BIASES AMONG CLASSES OF RANK-CRANK PARTITIONS (mod 11)

KATHRIN BRINGMANN AND BADRI VISHAL PANDEY

ABSTRACT. In this paper, we prove inequalities for ranks, cranks, and partitions among different classes modulo 11. These were conjectured by Borozenets.

1. INTRODUCTION AND STATEMENT OF RESULTS

We start by recalling Ramanujan's famous congruences for the partition function $p(n)$. He proved in [11] that for $n \in \mathbb{N}_0$,

$$p(5n + 4) \equiv 0 \pmod 5, \quad p(7n + 5) \equiv 0 \pmod 7, \quad p(11n + 6) \equiv 0 \pmod{11}. \qquad (1.1)$$

To combinatorically explain the first two congruences, Dyson [8] introduced the *rank* of a partition λ. This is given by

$$\mathrm{rank}(\lambda) := \text{largest part of } \lambda - \text{number of part of } \lambda.$$

Dyson also conjectured the existence of another partition statistic that would explain all of the three congruences, which he called "crank". Garvan discovered the so-called vector crank in [9], which was subsequently reformulated by Andrews and Garvan as the partition statistic crank [2]. This successfully completed the search for combinatorial decompositions of the three congruences in (1.1). If $o(\lambda)$ denotes the number of ones in λ, and $\mu(\lambda)$ is the number of parts strictly larger than $o(\lambda)$, then the *crank* is defined as

$$\mathrm{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if } o(\lambda) = 0, \\ \mu(\lambda) - o(\lambda) & \text{if } o(\lambda) > 0. \end{cases}$$

Let $M(m, n)$ (resp. $N(m, n)$) denote the number of partitions of n with crank (resp. rank) m. Moreover denote by $M(a, c; n)$ (resp. $N(a, c; n)$) the number of partitions of n with crank (resp. rank) congruent to $a \pmod c$. There are many inequalities known about ranks and cranks. For example Andrews and Lewis [3] proved several inequalities and conjectured the following:

$$M(0, 3; 3n) > M(1, 3; 3n), \qquad M(0, 3; 3n + 1) < M(1, 3; 3n + 1),$$
$$M(0, 3; 3n + 2) < M(1, 3; 3n + 2) \quad \text{unless } n \in \{1, 4, 5\},$$
$$N(0, 3; n) < N(1, 3; n) \quad \text{if } n \equiv 0, 2 \pmod 3,$$
$$N(0, 3; n) > N(1, 3, n) \quad \text{if } n \equiv 1 \pmod 3.$$

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The inequalities for the crank have been shown by Kane [10] and the ones for the rank by the first author [5]. Since then there has been many studies of such inequalities (see e.g. [6, 7]).

Recently, Borozenets [4] considered rank and crank statistic modulo 11, where he proved many inequalities among different classes of rank and crank modulo 11 as well as conjectured some new inequalities among them. The goal of this paper is to prove the following conjectures made in [4]. To state them, throughout, for sequences $a = \{a(n)\}_{n=1}^{\infty}$ and $b = \{b(n)\}_{n=1}^{\infty}$, we use the notation $a \leq_N b$ (resp. $a \leq b$) to mean that $a(n) \leq b(n)$ for $n \geq N$ (resp. $n \geq 0$). For $0 \leq j, d < 11$, define $N_{j,d}(n) := N(j, 11; 11n + d)$, $M_{j,d}(n) := M(j, 11; 11n + d)$, and $p_d(n) := p(11n + d)$.

Conjecture 1.1 (Conjecture 6.15 of [4]). *We have the following inequalities*

$$N_{5,0} \leq N_{4,0} \leq_2 N_{3,0} \leq M_{1,0} \leq \frac{p_0}{11} \leq M_{0,0} \leq_1 N_{2,0} \leq N_{1,0} \leq_3 N_{0,0},$$

$$N_{5,1} \leq N_{4,1} \leq N_{3,1} \leq_1 M_{0,1} \leq M_{2,1} \leq \frac{p_1}{11} \leq M_{1,1} \leq_1 N_{2,1} \leq N_{1,1} \leq N_{0,1},$$

$$N_{5,2} \leq N_{4,2} \leq N_{3,2} \leq M_{0,2} \leq \frac{p_2}{11} \leq M_{2,2} \leq_1 N_{2,2} \leq N_{1,2} \leq_3 N_{0,2},$$

$$N_{5,3} \leq N_{4,3} \leq N_{3,3} \leq M_{1,3} \leq \frac{p_3}{11} \leq M_{0,3} \leq N_{2,3} \leq_1 N_{1,3} \leq_2 N_{0,3},$$

$$N_{5,4} \leq_3 N_{4,4} \leq N_{3,4} \leq_1 M_{1,4} \leq \frac{p_4}{11} \leq M_{0,4} \leq_1 N_{2,4} \leq N_{1,4} \leq_3 N_{0,4},$$

$$N_{5,5} \leq N_{4,5} \leq_1 N_{3,5} \leq M_{2,5} \leq \frac{p_5}{11} \leq M_{0,5} \leq N_{2,5} \leq N_{1,5} \leq N_{0,5},$$

$$N_{5,6} \leq_1 N_{4,6} \leq N_{3,6} \leq \frac{p_6}{11} \leq N_{2,6} \leq N_{1,6} \leq_1 N_{0,6},$$

$$N_{5,7} \leq N_{4,7} \leq N_{3,7} \leq M_{0,7} \leq \frac{p_7}{11} \leq M_{1,7} \leq N_{2,7} \leq_1 N_{1,7} \leq N_{0,7},$$

$$N_{5,8} \leq N_{4,8} \leq N_{3,8} \leq M_{0,8} \leq_3 \frac{p_8}{11} \leq_3 M_{1,8} \leq N_{2,8} \leq N_{1,8} \leq_3 N_{0,8},$$

$$N_{5,9} \leq N_{4,9} \leq N_{3,9} \leq_1 M_{0,9} \leq \frac{p_9}{11} \leq M_{1,9} \leq N_{2,9} \leq N_{1,9} \leq_2 N_{0,9},$$

$$N_{5,10} \leq N_{4,10} \leq N_{3,10} \leq_1 M_{3,10} \leq \frac{p_{10}}{11} \leq M_{0,10} \leq N_{2,10} \leq N_{1,10} \leq_3 N_{0,10}.$$

In this paper we prove this conjecture.

Theorem 1.2. *Conjecture 1.1 is true.*

The paper is organized as follows: In Section 2 we state generating functions for our combinatorial objects. In Section 3 (resp. Section 4) we determine the asymptotic behavior of the crank (resp. rank) with an explicit error term. Section 5 is devoted to the proof of Theorem 1.2.

ACKNOWLEDGMENTS

2. GENERATING FUNCTIONS

In this section, using orthogonality of the roots of unity, we write the generating function for the crank (resp. rank) in an arithmetic progression in terms of the crank (resp. rank) function specialized at roots of unity. We consider the generating functions

$$C(\zeta; q) := \sum_{\substack{n \geq 0 \\ m \in \mathbb{Z}}} M(m, n) \zeta^m q^n = \frac{(q; q)_\infty}{(\zeta q; q)_\infty (\zeta^{-1} q; q)_\infty},$$

$$R(\zeta; q) := \sum_{\substack{n \geq 0 \\ m \in \mathbb{Z}}} N(m, n) \zeta^m q^n = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(\zeta q; q)_\infty (\zeta^{-1} q; q)_\infty},$$

where $(a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j)$, $a \in \mathbb{C}$, $n \in \mathbb{N}_0 \cup \{\infty\}$. Using orthogonality of roots of units $(\zeta_c := e^{\frac{2\pi i}{c}})$, we have

$$\sum_{n=0}^{\infty} M(a, c; n) q^n = \frac{1}{c} \sum_{n=0}^{\infty} p(n) q^n + \frac{1}{c} \sum_{j=1}^{c-1} \zeta_c^{-aj} C\left(\zeta_c^j; q\right),$$

and

$$\sum_{n=0}^{\infty} N(a, c; n) q^n = \frac{1}{c} \sum_{n=0}^{\infty} p(n) q^n + \frac{1}{c} \sum_{j=1}^{c-1} \zeta_c^{-aj} R\left(\zeta_c^j; q\right).$$

