IMPROVED BOUNDS FOR FOURIER COEFFICIENTS OF SIEGEL MODULAR FORMS

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1. INTRODUCTION AND STATEMENT OF RESULTS

The goal of this paper is to improve existing bounds for Fourier coefficients of higher genus Siegel modular forms of small weight.

To recall what is known for genus 1, let $\Delta(\tau) := q \prod_{n=1}^{\infty}(1 - q^n)^{24}$ ($q := e^{2\pi i \tau}$) be the classical Δ-function and denote by $\tau(n)$ its Fourier coefficients. The *Ramanujan conjecture* states that, for p prime,

$$|\tau(p)| \leq 2p^{\frac{11}{2}}.$$

This conjecture has been generalized for general, positive integral weight modular forms. The so-called *Ramanujan-Petersson conjecture* states that if $f(\tau) = \sum_{n=1}^{\infty} a(n)q^n$ is a weight k cusp form on a congruence subgroup, then, as $n \to \infty$,

$$(1.1) \qquad a(n) \ll_{\varepsilon,f} n^{\frac{k-1}{2}+\varepsilon} \qquad (\varepsilon > 0).$$

The estimate (1.1) follows from Deligne's proof of the Weil conjectures [7, 8], using highly complicated methods from algebraic geometry.

There are many related conjectures for more complicated types of automorphic forms. In this paper, we consider the case of Siegel modular form of genus $g > 1$. For this, let F be a cusp form of weight $k \in \mathbb{N}$ with respect to the Siegel modular group $\Gamma_g := \mathrm{Sp}_g(\mathbb{Z}) \subset \mathrm{GL}_{2g}(\mathbb{Z})$ with Fourier coefficients $a(T)$, where T is a positive definite symmetric half-integral $g \times g$ matrix. Then a conjecture of Resnikoff and Saldaña [16] says that

$$a(T) \ll_{\varepsilon,F} \det(T)^{\frac{k}{2} - \frac{(g+1)}{4}+\varepsilon} \qquad (\varepsilon > 0).$$

For $g = 1$ this is exactly the Ramanujan-Petersson conjecture. For higher genus g, however, there are counterexamples coming from lifts (cf. [13]).

For $k > g + 1$, the best known estimate is

$$(1.2) \qquad a(T) \ll_{\varepsilon,F} \det(T)^{\frac{k}{2} - c_g + \varepsilon} \qquad (\varepsilon > 0),$$

where

$$c_g := \begin{cases} \frac{13}{36} & \text{if } g = 2 \quad [13], \\ \frac{1}{4} & \text{if } g = 3 \quad [3], \\ \frac{1}{2g} + \left(1 - \frac{1}{g}\right)\alpha_g & \text{if } g > 3 \quad [2]. \end{cases}$$

Here

$$(1.3) \qquad \alpha_g^{-1} := 4(g-1) + 4 \left[\frac{g-1}{2} \right] + \frac{2}{g+2}.$$

In [4] and [5] it was shown that (1.2) still holds for $k = g+1$ and $k = g$, respectively. Moreover, for $(g+3)/2 < k < g$, we have [5]

$$(1.4) \qquad a(T) \ll_{\varepsilon,F} \det(T)^{\frac{k}{2} - \left(1 - \frac{1}{g}\right)\alpha_g + \varepsilon}.$$

In this paper we improve (1.4) and obtain

Theorem 1.1. *We have for $g/2 + 1 < k < g$*

$$a(T) \ll_{\varepsilon,F} \det(T)^{\frac{k}{2} + \frac{g-k}{2g(g-2)} - \frac{1}{2g} - \left(1 - \frac{1}{g}\right)\alpha_g + \varepsilon}.$$

Remark. Theorem 1.1 is indeed an improvement since

$$\frac{g-k}{2g(g-2)} - \frac{1}{2g} < 0.$$

Our proof follows the idea of [2] using a Jacobi decomposition of Siegel modular forms. Our main achievement is an improved bound for Kloosterman sums.

