for the coefficients of products of crank moment generating functions (see Section 3), we deduce the following rank moment asymptotics.

Theorem 1.2. *For* $k \geq 0$ *we have*

$$N_{2k}(n) = \pi\lambda_{2k}(24n-1)^{k-3/4}I_{3/2}(y_n) + \widetilde{\lambda}_{2k}(24n-1)^{k-5/4}I_{1/2}(y_n) + O\left(n^{k-7/4}\cdot n^{-1/4}\exp(y_n)\right),$$

where $\lambda_{2k} := \xi_{2k}$ *and*

$$\widetilde{\lambda}_{2k} := \begin{cases} -3\cdot 2k(2k-3)\xi_{2k} - \frac{3}{4}2k(2k-1)\xi_{2k-2} & k > 0, \\ 0 & k = 0. \end{cases}$$

Combining Theorems 1.1 and 1.2, we obtain an asymptotic expansion for the difference of the rank and crank moments.

Corollary 1.3. *For any* $k \geq 1$ *as* $n \to \infty$ *we have*

$$D_{2k}(n) \sim \frac{1}{2}\cdot 2k(2k-1)\xi_{2k-2}(24n-1)^{k-5/4}I_{1/2}(y_n).$$

Finally, the constants in the refined conjecture can now be computed using the fact that

$$p(n) \sim 2\pi(24n-1)^{-3/4}I_{3/2}(y_n).$$

Corollary 1.4. *The refined conjecture is true, with (positive) constants*

$$\alpha_{2k} = (-24)^k B_{2k}(1/2),$$
$$\beta_{2k} = \frac{\sqrt{6}}{\pi} \cdot 2k(2k-1)(-24)^{k-1} B_{2k-2}(1/2).$$

2. Crank Asymptotics and the proof of Theorem 1.1

Here we prove a modified version of Theorem 1.1, which reflects the Bessel function indices that arise most naturally when using the circle method.

Theorem 2.1. *For $k \geq 0$ we have*

$$\begin{aligned} M_{2k}(n) =& \pi\xi_{2k} \cdot (24n-1)^{k-3/4} I_{3/2-2k}(y_n) + \xi'_{2k} \cdot (24n-1)^{k-5/4} I_{3/2-2k+1}(y_n) \\ &+ O\left(n^{k-7/4} \cdot n^{-1/4} \exp(y_n)\right). \end{aligned}$$

Theorem 1.1 follows from Theorem 2.1 through a simple formula for shifting the indices of Bessel functions.

Lemma 2.2. *For $\ell \in \mathbb{Z}$, we have the relation*

$$I_{3/2-2\ell}(y_n) = I_{3/2}(y_n) - \frac{3}{\pi}(24n-1)^{-1/2}(2\ell)(2\ell-3) I_{1/2}(y_n) + O\left(n^{-1} I_{-1/2}(y_n)\right).$$

Proof. The total shift is the result of successive applications of the Bessel function relation [6]

$$I_{a-1}(x) = \frac{2a}{x} I_a(x) + I_{a+1}(x). \tag{2.1}$$

□

Remark. Formula (2.1) also implies the following simpler recurrence for the second order term in Theorem 1.1, which is also needed for the proof of Theorem 1.1:

$$I_{3/2-2\ell+1}(y_n) = I_{3/2-2(\ell-1)+1}(y_n) + O\left(n^{-1/2} I_{3/2-2\ell+2}(y_n)\right).$$

Proof of Theorem 2.1. The idea of the proof is to find a recursive formula for the leading order constants, which we then solve explicitly with Bernoulli polynomials, thus obtaining formulas for the constants ξ_{2k} and ξ'_{2k}. We begin with Atkin and Garvan's recurrence for the crank moment generating functions in terms of divisor sums, found as equation (4.6) in [7], namely

$$C_{2k}(q) = 2\sum_{j=1}^{k} \binom{2k-1}{2j-1} \Phi_{2j-1}(q) C_{2k-2j}(q). \tag{2.2}$$

