$\mathcal{W}$-algebras, false theta functions and quantum modular forms, I

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Abstract

In this paper, we study certain partial and false theta functions in connection to vertex operator algebras and conformal field theory. We prove a variety of results concerning the asymptotics of modified characters of irreducible modules of certain $\mathcal{W}$-algebras of singlet type, which allows us in particular to determine their (analytic) quantum dimensions. Our results are fully consistent with the previous conjectures on fusion rings for these vertex algebras. More importantly, we prove quantum modularity (à la Zagier) of the numerator part of irreducible characters of singlet algebra modules, thus demonstrating that quantum modular forms naturally appear in many "sufficiently nice" irrational vertex algebras. It is interesting that quantum modularity persists on the whole set of rationals as in the original Zagier's example coming from Kontsevich's "strange series". In the last part, slightly independent of all this, we also discuss Nahm-type q-hypergometric series in connection to tails of colored Jones polynomials of certain torus knots and characters of modules for the $(1,p)$-singlet algebra.

1 Introduction

It is well-known that characters of modules of rational vertex operator algebras (e.g. WZW models, lattice vertex algebras, Virasoro minimal models) often take the form (throughout $q = e^{2\pi i\tau}$)

$$\frac{f(\tau)}{\eta(\tau)^k},$$

where $f(\tau)$ is the "numerator" part, expressible as a (linear combination) of theta functions and their derivatives, while the denominator has a fixed power of $\eta(\tau) := q^{\frac{1}{24}}\prod_{j=1}^{\infty}(1-q^j)$ for all irreducible modules. The power k can be viewed as a *rank*, for instance the rank of an integral lattice. The numerator usually comes from considerations and specializations in the BGG-type resolutions of modules and may involve both positive and negative coefficients in the q-expansion. Alternatively, formulas of this sort also arise from free field realizations of modules, possibly with different k. For rational vertex operator algebras with certain additional properties (cf. Section 2.2), the numerator part (and of course the η-power) is always a modular form of weight $\frac{k}{2}$ on a

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suitable congruence subgroup [17]. This fact alone is extremely useful for purposes of getting the explicit S-matrix, very precise asymptotics, just to mention a few examples.

In rational conformal field theory, it is important to analyze the following quotient of characters

$$\frac{\mathrm{ch}_M(\tau)}{\mathrm{ch}_V(\tau)}, \tag{1.1}$$

where M is a module for the vertex operator algebra V, for the purposes of computing quantum dimensions of modules [14]. Similar quotients also appear in computations of correlations functions, where we usually divide with the partition function, which is roughly the size of $\mathrm{ch}_V(\tau)$ (cf. [29]).

In this article, and continuing further in [11], we are mostly concerned with different properties of characters, including their quotients (1.1), for certain families of irrational vertex algebras which lack the usual modular properties. We do not focus on vertex algebras whose numerator part of the character is just a finite q-series, as these algebras are too generic to carry nice fusion properties [30]. Instead, we put emphasis on vertex algebras for which the numerator is a certain q-series similar to theta series but with "wrong" signs, often dubbed *false theta series.* Examples of vertex algebras of this kind have already appeared in the literature on representation theory of W-algebras [12, 13, 30, **?**], and also implicitly in connection to affine Lie superalgebras in the work of Kac and Wakimoto [26]. Our starting point is an observation made in [12], and even earlier in [3, 30] that some numerators of atypical characters of the so-called $(1, p)$-singlet algebra are false theta series of Rogers [6]. Singlet vertex algebras are never C_2-cofinite (e.g. Zamolodchikov $\mathcal{W}$-algebra at $c = -2$) so its representations theory is more delicate compared to other C_2-cofinite and rational $\mathcal{W}$-algebra extensions. Yet, the singlet algebras are instrumental in connection to various models in (logarithmic) conformal field theory.

In the theory of modular forms, false theta series have played a peculiar role. On the one hand, they arise in investigations of q-series and partition identities. We would like to single out a beautiful identity of Warnaar for partial thetas [34] (see also Section 4) as well as other identities, some due to Ramanujan [6]. More recently, starting with [36], it was discovered that certain false theta series also appear in connection to Kontsevich's "strange series" coming from the Vassiliev invariants of 3-manfiolds, in the sense that they both share the same values and derivatives of all orders at all roots of unity. Functions of a similar type also occurred in [27] in connection with the Witten-Reshetikhin-Turaev invariants of knots. All these instances are formalized later in an influential work of Zagier [38] under the umbrella of *quantum modular forms.* A quantum modular form lives at the natural boundary of the upper-half space $\mathbb{P}^1_{\mathbb{Q}}$, is defined only asymptotically, rather than exactly, and has a transformation behavior of a quite different type with respect to some modular group. There is an additional requirement that such a function extends to an analytic function defined in both upper and lower half-plane. This way one obtains a fairly non-standard object: an analytic function in the upper half-plane which drips throughout the *quantum set* (typically a proper subset of $\mathbb{Q}$) into the lower half-plane. There are further interesting occurences of such functions, see [20] for one such example. This has led to a whole family of new examples of quantum modular forms [20]. There are also examples coming from negative index Jacobi forms [10], which are closely related to characters of representations of affine Lie superalgebras.

Our present work is motivated by several considerations, which are however not mutually exclusive. Firstly, we are presenting more evidence (after [12]) that characters of irrational vertex algebras such as $\mathcal{W}$-algebras also enjoy nice analytic properties, including interesting asymptotics. Secondly, we would like to furnish more examples of quantum modular forms coming from "natural constructions" (e.g. vertex algebra theory) with the quantum set being all of $\mathbb{Q}$. Thirdly, here and especially in [11], we give more indication that irrational $\mathcal{W}$-algebras are closely related to certain infinite-dimensional quantum groups at root of unity. There are perhaps other ramifications of this line of work such as possible connections with the knot invariants via tails of colored Jones polynomials of alternating knots, although at this stage we deem them as largely accidental.

Let us summarize our main results in two theorems whose proofs are given in Sections 2-6. We use $M_{r,s}$ to denote atypical irreducible modules of the $(1,p)$-singlet algebra (see Section 3, and [12]).

Theorem 1.1. *For $p \in \mathbb{N}$ with $p \geq 2$, we have:*

(a) For $1 \leq s \leq p$ and $r \in \mathbb{Z}$, $\dim_q(M_{r,s}) = s$. Moreover, quantum dimensions of irreducible $(1,p)$-singlet modules are bounded above by p.

(b) The numerator of $\mathrm{ch}_{M_{r,s}}(\tau)$ is a quantum modular form of weight $\frac{1}{2}$ with quantum set $\mathbb{Q}$.

(c) The full asymptotic expansion of $\mathrm{ch}_{M_{r,s}}(\tau)$ is given in Proposition 4.5 and Remark 4.8.

This result gives additional evidence for the correctness of the fusion rules conjectured in [12], including Kazhdan-Lusztig type correspondence with quantum groups at root of unity.

Next, we switch to the (p_+, p_-)-singlet algebra introduced in [4]. These series of vertex operator algebras are closely related to the more familiar rational Virasoro minimal models [33].

Theorem 1.2. *Let $p_+, p_- \geq 2$ be relatively prime, and let $\mathcal{J}_{r,s;n}$ be an atypical irreducible (p_+, p_-)-singlet module (see Section 6, and [31], [13]). Then the following properties hold:*

(a) We have $\dim_q(\mathcal{J}_{r,s;n}) = rs$. Moreover, quantum dimensions of irreducible (p_+, p_-)-modules are bounded above by p_+p_-.

(b) The numerator of $\mathrm{ch}_{\mathcal{J}_{r,s;0}}(\tau)$ is a quantum modular form of mixed weight $\frac{3}{2}$ and $\frac{1}{2}$, while for $n \neq 0$, it is a mixed quantum modular form (a linear combination of modular and quantum modular forms).

(c) The full asymptotic expansion of $\mathrm{ch}_{\mathcal{J}_{r,s;0}}(\tau)$ is given in Proposition 5.2.

Again, this result is in the agreement with [13] and [31].

In the remainder of the paper (Section 7) we discuss at great length q-series identities between certain Nahm-type sums and characters of the $(1,p)$-singlet modules. These identities dovetail nicely with similar Nahm-type sums which already appear in the context of knot theory [22]. We provide a somewhat uniform approach to all these identities without any reference to representation or knot theory.

In Part 2 of this series [11], we introduce higher "rank" generalizations of classical partial and false theta functions, now in the setup of ADE type root lattices and Lie algebras (the classical case corresponding to $\mathfrak{g} = \mathfrak{sl}_2$) and study their asymptotic properties with an eye on quantum modularity.

2 "Generalized" quantum dimensions and fusion rules

The aim of this section is to introduce and discuss several basic objects coming from considerations of representations of vertex (operator) algebras and their characters.

2.1 Quantum dimensions

Throughout this section V is a vertex operator algebra as defined in [28]. We use M or sometimes M_j (if the index set is given) to denote irreducible V-modules. Following [1], a vertex operator algebra is said to be *rational* if every admissible module is completely reducible (for definition of an admissible module see also [1]). Essentially the same definition of rationality was previously given by Zhu [39]. Given a vertex operator algebra V, we let $C_2(V) := \mathrm{Span}_{\mathbb{C}}\{a_{-2}b : a, b \in V\}$, where we use a_n, $n \in \mathbb{Z}$, to denote the Fourier modes of the vertex operator $Y(a, x)$. Then V is called C_2*-cofinite* [39] if the subspace $C_2(V)$ is of finite codimension in V. It is widely believed that rationality implies C_2-cofiniteness. On the other hand, C_2-cofiniteness alone does not imply rationality (see for instance [3]). Rationality and C_2-cofiniteness independently imply that there are, up to isomorphism, only finitely many irreducible modules [16].

To simplify the presentation, we say that V is *regular* if it is both C_2-cofinite and rational. We also say that V is *strongly regular* if it is regular, simple, self-dual (meaning that $V' \cong V$) and $\dim(V(0)) = 1$, where $V = \bigoplus_{n=0}^{\infty} V(n)$.

In this paper, a key role is played by the character (aka modified graded dimension) of an irreducible module M:

$$\mathrm{ch}_M(\tau) := \mathrm{tr}_M q^{L(0)-\frac{c}{24}} = \sum_{n=0}^{\infty} \dim(M(n)) q^{h_M+n-\frac{c}{24}}, \tag{2.1}$$

where $h_M \in \mathbb{C}$ denotes the *lowest conformal weight* of M, $M(n)$ the graded subspace of conformal weight $h_M + n$, and c is the central charge. We focus on vertex operator algebras and modules for which $\mathrm{ch}_M(\tau)$ is holomorphic in $\mathbb{H}$.

Characters of modules can be used to define other important gadgets such as quantum dimensions of modules. The analytic *quantum dimension* of a V-module M, or simply the q-dimension, is defined as [14]:

$$\dim_q(M) := \lim_{y\to 0^+} \frac{\mathrm{ch}_M(iy)}{\mathrm{ch}_V(iy)}. \tag{2.2}$$

Clearly, $\dim_q(V) = 1$, but for general M it is often nontrivial to compute (2.2). In fact, for arbitrary V and M, the limit might not even exist [30]. Observe that (2.2) only depends on the "numerators" of M and V as discussed in the introduction. What is interesting about (2.2), is that for vertex algebras where the category V-Mod is *braided* and *rigid* [24], it is expected to be related to another quantum dimension coming from purely categorical properties of V-Mod. This categorical dimension, which we denote by $\widetilde{\dim}_q$, typically a real number, behaves as a multiplicative character under the tensor product $\boxtimes$ in the category of V-modules (also known as the *fusion* product [24]):

$$\widetilde{\dim}_q(M_r \boxtimes M_j) = \widetilde{\dim}_q(M_r) \cdot \widetilde{\dim}_q(M_j).$$

If the category V-Mod is in addition semi-simple, then

$$M_r \boxtimes M_j = \sum_{k=1}^{n} N_{r,j}^k M_k,$$

where $N_{r,j}^k \in \mathbb{N}$ are the fusion rules, and we have

$$\widetilde{\dim}_q(M_r) \cdot \widetilde{\dim}_q(M_j) = \sum_{k=1}^{n} N_{r,j}^k \widetilde{\dim}_q(M_k),$$

meaning that $\widetilde{\dim}_q(\cdot)$ defines a character of the fusion ring. All these properties follow easily from the definition of $\widetilde{\dim}_q$ (see [9] for details). It is not clear whether the two quantum dimensions are equal - but as we shall see in a moment - for a sufficiently nice V this seems to be the case.

