

W-ALGEBRAS, HIGHER RANK FALSE THETA FUNCTIONS, AND QUANTUM DIMENSIONS

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ABSTRACT. Motivated by appearances of Roger's false theta functions in the representation theory of the singlet vertex operator algebra, for each finite-dimensional simple Lie algebra of ADE type, we introduce higher rank false theta functions as characters of atypical modules of certain W -algebras and compute asymptotics of irreducible characters which allows us to determine quantum dimensions of the corresponding modules. In the sl_2 -case, we recover many results from [10].

1. INTRODUCTION

It is well known (due to Frenkel-Kac and Segal) that any simply-laced affine Lie algebra $\widehat{\mathfrak{g}}$ admits a homogeneous realization in terms of vertex operators acting on a direct sum of infinitely many Fock spaces. More precisely, for every coset $\lambda + Q \in Q^0/Q$ (Q the root and Q^0 the weight lattice) there is an irreducible highest weight $\widehat{\mathfrak{g}}$ -module $V_{Q+\lambda}$. This construction can be reformulated (due to Borcherds [8]; see also [23]) in the language of vertex operator algebras. From this point of view, the module V_Q has a vertex operator algebra structure (lattice vertex algebra) while $V_{Q+\lambda}$ is an irreducible V_Q -module [29]. Under the standard grading, characters of these modules are known and given by

$$\text{ch}[V_{Q+\lambda}](\tau) = \frac{\sum_{\alpha \in Q+\lambda} q^{\frac{\|\alpha\|^2}{2}}}{\eta(\tau)^{\text{rank}(Q)}},$$

where $\|\alpha\|^2 := (\alpha, \alpha)$ and $\eta(\tau) := q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n)$ is Dedekind's η -function ($q := e^{2\pi i \tau}$ throughout), a modular form of weight $1/2$. Modular forms have appeared in abundance in the representation theory of rational vertex operator algebras [29], especially in the theory surrounding the moonshine phenomena [23].

In the last ten years, a lot of research has been devoted to understanding the so called logarithmic conformal field theories and related irrational vertex algebras [2, 4, 18, 19, 20, 24, 30, 32], etc. In these theories, modularity either holds with some modifications, as seen in C_2 -cofinite vertex operator algebras (VOAs), or is completely absent as is the case of non C_2 -cofinite VOAs. An interesting feature of irrational C_2 -cofinite vertex algebras is the lack of an affine Lie algebra action. One reason for this comes from the fact that all known examples of such vertex algebras appear as proper subalgebras of vertex algebras associated to dilated root lattices $\sqrt{p}Q$, with $p \in \mathbb{N}_{\geq 2}$ (here p is not necessarily prime!). Although these rootless lattices rarely appear in Lie theory, they do have relevance in vertex algebra theory.

In many ways this paper is a continuation of [10] (see also [11, 14, 30]) where the authors thoroughly studied properties of certain false theta functions coming from characters of modules of the singlet vertex operator algebra. The starting point in this line of work was an observation

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[10, 11] that the classical Roger's false function

$$\sum_{n \in \mathbb{Z}} \operatorname{sgn}(n) q^{p \left(n + \frac{j}{2p} \right)^2} = \sum_{n=0}^{\infty} q^{p \left(n + \frac{j}{2p} \right)^2} - \sum_{n=0}^{\infty} q^{p \left(n + \frac{n-j}{2p} \right)^2} \quad (1.1)$$

and other related false theta functions show up as “numerators” of characters of the $(1, p)$ -singlet W -algebra (see [1]).

In the theory of modular forms, false theta series have occurred in many important works, including [6, 28, 34, 36]. These are not modular but when passing to the lower half-plane radially one obtains mock modular forms which are holomorphic parts of so-called harmonic Maass forms (non-holomorphic modular forms annihilated by the Laplace operator). In particular, this yields an analytic function in the upper half-plane which drips into the lower half-plane.