Here we have denoted the j-th divisor function by

$$\Phi_j(q) := \sum_{n=1}^{\infty} \sigma_j(n) q^n,$$

where $\sigma_j(n) := \sum_{d|n} d^j$ is the j-th divisor sum. These functions can be written in terms of the classical Eisenstein series $E_k(q) := 1 - \frac{2k}{B_k}\Phi_{k-1}(q)$, as

$$\Phi_{k-1}(q) = -\frac{B_k}{2k} \cdot (E_k(q) - 1).$$

We next follow the arguments in [13] and use the Hardy-Ramanujan circle method, following the original techniques chronicled in [1, 21, 24]. We assume basic familiarity with these standard techniques, and in the following development, we use the fact that the dominant cuspidal contribution in the circle method occurs at $q = 1$. This is true because all rational cusps are (essentially) equivalent, which follows because the Eisenstein series are modular on $\mathrm{SL}_2(\mathbb{Z})$, and up to a 24-th root of unity, $\eta(q)$ is as well. To identify the main asymptotic contribution from the circle method, we therefore need only use the modular inversion transformation.

Let $q := e^{-2\pi z}$ with $\mathrm{Re}(z) > 0$, $q_1 := e^{-\frac{2\pi}{z}}$, which corresponds to setting $\tau = iz$. To capture the cusp at $q = 1$, we need only the transformation laws

$$E_k(q) = (iz)^{-k} E_k(q_1) \qquad \text{if } k > 2, \tag{2.3}$$
$$E_2(q) = (iz)^{-2} E_2(q_1) + \frac{6}{\pi z}.$$

In general, suppose that we wish to estimate the coefficients in an expression of the form

$$\sum_n a(n)q^n = c\,P(q)g(q_1)z^{-k} + \ldots, \tag{2.4}$$

where c is a constant and $g(q)$ has a holomorphic q-series expansion

$$g(q) = 1 + \sum_{n>0} b(n)q^n.$$

Then the asymptotic contribution to $a(n)$ due to the term displayed on the right side of (2.4) is

$$c \cdot 2\pi(24n-1)^{\frac{k}{2}-\frac{3}{4}} I_{\frac{3}{2}-k}(y_n).$$

This immediately implies the following recursive result, which allows us to keep track of asymptotic expansions as we successively multiply by Eisenstein series (note that (2.3) means that $E_2(q)$ yields an extra recursive term).

Lemma 2.3. *Suppose that $G(q) = \sum c(n)q^n = P(q)\widetilde{E}_{2k}(q)$, where $\widetilde{E}_{2k}(q) := E_{2a_1}(q)\cdots E_{2a_r}(q)$ has total weight $2k = 2a_1 + \cdots + 2a_r$. Then the coefficients $c(n)$ have an asymptotic expansion of the form*

$$c(n) = \pi\alpha(24n-1)^{k-3/4} I_{3/2-2k}(y_n) + \alpha'(24n-1)^{k-5/4} I_{5/2-2t}(y_n) + O\left(n^{k-2}e^{y_n}\right)$$

for some constants α and α'. Furthermore, the coefficients of $G(q) \cdot E_{2t}(q) =: \sum c_{2t}(n)q^n$ satisfy

$$c_{2t}(n) = \pi\alpha(-1)^t(24n-1)^{k-3/4+t} I_{3/2-2k-2t}(y_n)$$
$$+ \left(\alpha'(-1)^t + \delta_{t=1}\cdot 6\alpha\right)(24n-1)^{k-5/4+t} I_{5/2-2k-2t}(y_n) + O\left(n^{k-2+t}e^{y_n}\right).$$

We apply this lemma to the crank moments by first "unwinding" (2.2) to obtain the formula

$$C_{2k}(q) = 2P(q) \cdot \sum_{a_1+2a_2+\cdots+ka_k=k} \alpha_{a_1,a_2,\cdots,a_k} \Phi_1(q)^{a_1}\Phi_3(q)^{a_2}\cdots\Phi_{2k-1}(q)^{a_k} \tag{2.5}$$

for some integer constants α. After rewriting in terms of Eisenstein series, Lemma 2.3 now applies to each term in (2.5), which implies the existence of the asymptotic expansion in Theorem 2.1. Furthermore, (2.2) and Lemma 2.3 then also immediately imply the following recurrence:

$$\xi_{2k} = 2\sum_{j=1}^{k} \binom{2k-1}{2j-1} \frac{B_{2j}}{4j} (-1)^{j+1} \xi_{2k-2j}. \tag{2.6}$$