The concept of analytic quantum dimension has the following natural generalization that is relevant for vertex algebras studied in this paper. For $s = \frac{h}{k} \in \mathbb{P}^1_{\mathbb{Q}}$, we define "quantum dimension at s" as the limit towards $\frac{h}{k}$ in the upper half-plane:

$$\dim_q^s(M) := \lim_{y \to 0^+} \frac{\mathrm{ch}_M\,(s+iy)}{\mathrm{ch}_V\,(s+iy)}.$$

Assume that a representative of s is chosen with h and k relatively prime and let $x, y \in \mathbb{Z}$ satisfy $xh + zk = 1$. Consider $A := \left(\begin{smallmatrix} h & -z \\ k & x \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and its inverse A^{-1}. Also, let $\tau_{s,y} := s + iy$. Assume that V is regular and that the irreducible characters are given by $\mathrm{ch}_{M_r}(\tau)$, $r = 0, ..., n$, such that the zero label is used for the vertex algebra, meaning $V = M_0$. Then, by Zhu's modular invariance theorem [39], we have

$$\mathrm{ch}_{M_r}(B \cdot \tau) = \sum_{j=0}^{n} B_{r,j} \mathrm{ch}_{M_j}(\tau),$$

for any $B \in \mathrm{SL}_2(\mathbb{Z})$. If V is strongly regular, then this representation is realized by unitary matrices (cf. Proposition 3.7 and Corollary 3.8 in [17]). Hence $B^{-1}_{r,j} = \bar{B}_{j,r}$, and the entries $B_{r,j}$ lie in a cyclotomic extension of $\mathbb{Q}$.

Theorem 2.1. *Let V be regular, $s \in \mathbb{Q}$, and A be as above. Assume that $A_{0,0} \neq 0$ and that lowest conformal weights satisfy $h_{M_r} > 0$, $1 \leq r \leq n$. Then*

$$\dim_q^s(M_r) = \frac{A_{r,0}}{A_{0,0}}.$$

Proof: Clearly, $A^{-1} \cdot \tau_{s,y} = \frac{iyx + \frac{1}{k}}{-kiy}$. As $y \to 0^+$, the right hand-side tends to $i\infty$. Formula (2.1) implies that ch_{M_r} admits a q-expansion inside $q^{h_{M_r} - \frac{c}{24}} \mathbb{C}[[q]]$. We clearly have

$$\dim_q^s(M_r) = \lim_{y \to 0^+} \frac{\mathrm{ch}_{M_r}\left(A \cdot \left(A^{-1} \cdot (s+iy)\right)\right)}{\mathrm{ch}_{M_0}\left(A \cdot \left(A^{-1} \cdot (s+iy)\right)\right)} = \lim_{y \to 0^+} \frac{\sum_{j=0}^{n} A_{r,j} \mathrm{ch}_{M_j}\left(A^{-1} \cdot \tau_{s,y}\right)}{\sum_{j=0}^{n} A_{0,j} \mathrm{ch}_{M_j}\left(A^{-1} \cdot \tau_{s,y}\right)}.$$

We can eliminate the prefactor $q^{-\frac{c}{24}}$ from both the numerator and denominator. The result follows immediately because $h_{M_r} > 0$, so that only the $\mathrm{ch}_{M_0}(\tau)$ summand contributes to the limit. □

Of course, Theorem 2.1 is well-known for $s = 0$, for which $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is the usual S-matrix:

$$\dim_q(M_r) = \frac{S_{r,0}}{S_{0,0}},$$

(see [14], where more generally the zeroth label was replaced with a module with the minimal lowest conformal weight; see also [15]).

If V is strongly regular, an important theorem due to Huang [23, 24] says that the category V-Mod is *modular* (for more about modular tensor categories see [9] and loc.cit.). Under the assumption of Theorem 2.1 on M_r, it was recently shown in [17] that

$$\widetilde{\dim}_q(M_r) = \dim_q(M_r),$$

where the left hand side is known to be real and ≥ 1.

2.2 Asymptotics

In addition to obvious categorical considerations, another reason why quantum dimensions are useful has to do with the relative growth of dimensions of graded pieces of modules. Therefore we may ask to determine another limit

$$\lim_{n\to\infty} \frac{\dim(M(n))}{\dim(V(n))}, \tag{2.3}$$

simply from the knowledge of $\dim_q(M)$ and other properties of the vertex algebra. One expects that (2.3) and $\dim_q(M)$ are equal whenever one of the limits exists. For instance, if 0 is the "dominant cusp", this limit can be deduced by using a Tauberian result of Ingham [33].

Theorem 2.2. *Let $f(\tau) = q^\lambda \cdot \sum_{n=0}^\infty a_n q^n$ be a holomorphic function in $\mathbb{H}$ such that $a_n \le a_{n+1}$ for all $n \gg 0$. Assume there exist $c, d \in \mathbb{R}$ and $N \ge 0$ such that*

$$f(\tau) \sim c \cdot (-i\tau)^{-d} e^{\frac{2\pi i N}{\tau}}.$$

Then

$$a_n \sim \frac{c}{\sqrt{2}} N^{-\frac{1}{2}\left(d-\frac{1}{2}\right)} n^{\frac{1}{2}\left(d-\frac{3}{2}\right)} e^{4\pi\sqrt{Nn}}.$$

An application of this type of result was given by Kac and Peterson [25] in their study of the growth of graded dimensions of highest weight modules for affine Kac-Moody Lie algebras and related string functions (see also [33] for other applications).

Observe that for strongly regular vertex operator algebras, higher asymptotics of $\frac{\mathrm{ch}_M(\tau)}{\mathrm{ch}_V(\tau)}$ are easy to describe. In this case, $\mathrm{ch}_M(\tau)$ is a modular form of weight zero on a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (see Theorem I, [17]), and thus there exist $a_M \in \mathbb{Q}$ such that for all $n \in \mathbb{N}$

$$\mathrm{ch}_M(iy) \sim \alpha_M e^{-\frac{2\pi a_M}{y}} (1 + O(y^n)).$$

Thus,

$$\frac{\mathrm{ch}_M(iy)}{\mathrm{ch}_V(iy)} \sim \frac{\alpha_M}{\alpha_V} e^{-\frac{2\pi(a_M - a_V)}{y}} (1 + O(y^n)). \tag{2.4}$$

Under the assumptions of Theorem 2.1, we get that $a_M = a_V$ and $\dim_q(M) = \frac{\alpha_M}{\alpha_V}$.

If we deal, as in this paper, with non-rational vertex algebras, then clearly Theorem 2.1 cannot be used, and nothing like (2.4) holds true. Still we are interested in q-dimensions of irreducible representation, including their full asymptotic expansion. If the numerator of $\mathrm{ch}_M(iy)$ is a false like theta series, we first seek full asymptotics expansion:

$$f_M(iy) \sim y^a e^{by} \left(\sum_{k=0}^n a_k y^k + O\left(y^{n+1}\right) \right),$$

where $n \in \mathbb{N}_0$, and use it to determine asymptotics of $\mathrm{ch}_M(iy)$ and of $\frac{\mathrm{ch}_M(iy)}{\mathrm{ch}_V(iy)}$. We show that sufficiently "nice" irrational vertex operator algebra, such as the $(1,p)$-singlet algebra, (p_+, p_-)-singlet, and higher rank generalizations [11], all have q-dimensions which are real and ≥ 1. Figuring out asymptotics expansion and generalized quantum dimensions is directly related to quantum modularity addressed in Sections 4 and 6.

3 Virasoro, singlet, triplet, and lattice vertex operator algebras

In this section, which also contains much introductory material, we start by introducing the main objects of the paper. Although a fair amount of vertex algebra theory is needed to properly define specific algebras and their modules, we try to emphasize only those features that are directly related to characters of modules needed for our considerations. Specifically, we omit any detailed definitions pertaining to Virasoro algebra representations. The reader unfamiliar with the vertex algebra theory can simply skip all algebraic constructions and focus on explicit character formulas (see Sections 3.1 and 3.2).

For $p \in \mathbb{N}$ with $p \geq 2$, consider the rank one lattice $(L, \langle\ ,\ \rangle)$, where

$$L = \mathbb{Z}\alpha, \quad \langle \alpha, \alpha \rangle = 2p.$$

Alternatively, consider $L = \sqrt{2p}\mathbb{Z}$ with the usual multiplication. To this L, following [28] for instance, we associate the lattice vertex algebra $V_L := M(1) \otimes \mathbb{C}[L]$, where $M(1) := \mathbb{C}[\alpha(-1), \alpha(-2), \cdots]$ denotes the Heisenberg vertex algebra of rank one and $\mathbb{C}[L]$ is the group algebra of L generated by e^α, $\alpha \in L$. We are interested in central charges $c_{p,1} = 1 - \frac{6(p-1)^2}{p}, p \geq 2$, thus we choose the conformal vector

$$\omega = \frac{1}{4p}\alpha(-1)^2\mathbf{1} + \frac{p-1}{2p}\alpha(-2)\mathbf{1} \in M(1).$$

We also define certain rational numbers

$$h_{r,s}^{p,q} := \frac{(ps - rq)^2 - (p-q)^2}{4pq}.$$

With this central charge, the generalized vertex algebra V_{L°, where L° denotes the dual lattice of L, admits two *screenings*:

$$\widetilde{Q} = e_0^{-\frac{\alpha}{p}} \quad \text{and} \quad Q = e_0^{\alpha}.$$

Then we let

$$\mathcal{W}(p) := \mathrm{Ker}_{V_L} e_0^{-\frac{\alpha}{p}}.$$

We obtain a vertex subalgebra of V_L, called the *triplet algebra*. As shown in [4], $\mathcal{W}(p)$ is strongly generated by the conformal vector ω and three *primary* vectors

$$F = e^{-\alpha}, \qquad H = QF, \qquad E = Q^2 e^{-\alpha}.$$

There is another useful description of $\mathcal{W}(p)$ [4, 19]. As a module for the Virasoro algebra, V_L is not completely reducible but it has a semisimple filtration whose maximal semisimple part (or socle) is $\mathcal{W}(p)$. More precisely,

$$\mathcal{W}(p) = \mathrm{soc}_{Vir}(V_L) = \bigoplus_{n=0}^{\infty} \bigoplus_{j=0}^{2n} U(Vir).Q^j e^{-n\alpha} \cong \bigoplus_{n=0}^{\infty} (2n+1) L\left(c_{p,1}, h_{1,2n+1}^{p,1}\right),$$

where $L(c, h)$ denotes the highest weight Virasoro module of central charge c and lowest conformal weight h, and soc is the socle. Following the notation from [2], we can restrict the above kernel to the Heisenberg subalgebra and define

$$\mathcal{W}(2, 2p-1) := \mathrm{Ker}_{M(1)} \widetilde{Q} \subset M(1).$$

This structure is called the $(1,p)$-*singlet vertex algebra* of central charge $c_{p,1}$ and is usually denoted by $\mathcal{W}(2,2p-1)$ (we try to avoid using $\mathcal{W}(2,2p-1)$, as it is a bit cumbersome [1]). The vertex operator algebra $\mathcal{W}(2,2p-1)$ is clearly completely reducible as a Virasoro algebra module and the following decomposition holds:

$$\mathcal{W}(2,2p-1) \;=\; \mathrm{soc}_{Vir}(M(1)) = \bigoplus_{n=0}^{\infty} U(V\mathrm{ir}).Q^n e^{-\alpha n} \cong \bigoplus_{n=0}^{\infty} L\left(c_{p,1}, n^2 p + np - n\right).$$

3.1 Irreducible $\mathcal{W}(p)$-modules and their characters

The triplet $\mathcal{W}(p)$ is known to be C_2-cofinite but irrational [4]. It also admits precisely $2p$ inequivalent irreducible modules [4, 19], which are usually denoted by:

$$\Lambda(1), \ldots, \Lambda(p), \Pi(1), \ldots, \Pi(p),$$

where $\Lambda(1) := \mathcal{W}(p)$. Decomposition of irreducible representations into irreducible Virasoro modules follows the pattern as of $\mathcal{W}(p)$.