It is already known that the $(1, p)$ -singlet algebra is related in a non-trivial way to the Lie algebra \mathfrak{sl}_2 , so this raises the question of whether functions like (1.1) can be defined for other root systems such that they also give rise to characters of modules. As we previously announced in [30], with full details given here, this is indeed possible, at least in the simply-laced case.

To explain our construction, we first consider a sequence of embeddings of vertex algebras:

$$\mathcal{W}_{c_p}(\mathfrak{g}) \hookrightarrow W^0(p)_Q \hookrightarrow W(p)_Q \hookrightarrow V_{\sqrt{p}Q} ,$$

where \mathfrak{g} is the corresponding simple Lie algebra, $\mathcal{W}_{c_p}(\mathfrak{g})$ is the affine W -algebra associated to \mathfrak{g} [7, 15], and $W^0(p)_Q$ and $W(p)_Q$ are certain vertex algebras defined below. In the special case of $\mathfrak{g} = \mathfrak{sl}_2$, we recover a more familiar embedding of vertex algebras studied for examples in [2, 10, 11]:

$$L(c_{p,1}, 0) \hookrightarrow W(2, 2p-1) \hookrightarrow W(p) \hookrightarrow V_{\sqrt{2p}\mathbb{Z}} ,$$

where $W(2, 2p-1)$ is the $(1, p)$ -singlet vertex algebra [1].

It turns out that the most interesting q -series come from considerations of characters of $W^0(p)_Q$ -modules. The vertex algebra $W^0(p)_Q$ is not C_2 -cofinite and as such admits characters of modules with peculiar properties. In a special case of the vertex algebra $W^0(p)_Q$, we get the following character formula

$$\sum_{\alpha \in Q \cap P^+} \dim(V(\alpha)_0) \sum_{w \in W} (-1)^{\ell(w)} q^{\frac{1}{2} \|\sqrt{p}w(\alpha+\rho) - \frac{1}{\sqrt{p}}\rho\|^2} \frac{1}{\eta(\tau)^{\operatorname{rank}(Q)}} , \quad (1.2)$$

with $V(\alpha)_0$ the zero weight space of the irreducible highest weight \mathfrak{g} -module $V(\alpha)$, W is the Weyl group, $\ell(w)$ is the length of w , and P^+ denotes the set of dominant integral weights. The expression (1.2), after eliminating the η -factor, should be viewed as a higher rank generalization of the classical false theta functions. Indeed, the sign factor in (1.1) can be understood as $(-1)^{\ell(w)}$, where w is the non-unit Weyl group element of \mathfrak{sl}_2 . As far as we know, q -series like (1.2) have not been studied in the literature apart from the rank one case. We believe that they are closely related to certain limits of colored Jones polynomials of alternating knots colored by representations of \mathfrak{g} , studied in [25]. Connection between the $(1, p)$ singlet characters and certain torus knots colored by \mathfrak{sl}_2 representations was already established in [10]. Another, more conceptual, contact between this line of work and quantum knot (and link) invariants is expected to be obtained by establishing equivalences of categories of $W^0(p)_Q$ -modules and modules for “unrolled” quantum groups at root of unity. For the $(1, p)$ singlet algebra, this equivalence was recently studied in [13].

The aim of this paper is to study properties of characters of $W(p)_Q$ initiated in [18] and especially characters of $W^0(p)_Q$ -modules such as (1.2). We briefly outline the paper and indicate the main results. We introduce vertex operator algebras $W(p)_Q$ and $W^0(p)_Q$ in Section 2, following mostly [4, 18, 30], although similar constructions with screening operators can be traced to affine W -algebras [22]. In Section 3, we recall modular transformation properties of higher rank theta functions coming from lattices, while, in Section 4, we discuss characters of certain $W(p)_Q$ -modules following

closely the proposal in [18]. Our main result is Theorem 4.1 below, proving finite-dimensionality of the $\mathrm{SL}_2(\mathbb{Z})$ -closure of the space of $W(p)_Q$ -characters. We also work out in full detail modular transformation properties of $\mathrm{ch}[W(p)_{A_2}](\tau)$ (see Section 4.3). Section 5 is primarily concerned with characters of $W^0(p)_Q$ -modules and an explicit computation for $\mathrm{ch}[W^0(p)_{A_2}](\tau)$. The key aspect of this paper is in Section 6, where we study asymptotic expansions of characters of $W^0(p)_Q$ -modules. The main result here is Theorem 6.2, which can be restated as follows.