The paper is organized as follows. In Section 2 we recall basic facts about Jacobi forms and their relation to Siegel modular forms. In Section 3 we bound higher dimensional Kloosterman sums. Section 4 is devoted estimating coefficients of Poincaré series, in Section 5 we then conclude our main theorem.

2. Preliminaries

2.1. **Basic facts on Jacobi forms.**
Here we recall some basic facts about Jacobi cusp forms; for details we refer the reader to [9] and [19]. The Jacobi group $\Gamma_g^J := \mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^g \times \mathbb{Z}^g)$ acts on $\mathbb{H} \times \mathbb{C}^g$ in the usual way by $(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z}), (\lambda, \mu) \in \mathbb{Z}^g)$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \circ (\tau, z) := \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right).$$

Note that throughout vectors are viewed as columns unless noted otherwise. Let $k \in \mathbb{N}$, m be a positive definite symmetric half-integral $g \times g$ matrix, $\gamma = (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), (\lambda, \mu)) \in \Gamma_g^J$, and $\phi : \mathbb{H} \times \mathbb{C}^g \to \mathbb{C}$. Then we define the following *Jacobi slash action*

$$\phi|_{k,m}\gamma(\tau, z) := (c\tau + d)^{-k} e\left(-c(c\tau + d)^{-1} m[z + \lambda\tau + \mu] + m[\lambda]\tau + 2\lambda^T m z \right) \phi(\gamma \circ (\tau, z)),$$

where $e(w) := e^{2\pi i w}$ $(\forall w \in \mathbb{C})$, and where $A[B] := B^T A B$ for matrices A and B of compatible sizes.

2

A holomorphic function $\phi : \mathbb{H} \times \mathbb{C}^g \to \mathbb{C}$ is called a *Jacobi cusp form* of weight k and index m with respect to Γ_g^J, if, for all $\gamma \in \Gamma_g^J$, we have $\phi|_{k,m}\gamma = \phi$, and ϕ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{D>0} c(n, r)e\left(n\tau + r^T z\right),$$

where $D := \det \left(\begin{smallmatrix} 2n & r^T \\ r & 2m \end{smallmatrix}\right)$ with $n \in \mathbb{N}$ and $r \in \mathbb{Z}^g$. We denote by $J_{k,m}^{\mathrm{cusp}}$ the vector space of Jacobi cusp forms.

The space $J_{k,m}^{\mathrm{cusp}}$ is a finite dimensional Hilbert space with the Petersson scalar product

$$\langle \phi, \psi \rangle := \int_{\Gamma_g^J \backslash \mathbb{H} \times \mathbb{C}^g} \phi(\tau, z)\overline{\psi(\tau, z)} \exp\left(-4\pi m[y] \cdot v^{-1}\right) v^k dV_g^J,$$

where $dV_g^J := v^{-g-2}dudvdxdy$, $\tau = u + iv$, and $z = x + iy$.

2.2. Jacobi Poincaré series.

We next recall certain Jacobi Poincaré series, as considered in [2]. For $n \in \mathbb{Z}$, $r \in \mathbb{Z}^g$, and m a positive definite symmetric half-integral $g \times g$ matrix such that $4n > m^{-1}[r]$, define a *Poincaré series of exponential type* by

$$(2.1) \qquad P_{k,m;(n,r)}(\tau, z) := \sum_{\gamma \in \Gamma_{g,\infty}^J \backslash \Gamma_g^J} \left. e^{n,r}\right|_{k,m} \gamma(\tau, z),$$

where $e^{n,r}(\tau, z) := e^{2\pi i(n\tau + r^T z)}$ and $\Gamma_{g,\infty}^J := \{((\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}), (0, \mu))|n \in \mathbb{Z}, \mu \in \mathbb{Z}^g\}$ is the stabilizer group of $e^{n,r}$. For $k > g + 2$, $P_{k,m;(n,r)} \in J_{k,m}^{\mathrm{cusp}}$ and the Petersson coefficient formula holds ($\phi \in J_{k,m}^{\mathrm{cusp}}$ with Fourier coefficients c_ϕ),

$$(2.2) \qquad \left\langle \phi, P_{k,m;(n,r)} \right\rangle = \lambda_{k,m,D} c_\phi(n, r),$$

where

$$\lambda_{k,m,D} := 2^{-\frac{g}{2}} \Gamma\left(k - \frac{g}{2} - 1\right)(2\pi)^{-k+\frac{g}{2}+1} \det(2m)^{k-\frac{g+3}{2}} D^{-k+\frac{g}{2}+1}.$$