The characters of irreducible $\mathcal{W}(p)$-modules are computed in many papers on logarithmic conformal field theories starting with [19]. For $1 \le j \le p$, the formulas are

$$\mathrm{ch}_{\Lambda(j)}(\tau) = \frac{j\Theta_{p,p-j}(\tau) + 2\partial\Theta_{p,p-j}(\tau)}{p\eta(\tau)},$$
$$\mathrm{ch}_{\Pi(j)}(\tau) = \frac{j\Theta_{p,j}(\tau) - 2\partial\Theta_{p,j}(\tau)}{p\eta(\tau)},$$

where

$$\Theta_{j,p}(\tau) := \sum_{n\in\mathbb{Z}} q^{\frac{(2np+j)^2}{4p}}, \qquad \partial\Theta_{j,p}(\tau) := \sum_{n\in\mathbb{Z}} \left(n + \frac{j}{2p}\right) q^{\frac{(2np+j)^2}{4p}}.$$

Note that $\Lambda(p)$ and $\Pi(p)$ are precisely lattice vertex algebra modules.

3.2 Irreducible $(1,p)$-singlet modules and their characters

The aim here is to give explicit formula for irreducible $\mathcal{W}(2,2p-1)$ characters. We mostly follow [12] here. (Admissible) irreducible $\mathcal{W}(2,2p-1)$-modules fall into two categories:

- (Typical or generic) F_λ, those isomorphic to irreducible Virasoro Fock spaces, where λ does not satisfy a certain integrability condition [12].
- (Atypical or non-generic) $M_{r,s}$, certain subquotients of Fock spaces $F_{\frac{r-1}{2}\sqrt{2p}+\frac{s-1}{\sqrt{2p}}}$, $r \in \mathbb{Z}$, and $1 \le s \le p$ [12]. Each $M_{r,s}$ is isomorphic to an infinite direct sum of Virasoro irreducible representations.

The character of F_λ is simply

$$\mathrm{ch}_{F_\lambda}(\tau) = \frac{q^{\frac{\left(\lambda - \frac{\alpha_0}{2}\right)^2}{2}}}{\eta(\tau)},$$

[1] Another notation used in the literature is $\overline{M(1)}$ (see [2])

where $\alpha_0 := \sqrt{2p} - \sqrt{\frac{2}{p}}$.

Since $M_{r,s}$ decomposes as an infinite direct sum of irreducible Virasoro algebra (for explicit decomposition formulas see [3, 12]), we get

$$\mathrm{ch}_{M_{r,s}}(\tau) = \frac{\sum_{n=0}^{\infty}\left(q^{p\left(n+\frac{r}{2}-\frac{s}{2p}\right)^2} - q^{p\left(n+\frac{r}{2}+\frac{s}{2p}\right)^2}\right)}{\eta(\tau)}.$$

We can rewrite this as

$$\frac{P_{p,pr-s}(0,\tau) - P_{p,pr+s}(0,\tau)}{\eta(\tau)},$$

where

$$P_{a,b}(u,\tau) := \sum_{n=0}^{\infty} z^{n+\frac{b}{2a}} q^{a\left(n+\frac{b}{2a}\right)^2}, \quad z := e(u). \tag{3.1}$$

As the sum in (3.1) doesn't run over a full lattice, it is called a *partial theta function* and its properties are well-recorded in the literature [6]. In particular, for $M_{1,1} = \mathcal{W}(2, 2p-1)$, we get

$$\mathrm{ch}_{M_{1,1}}(\tau) = \frac{\sum_{n\in\mathbb{Z}} \mathrm{sgn}(n) q^{p\left(n+\frac{p-1}{2p}\right)^2}}{\eta(\tau)},$$

which is essentially the false theta function of Rogers. For brevity, let

$$F_{j,p}(\tau) := \sum_{n\in\mathbb{Z}} \mathrm{sgn}(n) q^{\left(n+\frac{j}{2p}\right)^2},$$

where we use the convention $\mathrm{sgn}(n) = 1$, for $n \geq 0$ and -1 otherwise. Observe that $F_{0,p}(\tau) = 1$ and

$$\mathrm{ch}_{M_{1,s}}(\tau) = \frac{F_{p-s,p}(p\tau)}{\eta(\tau)}.$$

Also,

$$\mathrm{ch}_{M_{r,p}}(\tau) = \frac{q^{p\left(\frac{r}{2}-\frac{1}{2}\right)^2}}{\eta(\tau)},$$

so $M_{r,p}$ can be regarded as a generic module.

Thus, for atypical modules, from now on, we may assume that $1 \leq s \leq p-1$ and $r \in \mathbb{Z}$. However, when we study characters, it is very useful to express them via false theta function $F_{j,p}$, with $1 \leq j \leq 2p-1$.

Proposition 3.1. *Every atypical character admits a decomposition*

$$\mathrm{ch}_{M_{r,s}}(\tau) = \frac{F_{j,p}(p\tau) + q_{r,s}(\tau)}{\eta(\tau)},$$

where $1 \leq j \leq 2p-1$ *and where* $q_{r,s}(\tau)$ *is a finite* q*-series (possibly zero).*

3.3 Verlinde-type formula for $(1, p)$-singlet modules

In [12], the second author and Creutzig formulated a Verlinde-type conjecture for the fusion of $(1, p)$-singlet modules motivated by computation of what they end up calling *regularized Verlinde algebra of characters*. As a consequence, they formulated

Conjecture 3.2. *The following relations are valid inside the (conjectural) Grothendieck ring of $\mathcal{W}(2, 2p-1)$-modules:*

$$[F_\lambda] \times [F_\mu] = \sum_{\ell=0}^{p-1} [F_{\lambda+\mu+\ell\alpha_-}],$$

$$[M_{r,s}] \times [F_\mu] = \sum_{\substack{\ell=-s+2 \\ \ell+s=0 \pmod 2}}^{s} [F_{\mu+\alpha_{r,\ell}}],$$

$$[M_{r,s}] \times [M_{r',s'}] = \sum_{\substack{\ell=|s-s'|+1 \\ \ell+s+s'=1 \pmod 2}}^{\min\{s+s'-1,p\}} [M_{r+r'-1,\ell}] + \sum_{\substack{\ell=p+1 \\ \ell+s+s'=1 \pmod 2}}^{s+s'-1} \Big([M_{r+r'-2,\ell-p}] + [M_{r+r'-1,2p-\ell}] + [M_{r+r',\ell-p}]\Big).$$

This conjecture seems to be difficult to prove by the existing methods unless we impose strong conditions on the category of modules, which are presumably even harder to verify (for instance, we do not even know if there is a braided category structure on $\mathcal{W}(2, 2p-1)$-Mod).

In this paper, we provide further evidence for correctness of this conjecture by examining asymptotic properties of characters and computation of their q-dimensions of modules.

4 Asymptotic properties of $\mathcal{W}(2, 2p-1)$ characters

In this section we derive the full asymptotic expansion for atypical $(1, p)$-singlet (by using two methods) and prove that the numerator part is a quantum modular form as announced earlier.

4.1 Quantum modularity of $\mathcal{W}(2, 2p-1)$ characters

Following [38], we say that $f : \mathbb{Q} \setminus S \to \mathbb{C}$ (S appropriate subset of $\mathbb{Q}$) is a *quantum modular form* of weight k and multiplier ε for $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, if for all $\gamma \in \Gamma$ the functions $h_\gamma : \mathbb{Q} \setminus \big(S \cup \gamma^{-1}(\infty)\big) \to \mathbb{C}$ defined by

$$h_\gamma(x) := f(x) - \varepsilon(\gamma)(cx+\tau)^{-k} f\left(\frac{ax+b}{cx+d}\right)$$

satisfies a "suitable" property of continuity or analyticity (now with respect to the real topology). This definition is purposely ambiguous to accommodate a variety of examples. Note that any finite q-series (with rational powers) can be viewed as a quantum modular form of any weight.

Note that

$$\begin{aligned}F_{j+2p,p}(\tau) &= -2q^{\frac{j^2}{4p^2}} + F_{j,p}(\tau),\\ F_{0,p}(\tau) &= \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n) q^{n^2} = 1,\\ F_{p,p}(\tau) &= \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n) q^{\left(n+\frac{1}{2}\right)^2} = \sum_{n\geq 0} q^{\left(n+\frac{1}{2}\right)^2} - \sum_{n<0} q^{\left(n+\frac{1}{2}\right)^2} = 0.\end{aligned}$$

Thus we may throughout assume $0 < j < 2p, j \neq p$. To state the main result of this section, we let

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4,\\ i & \text{if } d \equiv 3 \pmod 4.\end{cases}$$

We then prove

Theorem 4.1. *The function $F_{j,p}(\tau)$ is a strong quantum modular form of weight $\frac{1}{2}$, multiplier $\chi_j \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) := e^{\frac{\pi i a b j^2}{2p}} \left(\frac{pc}{d}\right) \varepsilon_d^{-3}$, group $\Gamma_1(4p)$, and quantum set $\mathbb{Q}$.*

Remark 4.2. *Theorem 4.1 implies that $F_{j,p}(p\tau)$ is also a quantum modular form on $\mathbb{Q}$.*

To prove Theorem 4.1, we proceed as in [36] and define for $\tau \in \mathbb{H}_- := \{\tau \in \mathbb{C}; \operatorname{Im}(\tau) < 0\}$ the non-holomorphic Eichler integral

$$F_{j,p}^*(\tau) := \sqrt{2} \int_{\overline{\tau}}^{i\infty} \frac{f_{j,p}(z)}{(-i(z-\tau))^{\frac{1}{2}}} dz,$$

where $f_{j,p}(z) := \sum_{n\in\mathbb{Z}} \left(n + \frac{j}{2p}\right) q^{\left(n+\frac{j}{2p}\right)^2}$. A key step in the proof of Theorem 4.1 is to show that $F_{j,p}(\tau)$ agrees for $\tau = \frac{h}{k} \in \mathbb{Q}$ with $F_{j,p}^*(\tau)$ up to infinite order and that $F_{j,p}^*(\tau)$ satisfies a nice transformation law. We start with the first claim. For this, we need the following lemma.

Lemma 4.3. *Let $C : \mathbb{Z} \to \mathbb{C}$ be a periodic function with mean value 0. Then the associated L-series $L(s, C) := \sum_{n=1}^\infty \frac{C(n)}{n^s}$ $(\operatorname{Re}(s) > 1)$ extends holomorphically to all of $\mathbb{C}$ and we have for $t > 0$*

$$\sum_{n=1}^\infty C(n) e^{-n^2 t} \sim \sum_{r=0}^\infty L(-2r, C) \frac{(-t)^r}{r!}.$$

If C is odd, then

$$\sum_{n=1}^\infty C(n) \Gamma\left(\frac{1}{2}; 2n^2 t\right) e^{n^2 t} \sim \sqrt{\pi} \sum_{r=0}^\infty L(-2r, C) \frac{t^r}{r!},$$

where $\Gamma(s;t) := \int_t^\infty e^{-u} u^{s-1} du$ denotes the incomplete gamma function. Moreover, if M is a period of C, then

$$L(-r, C) = -\frac{M^r}{r+1} \sum_{n=1}^M C(n) B_{r+1}\left(\frac{n}{M}\right) \tag{4.1}$$

with $B_\ell(x)$ the ℓth Bernoulli polynomial.

Proof: The claim on the analytic continuation of the L-series and the first asymptotic claim is given in [27]. The second claim can be proved as in [27]. We leave the details to the reader. $\square$

We next show that $F_{j,p}$ and $F^*_{j,p}$ asymptotically agree.