Theorem 1.1. *For $\beta \in L^0$, as $t \rightarrow 0^+$,*

$$\eta(it)^{\dim(Q)} \mathrm{ch}[W^0(p, \beta)_Q](it) \sim \frac{\dim(V(-\sqrt{p}\bar{\lambda}))}{p^{|\Delta_+|}} + O(\sqrt{t}).$$

where $\bar{\lambda}$ (depending on β) is defined in Section 5.1.

Although our method for computing asymptotic expansion of $W^0(p)_Q$ is very general, for brevity in Section 7 we only compute the full expansion of $\mathrm{ch}[W^0(p)_{A_2}](\tau)$. We obtain

Theorem 1.2. *Asymptotically, as $t \rightarrow 0^+$, we have with $p \in \mathbb{N}_{\geq 2}$,*

$$\eta(it)^2 \mathrm{ch}[W^0(p)_{A_2}](it) \sim \sum_{N \geq 0} \alpha(N) (-2\pi pt)^N,$$

where $\alpha(N)$ is defined in (7.6) below.

Remark 1. For $Q = A_1$, in [10], the authors obtained an asymptotic expansion of characters. This asymptotic expansion was then related to Eichler integrals which have a nice (mock transformation) behavior. We plan to investigate in future research whether the formulas in Theorem 1.2 can be used to find companions of the characters in the lower half-plane which is (mock) modular.

For regular vertex operator algebras (or more generally, modular tensor categories) a *quantum dimension* defines a homomorphism from the Grothendick ring of the category of modules to \mathbb{C} . If we denote by M_j , $0 \leq j \leq k$, all inequivalent irreducible V -modules, where we pick $V = M_0$, the quantum dimension of M_j is given by

$$\mathrm{qdim}(M_j) = \frac{S_{j,0}}{S_{0,0}},$$

where $S_{j,k}$ are the entries of the S -matrix. It is expected that this quantity plays an important role for irrational vertex algebras as well. Thus, in Section 8, based on results of Section 6, we compute *analytic* quantum dimensions of certain $W^0(p)_Q$ -modules. This quantum dimension is defined purely analytically as the limit

$$\mathrm{qdim}(M) := \lim_{t \rightarrow 0^+} \frac{\mathrm{ch}[M](it)}{\mathrm{ch}[V](it)}.$$

We stress that for regular vertex operator algebra these two definitions agree, at least if lowest conformal weights of M_j , $j \neq 0$, are positive. By using the last definition, we show in Section 8 that

Corollary 1.3. *For $\beta \in L^0$, whose coset representative in L^0/L is λ , we have*

$$\mathrm{qdim}(W^0(p, \beta)_Q) = \dim_{\mathbb{C}} V(-\sqrt{p}\bar{\lambda}),$$

where $\bar{\lambda}$ is defined in (4.1).

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2. VERTEX OPERATOR ALGEBRAS, $W(p)_Q$ AND $W^0(p)_Q$

Let $(Q, (\cdot, \cdot))$ be a rank n root lattice of ADE type, and $Q^0 = P$ its dual (weight) lattice. We are primarily concerned with dilated root lattices $L = \sqrt{p}Q$ and their duals L^0 , where $p \in \mathbb{N}_{\geq 2}$. Let α_j , $1 \leq j \leq n$, denote the simple roots of Q , and Δ_+ and Δ_- the set of positive and negative roots, respectively. We have $\|\alpha_j\|^2 = (\alpha_j, \alpha_j) = 2$ for all j .