For $k \leq g + 2$ the Poincaré series (2.1) diverge. However there is a way to analytically continue them, using the so-called Hecke trick. We denote the corresponding functions again by $P_{k,m;(n,r)}$. We have [2, 4, 5]:

Proposition 2.1. *For $k > g/2 + 2$, the functions $P_{k,m;(n,r)}$ are elements of $J_{k,m}^{\mathrm{cusp}}$. We have the Fourier expansions*

$$P_{k,m;(n,r)}(\tau, z) = \sum_{\substack{n' \in \mathbb{Z}, r' \in \mathbb{Z}^g \\ D'>0}} g_{k,m;(n,r)}^{\pm}(n', r')e\left(n'\tau + r'^T z\right),$$

where $D' := \det\left(\begin{smallmatrix} 2n' & r'^T \\ r' & 2m \end{smallmatrix}\right)$ and

$$g_{k,m;(n,r)}^{\pm}(n', r') := g_{k,m;(n,r)}(n', r') + (-1)^k g_{k,m;(n,r)}(n', -r')$$

with

$$(2.3) \quad g_{k,m;(n,r)}(n', r') := \delta_m(n, r, n', r') + 2\pi i^k \det(2m)^{-\frac{1}{2}} \left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{g}{4}-\frac{1}{2}}$$

$$\times \sum_{c \geq 1} e_{2c}\left(r^T m^{-1} r'\right) H_{m,c}(n, r, n', r') J_{k-\frac{g}{2}-1}\left(\frac{2\pi\sqrt{DD'}}{\det(2m)c}\right) c^{-\frac{g}{2}-1}.$$

3

Here $e_c(x) := e^{\frac{2\pi i x}{c}}$,

$$\delta_m(n, r, n', r') := \begin{cases} 1 & \text{if } D' = D \text{ and } r' - r \in 2m\mathbb{Z}^g, \\ 0 & \text{otherwise} \end{cases}$$

and the Kloosterman sums

$$H_{m,c}(n, r, n', r') := \sum_{\substack{\lambda \pmod c \\ d \pmod{c}^*}} e_c\left((m[\lambda] + r^T\lambda + n)\overline{d} + n'd + r'^T\lambda\right),$$

where by λ (mod c), we mean that all components run (mod c) and d (mod c)* sums only over d (mod c) which are coprime to c. Moreover formula (2.2) holds.

Remark. Note that in [2] the Kloosterman sums have a slightly different normalization.

Proposition 2.1 gives that for $k > g/2 + 2$ the $P_{k,m;(n,r)}$ are a generating system of $J_{k,m}^{\text{cusp}}$. We easily obtain, just using the Cauchy-Schwarz inequality

Lemma 2.2. *For $k > g/2 + 2$ and $\phi \in J_{k,m}^{\text{cusp}}$ with Fourier coefficients $c_\phi(n, r)$, we have*

$$|c_\phi(n, r)| \ll_k \left|b_{n,r}\left(P_{k,m;(n,r)}\right)\right|^{\frac{1}{2}} \frac{D^{\frac{k}{2} - \frac{g}{4} - \frac{1}{2}}}{\det(2m)^{\frac{k}{2} - \frac{1}{4}(g+3)}} \|\phi\|,$$

where $b_{n,r}(P_{k,m;(n,r)})$ denotes the (n, r)th Fourier coefficient of the Poincaré series $P_{k,m;(n,r)}$.