Now assume that c is odd. Noting that

$$C\left(\zeta_c^{-j}; q\right) = C\left(\zeta_c^j; q\right), \qquad R\left(\zeta_c^{-j}; q\right) = R\left(\zeta_c^j; q\right),$$

we may write

$$\sum_{n=0}^{\infty} M(a, c; n) q^n = \frac{1}{c} \sum_{n=0}^{\infty} p(n) q^n + \frac{2}{c} \sum_{j=1}^{\frac{c-1}{2}} \cos\left(\frac{2\pi aj}{c}\right) C\left(\zeta_c^j; q\right),$$

$$\sum_{n=0}^{\infty} N(a, c; n) q^n = \frac{1}{c} \sum_{n=0}^{\infty} p(n) q^n + \frac{2}{c} \sum_{j=1}^{\frac{c-1}{2}} \cos\left(\frac{2\pi aj}{c}\right) R\left(\zeta_c^j; q\right).$$

To prove Theorem 1.2, let

$$\mathcal{C}_a^{[1]}(n) := M(a, 11; n) - \frac{p(n)}{11}, \qquad \mathcal{C}_a^{[2]}(n) := N(a, 11; n) - \frac{p(n)}{11},$$

$$\mathcal{C}_{a_1,a_2}^{[3]}(n) := N(a_1, 11; n) - N(a_2, 11; n), \qquad \mathcal{C}_{a_1,a_2}^{[4]}(n) := N(a_1, 11; n) - M(a_2, 11; n),$$

$$\mathcal{C}_{a_1,a_2}^{[5]}(n) := M(a_1, 11; n) - M(a_2, 11; n).$$

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Write
$$C\left(\zeta_{11}^j; q\right) =: \sum_{n \geq 0} A_j(n) q^n, \qquad R\left(\zeta_{11}^j; q\right) =: \sum_{n \geq 0} B_j(n) q^n.$$

Then we have

$$\mathcal{C}_a^{[1]}(n) = \frac{2}{11} \sum_{j=1}^{5} \cos\left(\frac{2\pi a j}{11}\right) A_j(n), \qquad \mathcal{C}_a^{[2]}(n) = \frac{2}{11} \sum_{j=1}^{5} \cos\left(\frac{2\pi a j}{11}\right) B_j(n),$$

$$\mathcal{C}_{a_1,a_2}^{[3]}(n) = \frac{2}{11} \sum_{j=1}^{5} \left(\cos\left(\frac{2\pi a_1 j}{11}\right) - \cos\left(\frac{2\pi a_2 j}{11}\right)\right) B_j(n),$$

$$\mathcal{C}_{a_1,a_2}^{[4]}(n) = \frac{2}{11} \sum_{j=1}^{5} \left(\cos\left(\frac{2\pi a_1 j}{11}\right) B_j(n) - \cos\left(\frac{2\pi a_2 j}{11}\right) A_j(n)\right),$$

$$\mathcal{C}_{a_1,a_2}^{[5]}(n) = \frac{2}{11} \sum_{j=1}^{5} \left(\cos\left(\frac{2\pi a_1 j}{11}\right) - \cos\left(\frac{2\pi a_2 j}{11}\right)\right) A_j(n).$$

3. APPROXIMATING CRANKS

The goal of this section is to approximate $M(j, 11; n)$ with an explicit error. For h, k coprime, let h' be a solution to $hh' \equiv -1 \pmod{k}$ if k is odd and to $hh' \equiv -1 \pmod{2k}$ if k is even. For $c \mid k$, define the Kloosterman sum

$$B_{j,c,k}^*(n, m) := (-1)^{jk+1} \sin\left(\frac{\pi j}{c}\right) \sum_{h \pmod{k}^*} \frac{\omega_{h,k}}{\sin\left(\frac{\pi j h'}{c}\right)} e^{-\frac{\pi i j^2 k h'}{c^2}} e^{\frac{2\pi i}{k}(nh + mh')},$$

where the sum runs over all h modulo k that are coprime to k and we have (see [1, equation (5.2.4)])

$$\omega_{h,k} := \begin{cases} \left(\frac{-k}{h}\right) e^{-\pi i\left(\frac{1}{4}(2-hk-h)+\frac{1}{12}\left(k-\frac{1}{k}\right)\left(2h-h'+h^2h'\right)\right)} & \text{if } h \text{ is odd,} \\ \left(\frac{-h}{k}\right) e^{-\pi i\left(\frac{1}{4}(k-1)+\frac{1}{12}\left(k-\frac{1}{k}\right)\left(2h-h'+h^2h'\right)\right)} & \text{if } k \text{ is odd,} \end{cases}$$

Here $(\frac{\cdot}{\cdot})$ denotes the Kronecker symbol. Moreover, let $\delta_S := 1$ if a statement S is true and $\delta_S := 0$ otherwise.

Lemma 3.1. *We have, for $1 \leq j \leq 5$,*

$$M(j, 11; n) = M_j(n) + \sin\left(\frac{\pi j}{11}\right) \mathcal{E}_j^{[1]}(n),$$

where

$$M_j(n) := \frac{4\sqrt{3}}{\sqrt{24n-1}} \sinh\left(\frac{\pi \sqrt{24n-1}}{66}\right) \left(\frac{i B_{j,11,11}^*(-n, 0)}{\sqrt{11}} + 2\sin\left(\frac{\pi}{11}\right) \delta_{j \equiv 1 \pmod{11}}\right),$$

and where $\mathcal{E}_j^{[1]}(n)$ may be bound against

$$\left|\mathcal{E}_j^{[1]}(n)\right| \leq \mathcal{E}_1(n) := \frac{184}{11\sqrt{3}} n^{\frac{3}{4}} \sinh\left(\frac{\pi \sqrt{24n-1}}{132}\right) + 77491 n^{\frac{1}{4}} + 1324.9 n^{\frac{3}{8}} + 36620.2.$$

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3.1. **The asymptotic formula.** From Theorem 1.1 of [12] it is not hard to conclude that, for $\varepsilon > 0$, we have

$$A_{j,11}(n) = \frac{4\sqrt{3}i}{\sqrt{24n-1}} \sum_{\substack{1 \leq k \leq \sqrt{n} \\ 11|k}} \frac{B^*_{j,11,k}(-n,0)}{\sqrt{k}} \sinh\left(\frac{\pi\sqrt{24n-1}}{6k}\right) \tag{3.1}$$

$$+ \frac{8\sqrt{3}\sin\left(\frac{\pi j}{11}\right)}{\sqrt{24n-1}} \sum_{\substack{1 \leq k \leq \sqrt{n} \\ 11 \nmid k \\ jk \equiv \pm 1 \pmod{11}}} \frac{D_{j,11,k}\left(-n, m^+_{j,11,k,0}\right)}{\sqrt{k}} \sinh\left(\frac{\pi\sqrt{24n-1}}{66k}\right) + O\left(n^\varepsilon\right),$$

with $m^+_{j,11,k,0} \in \mathbb{Z}$ and

$$D_{j,c,k}(n,m) := (-1)^{jk+\ell} \sum_{h \pmod{k}^*} \omega_{h,k} e^{\frac{2\pi i}{k}(nh+mh')},$$

where $1 \leq \ell \leq 10$ is the unique solution to $\ell \equiv kj \pmod{11}$.