Similarly, we may deduce the following recurrence for ξ'_{2k},

$$\begin{aligned}\xi'_{2k} =&2\sum_{j=1}^{k-1}\binom{2k-1}{2j-1}\frac{B_{2j}}{4j}(-1)^{j+1}\xi'_{2k-2j}-2(2k-1)\frac{B_2}{4}\,6\,\xi_{2k-2}\\ =&2\sum_{j=1}^{k-1}\binom{2k-1}{2j-1}\frac{B_{2j}}{4j}(-1)^{j+1}\xi'_{2k-2j}-\frac{(2k-1)}{2}\xi_{2k-2}.\end{aligned} \tag{2.7}$$

We solve both of these recurrences by using the Bernoulli polynomial identity

$$\sum_{j=1}^{k}\binom{2k-1}{2j-1}\frac{B_{2j}}{4j}B_{2k-2j}(1/2)=-\frac{1}{2}B_{2k}(1/2). \tag{2.8}$$

This is a specialization of the general convolution sum

$$\sum_{j=0}^{n}\binom{n}{j}B_j(x)B_{n-j}(y)=-(n-1)B_n(x+y)+n(x+y-1)B_{n-1}(x+y),$$

which can be found (along with other relevant formulas) in [17]. The fact that $B_n(1/2)=0$ for all odd n implies that our formula (2.8) is equivalent to the case $x=0, y=1/2$. Applying induction to (2.6) and (2.8) and using the base case $\xi_0=2$ gives the claimed formula for ξ_{2k}.

To obtain the formula for ξ'_{2k}, note that $\binom{2k-1}{2j-1}=\frac{(k-1)(2k-1)}{(k-j)(2k-2j-1)}\binom{2k-3}{2j-1}$. Now the recurrence (2.6) inductively implies that the claimed formula for ξ'_{2k} is correct, as the right side of (2.7) evaluates to

$$\begin{aligned}&-\frac{1}{2}(2k-1)(k-1)\sum_{j=1}^{k-1}2\binom{2k-3}{2j-1}\frac{B_{2j}}{4j}(-1)^{j+1}\xi_{2(k-1)-2j}-\frac{(2k-1)}{2}\xi_{2k-2}\\ =&-\frac{1}{2}(2k-1)(k-1)\xi_{2(k-1)}-\frac{(2k-1)}{2}\xi_{2k-2}=-\frac{1}{2}k(2k-1)\xi_{2k-2}=\xi'_{2k}.\end{aligned}$$

□

We note for later purposes the identity

$$2\sum_{j=1}^{k-1}\binom{2k-1}{2j-1}\frac{B_{2j}}{4j}(-1)^{j+1}\xi'_{2k-2j}=\left(1-\frac{1}{k}\right)\xi'_{2k}. \tag{2.9}$$

3. Asymptotics for products of crank moments

The rank-crank PDE (see equation (4.1)) gives a recurrence for the rank moment generating functions that involves the triple products

$$C_{2\alpha}(q)C_{2\beta}(q)C_{2\gamma}(q)P(q)^{-2}=:\sum_n M_{2\alpha,2\beta,2\gamma}(n)q^n \tag{3.1}$$

for $\alpha,\beta,\gamma\geq 0$. In this section we will deal with the asymptotic evaluation of $M_{2\alpha,2\beta,2\gamma}(n)$.

Theorem 3.1. *If $\alpha+\beta+\gamma=k$, then*

$$\begin{aligned}M_{2\alpha,2\beta,2\gamma}(n)=&\pi\xi_{2\alpha,2\beta,2\gamma}(24n-1)^{k-3/4}I_{3/2-2k}(y_n)+\xi'_{2\alpha,2\beta,2\gamma}(24n-1)^{k-5/4}I_{3/2-2k+1}(y_n)\\ &+O\left(n^{k-7/4}\cdot n^{-1/4}\exp(y_n)\right),\end{aligned}$$

with $\xi_{2\alpha,2\beta,2\gamma} := \frac{1}{4}\xi_{2\alpha}\xi_{2\beta}\xi_{2\gamma}$ *and*

$$\xi'_{2\alpha,2\beta,2\gamma} := \frac{1}{4}\left(\xi'_{2\alpha}\xi_{2\beta}\xi_{2\gamma} + \xi_{2\alpha}\xi'_{2\beta}\xi_{2\gamma} + \xi_{2\alpha}\xi_{2\beta}\xi'_{2\gamma}\right).$$

We again use Lemma 2.2 to shift the indices for easier comparisons.