Thus, to get bounds for the Fourier coefficients of Jacobi forms, one only has to bound the Fourier coefficients of the Poincaré series which are explicitly given in Proposition 2.1. However, we also bound coefficients of Siegel modular forms, which requires estimating $\|\phi\|$. The connection between Siegel modular forms and Jacobi forms is described in the next subsection.

2.3. **Relation to Siegel modular forms.** Let $\mathbb{H}_g$ be the usual Siegel upper half space and write $Z \in \mathbb{H}_g$ as $Z = \left(\begin{smallmatrix} \tau & z^T \\ z & \tau' \end{smallmatrix}\right)$ with $\tau \in \mathbb{H}$, $z \in \mathbb{C}^{g-1}$, and $\tau' \in \mathbb{H}_{g-1}$. Then $F \in S_k(\Gamma_g)$, the space of Siegel cusp forms of weight k for Γ_g, has a so-called *Fourier Jacobi expansion* of the form

$$F(Z) = \sum_{m>0} \phi_m(\tau, z) e^{2\pi i \operatorname{tr}(m\tau')},$$

where tr denotes the trace of a matrix and where m runs through all positive definite symmetric half-integral $(g-1) \times (g-1)$ matrices. It is well-known, that the coefficients of ϕ_m are Jacobi cusp forms. So bounds for the Fourier coefficients of Siegel modular forms follow from the understanding of the coefficients of Jacobi forms.

3. Bounding Kloosterman sums

A first step in bounding Fourier expansions of Poincaré series is to estimate certain higher-dimensional Kloosterman sums which occur when restricting the Fourier coefficients of Jacobi Poincaré series to the diagonal $(n', r') = (n, r)$. To be more precise, we set

$$H_{m,c}^\pm(n, r) := H_{m,c}(n, r, n, \pm r).$$

To bound these, we require well-known evaluations of (generalized) *Gauss sums*

$$G(a, b; c) := \sum_{n \pmod c} e_c\left(an^2 + bn\right).$$

4

Lemma 3.1. *Let p be prime, $a, b \in \mathbb{Z}$, $\nu \in \mathbb{N}$, and $\alpha := \operatorname{ord}_p(a)$.*
(1) For $\alpha \geq \nu$, we have

$$G(a, b; p^\nu) = \begin{cases} p^\nu & \text{if } b \equiv 0 \pmod{p^\nu}, \\ 0 & \text{otherwise.} \end{cases}$$

(2) For $0 \leq \alpha < \nu$, $G(a, b; p^\nu) = 0$ unless $b \equiv 0 \pmod{p^\alpha}$ in which case we have the following evaluations:
(i) If $p \neq 2$ and $b \equiv 0 \pmod{p^\alpha}$, then

$$G(a, b; p^\nu) = p^{\frac{\alpha+\nu}{2}} \varepsilon_{p^{\nu-\alpha}} \left(\frac{a/p^\alpha}{p^{\nu-\alpha}} \right) e_{p^{\nu+\alpha}} \left(\overline{4a/p^\alpha} \right),$$

where $\bar{\ell}$ denotes the inverse of ℓ $(\bmod\ p^{\nu+a})$ and $\varepsilon_j = 1$ or i depending on whether $j \equiv 1 \pmod 4$ or $j \equiv 3 \pmod 4$, respectively.
(ii) If $p = 2$ and $b \equiv 0 \pmod{p^\alpha}$, then $G(a, b; p^\nu)$ equals

$$\begin{cases} 2^\nu & \text{if } \alpha = \nu - 1 \text{ and } b \not\equiv 0 \pmod{2^\nu}, \\ 2^{\frac{\nu+\alpha}{2}} \left(\frac{-2^{\nu-\alpha}}{a/2^\alpha} \right) \varepsilon_{\frac{a}{2^\alpha}} (1+i) e_{2^{\nu+\alpha+2}} \left(-b^2 \overline{a/2^\alpha} \right) & \text{if } b \equiv 0 \pmod{2^{\alpha+1}} \text{ and } \nu \equiv \alpha \pmod 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\bar{\ell}$ denotes the inverse of ℓ $(\bmod\ 2^{\nu+\alpha+2})$.