3.2. **The main term.** The term $k = 11$ from the first sum and the term $k = 1$ (if it occurs) from the second sum in (3.1) have the same size, and their sum is equal to

$$\frac{4\sqrt{3}i}{\sqrt{24n-1}} \frac{B^*_{j,11,11}(-n,0)}{\sqrt{11}} \sinh\left(\frac{\pi\sqrt{24n-1}}{66}\right)$$

$$+ \frac{8\sqrt{3}\sin\left(\frac{\pi j}{11}\right)}{\sqrt{24n-1}} \delta_{j \equiv 1 \pmod{11}} D_{j,11,1}\left(-n, m^+_{j,11,1,0}\right) \sinh\left(\frac{\pi\sqrt{24n-1}}{66}\right).$$

The main term in Lemma 3.1 follows by evaluating

$$D_{j,11,1}\left(-n, m^+_{j,11,1,0}\right) = (-1)^{j+\ell} = 1.$$

3.3. **Error bounds.** In this subsection we absorb the remaining terms into the error term. We first bound the contribution from the first term in (3.1) with $k \geq 22$. This can be estimated against

$$\frac{4\sqrt{3}}{\sqrt{24n-1}} \sin\left(\frac{\pi j}{11}\right) \sum_{\substack{22 \leq k \leq \sqrt{n} \\ 11|k}} \frac{1}{\sqrt{k}} \sum_{h \pmod{k}^*} \frac{1}{\left|\sin\left(\frac{\pi jh'}{11}\right)\right|} \sinh\left(\frac{\pi\sqrt{24n-1}}{6k}\right)$$

$$\leq \frac{4\sqrt{3}\sin\left(\frac{\pi j}{11}\right)\sinh\left(\frac{\pi\sqrt{24n-1}}{6\cdot 22}\right)}{\sqrt{24n-1}\sin\left(\frac{\pi}{11}\right)} \sum_{\substack{22 \leq k \leq \sqrt{n} \\ 11|k}} \sqrt{k} \leq \frac{4\sqrt{3}\sin\left(\frac{\pi j}{11}\right)\sinh\left(\frac{\pi\sqrt{24n-1}}{132}\right)}{\sqrt{24n-1}\sin\left(\frac{\pi}{11}\right)} \frac{2}{33} n^{\frac{3}{4}}$$

$$\leq \frac{8\sin\left(\frac{\pi j}{11}\right) n^{\frac{3}{4}}}{11\sqrt{3}\sqrt{24n-1}} \sinh\left(\frac{\pi\sqrt{24n-1}}{132}\right),$$

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where, in the penultimate step, we use that

$$\sum_{\substack{1\le k\le \sqrt{n}\\ c\mid k}} k^{\frac{1}{2}} \le \frac{2}{3c} n^{\frac{3}{4}}. \tag{3.2}$$

The contribution from the second sum in (3.1) with $k \ge 2$ can be bound against, using (3.2) with $c = 1$,

$$\frac{8\sqrt{3}\sin\left(\frac{\pi j}{11}\right)}{\sqrt{24n-1}}\sinh\left(\frac{\pi\sqrt{24n-1}}{132}\right)\sum_{1\le k\le\sqrt{n}}\sqrt{k} \le \frac{16\sin\left(\frac{\pi j}{11}\right)}{\sqrt{3}\sqrt{24n-1}}n^{\frac{3}{4}}\sinh\left(\frac{\pi\sqrt{24n-1}}{132}\right).$$

To make the error term in (3.1) explicit we follow the proof of [12] (which follows the proof of [5]) using the notation in there. From [12] the error that is obtained by bounding the non-principal part of Σ_1 is

$$|S_{\mathrm{err}}| \le 2e^{2\pi+\frac{\pi}{24}}\sin\left(\frac{\pi j}{11}\right)\left(c_2 + 2\left(1+\cos\left(\frac{\pi}{11}\right)\right)c_1(1+c_2)\right)\sum_{\substack{1\le k\le\sqrt{n}\\ 11\mid k}}k^{-\frac{3}{2}}\sum_{\substack{1\le h\le k\\ \gcd(h,k)=1}}\frac{1}{\left|\sin\left(\frac{\pi h}{11}\right)\right|},$$

where

$$c_1 := \sum_{m=1}^{\infty}\frac{e^{-\frac{\pi m(m+1)}{2}}}{1-e^{-\pi m}}, \qquad c_2 := \sum_{n=1}^{\infty}p(n)e^{-\pi n}.$$

We bound

$$c_1 \le \frac{1}{1-e^{-\pi}}\sum_{m=1}^{\infty}e^{-\pi m} = \frac{e^{-\pi}}{\left(1-e^{-\pi}\right)^2}.$$

Moreover, using that $p(n) < \exp(\pi\sqrt{\frac{2n}{3}})$, we estimate

$$p(n)e^{-\pi n} < e^{\pi\sqrt{\frac{2n}{3}}-\pi n} \le e^{-\frac{\pi n}{2}} \qquad \text{for } n \ge 3.$$

Thus we get

$$c_2 \le p(1)e^{-\pi} + p(2)e^{-2\pi} + \sum_{n=3}^{\infty}e^{-\frac{\pi n}{2}} = e^{-\pi} + 2e^{-2\pi} + \frac{e^{-\frac{3\pi}{2}}}{1-e^{-\frac{\pi}{2}}}. \tag{3.3}$$

Moreover, we have

$$\sum_{\substack{1\le k\le\sqrt{n}\\ 11\mid k}}k^{-\frac{3}{2}}\sum_{\substack{1\le h\le k\\ \gcd(h,k)=1}}\frac{1}{\left|\sin\left(\frac{\pi h}{11}\right)\right|} \le \frac{1}{\sin\left(\frac{\pi}{11}\right)}\sum_{\substack{1\le k\le\sqrt{n}\\ 11\mid k}}k^{-\frac{1}{2}} \le \frac{2n^{\frac{1}{4}}}{11\sin\left(\frac{\pi}{11}\right)},$$

using that

$$\sum_{\substack{1\le k\le\sqrt{n}\\ 11\mid k}}k^{-\frac{1}{2}} = 11^{-\frac{1}{2}}\sum_{1\le k\le\frac{\sqrt{n}}{11}}\frac{1}{k^{\frac{1}{2}}} \le 11^{-\frac{1}{2}}\int_0^{\frac{\sqrt{n}}{11}}x^{-\frac{1}{2}}dx = 11^{-\frac{1}{2}}2\sqrt{\frac{\sqrt{n}}{11}} = \frac{2n^{\frac{1}{4}}}{11}. \tag{3.4}$$

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Thus, we get

$$|S_{\mathrm{err}}| \leq 200.2 \sin\left(\frac{\pi j}{11}\right) n^{\frac{1}{4}}.$$

We next estimate the contribution from the non-principal part in Σ_2. From [12] this may be bound against

$$|T_{\mathrm{err}}| \leq 16 e^{2\pi} f(11) \sin\left(\frac{\pi j}{11}\right) n^{\frac{1}{4}}, \tag{3.5}$$

where

$$f(c) := \frac{1 + c_2 e^{\pi \delta_0(c)}}{1 - e^{-\frac{\pi}{c}}} + e^{\pi \delta_0(c)} c_1 (1 + c_2) + \frac{1}{2} e^{\pi \delta_0(c)} (c_2 + 1) c_3,$$

where

$$\delta_0(c) := \frac{1}{2c^2} + \frac{1}{24} - \frac{1}{2c}$$

and thus $\delta_0(11) = \frac{1}{2904}$. Moreover we have

$$c_3 := \sum_{m=2}^{\infty} \frac{e^{-\frac{\pi m(m+1)}{2}}}{1 - e^{\pi - \pi m}} \leq \frac{1}{1 - e^{-\pi}} \sum_{m=2}^{\infty} e^{-\frac{3\pi m}{2}} = \frac{e^{-3\pi}}{(1 - e^{-\pi})\left(1 - e^{-\frac{3\pi}{2}}\right)}.$$

Plugging these bounds into (3.5) we obtain

$$|T_{\mathrm{err}}| \leq 36928.5 \sin\left(\frac{\pi j}{11}\right) n^{\frac{1}{4}}.$$