Lemma 4.4. *We have as $t \to 0^+$ for $(h,k)=1$*

$$F_{j,p}\left(\frac{h}{k}+\frac{it}{2\pi}\right) \sim \sum_{r=0}^{\infty} L_j(h,k;-2r)\frac{\left(-\frac{t}{4p^2}\right)^r}{r!},$$

$$F^*_{j,p}\left(\frac{h}{k}-\frac{it}{2\pi}\right) \sim \sum_{r=0}^{\infty} L_j(h,k;-2r)\frac{\left(\frac{t}{4p^2}\right)^r}{r!},$$

where $L_j(h,k;-2r)$ is the analytic continuation of the L-series

$$L_j(h,k;s) := \sum_{\pm} \pm \sum_{\substack{n\geq 1\\ n\equiv \pm j \pmod{2p}}} \frac{e^{\frac{2\pi i n^2 h}{4p}}}{n^s}. \tag{4.2}$$

In particular, $F_{j,p}$ and $F^*_{j,p}$ "agree to infinite order" at rational points in the sense of Lawrence and Zagier (see p.103 of [27]).

Proof: Write for $t>0$

$$F_{j,p}\left(\frac{h}{k}+\frac{it}{2\pi}\right) = \sum_{\substack{n>0\\ n\equiv j \pmod{2p}}} e^{\frac{2\pi i n^2 h}{4p^2 k}-\frac{tn^2}{4p^2}} - \sum_{\substack{n>0\\ n\equiv -j \pmod{2p}}} e^{\frac{2\pi i n^2 h}{4p^2 k}-\frac{tn^2}{4p^2}} = \sum_{n\geq 1} C(n) e^{-\frac{tn^2}{4p^2}},$$

where we define

$$C(n) := \begin{cases} e^{\frac{2\pi i n^2 h}{4p^2 k}} & \text{if } n\equiv j \pmod{2p},\\ -e^{\frac{2\pi i n^2 h}{4p^2 k}} & \text{if } n\equiv -j \pmod{2p},\\ 0 & \text{otherwise.}\end{cases} \tag{4.3}$$

Clearly C is odd, periodic of period $2pk$ and has mean value 0 since

$$\sum_{\substack{n \pmod{2pk}\\ n\equiv j \pmod{2p}}} e^{\frac{2\pi i n^2 h}{2pk}} - \sum_{\substack{n \pmod{2pk}\\ n\equiv -j \pmod{2p}}} e^{\frac{2\pi i n^2 h}{4p^2 k}} = 0$$

by changing $n\to -n$. Lemma 4.3 thus gives the claim for $F_{j,p}$.

We next turn to $F^*_{j,p}$. A direct calculation gives that its Fourier expansion is given by ($\tau = x+iy \in \mathbb{H}^-$)

$$F^*_{j,p}(\tau) = \frac{1}{\sqrt{\pi}}\sum_{\pm}\pm \sum_{\substack{n>0\\ n\equiv \pm j \pmod{2p}}} q^{\frac{n^2}{4p^2}}\,\Gamma\left(\frac{1}{2}; -4\pi n^2 \frac{y}{4p^2}\right).$$

Thus for $t>0$,

$$F^*_{j,p}\left(\frac{h}{k}-\frac{it}{2\pi}\right) = \frac{1}{\sqrt{\pi}}\sum_{n>0} C(n) e^{\frac{n^2 t}{4p^2}}\,\Gamma\left(\frac{1}{2}; \frac{2n^2 t}{4p^2}\right)$$

with C given in (4.3). Now Lemma 4.3 finishes the claim about the asymptotic expansions. $\square$

We now obtain the following explicit asymptotic expansion.

Corollary 4.5. *We have*

$$F_{j,p}(it) \sim \sum_{r=0}^{\infty} \frac{2^{r+1}(-1)^{r+1}\pi^r}{r!(2r+1)} B_{2r+1}\left(\frac{j}{2p}\right) t^r.$$

In particular

$$\mathrm{ch}_{M_{1,s}}(it) = \frac{F_{p-s,p}(pit)}{\eta(it)} \sim \sqrt{t} e^{\frac{\pi}{12t}} F_{p-s,p}(pit).$$

Proof: The claim follows directly by using formula (4.1) and $B_n(1-r) = (-1)^n B_n(r)$. □

Theorem 4.1 now directly follows from the following transformation law for $F^*_{j,p}$.

Lemma 4.6. *We have for* $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(4p)$

$$F^*_{j,p}\left(\frac{a\tau+b}{c\tau+d}\right) \chi_j(M)^{-1}(c\tau+d)^{-\frac{1}{2}} - F^*_{j,p}(\tau) = r_{-\frac{d}{c}}(\tau)$$

with

$$r_\alpha(\tau) := \frac{1}{\sqrt{p}} \int_\alpha^{i\infty} \frac{f_{j,p}(pz)}{(z-\tau)^{\frac{1}{2}}} dz.$$

Proof: The proof is standard, using that by Proposition 2.1 of [32], we have for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(4p)$

$$f_{j,p}\left(p\frac{az+b}{cz+d}\right) = e\left(\frac{abj^2}{4p}\right)\left(\frac{pc}{d}\right)\varepsilon_d^{-3}\left(cz+d\right)^{\frac{3}{2}} f_{j,p}\left(pz\right).$$

□

The quantum modularity now follows since r_α extends to $\mathbb{R}$ and is real analytic on $\mathbb{R}\backslash\{\alpha\}$.

Corollary 4.7. *(Generalized quantum dimensions) If* $u = \frac{h}{k} \in \mathbb{P}^1_{\mathbb{Q}}$, *then*

$$\dim_q^u(M_{1,s}) = \frac{L_{p-s}(h,k;0)}{L_{p-1}(h,k;0)},$$

provided $L_{p-1}(h,k;0) \neq 0$, *where* $L_j(j,k;r)$ *is defined in (4.2) (analytically extended to* $r=0$*).*

Now we consider characters of $M_{r,s}$ for general r. To prove quantum modularity of the numerator part of $\mathrm{ch}_{M_{r,s}}(\tau)$, it suffices to make a few trivial observations:

(i) If $n \in \mathbb{N}$, then any polynomial in $q^{\frac{1}{n}}$ is a quantum modular form of arbitrary weight.

(ii) By Proposition 3.1, $\mathrm{ch}_{M_{r,s}}(\tau)$ can be expressed as $\frac{1}{\eta(\tau)}(F_{j,p}(p\tau) + q_{r,s}(\tau))$, where $q_{r,s}$ is a polynomial in $q^{\frac{1}{4p}}$ and $0 < j < 2p$. Quantum modularity still holds by adding any polynomial to $F_{j,p}$.

We also aim to determine

$$\dim_q(M_{r,s}) = \lim_{y\to 0} \frac{P_{p,pr-s}(iy) - P_{p,pr+s}(iy)}{F_{p-1,p}(ipy)},$$

where we assume $r > 0$. Lemma 4.4 gives the leading asymptotics

$$F_{j,p}(ipy) \sim L_{0,1,j}(-2r) = B_1\left(\frac{2p-j}{2p}\right) - B_1\left(\frac{j}{2p}\right) = 1 - \frac{j}{p}.$$

Similarly,

$$P_{p,pr-s}(iy) - P_{p,pr+s}(iy) \sim B_1\left(\frac{rp+s}{2p}\right) - B_1\left(\frac{rp-s}{2p}\right) = \frac{s}{p}. \tag{4.4}$$

Remark 4.8. *A full asymptotic expansion of* $\mathrm{ch}_{M_{r,s}}(\tau)$ *can be easily obtained directly from Proposition 4.5 by adding the* $q_{r,s}(\tau)$ *part.*

Corollary 4.9. *We have*

$$\dim_q(M_{r,s}) = s.$$

Also, for all λ*,*

$$\dim_q(F_\lambda) = p.$$

Proposition 4.10. *Conjecture 3.2 is true at the level of quantum dimensions.*

Proof: Generic modules have quantum dimension p and this is obviously compatible with the first two formulas in (3.2). So we only consider the third formula for atypical modules. Let $0 \le k, \ell \le p-1$ and assume first that $\min\{k+\ell-1, 2p-k-\ell-1\} = k+\ell-1$ with $k \ge \ell$. Thus $k+\ell \le p$ and we have

$$\sum_{j=(k-\ell)+1;2}^{k+\ell-1} j = (k-\ell)\ell + \sum_{j=1}^{2\ell-1} j = k\ell - \ell^2 + \ell^2 = k\ell$$

as predicted. Here $;2$ indicates that the increment in the summation is 2 rather than 1. Similarly, we verify the case $\min\{k+\ell-1, 2p-k-\ell-1\} = 2p-k-\ell-1$. □

Using Theorem 2.2 and the asymptotics (4.4) immediately gives the following result.

Proposition 4.11. *As in Section 2.3, denote by* $M_{r,s}(n)$ *the* n*-th graded component of the module* $M_{r,s}$*. Then we have*

$$\lim_{n\to+\infty} \frac{M_{r,s}(n)}{M_{1,1}(n)} = s.$$

Proof: By definition $M_{r,s} = \coprod_{n=0}^{\infty} M_{r,s}(n)$, where

$$M_{r,s}(n) := \{v \in M_{r,s} : L(0)v = (n + h_{r,s})v\}$$

and $h_{r,s}$ denotes the lowest weight of $M_{r,s}$. We are done once we show that the graded dimensions are eventually increasing. Consider the Virasoro operator

$$L(-1) : M_{r,s}(n) \to M_{r,s}(n+1).$$

Suppose that $\mathrm{Ker}(L(-1)) \neq 0$. Then $L(-1)v = 0$ for some $v \in M_{r,s}(n)$, so v is vacuum-like. As $M_{r,s}$ is irreducible, such a v must be a highest weight vector with $h_{r,s} = 0$. An easy inspection shows that this can occur only if $(r,s) = (1,1)$ and $n = 0$ (the vertex algebra). Therefore, in the Fourier expansion of

$$\mathrm{ch}_{M_{r,s}}(\tau) = q^{h_{r,s}-\frac{c_{p,1}}{24}} \sum_{n=0}^{\infty} \dim(M_{r,s}(n)) q^n$$

we have $\dim(M_{r,s}(n)) \leq \dim(M_{r,s}(n+1))$ except for $n = 0$ and $(r,s) = (1,1)$. Hence, the only instance where the graded dimension is decreasing occurs at the first term in the q-expansion of $\mathrm{ch}_{M_{1,1}}$:

$$\mathrm{ch}_{M_{1,1}}(\tau) = q^{-\frac{c_{p,1}}{24}} \left(1 + q^2 + \cdots\right).$$

Ingham's Theorem 2.2 applies and gives

$$M_{r,s}(n) \sim \frac{s}{p} \cdot p(n),$$

where $p(n)$ denotes the number of partitions of n. The proof now follows from (4.4). □

Remark 4.12. There is another approach to quantum modularity of $F_{j,p}$ that we would like to emphasize here, which leads to a strictly smaller quantum set, but allows us to explicitly compute the radial limit. We let

$$S_p := \left\{ \frac{h}{k} \in \mathbb{Q} : \ (h,k) = 1, 2p \mid k \right\}.$$

We can find a representation for $F_{j,p}$ which exists at a certain subset of $\mathbb{Q}$. For this, we use the following identity (see [34])

$$\sum_{n=0}^{\infty} \frac{\left(q;q^2\right)_n \left(aq;q^2\right)_n (aq)^n}{(-aq)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n a^n q^{n(n+1)}.$$

This gives that

$$F_{j,p}(\tau) = q^{\left(\frac{j}{2p}\right)^2} \sum_{n=0}^{\infty} \frac{\left(q;q^2\right)_n \left(-q^{\frac{j}{p}};q^2\right)_n (-1)^n q^{\frac{jn}{p}}}{\left(q^{\frac{j}{p}}\right)_{2n+1}} - q^{\left(1-\frac{j}{2p}\right)^2} \sum_{n=0}^{\infty} \frac{\left(q;q^2\right)_n \left(-q^{2-\frac{j}{p}};q^2\right)_n (-1)^n q^{n\left(2-\frac{j}{p}\right)}}{\left(q^{2-\frac{j}{p}}\right)_{2n+1}}.$$

A direct calculation now shows that these sums terminate for $\frac{h}{k} \in S_p$.