We are now ready to bound the higher-dimensional Kloosterman sums.

Lemma 3.2. *We have*

$$H_{m,c}^\pm(n, r) \ll 2^{\omega(c)} (D, c) \, c^{\frac{g+1}{2}} \det(2m)^{\frac{1}{2}},$$

where $\omega(c)$ denotes the number of prime divisors of c.

Proof: Our proof closely follows the one in [2]. There it was shown on page 507 that, for $c = c_1 c_2$ with $(c_1, c_2) = 1$,

$$H_{m,c}^\pm(n, r) = H_{c_1 m, c_2}^\pm(n\overline{c_1}, r) H_{c_2 m, c_1}^\pm(n\overline{c_2}, r),$$

where $\overline{c_1}$ and $\overline{c_2}$ are inverses of c_1 and c_2 modulo c_2 and c_1, respectively. Thus we may assume that $c = p^\nu$ with p prime and $\nu \in \mathbb{N}$ and for simplicity we for now restrict to $p \neq 2$. The modifications required for $p = 2$ follow along the same lines as in [2].

Since a non-degenerate binary quadratic form over $\mathbb{Z}_p$ $(p \neq 2)$ is diagonalizable, we may assume that $m = \operatorname{diag}(m_1, \ldots, m_g)$ is a diagonal matrix. Set $\mu_j := \operatorname{ord}_p(m_j)$ $(1 \leq j \leq g)$. We assume without loss of generality that $\nu \leq \mu_j$ for $1 \leq j \leq \ell$ and $\nu > \mu_j$ for $\ell+1 \leq j \leq g$. Write $r = (r_1, \ldots, r_g)$. From (18) of [2], we conclude that

$$H_{m,p^\nu}^\pm(n, r) = \sum_{d \pmod{p^\nu}^*} e_{p^\nu} \left(n \left(d + \bar{d} \right) \right) \prod_{j=1}^g \sum_{\lambda_j \pmod{p^\nu}} e_{p^\nu} \left(\left(m_j \lambda_j^2 + r_j \lambda_j \right) \bar{d} \pm r_j \lambda_j \right).$$

The sum on λ_j equals $G(m_j, r_j(\bar{d} \pm 1); p^\nu)$ and we may use Lemma 3.1 to evaluate it. For $1 \leq j \leq \ell$, we have

$$G\left(m_j, r_j \left(\bar{d} \pm 1 \right); p^\nu \right) = \begin{cases} p^\nu & \text{if } r_j \left(\bar{d} \pm 1 \right) \equiv 0 \pmod{p^\nu}, \\ 0 & \text{if } r_j \left(\bar{d} \pm 1 \right) \not\equiv 0 \pmod{p^\nu}. \end{cases}$$

5

For $\ell + 1 \leq j \leq g$, the Gauss sum equals

$$\begin{cases} p^{\frac{\nu+\mu_j}{2}} \varepsilon_{p^{\nu-\mu_j}} \left(\frac{m_j/p^{\mu_j}}{p^{\nu-\mu_j}} \right) e_{p^{\nu+\mu_j}} \left(-r_j^2 \left(\overline{d} \pm 1\right)^2 \overline{4\frac{m_j}{p^{\mu_j}}} \right) & \text{if } r_j \left(\overline{d} \pm 1\right) \equiv 0 \pmod{p^{\nu_j}}, \\ 0 & \text{if } r_j \left(\overline{d} \pm 1\right) \not\equiv 0 \pmod{p^{\nu_j}}. \end{cases}$$

Thus $H_{m,p^\nu}^\pm(n,r)$ becomes
(3.1)

$$p^{\nu\ell} \sum_{\substack{d \pmod{p^\nu} \\ r_j(\overline{d}\pm 1)\equiv 0 \pmod{p^\nu}(1\leq j\leq \ell) \\ r_j(\overline{d}\pm 1)\equiv 0 \pmod{p^{\mu_j}}(\ell+1\leq j\leq g)}} \prod_{j=\ell+1}^{g} \varepsilon_{p^{\nu-\mu_j}} \left(\frac{m_j/p^{\nu_j}}{p^{\nu-\mu_j}} \right) p^{\frac{\nu+\mu_j}{2}} e_{p^{\nu+\mu_j}} \left(-r_j^2 \left(\overline{d} \pm 1\right)^2 \overline{4\frac{m_j}{p^{\mu_j}}} \right).$$

We now consider whether $p^\nu \mid D$ or not.