Corollary 3.2. *The asymptotic expansion of the triple crank product is given by*

$$M_{2\alpha,2\beta,2\gamma}(n) = \pi\xi_{2\alpha,2\beta,2\gamma}(24n-1)^{k-3/4}I_{3/2}(y_n) + \widetilde{\xi}_{2\alpha,2\beta,2\gamma}(24n-1)^{k-5/4}I_{1/2}(y_n) + O\left(n^{k-7/4}\cdot n^{-1/4}\exp(y_n)\right),$$

where

$$\widetilde{\xi}_{2\alpha,2\beta,2\gamma} := -3(2k)\,(2k-3)\,\xi_{2\alpha,2\beta,2\gamma} + \xi'_{2\alpha,2\beta,2\gamma}.$$

Proof of Theorem 3.1. Using (2.2) for each term in the product of $C_{2\alpha}C_{2\beta}C_{2\gamma}P^{-2}$ as well as a minor modification of Lemma 2.3 we obtain the recursion

$$\xi_{2\alpha,2\beta,2\gamma} = -2^3 \sum_{\substack{1\le j\le\alpha\\1\le i\le\beta\\1\le\ell\le\gamma}} \binom{2\alpha-1}{2j-1}\binom{2\beta-1}{2i-1}\binom{2\gamma-1}{2\ell-1}\frac{B_{2j}}{4j}\frac{B_{2i}}{4i}\frac{B_{2\ell}}{4\ell}(-1)^{i+j+\ell}\xi_{2\alpha-2j,2\beta-2i,2\gamma-2\ell}. \tag{3.2}$$

Induction and three applications of the recurrence (2.6) shows that $\xi_{2\alpha,2\beta,2\gamma} = \frac{1}{4}\xi_{2\alpha}\xi_{2\beta}\xi_{2\gamma}$ is the unique solution to (3.2).

We next turn to $\xi'_{2\alpha,2\beta,2\gamma}$. Following similar arguments as those that led to the recursion for ξ'_{2k}, we obtain

$$\begin{aligned}\xi'_{2\alpha,2\beta,2\gamma} =& 8\sum_{\substack{1\le j\le\alpha\\1\le i\le\beta\\1\le\ell\le\gamma}} \binom{2\alpha-1}{2j-1}\binom{2\beta-1}{2i-1}\binom{2\gamma-1}{2\ell-1}\frac{B_{2j}}{4j}\frac{B_{2i}}{4i}\frac{B_{2\ell}}{4\ell}(-1)^{i+j+\ell+3}\xi'_{2\alpha-2j,2\beta-2i,2\gamma-2\ell}\\ &+8\sum_{\substack{1\le i\le\beta\\1\le\ell\le\gamma}}(2\alpha-1)\binom{2\beta-1}{2i-1}\binom{2\gamma-1}{2\ell-1}6\,\frac{B_2}{4}\frac{B_{2i}}{4i}\frac{B_{2\ell}}{4\ell}(-1)^{i+\ell+3}\xi_{2\alpha-2,2\beta-2i,2\gamma-2\ell}\\ &+8\sum_{\substack{1\le j\le\alpha\\1\le\ell\le\gamma}}\binom{2\alpha-1}{2j-1}(2\beta-1)\binom{2\gamma-1}{2\ell-1}6\,\frac{B_{2j}}{4j}\frac{B_2}{4}\frac{B_{2\ell}}{4\ell}(-1)^{j+\ell+3}\xi_{2\alpha-2j,2\beta-2,2\gamma-2\ell}\\ &+8\sum_{\substack{1\le j\le\alpha\\1\le i\le\beta}}\binom{2\alpha-1}{2j-1}\binom{2\beta-1}{2i-1}(2\gamma-1)6\,\frac{B_{2j}}{4j}\frac{B_{2i}}{4i}\frac{B_2}{4}(-1)^{i+j+3}\xi_{2\alpha-2j,2\beta-2i,2\gamma-2}.\end{aligned}$$