Next the error obtained by symmetrizing the first main contribution may be bound again, using (3.4),

$$\left|S_{\mathrm{err}}^{[1]}\right| \leq \sin\left(\frac{\pi j}{11}\right) e^{2\pi + \frac{\pi}{12}} n^{\frac{1}{2}} \frac{2}{\lfloor\sqrt{n}\rfloor + 1} \sum_{\substack{0 \leq h < k \leq \sqrt{n} \\ \gcd(h,k)=1 \\ 11 \mid k}} \frac{1}{k^{\frac{3}{2}}} \frac{1}{\left|\sin\left(\frac{\pi h}{11}\right)\right|}$$

$$\leq 2 \sin\left(\frac{\pi j}{11}\right) e^{2\pi + \frac{\pi}{12}} \frac{1}{\sin\left(\frac{\pi}{11}\right)} \frac{2}{11} n^{\frac{1}{4}} \leq 898.1 \sin\left(\frac{\pi j}{11}\right) n^{\frac{1}{4}}.$$

The error obtained by symmetrizing the second main term contribution may be estimated against

$$\left|S_{\mathrm{err}}^{[2]}\right| \leq 32 e^{2\pi} \sin\left(\frac{\pi j}{11}\right) \frac{e^{2\pi \delta_0}}{1 - e^{-\frac{2\pi}{11}}} n^{\frac{1}{4}} \leq 39464.1 \sin\left(\frac{\pi j}{11}\right) n^{\frac{1}{4}}.$$

Next we have the errors obtained by integrating against the smaller arc

$$\frac{4\left(\frac{4}{3} + 2^{\frac{5}{4}}\right) \sin\left(\frac{\pi j}{11}\right) (1 + \log(5)) e^{2\pi + \frac{\pi}{12}}}{11\pi \left(1 - \frac{\pi^2}{24}\right)} n^{\frac{3}{8}} \leq 1324.9 \sin\left(\frac{\pi j}{11}\right) n^{\frac{3}{8}},$$

$$8\left(\frac{4}{3} + 2^{\frac{5}{4}}\right) \sin\left(\frac{\pi j}{11}\right) \frac{e^{2\pi \delta_0 + 2\pi}}{1 - e^{-\frac{2\pi}{11}}} \leq 36620.2 \sin\left(\frac{\pi j}{11}\right).$$

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Combining gives that overall error can be bound against

$$\sin\left(\frac{\pi j}{11}\right)\left(\frac{8n^{\frac{3}{4}}}{11\sqrt{3}\sqrt{24n-1}}\sinh\left(\frac{\pi\sqrt{24n-1}}{132}\right)+\frac{16n^{\frac{3}{4}}}{\sqrt{3}\sqrt{24n-1}}\sinh\left(\frac{\pi\sqrt{24n-1}}{132}\right)\right.$$

$$\left.+200.2n^{\frac{1}{4}}+36928.5n^{\frac{1}{4}}+898.1n^{\frac{1}{4}}+39464.1n^{\frac{1}{4}}+1324.9n^{\frac{3}{8}}+36620.2\right).$$

Simplifying gives the claim.

4. Approximating ranks

The goal of this section is to show the following approximation of $B_j(n)$.

Lemma 4.1. *We have, for $1 \le j \le 5$,*

$$B_j(n) = \frac{8\sqrt{3}\sin\left(\frac{\pi}{11}\right)}{\sqrt{24n-1}}\delta_{j\equiv1 \pmod{11}}\sinh\left(\frac{5\pi\sqrt{24n-1}}{66}\right)+\sin\left(\frac{\pi j}{11}\right)\mathcal{E}_j^{[2]}(n),$$

where $\mathcal{E}_j^{[2]}$ may be bound against

$$\left|\mathcal{E}_j^{[2]}(n)\right| \le \mathcal{E}_2(n) := \frac{10.74n^{\frac{3}{4}}}{\sqrt{24n-1}}\sinh\left(\frac{\pi\sqrt{24n-1}}{132}\right)+18092.8n^{\frac{1}{4}}+40707.2n^{\frac{5}{4}}.$$

4.1. The asymptotic formula. Using Theorem 1.1 of [5] it is not hard to show that

$$B_j(n) = \frac{4\sqrt{3}i}{\sqrt{24n-1}}\sum_{\substack{1\le k\le\sqrt{n}\\11|k}}\frac{B_{j,11,k}(-n,0)}{\sqrt{k}}\sinh\left(\frac{\pi\sqrt{24n-1}}{6k}\right) \tag{4.1}$$

$$+\frac{8\sqrt{3}\sin\left(\frac{\pi j}{11}\right)}{\sqrt{24n-1}}\sum_{\substack{1\le k\le\sqrt{n}\\11\nmid k\\jk\equiv\pm1 \pmod{11}}}\frac{D_{j,11,k}(-n,m_{j,11,k,0})}{\sqrt{k}}\sinh\left(\frac{5\pi\sqrt{24n-1}}{66k}\right)+O\left(n^\varepsilon\right).$$

Here $m_{j,11,k,0} \in \mathbb{Z}$, and for $c \mid k$, we have

$$B_{j,c,k}(n,m) := (-1)^{jk+1}\sin\left(\frac{\pi j}{c}\right)\sum_{h \pmod{k}^*}\frac{\omega_{h,k}}{\sin\left(\frac{\pi jh'}{c}\right)}e^{-\frac{3\pi ij^2kh'}{c}}e^{\frac{2\pi i}{k}(nh+mh')},$$

and

$$D_{j,c,k}(n,m) := (-1)^{jk+\ell}\sum_{h \pmod{k}^*}\omega_{h,k}e^{\frac{2\pi i}{k}(nh+mh')},$$

where, just like in the case of cranks, $1 \le \ell \le 10$ is the unique solution to $jk \equiv \ell \pmod{11}$.

4.2. The dominant term. The main contribution in Lemma 4.1 comes from the term $k = 1$, which only occurs if $\ell = 1$ and thus $j = 1$.

4.3. **Error bounds.** As in the case of the crank, we first bound the first sum in (4.1) against

$$\frac{4\sqrt{3}}{\sqrt{24n-1}} \sum_{\substack{1\leq k\leq\sqrt{n}\\11|k}} \frac{|B_{j,11,k}(-n,0)|}{\sqrt{k}} \sinh\left(\frac{\pi\sqrt{24n-1}}{6k}\right) \leq \frac{1.5\sin\left(\frac{\pi j}{11}\right)n^{\frac{3}{4}}}{\sqrt{24n-1}} \sinh\left(\frac{\pi\sqrt{24n-1}}{66}\right),$$

using again (3.2).

Similarly, the contribution from $k \neq 1$ from the second sum can be bound against

$$\frac{16\sin\left(\frac{\pi j}{11}\right)}{\sqrt{3}\sqrt{24n-1}} n^{\frac{3}{4}} \sinh\left(\frac{5\pi\sqrt{24n-1}}{132}\right).$$

We now look at the O-terms and make these explicit. We follow the proof and the notation of [5]. We bound the Kloosterman sums trivially. The contribution from S_2 may be estimated against

$$|S_2| \leq \sin\left(\frac{\pi j}{11}\right) \sum_{\substack{0\leq h<k\leq\sqrt{n}\\11|k\\ \gcd(h,k)=1}} \frac{1}{\left|\sin\left(\frac{\pi jh'}{11}\right)\right|} \frac{2}{k\left(\lfloor\sqrt{n}\rfloor+1\right)} \frac{\sqrt{n}}{\sqrt{k}} e^{\frac{2\pi}{k}\frac{k}{n}\left(n-\frac{1}{24}\right)} \sum_{r\geq 1} |a(r)|e^{-\frac{2\pi}{k}\left(r-\frac{1}{24}\right)\frac{k}{2}}$$

$$\leq \frac{2e^{2\pi}\sqrt{n}\sin\left(\frac{\pi j}{11}\right)}{\left(\lfloor\sqrt{n}\rfloor+1\right)\sin\left(\frac{\pi}{11}\right)} \sum_{r\geq 1} |a(r)|e^{-\pi\left(r-\frac{1}{24}\right)} \sum_{\substack{1\leq k\leq\sqrt{n}\\11|k}} \frac{1}{k^{\frac{1}{2}}}.$$