4.2 Asymptotic formulas via Jacobi forms

Here we use a second approach to determine asymptotics of $F_{j,p}$ by employing Jacobi forms. For $x := e^{2\pi i z}$ let

$$F_{j,p}(\tau,z) := \sum_{n=0}^{\infty} \left(x^n + \ldots + 1 + \ldots + x^{-n}\right) \left(q^{\left(n+\frac{j}{2p}\right)^2} - q^{\left(n+\frac{2p-j}{2p}\right)^2}\right),$$

which can be also viewed as the numerator part of the full character of a certain triplet algebra module. Obviously

$$F_{j,p}(\tau) = \mathrm{CT}_x \left[F_{j,p}(\tau,z)\right],$$

where CT_x stands for the constant term. Recall the *Jacobi theta function*

$$\Theta_{\alpha,\beta}(\tau,z) := \sum_{n\in\mathbb{Z}} q^{\beta\left(n+\frac{\alpha}{2\beta}\right)^2} x^{2n+\alpha}.$$

Using the theta transformation law, we obtain

$$\Theta_{\pm j,p}(\tau,z) = (-2pi\tau)^{-\frac{1}{2}} e^{\pm 2\pi ijz} \Theta_{-2z,\pm\frac{j}{2p}}\left(-\frac{1}{\tau}, \frac{1}{2p}\right). \tag{4.5}$$

Now we easily see

$$F_{j,p}(\tau) = \mathrm{CT}_x \left[\frac{x^{2-j}\Theta_{j,p}\left(\frac{\tau}{p}, z\right) - x^j \Theta_{-j,p}\left(\frac{\tau}{p}, z\right)}{x^2-1}\right].$$

To write the right hand-side as an integral, observe that this function is invariant under $z \mapsto z+\frac{1}{2}$ and the pole in the denominator in canceled by a root in the numerator. Thus, by Cauchy,

$$F_{j,p}(\tau) = 2\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{e^{2\pi i(2-j)z}\Theta_{j,p}\left(\frac{\tau}{p}, z\right) - e^{2\pi ijz}\Theta_{-j,p}\left(\frac{\tau}{p}, z\right)}{e^{4\pi iz}-1} dz.$$

We next use (4.5) to get

$$F_{j,p}(it) = 2(2t)^{-\frac{1}{2}} \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{1}{e^{4\pi iz}-1}\left(e^{4\pi iz}\Theta_{-2z,\frac{j}{2p}}\left(\frac{ip}{t}, \frac{1}{2p}\right) - \Theta_{-2z,-\frac{j}{2p}}\left(\frac{ip}{t}, \frac{1}{2p}\right)\right) dz.$$

It is not hard to see that

$$\Theta_{-2z,\pm\frac{j}{2p}}\left(\frac{ip}{t}, \frac{1}{2p}\right) \sim e^{-\frac{2\pi z^2}{t} \mp \frac{2\pi ijz}{p}},$$

where throughout the following terms in the asymptotic expansion are exponentially smaller. Thus

$$F_{j,p}(it) \sim 2(2t)^{-\frac{1}{2}} \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{\left(e^{4\pi iz - \frac{2\pi ijz}{p}} - e^{\frac{2\pi ijz}{p}}\right) e^{-\frac{2\pi z^2}{t}}}{e^{4\pi iz}-1} dz.$$

Turning the integral into an integral over $\mathbb{R}$ only introduces an exponentially small error. Thus

$$F_{j,p}(it) \sim 2(2t)^{-\frac{1}{2}} \int_{\mathbb{R}} \frac{\left(e^{4\pi iz - \frac{2\pi ijz}{p}} - e^{\frac{2\pi ijz}{p}}\right)}{e^{4\pi iz}-1} e^{-\frac{2\pi z^2}{t}} dz.$$

We next rewrite

$$\frac{e^{4\pi iz - \frac{2\pi ijz}{p}} - e^{\frac{2\pi ijz}{p}}}{e^{4\pi iz}-1} = \sum_{\ell=0}^{\infty} \frac{\left(B_{2\ell+1}\left(1-\frac{j}{2p}\right) - B_{2\ell+1}\left(\frac{j}{2p}\right)\right)}{(2\ell+1)!}(4\pi iz)^{2\ell},$$

and easily obtain

$$F_{j,p}(it) \sim \sum_{\ell=0}^{\infty} B_{2\ell+1}\left(\frac{j}{2p}\right) \frac{2^{\ell+1}\pi^{\ell}(-1)^{\ell+1}}{(2\ell+1)\ell!} t^{\ell},$$

in agreement with Corollary 4.5.

Remark 4.13. *Although the above asymptotic expansion was previously obtained via quantum modularity, we leave this independent derivation as it generalizes nicely to higher ranks [11].*

5 The (p_+, p_-)-singlet vertex algebras and its characters

In this section, we consider a two-parameter generalization of the singlet algebra $\mathcal{W}(2, 2p-1)$. To avoid further notational difficulties, we refer to these new algebras as (p_+, p_-)-singlets. Let us briefly recall their construction following [4] (also [13]).

5.1 Characters of irreducible (p_+, p_-)-modules

Let p_+ and p_- be *relatively prime* integers ≥ 2. Consider again the vertex algebra V_L, where $L = \mathbb{Z}\alpha$ but now with $\langle \alpha, \alpha \rangle = 2p_+p_-$ and let L^0 be the dual lattice. In the generalized vertex algebra V_{L^0} we have two screening operators e_0^{α/p_+} and $e_0^{-\alpha/p_-}$. Now consider these operators acting on $M(1) \subset V_L \subset V_{L^\circ}$. Then the (p_+, p_-)-singlet is defined as

$$\left(\mathrm{Ker}_{M(1)} e_0^{\frac{\alpha}{p_+}} \right) \cap \left(\mathrm{Ker}_{M(1)} e_0^{-\frac{\alpha}{p_-}} \right).$$

This is a conformal vertex algebra of central charge $c_{p_+,p_-} = 1 - \frac{6(p_+-p_-)^2}{p_-p_+}$ generated by the Virasoro vector and a primary vector of degree $(2p_+ - 1)(2p_- - 1)$, thus it is of singlet type. Furthermore, this vertex algebra is an extension of the well studied (p_+, p_-) Virasoro minimal model VOA [34].

Irreducible singlet (p_+, p_-)-modules can be classified and fall into three categories:
(i) (atypical) Modules $\mathcal{J}_{r,s;n}$, where $1 \leq r \leq p_+$, $1 \leq s \leq p_-$, and $n \in \mathbb{Z}$. They come with the decomposition

$$\mathcal{J}_{r,s;n} = \bigoplus_{k=0}^{\infty} L\left(c_{p_+,p_-}, \Delta\left(r, p_- - s; |n| + 2k + 1\right) - \frac{(p_- - p_+)^2}{4p_-p_+} \right), \tag{5.1}$$

where

$$\Delta(r, s; m) := \frac{(p_+s - p_-r + mp_+p_-)^2}{4p_+p_-}.$$

(ii) (small) Minimal (p_+, p_-) Virasoro models of central charge $1 - \frac{6(p_+-p_-)^2}{p_-p_+}$ [33]:

$$\mathcal{L}_{r,s}; \quad 1 \leq r \leq p_+ - 1,\ 1 \leq s \leq p_- - 1.$$

Due to symmetry, there are precisely $\frac{1}{2}(p_+ - 1)(p_- - 1)$ of these.
(iii) (typical) Irreducible Fock spaces F_λ, as introduced in Section 2.

Next, we discuss atypical characters. The characters of irreducible Virasoro modules appearing in the decomposition of $\mathcal{J}_{r,s;n}$ are known (but interestingly much less studied in the literature, perhaps due to their "false" behavior). For sake of brevity we first let

$$\chi_{r,s;n}(\tau) := \mathrm{ch}_{\mathcal{J}_{r,s;n}}(\tau).$$

Then the formula (5.1) yields

$$\begin{aligned}
\eta(\tau)\chi_{r,s;n}(\tau) \;=\; & \sum_{m=0}^{\infty} m q^{p_-p_+\left(m + \frac{|n|}{2} - \frac{p_+s+p_-r}{2p_+p_-}\right)^2} + \sum_{m=0}^{\infty} m q^{p_-p_+\left(m + \frac{|n|}{2} + \frac{p_+s+p_-r}{2p_+p_-}\right)^2} \\
- & \sum_{m=0}^{\infty} m q^{p_-p_+\left(m + \frac{|n|}{2} + \frac{p_+s-p_-r}{2p_+p_-}\right)^2} + \sum_{m=0}^{\infty} m q^{p_-p_+\left(m + \frac{|n|}{2} - \frac{p_+s-p_-r}{2p_+p_-}\right)^2}.
\end{aligned}$$

The character $\chi_{r,s;n}$ with $r = p_+$ (resp. $s = p_-$) can be written in the form $\frac{F_{j,p}(p\tau)+h(\tau)}{\eta(\tau)}$, where $p = p_-$ (resp. p_+), with $h(\tau)$ a finite q-series. Therefore their numerators are already quantum modular by virtue of results from Section 4. Similarly, we can easily derive full asymptotics.

Thus we can assume $1 \leq r \leq p_+ - 1$ and $1 \leq s \leq p_- - 1$. If we let $n = 0$, then this character simplifies to

$$\chi_{r,s;0}(\tau) = \frac{\sum_{m\in\mathbb{Z}} |m| q^{\Delta(r,p_- - s;2m-1)} - \sum_{m\in\mathbb{Z}} |m| q^{\Delta(r,s;2m)}}{\eta(\tau)}.$$

Characters with $n = 0$ are somewhat distinguished as they correspond to certain "top" components of Fock spaces.

For $n \neq 0$ we distinguish between n even and n odd. For n even, we can write

$$\eta(\tau)\chi_{r,s;n}(\tau) = \eta(\tau)\chi_{r,s;0}(\tau) + f_{r,s;n}(\tau) - \frac{|n|}{2}\left(\sum_{m\in\mathbb{Z}} q^{p_+p_-\left(m+\frac{p_+s+p_-r}{2p_+p_-}\right)^2} - \sum_{m\in\mathbb{Z}} q^{p_+p_-\left(m+\frac{p_+s-p_-r}{2p_+p_-}\right)^2}\right),$$

where $f_{r,s;n}(\tau)$ is a finite q-series. For n odd and positive we first observe that

$$\eta(\tau)\chi_{r,s;n}(\tau) = -\eta(\tau)\chi_{r,p_--s;n-1}(\tau) + \sum_{m=0}^{\infty}\left(q^{p_+p_-\left(m+\left(\frac{n}{2}-\frac{1}{2}\right)-\frac{p_+s+p_-r}{2p_+p_-}\right)^2} - q^{p_+p_-\left(m+\left(\frac{n}{2}-\frac{1}{2}\right)-\frac{p_+s-p_-r}{2p_+p_-}\right)^2}\right).$$

Furthermore, keeping n odd, we get

$$\sum_{m=0}^{\infty}\left(q^{p_+p_-\left(m+\left(\frac{n}{2}-\frac{1}{2}\right)-\frac{p_+s+p_-r}{2p_+p_-}\right)^2} - q^{p_+p_-\left(m+\left(\frac{n}{2}-\frac{1}{2}\right)-\frac{p_+s-p_-r}{2p_+p_-}\right)^2}\right) = g_{r,s;n}(\tau) + \frac{1}{2}\left(A(\tau) + B(\tau)\right),$$

where $g_{r,s;n}(\tau)$ is a finite q-series and

$$A(\tau) := \sum_{m=0}^{\infty} q^{p_-p_+\left(m-\frac{p_+s+p_-r}{2p_+p_-}\right)^2} - \sum_{m=0}^{\infty} q^{p_-p_+\left(m+\frac{p_+s+p_-r}{2p_+p_-}\right)^2} + \sum_{m=0}^{\infty} q^{p_-p_+\left(m+\frac{p_+s-p_-r}{2p_+p_-}\right)^2} - \sum_{m=0}^{\infty} q^{p_-p_+\left(m-\frac{p_+s-p_-r}{2p_+p_-}\right)^2},$$

$$B(\tau) := \sum_{m\in\mathbb{Z}} q^{p_-p_+\left(m+\frac{p_+s+p_-r}{2p_+p_-}\right)^2} - \sum_{m\in\mathbb{Z}} q^{p_-p_+\left(m+\frac{p_+s-p_-r}{2p_+p_-}\right)^2}.$$

By using results from Section 4, we see that A is a sum of two quantum modular forms while B is a modular form of weight $\frac{1}{2}$. To summarize, in order to prove mixed quantum modularity of $\chi_{r,s;n}$ it is sufficient to prove quantum modularity of $\chi_{r,s;0}$. This is accomplished in Section 6.