Again using the formula $\xi_{2\alpha,2\beta,2\gamma} = \frac{1}{4}\xi_{2\alpha}\xi_{2\beta}\xi_{2\gamma}$ and (2.6), we may simplify the second line (and analogously the third and fourth) as

$$\begin{aligned}&-\frac{2(2\alpha-1)}{16}\xi_{2\alpha-2}\left(2\sum_{1\le i\le\beta}\binom{2\beta-1}{2i-1}\frac{B_{2i}}{4i}(-1)^{i+1}\xi_{2\beta-2i}\right)\left(2\sum_{1\le\ell\le\gamma}\binom{2\gamma-1}{2\ell-1}\frac{B_{2\ell}}{4\ell}(-1)^{\ell+1}\xi_{2\gamma-2\ell}\right)\\ &\qquad= -\frac{2(2\alpha-1)}{16}\xi_{2\alpha-2}\xi_{2\beta}\xi_{2\gamma} = \frac{1}{4\alpha}\xi'_{2\alpha}\xi_{2\beta}\xi_{2\gamma}.\end{aligned}$$

This yields the following recursion for $\xi'_{2\alpha,2\beta,2\gamma}$

$$\xi'_{2\alpha,2\beta,2\gamma}=8\sum_{\substack{1\le j\le\alpha\\1\le i\le\beta\\1\le\ell\le\gamma}}\binom{2\alpha-1}{2j-1}\binom{2\beta-1}{2i-1}\binom{2\gamma-1}{2\ell-1}\frac{B_{2j}}{4j}\frac{B_{2i}}{4i}\frac{B_{2\ell}}{4\ell}(-1)^{i+j+\ell+3}\xi'_{2\alpha-2j,2\beta-2i,2\gamma-2\ell}$$
$$+\frac{1}{4\alpha}\xi'_{2\alpha}\xi_{2\beta}\xi_{2\gamma}+\frac{1}{4\beta}\xi_{2\alpha}\xi'_{2\beta}\xi_{2\gamma}+\frac{1}{4\gamma}\xi_{2\alpha}\xi_{2\beta}\xi'_{2\gamma}. \tag{3.3}$$

Our claimed formula $\xi'_{2\alpha,2\beta,2\gamma}=\frac{1}{4}(\xi'_{2\alpha}\xi_{2\beta}\xi_{2\gamma}+\xi_{2\alpha}\xi'_{2\beta}\xi_{2\gamma}+\xi_{2\alpha}\xi_{2\beta}\xi'_{2\gamma})$ solves this recurrence, as using (2.6) and (2.9) to evaluate the right side of (3.3) gives the expected result:

$$\frac{1}{4}\left(1-\frac{1}{\alpha}\right)\xi'_{2\alpha}\xi_{2\beta}\xi_{2\gamma}+\frac{1}{4}\left(1-\frac{1}{\beta}\right)\xi_{2\alpha}\xi'_{2\beta}\xi_{2\gamma}+\frac{1}{4}\left(1-\frac{1}{\gamma}\right)\xi_{2\alpha}\xi_{2\beta}\xi'_{2\gamma}$$
$$+\frac{1}{4\alpha}\xi'_{2\alpha}\xi_{2\beta}\xi_{2\gamma}+\frac{1}{4\beta}\xi_{2\alpha}\xi'_{2\beta}\xi_{2\gamma}+\frac{1}{4\gamma}\xi_{2\alpha}\xi_{2\beta}\xi'_{2\gamma}$$
$$=\frac{1}{4}\left(\xi'_{2\alpha}\xi_{2\beta}\xi_{2\gamma}+\xi_{2\alpha}\xi'_{2\beta}\xi_{2\gamma}+\xi_{2\alpha}\xi_{2\beta}\xi'_{2\gamma}\right)=\xi'_{2\alpha,2\beta,2\gamma}.$$

□

4. Rank Asymptotics

In this section we prove the asymptotic expansion for $N_{2k}(n)$ given in Theorem 1.2. In order to relate the rank moments to the crank moments that we have already calculated, we use Atkin and Garvan's rank-crank PDE [7], which states that