To bound the sum on k, we use (3.4). For the sum on r, we require the estimate from Section 5 of [5]

$$\sum_{r\geq 1} |a(r)|e^{-\pi r} \leq 0.06.$$

Combining gives that

$$|S_2| \leq \frac{2\cdot 0.06\sin\left(\frac{\pi j}{11}\right)\cdot 2}{11\sin\left(\frac{\pi}{11}\right)} e^{2\pi+\frac{\pi}{24}} n^{\frac{1}{4}} \leq 47.3\sin\left(\frac{\pi j}{11}\right)n^{\frac{1}{4}}.$$

We next bound the non-principal part from $\sum_2$ against

$$4\sin\left(\frac{\pi j}{11}\right) \sum_{\substack{0\leq h<k\leq\sqrt{n}\\\gcd(h,k)=1\\11\nmid k}} \frac{2}{k\left(\lfloor\sqrt{n}\rfloor+1\right)} \frac{\sqrt{n}}{\sqrt{k}} e^{\frac{2\pi}{n}\left(n-\frac{1}{24}\right)} \sum_{r\geq 1} |b(r)|e^{-\frac{\pi r}{12k\cdot 11^2}\frac{k}{2}}, \qquad (4.2)$$

where $b(r)$ is defined via

$$e^{-\frac{2\pi ih'sa}{c}-\frac{3\pi ih'a^2k_1}{cc_1}+\frac{6\pi ih'\ell a}{cc_1}+\frac{\pi}{12kz}} q_1^{\frac{s(\ell,11)\ell}{c_1}-\frac{3\ell^2}{2c_1^2}} N\left(ah',\frac{\ell c}{c_1},c;q_1\right) =: \sum_{r\geq r_0} b(r)e^{\frac{2\pi im_r h'}{k}} e^{-\frac{\pi ir}{12kc^2 z}}, \quad (4.3)$$

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where $s(\ell, c)$ is defined in (1.9) of [5]. We first show that

$$\sum_{r \geq 1} |b(r)| e^{-\frac{\pi r}{2904}} \leq 2.8, \tag{4.4}$$

By taking $k = 1, c = 1, h' = 0$ in (4.3) we have, for $q \in \mathbb{R}$,

$$q^{-\frac{1}{24} + \frac{s(\ell,11)\ell}{11} - \frac{3\ell^2}{2 \cdot 11^2}} N(0, \ell, 11; q) = \sum_{r \geq 0} b(r) q^{\frac{r}{24 \cdot 11^2}}.$$

By definition

$$N(0, \ell, 11; q)$$

$$= \frac{i}{2(q;q)_\infty} \left(\sum_{m \geq 0} \frac{(-1)^m q^{\frac{m}{2}(3m+1) + ms(\ell,11) + \frac{\ell}{22}}}{1 - q^{m + \frac{\ell}{11}}} - \sum_{m \geq 1} \frac{(-1)^m q^{\frac{m}{2}(3m+1) - ms(\ell,11) - \frac{\ell}{22}}}{1 - q^{m - \frac{\ell}{11}}} \right).$$

Thus, we have

$$\sum_{r \geq 0} |b(r)| q^{\frac{r}{24 \cdot 11^2} + \frac{1}{24} - \frac{s(\ell,11)\ell}{11} + \frac{3\ell^2}{2 \cdot 11^2}}$$

$$= \frac{q^{\frac{\ell}{22}}}{2} P(q) \sum_{m \geq 0} \frac{q^{\frac{m(3m+1)}{2} + ms(\ell,11)}}{1 - q^{m + \frac{\ell}{11}}} + \frac{q^{-\frac{\ell}{22}}}{2} P(q) \sum_{m \geq 1} \frac{q^{\frac{m(3m+1)}{2} - ms(\ell,11)}}{1 - q^{m - \frac{\ell}{11}}},$$

where $P(q) := \frac{1}{(q;q)_\infty} = \sum_{n \geq 0} p(n) q^n$. We bound the first sum, for $0 < q < 1$,

$$\sum_{m \geq 0} \frac{q^{\frac{m(3m+1)}{2} + ms(\ell,11)}}{1 - q^{m + \frac{\ell}{11}}} \leq \frac{1}{\left(1 - q^{\frac{\ell}{11}}\right)\left(1 - q^{\frac{1}{2} + s(\ell,11)}\right)}.$$

For the second sum, we estimate

$$\sum_{m \geq 1} \frac{q^{\frac{m(3m+1)}{2} - ms(\ell,11)}}{1 - q^{m - \frac{\ell}{11}}} \leq \frac{q^{2 - s(\ell,11)}}{1 - q^{1 - \frac{\ell}{11}}} + \frac{q^{2\left(\frac{7}{2} - s(\ell,11)\right)}}{\left(1 - q^{2 - \frac{\ell}{11}}\right)\left(1 - q^{\frac{7}{2} - s(\ell,11)}\right)}.$$

Thus, for $q = e^{-\pi}$ and using the bound (3.3), we get

$$\sum_{r \geq 0} |b(r)| e^{-\frac{\pi r}{2904}} \leq e^{-\pi\left(-\frac{1}{24} + \frac{s(\ell,11)\ell}{11} - \frac{3\ell^2}{2 \cdot 11^2}\right)} \frac{1}{2} \left(1 + e^{-\pi} + 2e^{-2\pi} + \frac{e^{-\frac{3\pi}{2}}}{1 - e^{-\frac{\pi}{2}}}\right)$$

$$\times \left(\frac{e^{-\frac{\pi\ell}{22}}}{\left(1 - e^{-\frac{\pi\ell}{11}}\right)\left(1 - e^{-\pi\left(\frac{1}{2} + s(\ell,11)\right)}\right)} \right.$$

$$\left. + e^{\frac{\pi\ell}{22}} \left(\frac{e^{-\pi(2 - s(\ell,11))}}{1 - e^{-\pi\left(1 - \frac{\ell}{11}\right)}} + \frac{e^{-2\pi\left(\frac{7}{2} - s(\ell,11)\right)}}{\left(1 - e^{-\pi\left(2 - \frac{\ell}{11}\right)}\right)\left(1 - e^{-\pi\left(\frac{7}{2} - s(\ell,11)\right)}\right)} \right) \right).$$

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We now see that this contribution is maximized for $\ell = 1$ (in which case $s(\ell, 11) = 0$) and in this case can be bound against ≤ 2.8. This gives (4.4).

Thus (4.2) may be estimated by

$$\frac{4\sin\left(\frac{\pi j}{11}\right)2\sqrt{n}}{\lfloor\sqrt{n}\rfloor + 1}e^{2\pi}\cdot 2.8 \sum_{1\leq k\leq\sqrt{n}}\frac{1}{\sqrt{k}} \leq 4\sin\left(\frac{\pi j}{11}\right)2e^{2\pi}\cdot 2.8\cdot 2n^{\frac{1}{4}} \leq 8567.9\sin\left(\frac{\pi j}{11}\right)n^{\frac{1}{4}},$$

bounding the sum on k against

$$\int_0^{\sqrt{n}} x^{-\frac{1}{2}}dx = 2n^{\frac{1}{4}}. \tag{4.5}$$

Next the contribution from S_{12} gives (S_{13} yields a contribution of exactly the same size)

$$|S_{12}| \leq \sin\left(\frac{\pi j}{11}\right)\sum_{\substack{0\leq h<k\leq\sqrt{n}\\11|k\\\gcd(h,k)=1}}\frac{1}{\left|\sin\left(\frac{\pi jh'}{11}\right)\right|}\frac{2}{k\left(\lfloor\sqrt{n}\rfloor + 1\right)}\frac{\sqrt{n}}{\sqrt{k}}e^{\frac{2\pi}{k}\frac{k}{n}\left(n-\frac{1}{24}\right)}e^{\frac{\pi}{12k}k}$$