If $p^\nu \mid D$, then we have $(D, p^\nu) = p^\nu$. We bound (3.1) trivially, yielding

$$\left| H_{m,p^\nu}^\pm(n,r) \right| \leq p^{\nu\ell} \cdot p^\nu \cdot p^{\frac{\nu}{2}(g-\ell)} p^{\frac{1}{2}\sum_{j=\ell+1}^{g} \mu_j} = (D, p^\nu) p^{\frac{\nu g}{2} + \frac{\nu\ell}{2} + \frac{1}{2}\sum_{j=\ell+1}^{g} \mu_j}.$$

Now

$$p^{\frac{\nu g}{2}+\frac{\nu}{2}} = c^{\frac{g+1}{2}}, \quad p^{\frac{\nu\ell}{2}+\frac{1}{2}\sum_{j=\ell+1}^{g}\mu_j} \leq p^{\frac{1}{2}\sum_{j=1}^{g}\mu_j} \leq \det(2m)^{\frac{1}{2}}, \quad p^{-\frac{\nu}{2}} \leq 1,$$

giving the claim in this case.

If $p^\nu \nmid D$, then we use that

$$(3.2) \qquad\qquad D = \frac{1}{2}\det(2m)\left(4n - m^{-1}[r]\right),$$

which follows from the Jacobi decomposition. This gives that p^ν divides at most one of the m_j. There are two cases to distinguish depending on whether p^ν divides one of the m_j or none.

We first assume $\nu > \mu_j$ for $1 \leq j \leq g$. In (27) of [2] it was shown that

$$\left| H_{m,c}^\pm(n,r) \right| \leq 2p^{\frac{\nu(g+1)}{2}} (D, p^\nu) \leq 2c^{\frac{g+1}{2}} (D, p^\nu).$$

This implies the claim in this case.

Finally we consider the case that p^ν divides exactly one m_j and we may assume without loss of generality that $\mu_g \geq \nu$. Let $\kappa := \max\{0, \nu - \rho_g, \mu_1 - \rho_1, \cdots, \mu_{g-1} - \rho_{g-1}\}$, where $\rho_j = \text{ord}(r_j)(1 \leq j \leq g)$ and let $\lambda := \text{ord}_p(D)$. It is shown in the first displayed formula on page 509 of [2] that

$$(3.3) \qquad\qquad \left| H_{m,c}^\pm(n,r) \right| \leq p^{2\nu-\kappa} \prod_{j=1}^{g-1} p^{\frac{1}{2}(\nu+\mu_j)} = p^{\frac{\nu(g+1)}{2}+\nu-\kappa} p^{\frac{1}{2}\sum_{j=1}^{g-1}\mu_j}.$$

We next analyze (3.2). Since $m^{-1} = \text{diag}(m_1^{-1}, \ldots, m_g^{-1})$, we obtain, since $p^\nu | m_g$,

$$D = 2^{g+1}n\prod_{j=1}^{g} m_j - 2^{g-1}\sum_{j=1}^{g} r_j^2 \prod_{\substack{\ell=1 \\ \ell\neq j}}^{g} m_j \equiv -2^{g-1}r_g^2 \prod_{j=1}^{g-1} m_j \pmod{p^\nu}.$$