$$\sum_{i=0}^{k-1}\binom{2k}{2i}\sum_{\substack{2\alpha+2\beta+2\gamma=2k-2i\\\alpha,\beta,\gamma\ge0}}\binom{2k-2i}{2\alpha,2\beta,2\gamma}C_{2\alpha}(q)C_{2\beta}(q)C_{2\gamma}(q)P(q)^{-2}-3(2^{2k-1}-1)C_2(q)$$
$$=\frac{1}{2}(2k-1)(2k-2)R_{2k}(q)+6\sum_{i=1}^{k-1}\binom{2k}{2i}(2^{2i-1}-1)\delta_q(R_{2k-2i}(q))$$
$$+\sum_{i=1}^{k-1}\left(\binom{2k}{2i+2}(2^{2i+1}-1)-2^{2i}\binom{2k}{2i+1}+\binom{2k}{2i}\right)R_{2k-2i}(q), \tag{4.1}$$

where $\delta_q:=q\frac{d}{dq}$. We once more argue inductively in order to find recurrences for λ_{2k} and $\widetilde{\lambda}_{2k}$ as defined in Theorem 1.2.

Proof of Theorem 1.2. The rank-crank PDE and Theorems 1.1 and 3.1 inductively imply that there is an asymptotic expansion for the rank moments of the form

$$N_{2k}(n)=\pi\lambda_{2k}(24n-1)^{k-3/4}I_{3/2-2k}(y_n)+\lambda'_{2k}(24n-1)^{k-5/4}I_{3/2-2k-1}(y_n)+O\left(n^{k-2}e^{y_n}\right).$$

Lemma 2.2 then implies that

$$N_{2k}(n)=\pi\lambda_{2k}(24n-1)^{3/4}I_{3/2}(y_n)+\widetilde{\lambda}_{2k}(24n-1)^{k-5/4}I_{1/2}(y_n)+O\left(n^{k-2}e^{y_n}\right),$$

with $\widetilde{\lambda}_{2k}:=-3(2k)(2k-3)\lambda_{2k}+\lambda'_{2k}$.

The only terms that contribute to the leading order on the crank side of (4.1) are

$$\sum_{\alpha+\beta+\gamma=k} \binom{2k}{2\alpha, 2\beta, 2\gamma} \xi_{2\alpha,2\beta,2\gamma} \tag{4.2}$$

while the only terms contributing to the main term on the rank side of the equation are

$$\binom{2k-1}{2} \lambda_{2k} + \binom{2k}{2} \frac{1}{4} \lambda_{2k-2}. \tag{4.3}$$

We can now show inductively that $\lambda_{2k-2} = \xi_{2k-2}$. Indeed, adding the terms from (4.2) and (4.3) and using the formula for $\xi_{2\alpha,2\beta,2\gamma}$ shows that the claimed equality is equivalent to the triple product summation

$$\sum_{\alpha,\beta,\gamma} \binom{2k}{2\alpha, 2\beta, 2\gamma} B_{2\alpha}(1/2) B_{2\beta}(1/2) B_{2\gamma}(1/2) = \binom{2k-1}{2} B_{2k}(1/2) - \binom{2k}{2} \frac{1}{4} B_{2k-2}(1/2).$$

Like (2.8), this is also a specialization of a general Bernoulli polynomial identity that can be found in [17]. As a consequence of the equality between ξ_{2k} and λ_{2k}, we now have

$$\sum_{\alpha+\beta+\gamma=k} \binom{2k}{2\alpha, 2\beta, 2\gamma} \xi_{2\alpha,2\beta,2\gamma} = \binom{2k-1}{2} \xi_{2k} + \binom{2k}{2} \frac{1}{4}\, \xi_{2k-2}. \tag{4.4}$$

We next consider the second leading term. Using similar reasoning as above, we equate the terms of second highest order from (4.1), obtaining

$$\sum_{\alpha+\beta+\gamma=k} \binom{2k}{2\alpha, 2\beta, 2\gamma} \widetilde{\xi}_{2\alpha,2\beta,2\gamma} = \binom{2k-1}{2} \widetilde{\lambda}_{2k} + \binom{2k}{2} \frac{1}{4} \widetilde{\lambda}_{2k-2}. \tag{4.5}$$

Expanding the left side of (4.5) gives

$$\sum_{\alpha+\beta+\gamma=k} \binom{2k}{2\alpha, 2\beta, 2\gamma} \widetilde{\xi}_{2\alpha,2\beta,2\gamma} = \sum_{\alpha+\beta+\gamma=k} \binom{2k}{2\alpha, 2\beta, 2\gamma} \left(-3 \cdot 2k\,(2k-3)\, \xi_{2\alpha,2\beta,2\gamma} + \xi'_{2\alpha,2\beta,2\gamma}\right). \tag{4.6}$$