The (p_+, p_-)-singlet vertex algebra is no longer simple, instead it is an extension of $\mathcal{L}_{1,1}$ and $\mathcal{J}_{1,1;0}$ (its maximal ideal) and is denoted by $\mathcal{K}_{1,1}$. On the level of characters we can write this

$$\mathrm{ch}_{\mathcal{K}_{1,1}}(\tau) = \mathrm{ch}_{\mathcal{L}_{1,1}}(\tau) + \chi_{1,1;0}(\tau).$$

5.2 Asymptotic properties of $\chi_{r,s;n}$

In this section we determine the asymptotic behavior of $\chi_{r,s;n}(it)$ as $t \to 0^+$. For this, we first note that we may rewrite

$$\chi_{r,s;n}\left(\frac{it}{2\pi}\right)\eta\left(\frac{it}{2\pi}\right) = \sum_{m=0}^{\infty} me^{-p_-p_+t\left(m+\frac{|n|}{2}-\frac{p_+s+p_-r}{2p_+p_-}\right)^2} + \sum_{m=0}^{\infty} me^{-p_-p_+t\left(m+\frac{|n|}{2}+\frac{p_+s+p_-r}{2p_+p_-}\right)^2}$$
$$- \sum_{m=0}^{\infty} me^{-p_-p_+t\left(m+\frac{|n|}{2}+\frac{p_+s-p_-r}{2p_+p_-}\right)^2} - \sum_{m=0}^{\infty} me^{-p_-p_+t\left(m+\frac{|n|}{2}-\frac{p_+s-p_-r}{2p_+p_-}\right)^2}.$$

Via symmetry, we may assume that $\frac{s}{p_-} \geq \frac{r}{p_+} > 0$. We require a general lemma [37], which follows from the Euler McLaurin summation formula. Suppose that $f : (0, \infty) \to \mathbb{C}$ has an asymptotic expansion

$$f(t) = \sum_{n=0}^{\infty} b_n t^n \quad \left(t \to 0^+\right),$$

where by "=" we mean that $f(t) = \sum_{n=0}^{N} b_n t^n + O(t^{N+1})$ for any $N \in \mathbb{N}$. Define

$$I_f := \int_0^{\infty} f(u)du.$$

Lemma 5.1. *If $I_f < \infty$, then we have for $a > 0$*

$$\sum_{m\geq 0} f\left((m+a)t\right) = \frac{I_f}{t} - \sum_{n\geq 0} b_n \frac{B_{n+1}(a)}{n+1} t^n.$$

For our purposes, define for $\alpha > 0$

$$\mathcal{F}_\alpha(t) := \sum_{m\geq 0} me^{-t(m+\alpha)^2}.$$

We aim to first determine the asymptotic behavior of $\mathcal{F}_\alpha$ and write

$$\mathcal{F}_\alpha(t) = \frac{1}{\sqrt{t}} \sum_{m\geq 0} f_1\left(\sqrt{t}(m+\alpha)\right) - \alpha \sum_{m\geq 0} f_2\left(\sqrt{t}(m+\alpha)\right)$$

with

$$f_1(x) := xe^{-x^2} = x + O\left(x^3\right),$$
$$f_2(x) := e^{-x^2} = 1 + O\left(x^2\right).$$

Now, by Lemma 5.1,

$$\mathcal{F}_\alpha(t) = \frac{I_{f_1}}{t} - \frac{1}{2}B_2(\alpha) - \frac{\alpha I_{f_2}}{\sqrt{t}} + \alpha B_1(\alpha) + O\left(\sqrt{t}\right),$$

and hence

$$\sum_{m=0}^{\infty} me^{-t(m+n-\alpha)^2} + \sum_{m=0}^{\infty} me^{-t(m+n+\alpha)^2} = \frac{2I_{f_1}}{t} - \frac{2nI_{f_2}}{\sqrt{t}} + n^2 + \alpha^2 - \frac{1}{6} + O\left(\sqrt{t}\right).$$

Now we can compute the leading term in the asymptotic expansion of $\eta(it)\chi_{r,s;n}(it)$. The terms $\frac{2I_{f_1}}{t}$ and $\frac{2nI_{f_2}}{\sqrt{t}}$ cancel out, so we can freely substitute $2\pi p_+p_-t$ for t. The result is

$$\eta(it)\chi_{r,s;n}(it) \sim \frac{rs}{p_-p_+}.$$

We also record

$$\lim_{t\to 0^+} \frac{\chi_{r,s;n}(it)}{\chi_{1,1;0}(it)} = rs. \tag{5.2}$$

Proposition 5.2. *Asymptotically,*

$$\eta(it)\chi_{r,s;0}(it) = \sum_{n=0}^{\infty} \frac{(-1)^n(2\pi p_+p_-)^n t^n}{n!} \left(\frac{2\alpha B_{2n+1}(\alpha) - 2\beta B_{2n+1}(\beta)}{2n+1} - \frac{B_{2n+2}(\alpha) - B_{2n+2}(\beta)}{n+1} \right),$$

where $\alpha := \frac{p_+s+p_-r}{2p_+p_-}$ *and* $\beta := \frac{p_+s-p_-r}{2p_+p_-}$.

Clearly, the previous proposition and formulas in Section 5.1 can be now used to write down asymptotic expansion for $\eta(it)\chi_{r,s;n}(it)$. As the formula is rather messy we omit it here.

Next, we consider the remaining irreducible modules. The following result pertaining to minimal models can be found for example in [34]:

Lemma 5.3. *The characters of* (p_+, p_-) *minimal models are given by:*

$$\mathrm{ch}_{\mathcal{L}_{r,s}}(\tau) = \frac{1}{\eta(\tau)} \left(\theta_{p_+s-p_-r}(\tau) - \theta_{p_+s+p_-r}(\tau) \right),$$

where r *and* s *are as before. Moreover, we have the following asymptotics*

$$\mathrm{ch}_{\mathcal{L}_{r,s}}(it) \sim A^{r,s}_{p_+,p_-} e^{\frac{\pi d^{p_+,p_-}}{12t}} \quad (t \to 0^+).$$

Here $d^{p_+,p_-} := 1 - \frac{6}{p_+p_-}$ *and*

$$A^{r,s}_{p_+,p_-} := (-1)^{(r+s)(r'+s')} \sqrt{\frac{8}{p_+p_-}} \sin\left(\frac{\pi r r'}{p_-}(p_+ - p_-) \right) \sin\left(\frac{\pi s s'}{p_+}(p_+ - p_-) \right),$$

with (r', s') *the unique integers satisfying* $1 \le r' \le p_- - 1$ *and* $1 \le s' \le p_+ - 1$ *and* $r'p_+ - s'p_- = 1$.

We summarize everything into the following result

Theorem 5.4. *We have*

(i) For all $n \in \mathbb{Z}$,

$$\dim_q(\mathcal{J}_{r,s;n}) = rs.$$

(ii) For all r, s *we have*

$$\dim_q(\mathcal{L}_{r,s}) = 0.$$

(iii) For all $\lambda \in \mathbb{C}$,

$$\dim_q(F_\lambda) = p_+p_-.$$

Proof: Assertion (i) essentially follows directly from formula (5.2). We are interested in

$$\dim_q(\mathcal{J}_{r,s;n}) = \lim_{t\to 0^+} \frac{\chi_{r,s;n}(it)}{\chi_{1,1;0}(it) + \mathrm{ch}_{\mathcal{L}_{1,1}}(it)}.$$

Simplifying and noting that $\eta(it)\mathrm{ch}_{\mathcal{L}_{1,1}}(it) \sim A^{1,1}_{p_+,p_-}\frac{1}{\sqrt{t}}e^{-\frac{\pi p_+ p_-}{2t}}$, the additional term in the denominator does not contribute, so as before in (5.2), we get

$$\dim_q(\mathcal{J}_{r,s;n}) = \lim_{t\to 0^+} \frac{\chi_{r,s;n}(it)}{\chi_{1,1;0}(it)} = rs.$$

For (ii), we used previously $\chi_{1,1;0}(it) \sim \frac{1}{p_-p_+}\sqrt{t}e^{\frac{\pi}{12t}}$ and $\mathrm{ch}_{\mathcal{L}_{r,s}}(it) \sim A^{r,s}_{p_+,p_-}e^{\frac{\pi d^{p_+,p_-}}{12t}}$. It is sufficient to observe that $\sqrt{t}$ tends to zero much slower compared to $e^{-\lambda t}$ for $\lambda > 0$, so the limit is zero. Part (iii) follows immediately once we observe that $\mathrm{ch}_{F_\lambda}(\tau) = \frac{q^{\widetilde{\lambda}}}{\eta(\tau)}$, for some $\widetilde{\lambda}$. □

Remark 5.5. Our results in Theorem 5.4 are in agreement with the proposed fusion rules among irreducible (p_+,p_-)-singlet modules given in [31] and [13], in the sense that $\dim_q(\cdot)$ defines a representation of the (conjectural) Verlinde algebra. Let us illustrate this in a few cases.

Suppose first that $r + r' \leq p_\pm + 1$ and $s + s' \leq p_\pm + 1$. Then the fusion rules obtained by a Verlinde-type formula in [31] read

$$[\mathcal{J}_{r,s;n}] \times [\mathcal{J}_{r',s';n'}] = \sum_{\ell=|r-r'|+1;2}^{r+r'-1} \sum_{j=|s-s'|+1;2}^{s+s'-1} [\mathcal{J}_{\ell,j;n+n'}].$$

By using Theorem 5.4 (i), we easily infer that the q-dimension of the right-hand side equals $rr'ss'$. Similarly, one checks the other cases in formula (4.24a) in [13, 31].

For representations in (ii), we expect that

$$[\mathcal{L}_{r,s}] \times [\mathcal{J}_{r',s';n}] = 0,$$

which again agrees with our result. In fact, the minimal models $\mathcal{L}_{r,s}$ are expected to form a full tensor ideal in $\mathcal{K}_{1,1}$-Mod, which is clearly consistent with our formula as two minimal models are expected to fuse nontrivially only among themselves (producing another module with q-dimension zero!).

For representations in (iii), the proposed fusion rules among $\mathcal{J}_{r,s;n}$ and F_λ can be found in formula (4.14) in [13, 31]. We omit details and only note that $[\mathcal{J}_{r,s;n}] \times [F_\lambda]$ decomposes as a sum of rs generic modules each with q-dimension p_+p_-.