$$\leq \frac{2\sin\left(\frac{\pi j}{11}\right)e^{2\pi+\frac{\pi}{12}}}{\sin\left(\frac{\pi}{11}\right)}\sum_{\substack{1\leq k\leq\sqrt{n}\\11|k}}\frac{1}{\sqrt{k}} \leq \frac{2\sin\left(\frac{\pi j}{11}\right)e^{2\pi+\frac{\pi}{12}}}{\sin\left(\frac{\pi}{11}\right)}\frac{2}{11}n^{\frac{1}{4}} \leq 898.1\sin\left(\frac{\pi j}{11}\right)n^{\frac{1}{4}},$$

where we use (3.4) to estimate the sum on k. Next the corresponding contribution from $\sum_2$ may be bound against

$$\sin\left(\frac{\pi j}{11}\right)\sum_{\substack{0\leq h<k\leq\sqrt{n}\\\gcd(h,k)=1\\11\nmid k\\r\geq 0\\\delta_{11,k,r}>0}}\frac{2}{k\left(\lfloor\sqrt{n}\rfloor + 1\right)}\frac{\sqrt{n}}{\sqrt{k}}e^{\frac{2\pi}{n}\left(n-\frac{1}{24}\right)}e^{\frac{2\pi\delta_{11,k,r}}{k}k},$$

where $\delta_{c,k,r}$ is defined in (1.13) of [5]. We determine for which values $\delta_{11,k,r} > 0$, namely $jk \equiv \pm 1 \pmod{11}$, $r = 0$, and $\delta_{11,k,0} = \frac{25}{2904}$ in both cases. Thus we bound the above against, using (4.5),

$$2\sin\left(\frac{\pi j}{11}\right)e^{2\pi\frac{25}{2904}+2\pi}\sum_{\substack{1\leq k\leq\sqrt{n}\\11\nmid k\\jk\equiv\pm 1\pmod{11}}}\frac{1}{\sqrt{k}} \leq 4\sin\left(\frac{\pi j}{11}\right)n^{\frac{1}{4}}e^{2\pi\frac{25}{2904}+2\pi} \leq 2261.1\sin\left(\frac{\pi j}{11}\right)n^{\frac{1}{4}}.$$

On the smaller arcs we have the same estimates as on S_{12} and we obtain a contribution of the sine.

Finally Σ_3 may be bounded against

$$\left|\sum_3\right| \leq 2\sin\left(\frac{\pi j}{11}\right)^2\sum_{\substack{0\leq h<k\leq\sqrt{n}\\\gcd(h,k)=1}}\frac{1}{k}\sum_{\nu=0}^{k-1}\left|\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}}e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)}z^{\frac{1}{2}}I_{j,11,k,\nu}(z)d\Phi\right|,$$

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where $z := \frac{k}{n} + k\Phi i$ and where the Mordell integral $I_{j,11,k,\nu}(z)$ is defined in [5]. We next bound the Mordell integrals, making Lemma 3.1 of [5] explicit as ($H_{j,c}$ is defined in [5])

$$\left| z^{\frac{1}{2}} I_{j,11,k,\nu}(z) \right| \leq \frac{k}{\pi |z|^{\frac{1}{2}}} \int_{\mathbb{R}} \left| e^{-\frac{3kt^2}{\pi z}} \right| \left| H_{j,c}\left(\frac{\pi i \nu}{k} - \frac{\pi i}{6k} - t \right) \right| dt$$

$$= \frac{k}{\pi |z|^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{3k}{\pi}\mathrm{Re}\left(\frac{1}{z}\right)t^2} \left| \frac{\cosh\left(\frac{\pi i \nu}{k} - \frac{\pi i}{6k} - t\right)}{\sinh\left(\frac{\pi i \nu}{k} - \frac{\pi i}{6k} - t + \frac{\pi i j}{11}\right) \sinh\left(\frac{\pi i \nu}{k} - \frac{\pi i}{6k} - t - \frac{\pi i j}{11}\right)} \right| dt.$$

Now, for $a, b \in \mathbb{R}$,

$$\left| \cosh\left(\frac{\pi i \nu}{k} - \frac{\pi i}{6k} - t \right) \right| \leq \cosh(t), \qquad |\sinh(a+ib)|^2 \geq \begin{cases} \frac{e^{2|a|}}{6} & \text{if } |a| \geq 1, \\ \sin(b)^2 & \text{if } |a| \leq 1. \end{cases}$$

Thus, using that $\mathrm{Re}(\frac{1}{z}) > \frac{k}{2}$

$$\left| z^{\frac{1}{2}} I_{j,11,k,\nu}(z) \right| \leq \frac{k}{\pi |z|^{\frac{1}{2}}} \left(6 \int_{|t|\geq 1} e^{-\frac{3t^2}{2\pi}} \cosh(t) e^{-2|t|} dt \right.$$

$$\left. + \int_{|t|\leq 1} \frac{e^{-\frac{3t^2}{2\pi}} \cosh(t)}{\left|\sin\left(\frac{\pi\nu}{k} - \frac{\pi}{6k} + \frac{\pi j}{11}\right)\right| \left|\sin\left(\frac{\pi\nu}{k} - \frac{\pi}{6k} - \frac{\pi j}{11}\right)\right|} dt \right)$$

$$\leq \frac{k}{|z|^{\frac{1}{2}}} \left(0.21 + \frac{0.64}{\left|\sin\left(\frac{\pi\nu}{k} - \frac{\pi}{6k} + \frac{\pi j}{11}\right)\right| \left|\sin\left(\frac{\pi\nu}{k} - \frac{\pi}{6k} - \frac{\pi j}{11}\right)\right|} \right).$$

Now note that $\frac{\nu}{k} - \frac{1}{6k} \pm \frac{j}{11} \notin \mathbb{Z}$. Thus we bound

$$\left| \sin\left(\frac{\pi\nu}{k} - \frac{\pi}{6k} + \frac{\pi j}{11} \right) \right| \left| \sin\left(\frac{\pi\nu}{k} - \frac{\pi}{6k} - \frac{\pi j}{11} \right) \right| \geq \sin\left(\frac{\pi}{66k}\right) \sin\left(\frac{\pi}{66k} + \frac{\pi}{11}\right)$$

$$\geq \frac{\pi}{66k} \sin\left(\frac{\pi}{11}\right),$$

using that for $0 \leq x \leq \frac{\pi}{2}$, $\sin(x)\sin(x + \frac{\pi}{11}) \geq x \sin(\frac{\pi}{11})$. This gives that

$$\left| z^{\frac{1}{2}} I_{j,11,k,\nu}(z) \right| \leq \frac{k}{|z|^{\frac{1}{2}}} (0.21 + 47.73k) \leq \frac{48k^2}{|z|^{\frac{1}{2}}}.$$

Using $|z| \geq \frac{k}{n}$ we obtain

$$\left| \sum_3 \right| \leq 2\sin\left(\frac{5\pi}{11}\right) \sin\left(\frac{\pi j}{11}\right) 48 \sum_{\substack{0 \leq h < k \leq \sqrt{n} \\ \gcd(h,k)=1}} \frac{1}{k} k^2 \left(\frac{k}{n}\right)^{-\frac{1}{2}} \sum_{\nu=0}^{k-1} \frac{2}{k(\lfloor\sqrt{n}\rfloor + 1)} e^{\frac{2\pi}{k}\frac{k}{n}\left(n - \frac{1}{24}\right)}$$

$$\leq \frac{4 \cdot 48 \sin\left(\frac{5\pi}{11}\right) \sin\left(\frac{\pi j}{11}\right) \sqrt{n} e^{2\pi}}{\lfloor\sqrt{n}\rfloor + 1} \sum_{1 \leq k \leq \sqrt{n}} k^{\frac{3}{2}} \leq 40707.2 \sin\left(\frac{\pi j}{11}\right) n^{\frac{5}{4}}.$$

Combining gives as error

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$$\sin\left(\frac{\pi j}{11}\right)\left(\frac{1.5n^{\frac{3}{4}}}{\sqrt{24n-1}}\sinh\left(\frac{\pi\sqrt{24n-1}}{66}\right)+\frac{16}{\sqrt{3}\sqrt{24n-1}}n^{\frac{3}{4}}\sinh\left(\frac{5\pi\sqrt{24n-1}}{132}\right)\right.$$

$$\left.+47.3n^{\frac{1}{4}}+8567.9n^{\frac{1}{4}}+3\cdot898.1n^{\frac{1}{4}}+3\cdot2261.1n^{\frac{1}{4}}+40707.2n^{\frac{5}{4}}\right).$$

Simplifying gives the claim.