The first term of (4.6) can be evaluated using (4.4):

$$\begin{aligned} -3 \cdot 2k\,(2k-3) \sum_{\alpha+\beta+\gamma=k} \binom{2k}{2\alpha, 2\beta, 2\gamma} \xi_{2\alpha,2\beta,2\gamma} &= -3 \cdot 2k\,(2k-3) \left(\binom{2k-1}{2} \xi_{2k} + \binom{2k}{2} \frac{1}{4} \xi_{2k-2} \right) \\ &= -\frac{3}{2}(2k)(2k-1)(2k-2)(2k-3)\xi_{2k} - \frac{3}{8}(2k)^2(2k-1)(2k-3)\xi_{2k-2}, \end{aligned} \tag{4.7}$$

and the second term of (4.6) is

$$\begin{aligned} \sum_{\alpha+\beta+\gamma=k} \binom{2k}{2\alpha, 2\beta, 2\gamma} \xi'_{2\alpha,2\beta,2\gamma} &= \frac{1}{4} \sum_{\alpha+\beta+\gamma=k} \binom{2k}{2\alpha, 2\beta, 2\gamma} \left(\xi'_{2\alpha}\xi_{2\beta}\xi_{2\gamma} + \xi_{2\alpha}\xi'_{2\beta}\xi_{2\gamma} + \xi_{2\alpha}\xi_{2\beta}\xi'_{2\gamma} \right) \\ &= \frac{1}{4} \sum_{\alpha+\beta+\gamma=k} \binom{2k}{2\alpha, 2\beta, 2\gamma} \left(-\frac{1}{4}(2\alpha)(2\alpha-1)\xi_{2\alpha-2}\xi_{2\beta}\xi_{2\gamma} - \frac{1}{4}(2\beta)(2\beta-1)\xi_{2\alpha}\xi_{2\beta-2}\xi_{2\gamma} \right. \\ &\qquad \left. -\frac{1}{4}(2\gamma)(2\gamma-1)\xi_{2\alpha}\xi_{2\beta}\xi_{2\gamma-2} \right). \end{aligned} \tag{4.8}$$

After using the identity $\binom{2k}{2\alpha,2\beta,2\gamma}(2\alpha)(2\alpha-1) = 2k(2k-1)\binom{2k-2}{2\alpha-2,2\beta,2\gamma}$ (with similar analogues for β and γ) and making a shift of indices, each of the three terms of (4.8) become alike. Using the definition of $\xi_{2\alpha,2\beta,2\gamma}$ this gives a total of

$$
\begin{aligned}
(4.9) \qquad & -\frac{3}{4}(2k)(2k-1)\sum_{\alpha+\beta+\gamma=k-1}\binom{2k-2}{2\alpha,2\beta,2\gamma}\xi_{2\alpha,2\beta,2\gamma} \\
&= -\frac{3}{4}(2k)(2k-1)\left(\binom{2k-3}{2}\xi_{2k-2} + \binom{2k-2}{2}\frac{1}{4}\xi_{2k-4}\right) \\
&= -\frac{3}{8}(2k)(2k-1)(2k-3)(2k-4)\xi_{2k-2} - \frac{3}{32}(2k)(2k-1)(2k-2)(2k-3)\xi_{2k-4}.
\end{aligned}
$$

Combining (4.7) and (4.9) gives

$$
\begin{aligned}
\sum_{\alpha+\beta+\gamma=k}\binom{2k}{2\alpha,2\beta,2\gamma}\widetilde{\xi}_{2\alpha,2\beta,2\gamma} &= -\frac{3}{2}(2k)(2k-1)(2k-2)(2k-3)\xi_{2k} \\
&\quad - \frac{3}{4}(2k)(2k-1)(2k-2)(2k-3)\xi_{2k-2} - \frac{3}{32}(2k)(2k-1)(2k-2)(2k-3)\xi_{2k-4}.
\end{aligned}
$$

It is now easy to see that the expression

$$\widetilde{\lambda}_{2k} = -3\cdot 2k(2k-3)\xi_{2k} - \frac{3}{4}2k(2k-1)\xi_{2k-2}$$

from the theorem statement is the unique solution to (4.5). □

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