5.3 Example: $(2,3)$-singlet

Here we analyze in more details the simplest (p_+,p_-)-singlet with $p_+ = 2$ and $p_- = 3$. This case is clearly degenerate as $\mathcal{L}_{1,1} = \mathbb{C}$ (the trivial vertex algebra!). Our vertex algebra $\mathcal{K}_{1,1}$ is thus an extension of the one-dimensional space $\mathcal{L}_{1,1}$ and $\mathcal{J}_{1,1;0}$, whose character is given by

$$\chi_{1,1;0}(\tau) = \frac{q^{\frac{1}{24}}\sum_{n\in\mathbb{Z}}|n|q^{6n^2-5n+1} - q^{\frac{1}{24}}\sum_{n\in\mathbb{Z}}|n|q^{6n^2+n}}{\eta(\tau)}.$$

We can rewrite the numerator as

$$q^{\frac{1}{24}}\left(-\sum_{n\in\mathbb{Z}}(-1)^n\left|\frac{n}{2}+\frac{1}{12}\right|q^{\frac{(3n^2+n)}{2}}+\frac{1}{12}\sum_{n\in\mathbb{Z}}\operatorname{sgn}(n)q^{6n^2+n}+\frac{5}{12}\sum_{n\in\mathbb{Z}}\operatorname{sgn}(n)q^{6n^2+7n+2}\right).$$

The first sum is known to be a quantum modular form of weight $\frac{3}{2}$ for $\mathrm{SL}_2(\mathbb{Z})$ with quantum set $\mathbb{Q}$ as it already appeared in the analysis of Kontsevich's "strange q-series" (after Zagier's paper [36]). After inclusion of the fudge factor $q^{\frac{1}{24}}$, the second and third sum now read

$$\sum_{n\in\mathbb{Z}}\operatorname{sgn}(n)q^{6\left(n+\frac{1}{12}\right)^2}\quad\text{and}\quad\sum_{n\in\mathbb{Z}}\operatorname{sgn}(n)q^{6\left(n+\frac{7}{12}\right)^2},$$

respectively. Both are quantum modular forms of weight $\frac{1}{2}$ with quantum set $\mathbb{Q}$ (see Section 4). In the next section we prove a more general statement for all $\mathcal{J}_{r,s;n}$ modules.

6 Quantum modularity of $\chi_{r,s;0}$

In this section we consider quantum modularity of the numerator $\eta\chi_{r,s;0}$.

Theorem 6.1. *The numerator $\eta\chi_{r,s;0}$ is a quantum modular form of mixed weight $\frac{1}{2}$ and $\frac{3}{2}$, with quantum set $\mathbb{Q}$. For $n\neq 0$, $\eta\chi_{r,s;n}$ is a mixed quantum modular form.*

Remark 6.2. *Combined with results from Section 5, this also proves mixed quantum modularity of $\eta\chi_{r,s;n}$ for all n.*

For the proof, we define the Gauss sum

$$G(a,b,c):=\sum_{m \pmod c} e^{\frac{2\pi i}{c}\left(am^2+bm\right)}.$$

We require the following properties.

Lemma 6.3. *(i) If $b=0$ and $(a,c)=1$, then we have*

$$G(a,0,c)=\begin{cases}0 & \text{if } c\equiv 2 \pmod 4,\\ \varepsilon_c\sqrt{c}\left(\frac{a}{c}\right) & \text{if } c \text{ is odd},\\ (1+i)\varepsilon_a^{-1}\sqrt{c}\left(\frac{c}{a}\right) & \text{if } a \text{ is odd}, 4|c.\end{cases}$$

(ii) If $4|c,(a,c)=1$, and b is odd, then $G(a,b,c)=0$.

(iii) We have for $(c,d)=1$ that

$$G(a,b,cd)=G(ac,b,d)G(ad,b,c).$$

(iv) If $(a,c)>1, G(a,b,c)=0$ unless $(a,c)|b$, in which case

$$G(a,b,c)=(a,c)G\left(\frac{a}{(a,c)},\frac{b}{(a,c)},\frac{c}{(a,c)}\right).$$

Proof of Theorem 6.1: Via symmetry, we may assume that $p_+s - p_-r \geq 0$. We may write

$$\eta(\tau)\chi_{r,s;0}(\tau) = \frac{1}{P}\sum_{\substack{j\in\{1,2\}\\ \pm}} (-1)^{j+1} \sum_{\substack{m\geq 0\\ m\equiv \pm A_j \pmod{P}}} mq^{\frac{m^2}{2P}} + \frac{1}{P}\sum_{j\in\{1,2\}} (-1)^j A_j \sum_{m\in\mathbb{Z}} \operatorname{sgn}(m) q^{\frac{P}{2}\left(m+\frac{A_j}{P}\right)^2},$$

where $P := 2p_+p_-$, $A_1 := p_+s + p_-r$, and $A_2 := p_+s - p_-r$. Let us replace τ with 2τ for simplicity. Then the second term on the right hand-side equals

$$\frac{1}{P}\sum_{j\in\{1,2\}} (-1)^j A_j F_{A_j,\frac{P}{2}}(P\tau)$$

and is thus quantum modular by the previous results. Thus we are left to show that

$$G(\tau) := \sum_{\substack{j\in\{1,2\}\\ \pm}} (-1)^j \sum_{\substack{m\geq 0\\ m\equiv \pm A_j \pmod{P}}} mq^{\frac{m^2}{P}}$$

has quantum set $\mathbb{Q}$. For this, let $\frac{h}{k}\in\mathbb{Q}$ with $(h,k)=1$. Then for $t>0$

$$G\left(\frac{h}{k}+it\right) = \sum_{m\geq 0} m\gamma(m) e^{-\frac{2\pi t m^2}{P}}$$

with

$$\gamma(m) := \begin{cases} e^{\frac{2\pi i h m^2}{Pk}} & \text{if } m\equiv \pm A_1 \pmod{P},\\ -e^{\frac{2\pi i h m^2}{Pk}} & \text{if } m\equiv \pm A_2 \pmod{P}.\end{cases}$$

Theorem 6.1 is proven once we show that γ has mean value 0, which is equivalent to

$$0 = \sum_{\substack{j\in\{1,2\}\\ \pm}} (-1)^j e^{\frac{2\pi i h A_j^2}{Pk}} \sum_{m \pmod{k}} e^{\frac{2\pi i h}{k}\left(Pm^2 \pm 2A_j m\right)}. \tag{6.1}$$

Equation (6.1) follows once we show that

$$e\left(\frac{hA_1^2}{Pk}\right) G\left(hP, \pm 2hA_1, k\right) = e\left(\frac{hA_2^2}{Pk}\right) G\left(hP, \pm 2hA_2, k\right),$$

where $e(x) := e^{2\pi i x}$. Changing in the Gauss sum $m\to -m$ we only have to consider the $+$ sign. Now write $k = 2^\nu k'$ ($\nu\in\mathbb{N}_0, k'$ odd). By Lemma 6.3 (iii), we have

$$G\left(hP, 2hA_j, k\right) = G\left(2^\nu hP, 2hA_j, k'\right) G\left(k'hP, 2hA_j, 2^\nu\right). \tag{6.2}$$

We evaluate both factors. For the first factor, let $\ell := (k', P)$. Using the conditions on p_- and p_+, it is not hard to see that $\ell | A_1 \Leftrightarrow \ell | A_2$. Thus we may assume that $\ell | A_j$ since otherwise the Gauss sums vanish by Lemma 6.3 (iv). Completing the square, we obtain by Lemma 6.3 (i),

$$G\left(2^\nu hP, 2hA_j, k'\right) = \ell e\left(-\frac{h\left[2^\nu \frac{P}{\ell}\right]_{\frac{k'}{\ell}} \left(\frac{A_j}{\ell}\right)^2}{\frac{k'}{\ell}}\right) \varepsilon_{\frac{k'}{\ell}} \sqrt{\frac{k'}{\ell}} \left(\frac{2^\nu h\frac{P}{\ell}}{\frac{k'}{\ell}}\right),$$

where throughout $[a]_b$ denotes the inverse of $a \pmod b$. Note that only the exponential factor is relevant as the other contribution is independent of A_j.

We next consider the second Gauss sum (if $\nu \geq 1$) in (6.2). Let $2^n := (2^\nu, P)$. As before $2^n|2A_1 \Leftrightarrow 2^n|2A_2$. By Lemma 6.3 (iv), we may thus assume that $2^n|2A_j$. Then

$$G\left(k'hP, 2hA_j, 2^\nu\right) = 2^n G\left(k'h\frac{P}{2^n}, h\frac{A_j}{2^{n-1}}, 2^{\nu-n}\right).$$

We now first consider the case $2^n \nmid A_1 \Leftrightarrow 2^n \nmid A_2$. By Lemma 6.3 (ii), we obtain that the Gauss sum vanishes unless $\nu = n$ or $\nu = n+1$ in which case it equals 1 or 2, respectively. Next if $2^n|A_j$, we may again complete the square. The sum on m then becomes

$$e\left(-\frac{h\left[\frac{P}{2^n}k'\right]_{2^{\nu-n}}\left(\frac{A_j}{2^n}\right)^2}{2^{\nu-n}}\right)\sum_{m \pmod{2^{\nu-n}}} e\left(\frac{hk'\frac{P}{2^n}m^2}{2^{\nu-n}}\right).$$

The sum on m now equals 1 if $\nu = n$, vanishes if $\nu = n+1$, and otherwise equals

$$(1+i)\varepsilon_{hk'\frac{P}{2^n}}^{-1}\sqrt{2^{\nu-n}}\left(\frac{2^{\nu-n}}{hk'\frac{P}{2^n}}\right),$$

so is in any case independent of A_j.

We now distinguish two cases. Firstly if $2^n \nmid A_j$, we have to show that

$$\frac{hA_j^2}{Pk} - \frac{h\left[2^\nu\frac{P}{\ell}\right]_{\frac{k'}{\ell}}\left(\frac{A_j}{\ell}\right)^2}{k'} =: f_j$$

is independent of $A_j \pmod{\mathbb{Z}}$. Write

$$f_j = \frac{h\left(\frac{A_j}{2^{n-1}\ell}\right)^2}{4\left(\frac{P}{2^n\ell}\right)\left(\frac{k}{2^n\ell}\right)}\left(1 - \left[2^\nu\frac{P}{\ell}\right]_{\frac{k'}{\ell}} 2^\nu\frac{P}{\ell}\right).$$

Now $\left(\frac{P}{2^n\ell}, \frac{k}{2^n\ell}\right) = 1$ and a direct calculation gives that

$$\begin{aligned}
&f_1 \equiv f_2 \pmod{\frac{P}{2^n\ell}},\\
&f_1 \equiv f_2 \pmod{\frac{k'}{\ell}}, && (6.3)\\
&f_1 \equiv f_2 \pmod{4\cdot\operatorname{lcm}\left(2^{\nu-n}, 2^{\operatorname{ord}_2\left(\frac{P}{2^n}\right)}\right)}. && (6.4)
\end{aligned}$$

If $2^n|A_j$, we require that

$$\frac{hA_j^2}{Pk} - \frac{h\left[2^\nu\frac{P}{\ell}\right]_{\frac{k'}{\ell}}\left(\frac{A_j}{\ell}\right)^2}{k'} - \frac{h\left[\frac{P}{2^n}k'\right]_{2^{\nu-n}}\left(\frac{A_j}{2^n}\right)^2}{2^{\nu-n}} =: g_j$$

is independent of $A_j \pmod{\mathbb{Z}}$. Write

$$g_j = \frac{h\left(\frac{A_j}{2^n\ell}\right)^2}{\left(\frac{P}{2^n\ell}\right)\left(\frac{k}{2^n\ell}\right)}\left(1 - \left[2^\nu \frac{P}{\ell}\right]_{\frac{k'}{\ell}} 2^\nu \frac{P}{\ell} - \left[\frac{P}{2^n}k'\right]_{2^{\nu-n}} \frac{P}{2^n}k'\right).$$

We have to show that

$$g_1 \equiv g_2 \pmod{\frac{k}{2^n\ell}},$$
$$g_1 \equiv g_2 \pmod{\frac{P}{2^n\ell}},$$

which again follows by a direct calculation. This finishes the claim. □

7 The tail of colored Jones polynomials, Nahm-type sums, and false theta functions

In this part we investigate various Nahm-type expressions for characters of the $(1,p)$-singlet algebra modules in connection with knot invariants.

As already mentioned in [12], there is a close relationship between characters of the $(1,p)$-singlet vertex algebra and "limits" of normalized colored Jones polynomials of certain alternating knots. As shown in [21], alternating knots have the remarkable property that their Jones polynomials $J_{n,K}(q)$ satisfy the stability property, that is, suitably normalized polynomials $\hat{J}_{n,K}(q)$ approach to a fixed infinite q-series $\Phi_K(q)$, called the *tail* of K. In [21], many examples of tails were computed. The concept of tail can be used to prove various q-series identities. This was first utilized by Armond and Dasbach in [7], where they showed that the Andrews-Gordon identity can be proven using two methods for computing the tail of the $(2, 2p+1)$ torus knots.