5. Proof of Theorem 1.2

We are now ready to prove Theorem 1.2.

5.1. $\mathcal{C}_a^{[1]}(n)$. We need to show that

$$M_{a,d}(n)-\frac{p_{11}(n)}{11}\begin{cases}>0 & \text{if }(a,d)\in\mathcal{S}_1,\\<0 & \text{if }(a,d)\in\mathcal{S}_2,\end{cases}$$

where

$$\mathcal{S}_1:=\{(0,0),(1,1),(2,2),(0,3),(0,4),(0,5),(1,7),(1,8),(1,9),(0,10)\},$$
$$\mathcal{S}_2:=\{(1,0),(2,1),(0,2),(1,3),(1,4),(2,5),(0,7),(0,8),(0,9),(3,10)\}.$$

For $(j,d)\in\{(0,8),(1,8)\}$, we need to additionally assume that $n\geq41$. To show the claim, we require

$$2\sum_{j=1}^{5}\cos\left(\frac{2\pi aj}{11}\right)A_j(n)\begin{cases}>0 & \text{if }(a,d)\in\mathcal{S}_1,\\<0 & \text{if }(a,d)\in\mathcal{S}_2,\end{cases}$$

where $n\equiv d\pmod{11}$. A computer check gives that the claim holds for $n\leq1779$ (up to the above mentioned exceptions).

For $n\geq1780$, we use Lemma 3.1 so that the left-hand side equals

$$\frac{4\sqrt{3}}{\sqrt{24n-1}}\sinh\left(\frac{\pi\sqrt{24n-1}}{66}\right)\left(\frac{2i}{\sqrt{11}}\sum_{j=1}^{5}\cos\left(\frac{2\pi aj}{11}\right)B_{j,11,11}^*(-n,0)+4\sin\left(\frac{\pi}{11}\right)\cos\left(\frac{2\pi a}{11}\right)\right)$$

$$+2\sum_{j=1}^{5}\sin\left(\frac{\pi j}{11}\right)\cos\left(\frac{2\pi aj}{11}\right)\mathcal{E}_j^{[1]}(n).$$

A numerical check gives that for $(a,d)\in\mathcal{S}_1$, the first summand is positive and for $(a,d)\in\mathcal{S}_2$, it is negative.

Let for $(a,d)\in\mathcal{S}_1\cup\mathcal{S}_2$

$$g_{a,d}(n):=4\sqrt{3}\left(\frac{2i}{\sqrt{11}}\sum_{j=1}^{5}\cos\left(\frac{2\pi aj}{11}\right)B_{j,11,11}^*(-11n-d,0)+4\sin\left(\frac{\pi}{11}\right)\cos\left(\frac{2\pi a}{11}\right)\right).$$

We compute

$$iB_{j,11,11}^*(-m,0)$$

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$$= 2(-1)^j \sin\left(\frac{\pi j}{11}\right)\left(-\frac{\cos\left(\frac{2\pi}{11}\left(m - \frac{j^2}{2} + 10\right)\right)}{\sin\left(\frac{\pi j}{11}\right)} - \frac{\cos\left(\frac{2\pi}{11}\left(2m + 5\frac{j^2}{2} + 7\right)\right)}{\sin\left(\frac{5\pi j}{11}\right)}\right.$$

$$\left. - \frac{\cos\left(\frac{2\pi}{11}(3m - 2j^2 + 2)\right)}{\sin\left(\frac{4\pi j}{11}\right)} - \frac{\cos\left(\frac{2\pi}{11}\left(4m - 3\frac{j^2}{2} + 2\right)\right)}{\sin\left(\frac{3\pi j}{11}\right)} + \frac{\cos\left(\frac{2\pi}{11}(5m + j^2 + 4)\right)}{\sin\left(\frac{2\pi j}{11}\right)}\right).$$

A computer check gives that for $n \geq 1780$, we have

$$\frac{2\sqrt{24(11n + d) - 1}\,\mathcal{E}_1(11n + d)}{\sinh\left(\frac{\pi\sqrt{24(11n+d)-1}}{66}\right)} \max_{(a,d)\in\mathcal{S}_1\cup\mathcal{S}_2} \frac{\sum_{j=1}^5 \left|\sin\left(\frac{\pi j}{11}\right)\cos\left(\frac{2\pi a j}{11}\right)\right|}{g_{a,d}(n)} < 1.$$

This gives the claim.

5.2. $\mathcal{C}_a^{[2]}(n)$.

We need to show that

$$N_{3,6}(n) \leq \frac{p_{11}(n)}{11} \leq N_{2,6}(n)$$

$$\Leftrightarrow 2\sum_{j=0}^5 \cos\left(\frac{6\pi j}{11}\right) B_j(11n+6) < 0, \quad 2\sum_{j=0}^5 \cos\left(\frac{4\pi j}{11}\right) B_j(11n+6) > 0 \quad (n \equiv 6 \pmod{11}).$$

A computer check gives that this is true for $n \leq 45$.

For $n \geq 46$, we use Lemma 4.1. This gives for the sums of interest ($a \in \{2, 3\}$)

$$2\cos\left(\frac{2\pi a}{11}\right) \frac{8\sqrt{3}\sin\left(\frac{\pi}{11}\right)}{\sqrt{24(11n + 6) - 1}} \sinh\left(\frac{5\pi\sqrt{24(11n + 6) - 1}}{66}\right)$$

$$+ 2\sum_{j=1}^5 \cos\left(\frac{2\pi a j}{11}\right) \sin\left(\frac{\pi j}{11}\right) \mathcal{E}_j^{[2]}(11n + 6).$$

So the sign of the main term is dictated by $\cos(\frac{2\pi a}{11})$ which is positive for $a = 2$ and negative for $a = 3$ as desired.

To finish the claim, one can show that for $n \geq 46$,

$$\frac{2\mathcal{E}_2(11n + 6) \sum_{j=1}^5 \left|\cos\left(\frac{2\pi a j}{11}\right)\right| \sin\left(\frac{\pi j}{11}\right)}{2\left|\cos\left(\frac{2\pi a}{11}\right)\right| \frac{8\sqrt{3}\sin\left(\frac{\pi}{11}\right)}{\sqrt{24(11n+6)-1}} \sinh\left(\frac{5\pi\sqrt{24(11n+6)-1}}{66}\right)} < 1.$$

5.3. $\mathcal{C}_{a_1,a_2}^{[3]}(n)$.