Thus, since $\lambda < \nu$,

$$\lambda = \sum_{j=1}^{g-1} \mu_j + 2\rho_g.$$

6

Moreover, from the definition of κ, we obtain that $\kappa \geq \nu - \rho_g$. Thus, by (3.3),

$$\left| H_{m,c}^{\pm}(n,r) \right| \leq p^{\frac{\nu}{2}(g+1) + \frac{1}{2} \sum_{j=1}^{g-1} \mu_j + \rho_g} = p^{\frac{\nu}{2}(g+1) + \frac{\lambda}{2}} = c^{\frac{g+1}{2}}(D,C)^{\frac{1}{2}} \leq c^{\frac{g+1}{2}}(D,c).$$

This finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

4. Bounding coefficients of Poincaré series

In this section, we estimate the Fourier coefficients $b_{n,r}$ of $P_{k,m;(n,r)}$. This is of independent interest for obtaining bounds for Fourier coefficients of Jacobi forms.

Theorem 4.1. *Assume that $k \in \mathbb{N}$ satisfies $(g+3)/2 < k < g$. Then, with notation as above,*

$$b_{n,r}\left(P_{k,m;(n,r)}\right) \ll \left(1 + \frac{D^{\frac{g}{2}+\varepsilon}}{\det(2m)^{\frac{g+1}{2}}}\left(1 + D^{k-g-1}\det(2m)^{-k+g+1+\frac{1}{g-1}(-k+g+1)+\varepsilon}\right)\right).$$

Proof: We use the explicit representation of $b_{n,r}$ given in Proposition 2.1. Recall that $n' = n$ and $r' = r$ and thus $D' = D$. The first term in (2.3) yields the first term in the bound in Theorem 4.1. Thus, we have to bound

$$f_m(n,r) := \sum_{c \geq 1} \left| H_{m,c}^{\pm}(n,r) \right| J_{k-\frac{g}{2}-1}\left(\frac{2\pi D}{\det(2m)c}\right) c^{-\frac{g}{2}-1}.$$

We may rewrite

$$f_m(n,r) = \sum_{d \mid D} \sum_{\substack{c \geq 1 \\ \left(c,\frac{D}{d}\right)=1}} (cd)^{-\frac{g}{2}-1} \left| H_{m,dc}^{\pm}(n,r) \right| J_{k-\frac{g}{2}-1}\left(\frac{A}{c}\right),$$

where $A = A_d := \frac{2\pi D}{d\det(2m)}$. To bound the inner sum, we split it into three pieces: a part with $c \leq A$, a contribution from $A \leq c \leq B$, and a piece with $c \geq B$, with B to be determined later. Note that the range of any of these sums is allowed to be empty. We require the bounds for Kloosterman sums from Section 3 as well as the following estimates

$$\tag{4.1} \left| H_{m,c}^{\pm}(n,r) \right| \ll c^{g+\varepsilon}(D,c),$$

$$\tag{4.2} J_\ell(t) \ll_\ell \min\left\{t^{-\frac{1}{2}}, t^\ell\right\}.$$

The bound (4.1) follows from Lemma 2 of [2] whereas (4.2) is standard.

To bound the part with $c \leq A$, we use (4.1) with dc instead of c and the first estimate in (4.2). This gives the contribution for the sum on c

$$A^{-\frac{1}{2}}d^{\frac{g}{2}+\varepsilon} \sum_{1 \leq c \leq A} c^{\frac{g}{2}-\frac{1}{2}+\varepsilon} \ll A^{\frac{g}{2}+\varepsilon}d^{\frac{g}{2}+\varepsilon} \ll \left(\frac{D}{\det(2m)}\right)^{\frac{g}{2}+\varepsilon}.$$

Upon multiplying by $\det(2m)^{-\frac{1}{2}}$, this yields the second summand in the bound in Theorem 4.1.