We equip V_L with a vertex algebra structure by choosing an appropriate 2-cocycle as in the case of V_Q [4, 29, 31]. For the conformal vector we choose [2, 18, 30, 31]

$$\omega = \omega_0 + \frac{p-1}{2\sqrt{p}} \sum_{\alpha \in \Delta_+} \alpha(-2)\mathbf{1},$$

where ω_0 denotes the canonical (quadratic) Virasoro generator [29]. Then V_L is a conformal vertex algebra of central charge

$$\text{rank}(Q) + 12(\rho, \rho) \left(2 - p - \frac{1}{p} \right),$$

where ρ denotes the half-sum of positive roots. Next, we consider screening operators $e_0^{\sqrt{p}\alpha_j}$, $e_0^{-\frac{\alpha_k}{\sqrt{p}}}$ ($1 \leq j, k \leq n$) acting among V_L and certain V_L -modules. It is known that for every j and k the operators $e_0^{\sqrt{p}\alpha_j}$ and $e_0^{-\frac{\alpha_k}{\sqrt{p}}}$ commute with each other, and they both commute with the Virasoro algebra [4, 18, 31].

The connection to affine W -algebras is achieved via screening operators. Recall that there are two methods for defining universal affine W -algebras. The affine W -algebra associated to $\hat{\mathfrak{g}}$ at level $k \neq -h^\vee$, denoted by $\mathcal{W}_k(\mathfrak{g})$, is usually defined as the cohomology group obtained via a quantized BRST complex for the Drinfeld-Sokolov hamiltonian reduction [22]. Cohomology of this complex is nontrivial only in degree zero [22]. Moreover, it is known that $\mathcal{W}_k(\mathfrak{g})$ is freely generated by primary fields as a vertex algebra (thus the name W -algebra), in the sense that it is strongly freely generated by $\text{rank}(\mathfrak{g})$ primary fields, not counting the conformal vector. On the other hand, an important result by B. Feigin and E. Frenkel [22] says that for k generic and \mathfrak{g} simply-laced, there is an alternative description of $\mathcal{W}_k(\mathfrak{g})$ in terms of free fields. We use F_0 to denote the rank n vacuum Fock space, which is also a vertex algebra, and by F_λ Fock modules parametrized by \mathfrak{h} . Let $\nu = k + h^\vee$, where k is generic and consider the following operators [22]:

$$e_0^{-\frac{\alpha_j}{\sqrt{\nu}}} : F_0 \longrightarrow F_{-\frac{\alpha_j}{\sqrt{\nu}}} \quad (1 \leq j \leq n).$$

Then

$$\mathcal{W}_\nu(\mathfrak{g}) = \bigcap_{j=1}^n \text{Ker}_{F_0} \left(e_0^{-\frac{\alpha_j}{\sqrt{\nu}}} \right).$$

Moreover, we also have the following important duality

$$\mathcal{W}_\nu(\mathfrak{g}) = \bigcap_{j=1}^n \text{Ker}_{F_0} \left(e_0^{\sqrt{\nu}\alpha_j} \right).$$

For non-generic values, screening operators can still be used to define vertex algebras which we now describe. For instance, $p = k + h^\vee \in \mathbb{N}_{\geq 2}$ is known to be non-generic. So we let

$$W^0(p)_Q := \bigcap_{j=1}^n \text{Ker}_{F_0} e_0^{-\frac{\alpha_j}{\sqrt{p}}},$$

a vertex operator algebra containing $\mathcal{W}_p(\mathfrak{g})$ as a proper subalgebra [4, 12, 18]. In particular, for $Q = A_1$, $W(p)_Q$ is precisely the $(1, p)$ -singlet algebra discussed in [10, 14]. Because of the integrality condition, the previous algebra can be maximally extended within the lattice vertex algebra leading to [2, 12, 18, 32]

$$W(p)_Q := \bigcap_{j=1}^n \text{Ker}_{V_L} e_0^{-\frac{\alpha_j}{\sqrt{p}}}.$$

Again, if we let $Q = A_1$, this is the well-known triplet vertex operator algebra $W(p)$. Similarly, we can construct irreducible $W(p)_Q$ -modules. The following conjecture was mentioned in [4, 18].