We need to show that

$$N_{5,d}(n) \leq N_{4,d}(n) \leq N_{3,d}(n), \qquad N_{2,d}(n) \leq N_{1,d}(n) \leq N_{0,d}(n).$$

If $(d, a_1, a_2) \in \{(3, 1, 2), (5, 3, 4), (6, 4, 5), (6, 0, 1), (7, 0, 1)\}$, then we need to additionally assume that $n > 11$, if $(d, a_1, a_2) \in \{(0, 3, 4), (3, 0, 1), (9, 0, 1)\}$, then we require $n > 22$, and

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if $(d, a_1, a_2) \in \{(0,0,1), (2,0,1), (4,3,4), (4,0,1), (8,0,1), (10,0,1)\}$, then we let $n > 33$. The claim follows if we prove that

$$N_{a,d}(n) - N_{a+1,d}(n) \geq 0 \text{ for } 0 \leq a \leq 4.$$

Thus we need

$$\mathcal{C}^{[3]}_{a,a+1}(11n + d) \geq 0 \Leftrightarrow 2 \sum_{j=1}^{5} \left(\cos\left(\frac{2\pi a j}{11}\right) - \cos\left(\frac{2\pi(a+1)j}{11}\right) \right) B_j(11n + d) \geq 0.$$

A computer check gives that this is true for $n \leq 46$ (up to the above mentioned exception). For $n \geq 47$, we again use Lemma 4.1. The overall difference is

$$\frac{8\sqrt{3}}{\sqrt{24(11n+d)-1}} \sinh\left(\frac{5\pi}{66}\sqrt{24(11n+d)-1}\right) 2 \left(\cos\left(\frac{2\pi a}{11}\right) - \cos\left(\frac{2\pi(a+1)}{11}\right)\right) \sin\left(\frac{\pi}{11}\right)$$

$$+ 2 \sum_{j=1}^{5} \left| \cos\left(\frac{2\pi a j}{11}\right) - \cos\left(\frac{2\pi(a+1)j}{11}\right) \right| \sin\left(\frac{\pi j}{11}\right) \mathcal{E}_j^{[2]}(11n + d).$$

Now

$$\cos\left(\frac{2\pi a}{11}\right) - \cos\left(\frac{2\pi(a+1)}{11}\right) > 0.$$

To finish the claim, we show that, for $0 \leq a \leq 4$, and $n \geq 47$,

$$\frac{2\mathcal{E}_2(11n + d) \sum_{j=1}^{5} \left| \cos\left(\frac{2\pi a j}{11}\right) - \cos\left(\frac{2\pi(a+1)j}{11}\right) \right| \sin\left(\frac{\pi j}{11}\right)}{\frac{8\sqrt{3}}{\sqrt{24(11n+d)-1}} \sinh\left(\frac{5\pi}{66}\sqrt{24(11n+d)-1}\right) \cdot 2 \sin\left(\frac{\pi}{11}\right)\left(\cos\left(\frac{2\pi a}{11}\right) - \cos\left(\frac{2\pi(a+1)}{11}\right)\right)} < 1.$$

5.4. $\mathcal{C}^{[4]}_{a_1,a_2}(n)$. We need to show that

$$M_{a_1,d}(n) - N_{a_2,d}(n) \begin{cases} > 0 & \text{if } (a_1, a_2, d) \in \mathbb{S}_1, \\ < 0 & \text{if } (a_1, a_2, d) \in \mathbb{S}_2, \end{cases}$$

where

$$\mathbb{S}_1 := \{(1,3,0), (0,3,1), (0,3,2), (1,3,3), (1,3,4), (2,3,5), (0,3,7), (0,3,8),$$
$$(0,3,9), (3,3,10)\},$$
$$\mathbb{S}_2 := \{(0,2,0), (1,2,1), (2,2,2), (0,2,3), (0,2,4), (0,2,5), (1,2,7), (1,2,8),$$
$$(1,2,9), (0,2,10)\}.$$

If $(a_1, a_2, d) \in \{(0,2,0), (0,3,1), (1,2,1), (2,2,2), (1,3,4), (0,2,4), (0,3,9), (3,3,10)\}$, then we need to additionally assume that $n \geq 12$. A computer check shows that the claim holds for $n \leq 44$ (up to the above mentioned exception).

We next turn to $n \geq 45$. Note that from the asymptotics above the main term from the rank is dominant. Thus main term contributes

$$\frac{16\sqrt{3}}{\sqrt{24(11n+d)-1}} \sinh\left(\frac{5\pi}{66}\sqrt{24(11n+d)-1}\right) \sin\left(\frac{\pi}{11}\right) \cos\left(\frac{2\pi a_2}{11}\right).$$

Note that a_2 determines whether we are in $\mathbb{S}_1$ or $\mathbb{S}_2$. Since

$$\cos\left(\frac{4\pi}{11}\right) > 0, \qquad \cos\left(\frac{6\pi}{11}\right) < 0,$$

the above expresssion is positive if $a_2 = 2$ and negative if $a_2 = 3$.

Using Lemmas 3.1 and 4.1 the error term is bounded by

$$2\mathcal{E}_1(11n+d)\sum_{j=1}^{5}\left|\cos\left(\frac{2\pi a_1 j}{11}\right)\right|\sin\left(\frac{\pi j}{11}\right) + 2\mathcal{E}_2(11n+d)\sum_{j=1}^{5}\left|\cos\left(\frac{2\pi a_2 j}{11}\right)\right|\sin\left(\frac{\pi j}{11}\right)$$

$$+ 2\sum_{j=1}^{5}\left|\cos\left(\frac{2\pi a_2 j}{11}\right)\right||M_j(11n+d)|.$$

We estimate

$$2\sum_{j=1}^{5}\left|\cos\left(\frac{2\pi a_2 j}{11}\right)\right||M_j(11n+d)| \le \frac{8\sqrt{3}\sinh\left(\frac{\pi\sqrt{24(11n+d)-1}}{66}\right)}{\sqrt{24(11n+d)-1}}$$

$$\times \max_{a_2\in\{2,3\}}\left(\frac{\sqrt{11}}{\sin\left(\frac{\pi}{11}\right)}\sum_{j=1}^{5}\left|\cos\left(\frac{2\pi a_2 j}{11}\right)\right|\sin\left(\frac{\pi j}{11}\right) + 2\sin\left(\frac{\pi}{11}\right)\left|\cos\left(\frac{2\pi a_2}{11}\right)\right|\right).$$

The maximum is obtained for $a_2 = 3$ and this can be bound against

$$\frac{374.1\sinh\left(\frac{\pi\sqrt{24(11n+d)-1}}{66}\right)}{\sqrt{24(11n+d)-1}}.$$

Similarly

$$2\max_{a_2\in\{2,3\}}\sum_{j=1}^{5}\left|\cos\left(\frac{2\pi a_2 j}{11}\right)\right|\sin\left(\frac{\pi j}{11}\right)$$

is obtained for $a_2 = 3$. It can be bound against 4.58. Finally

$$2\max_{0\le a_1\le 3}\sum_{j=1}^{5}\left|\cos\left(\frac{2\pi a_1 j}{11}\right)\right|\sin\left(\frac{\pi j}{11}\right)$$

is obtained for $a_1 = 0$ and is bound in this case by 7. Thus the error can be estimated against

$$7\mathcal{E}_1(11n+d) + 4.58\mathcal{E}_2(11n+d) + \frac{374.1\sinh\left(\frac{\pi\sqrt{24(11n+d)-1}}{66}\right)}{\sqrt{24(11n+d)-1}}.$$

A computer calculation shows that for $a_2 \in \{2,3\}$ and $n \geq 45$

$$\frac{7\mathcal{E}_1(11n+d) + 4.58\mathcal{E}_2(11n+d) + \dfrac{374.1\sinh\left(\frac{\pi\sqrt{24(11n+d)-1}}{66}\right)}{\sqrt{24(11n+d)-1}}}{\dfrac{16\sqrt{3}}{\sqrt{24(11n+d)-1}}\sinh\left(\frac{5\pi}{66}\sqrt{24(11n+d)-1}\right)\sin\left(\frac{\pi}{11}\right)\left|\cos\left(\frac{2\pi a_2}{11}\right)\right|} < 1.$$

5.5. $\mathcal{C}_{a_1,a_2}^{[5]}(n)$. We need to show that for all $n \in \mathbb{N}_0$

$$M_{0,1}(n) \leq M_{2,1}(n).$$

This follows by Corollary 4.1 (4) of [12].

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University of Cologne, Department of Mathematics and Computer Science, Weyertal 86-90, 50931 Cologne, Germany

Email address: kbringma@math.uni-koeln.de

Email address: bpandey@uni-koeln.de, badrivishal9451@gmail.com