Example 7.1. *(Torus knot $(2,2p)$) The tail of the Jones polynomial of the $(2,2p)$ torus knot is given by (see [21], [22] for instance):*

$$\Phi_p(q) = \sum_{n\in\mathbb{Z}} \operatorname{sgn}(n) q^{pn^2+(p-1)n}.$$

By using skein theoretical techniques, very recently Hajij in [22] obtained another representation of $\Phi_p(q)$ as a multi q-hypergeometric series. It is not clear to us if this is the same Nahm-type sum that comes from the work of Garoufalidis and Le [21]. In any event, Hajij's result reads as

$$\Phi_p(q) = (q;q)_\infty \sum_{(n_1,\dots,n_{p-1})\in\mathbb{N}_0^{p-1}} \frac{q^{N_1^2+\cdots+N_{p-1}^2+N_1+\cdots+N_{p-1}}}{(q;q)_{n_{p-1}}^2 (q;q)_{n_1}\cdots(q;q)_{n_{p-2}}}, \tag{7.1}$$

where $N_j := \sum_{k=j}^{p-1} n_k$ and $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$. In particular, for $p=2$, one gets

$$\Phi_2(q) = (q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n^2},$$

an identity going back to Ramanujan.

We have already seen in Section 2 that

$$\mathrm{ch}_{W(2,2p-1)}(\tau)=\frac{\sum_{n\in\mathbb{Z}}\mathrm{sgn}(n)q^{p\left(n+\frac{p-1}{2p}\right)^2}}{\eta(\tau)}.$$

To get rid off all non-integral q-powers, we normalize $\mathrm{ch}_{W(2,2p-1)}(q)$ by letting

$$\widetilde{\mathrm{ch}}_{W(2,2p-1)}(q):=\frac{\sum_{n\in\mathbb{Z}}\mathrm{sgn}(n)q^{pn^2+(p-1)n}}{(q;q)_\infty}=\frac{\Phi_p(q)}{(q;q)_\infty}.$$

Thus, by (7.1),

$$\widetilde{\mathrm{ch}}_{W(2,2p-1)}(q)=\sum_{(n_1,\ldots,n_{p-1})\in\mathbb{N}_0^{p-1}}\frac{q^{N_1^2+\cdots+N_{p-1}^2+N_1+\cdots+N_{p-1}}}{(q;q)_{n_{p-1}}^2(q;q)_{n_1}\cdots(q;q)_{n_{p-2}}},\tag{7.2}$$

where the N_j are as above. Recall also characters of related $W(2,2p-1)$-modules $M_{1,s}$ and their modified graded dimension (here $1\leq\lambda\leq p$):

$$\widetilde{\mathrm{ch}}_{M_{1,p-\lambda}}(q)=\frac{\sum_{n\in\mathbb{Z}}\mathrm{sgn}(n)q^{pn^2+\lambda n}}{(q;q)_\infty}\in\mathbb{Z}[[q]].\tag{7.3}$$

Related to (7.2), but different, q-series identities also appeared a few years earlier in [18] in the context of vertex algebras. The main result in [18] gives q-hypergeometric type expressions for certain characters of the so-called *doublet* vertex algebra. These character formulas come from (combinatorial) quasiparticle bases of modules. Writing the generating function for quasi-particle bases results in the following identity: For $1\leq\lambda\leq p-1$, we have

$$\begin{aligned}\widetilde{\mathrm{ch}}_{\Lambda(p-\lambda)\oplus\Pi(p-\lambda)}(q)&=\frac{\sum_{n\in\mathbb{Z}}nq^{\frac{p}{4}n^2-\frac{\lambda n}{2}}}{(q;q)_\infty}\\&=\sum_{(n_+,n_-,n_1,\ldots,n_{p-1})\in\mathbb{N}_0^{p+1}}\frac{q^{\frac{p}{4}(n_++n_-)^2+\frac{\lambda}{2}(n_++n_-)+N_1^2+\cdots+N_{p-1}^2+N_{p-\lambda}+\cdots+N_{p-1}+(n_++n_-)(N_1+\cdots+N_{p-1})}}{(q;q)_{n_-}(q;q)_{n_+}(q;q)_{n_1}\cdots(q;q)_{n_{p-1}}}.\end{aligned}\tag{7.4}$$

The next argument, relying heavily on [18], uses the previous formula to compute Nahm-type representation of (7.3). There are two observations to be made here. First, a basis of $\Lambda(p-\lambda)\oplus\Pi(p-\lambda)$ is obtained by using generators of the doublet vertex algebra a^+, a^- and the Virasoro generator (for details see [18], [5]). The $\mathbb{Z}$-grading on a basis element is defined as the number of appearances of a^+ minus the number of a^-. This is analogous to the charge grading for the lattice vertex algebra V_L and also for free fermions. The way the doublet modules decompose into $\mathbb{Z}$-graded pieces coincides with the decomposition with respect to the singlet algebra. Thus we are only interested in the graded dimension of the subspace of $\Lambda(p-\lambda)\oplus\Pi(p-\lambda)$, where the number of a^+ appearances equals the number of a^-. Finally, we note that n_+ variable in the summation of (7.4) controls the number of a^+, while n_- takes care of a^-. We are essentially done, because we only have to put $n_+=n_-=n$ in the above character (details can be found in [18]). We conclude

$$\widetilde{\mathrm{ch}}_{M_{1,p-\lambda}}(q)=\sum_{(n,n_1,\dots,n_{p-1})\in\mathbb{N}_0^p}\frac{q^{pn^2+\lambda n+N_1^2+\cdots+N_{p-1}^2+N_{p-\lambda}+\cdots+N_{p-1}+2n(N_1+\cdots+N_{p-1})}}{(q;q)_n^2(q;q)_{n_1}\cdots(q;q)_{n_{p-1}}}. \tag{7.5}$$

In particular, for $\lambda=p-1$, where $M_{1,1}=\mathcal{W}(2,2p-1)$, we have

$$\widetilde{\mathrm{ch}}_{W(2,2p-1)}(q)=\sum_{(n,n_1,\dots,n_{p-1})\in\mathbb{N}_0^p}\frac{q^{pn^2+(p-1)n+N_1^2+\cdots+N_{p-1}^2+N_{p-\lambda}+\cdots+N_{p-1}+2n(N_1+\cdots+N_{p-1})}}{(q;q)_n^2(q;q)_{n_1}\cdots(q;q)_{n_{p-1}}}. \tag{7.6}$$

To proceed, we make several observations. Clearly (7.2) and (7.6) give two *different* Nahm-type expressions for the same character, at least if $\lambda=p-1$. Observe that for fixed $p\geq 2$, formula (7.2) involves $p-1$-fold summation but (7.2) gives a p-fold summation. Second, both proofs of identities are based on non-elementary techniques. For instance, (7.5) requires a variety of techniques from vertex algebra theory and is by no means elementary. On the other hand Hajij's proof employs many knot theoretical results.

The next result proves both formulas in a somewhat unifying manner without relying on anything but Andrews-Gordon identities and some manipulations with q-series. Also it explains why (7.2) simplifies, compared to (7.6), only for $\lambda=p-1$.

Proposition 7.2. *Formulas (7.2) and (7.5) hold.*

Proof: Our proof is based on some ideas of Warnaar [35] in his proof of Flohr-Grabow-Koehn's conjectures for the characters of irreducible modules for the triplet vertex algebra. Following Andrews' notation, we let

$$Q_{p,\ell}(x):=\frac{1}{(xq;q)_\infty}\sum_{j=0}^{\infty}(-1)^jq^{\binom{j}{2}+pj^2+(p-\ell+1)p}\left(1-x^\ell q^{\ell(2j+1)}\right)\frac{(xq;q)_j}{(q;q)_j}.$$

The Andrews-Gordon identities [8] then give that

$$Q_{p,\ell}(x)=\sum_{(n_1,\dots,n_{p-1})\in\mathbb{N}_0^{p-1}}\frac{x^{N_1+\cdots+N_{p-1}}q^{N_1^2+\cdots+N_{p-1}^2+N_\ell+\cdots+N_{p-1}}}{(q;q)_{n_1}\cdots(q;q)_{n_{p-1}}}.$$

To prove (7.5), we consider $Q_{p,p-\lambda}(q^{2n})$ and observe that its right hand-side can be written as

$$\sum_{n=0}^{\infty}\frac{q^{pn^2+n\lambda}}{(q;q)_n^2}Q_{p,p-\lambda}\left(q^{2n}\right).$$

Rewriting gives

$$\frac{1}{(q^{2n+1};q)_\infty}\sum_{j,n\geq 0}(-1)^j\frac{q^{pn^2+n\lambda+2npj+\binom{j}{2}+pj^2+(\lambda+1)j}\left(1-q^{(p-\lambda)(2j+2n+1)}\right)}{(q;q)_n^2}\frac{\left(q^{2n+1};q\right)_j}{(q;q)_j}$$

$$=\frac{1}{(q;q)_\infty}\sum_{j,n\geq 0}(-1)^j\frac{q^{p(n+j)^2+\lambda(n+j)+\binom{j}{2}+j}\left(1-q^{(p-\lambda)(2(j+n)+1)}\right)}{(q;q)_n^2}\frac{\left(q^{2n+1};q\right)_j}{(q;q)_j}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j \frac{q^{pn^2+\lambda n+\binom{j}{2}+j} \left(1-q^{(p-\lambda)(2n+1)}\right)}{(q;q)_{n-j}^2} \frac{(q;q)_{2n-j}}{(q;q)_j},$$

where for the last equality we changed n into $n+j$ and sum j from 0 to n.

Now we utilize the identity

$$1 = \sum_{j=0}^{n} (-1)^j q^{\frac{j(j+1)}{2}} \begin{bmatrix} 2n-j \\ n-j \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q = \sum_{j=0}^{n} (-1)^j \frac{q^{\frac{j(j+1)}{2}} (q;q)_{2n-j}}{(q;q)_j (q;q)_{n-j}^2},$$

which follows from a version of q-Chu-Vandermonde summation, where $\begin{bmatrix} n \\ j \end{bmatrix}_q$ denotes the q-binomial. This gives the desired formula.

To prove (7.2), we first isolate the n_{p-1} variable from $N_1^2 + N_1 + \cdots + N_{p-1}^2 + N_{p-1}$ and obtain $N_1^2 + \cdots N_{p-2}^2 + N_1 + \cdots + N_{p-2}^2 + (p-1)n_{p-1}^2 + (p-1)n_{p-1}$. We also write n for n_{p-1} for simplicity. We now proceed as before by using $Q_{p-1,1}(q^{2n})$, so the right hand-side in (7.2) equals (we omit some details like shifting the sum in n)

$$\sum_{n=0}^{\infty} \frac{q^{(p-1)n^2+(p-1)n}}{(q;q)_n^2} Q_{p-1,1}\left(q^{2n}\right)$$
$$= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j q^{(p-1)n^2+(p-1)n+\binom{j}{2}} \left(1-q^{2n+1}\right) \frac{(q;q)_{2n-j}}{(q;q)_{n-j}^2 (q;q)_j}. \quad (7.7)$$

We also record another consequence of q-Chu-Vandermonde summation

$$q^{n^2} = \sum_{j=0}^{j} (-1)^j q^{\frac{j(j-1)}{2}} \begin{bmatrix} 2n-j \\ n-j \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q = \sum_{j=0}^{n} (-1)^j \frac{q^{\frac{j(j-1)}{2}} (q;q)_{2n-j}}{(q;q)_j (q;q)_{n-j}^2}.$$

Thus, the expression in (7.7) equals

$$\sum_{n=0}^{\infty} q^{(p-1)n^2+n^2+(p-1)n} \left(1-q^{2n+1}\right) = \frac{\sum_{k\in\mathbb{Z}} \operatorname{sgn}(n) q^{pn^2+(p-1)n}}{(q;q)_\infty},$$

finishing the proof. It is now clear that similar "compressed" identities cannot be obtained from $Q_{p-1,\ell}(q^{2n})$ for $\ell \neq 1$.

□

Remark 7.3. *It is an open problem to find a knot theoretical interpretation of atypical (p_+, p_-)-characters and to determine their q-hypergeometric formulas.*

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