Next we estimate the part with $A \leq c \leq B$. For this, we use (4.1) and the second estimate in (4.2). This gives the contribution, using that $k < g$,

$$A^{k-\frac{g}{2}-1}d^{\frac{g}{2}+\varepsilon} \sum_{A \leq c \leq B} c^{-k+g+\varepsilon} \ll A^{k-\frac{g}{2}-1}d^{\frac{g}{2}+\varepsilon}B^{-k+g+1+\varepsilon}.$$

Finally, we estimate the piece with $c \geq B$. For this, we use Lemma 3.2 and the second estimate in (4.2). This gives the bound

$$A^{k-\frac{g}{2}-1}d^{\frac{1}{2}}\det(2m)^{\frac{1}{2}}\sum_{c\geq B}2^{\omega(c)}c^{\frac{g+1}{2}-k} \ll B^{\frac{g+3}{2}-k+\varepsilon}A^{k-\frac{g}{2}-1}d^{\frac{1}{2}}\det(2m)^{\frac{1}{2}}$$

since $k > (g+3)/2$.

Now, to minimize the error, we choose B such that the second and third error agree (up to ε exponents). One can show that this is the case for

$$B = d^{-1}\det(2m)^{\frac{1}{g-1}}.$$

Plugging back in gives the claim after multiplying by $\det(2m)^{-\frac{1}{2}}$. $\qquad\square$

5. Proof of theorem 1.1

In this section we use the previous bounds with $g \mapsto g-1$ and $\phi = \phi_m$, where ϕ_m comes from the Jacobi coefficients of a Siegel modular form. We recall the following bound from Proposition 2 of [2].

Lemma 5.1. *If ϕ_m is the mth Fourier-Jacobi coefficient of a Siegel modular form F, then*

$$||\phi_m|| \ll_{\varepsilon,F} \det(2m)^{\frac{k}{2}-\alpha_g+\varepsilon} \qquad (\varepsilon > 0),$$

where α_g is defined in (1.3).

We are now ready to prove Theorem 1.1

Proof of Theorem 1.1: By Lemma 2.2 and Lemma 5.1,

$$a(T) \ll \left|b_{n,r}\left(P_{k,m;(n,r)}\right)\right|^{\frac{1}{2}} D^{\frac{k}{2}-\frac{g}{4}-\frac{1}{4}}\det(2m)^{\frac{g}{4}+\frac{1}{2}-\alpha_g+\varepsilon}.$$

Theorem 4.1 then yields

$$a(T) \ll \left(\det(2m)^{-\frac{g}{2}}f(m,D)\right)^{\frac{1}{2}} D^{\frac{k}{2}-\frac{g}{4}-\frac{1}{4}}\det(2m)^{\frac{g}{4}+\frac{1}{2}-\alpha_g+\varepsilon},$$

where

$$f(m,D) := \det(2m)^{\frac{g}{2}} + D^{\frac{g-1}{2}+\varepsilon}\left(1 + D^{k-g}\det(2m)^{-k+g+\frac{1}{g-2}(-k+g)+\varepsilon}\right).$$

Define

$$m_{g-1}(T) := \min\left\{T[U]|_{g-1} \mid U \in \mathrm{GL}_g(\mathbb{Z})\right\},$$

where $T[U]|_{g-1}$ denotes the determinant of the leading $(g-1)$-rowed submatrix of $T[U]$. Since both sides of the bound in Theorem 1.1 are invariant under replacing T by $T[U]$ ($U \in \mathrm{GL}_g(\mathbb{Z})$), we may assume that $T = \left(\begin{smallmatrix} n & r^T/2 \\ r/2 & m \end{smallmatrix}\right)$ with $\det(m) = m_{g-1}(T)$. Now, by reduction theory,

$$\det(m) = m_{g-1}(T) \ll D^{1-\frac{1}{g}}.$$

It is easy to see that the powers of $\det(m)$ in $f(m,D)$ are all non-negative. So we may replace $\det(m)$ by $D^{1-\frac{1}{g}}$ in this expression. One can then show that the last term in $f(m,D)$ is dominant. This gives

$$a(T) \ll \det(2m)^{\frac{1}{2}-\alpha_g+\varepsilon}D^{\frac{k}{2}-\frac{1}{2}+\frac{g-k}{2g(g-2)}+\varepsilon}.$$

This yields the claim of the theorem since $1/2 - \alpha_g > 0$. $\qquad\square$

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