Conjecture 1. *The vertex operator algebra $W(p)_Q$ is C_2 -cofinite.*

Remark 2. Due to the complicated embedding structure of higher Feigin-Fuchs modules, there are essentially no results on the representation theory of $W^0(p)_Q$ and of $W(p)_Q$, for $\text{rank}(Q) \geq 2$. Still, in [18], the authors proposed an interesting geometric method for studying $W(p)_Q$ -modules and their characters. In particular, it is expected that all irreducible representations of $W(p)_Q$ can be understood as subquotients of $V_{\sqrt{p}Q}$ -modules [4, 18, 30]. Since we are only interested in properties of characters of modules, from now on, we assume that the modules in [18] provide a complete list of irreducible $W(p)_Q$ -modules and characters (this is yet to be proven by using methods of vertex algebras as in [2]). We also conjecture that all atypical irreducible representations of $W^0(p)_Q$ appear in the decomposition of irreducible $W(p)_Q$ -modules. Typical representations are modules isomorphic to higher rank Fock spaces.

3. THETA FUNCTIONS ASSOCIATED TO LATTICES

This short section deals with higher rank theta functions. All results presented here are already contained in the literature, but we recall them for the reader's convenience.

For $\tau \in \mathbb{H}$, $M \in \text{Pos}(n; \mathbb{R})$ (positive definite matrices with real entries) and $a, b \in \mathbb{C}^n$, we define

$$\Theta_{a,b}(\tau; M) := \sum_{r \in \mathbb{Z}^n} e^{\pi i \tau (r+a)^T M (r+a) + 2\pi i (r+a)^T b}.$$

Then

$$\Theta_{-b,a} \left(-\frac{1}{\tau}; M^{-1} \right) = (-i\tau)^{\frac{n}{2}} \sqrt{\det(M)} e^{-2\pi i a^T b} \Theta_{a,b}(\tau; M). \quad (3.1)$$

We also present essentially the same result using different notation. Let $(L, (\cdot, \cdot))$ be a positive definite integral lattice and L° denotes its dual. For $g \in \mathbb{Q} \otimes L$, define

$$\vartheta_{L,g}(u; \tau) := \sum_{\alpha \in L} e^{\pi i \tau \|\alpha+g\|^2 + 2\pi i (\alpha+g, u)},$$

where $u \in \mathbb{C} \otimes L \cong \mathbb{C}^{\text{rank}(L)}$ and $\|\gamma\|^2 := (\gamma, \gamma)$. Denoting by A the Gram matrix of L , we have

$$\Theta_{g,u}(\tau; A) = \vartheta_{L,g}(u; \tau),$$

where $n := \text{rank}(L)$. The next result is well-known (see e.g. [26]).

Theorem 3.1. *For $[g] = g + L \in L^\circ/L$, we have*

$$\vartheta_{L,g} \left(u; -\frac{1}{\tau} \right) = \frac{(-i\tau)^{\frac{m}{2}}}{\sqrt{\det(A)}} e^{\pi i u^2} \sum_{k=0}^{m-1} \zeta_n^k \vartheta_{L_g, kr}(\tau u; \tau),$$

where $m := o([g])$ (order of $[g]$ in L°/L), r satisfies $(r, g) \equiv \frac{1}{m} \pmod{1}$ and $L_g := \{\alpha \in L : (\alpha, g) \in \mathbb{Z}\}$, an intermediate lattice of L and L^0 .

4. CHARACTERS OF $W(p)_Q$ -MODULES

4.1. The Weyl character formula. As in Section 2, let $L = \sqrt{p}Q$, where Q is an ADE root lattice and ρ is the half-sum of positive roots. Also, denote by W the Weyl group. Set

$$\Delta := \prod_{\alpha \in \Delta_-} (1 - e^\alpha),$$

the (formal) Weyl denominator. We can also view Δ as a complex valued function. We introduce complex variables z_j and let $\mathbf{z}^\alpha = z_1^{(\omega_1, \alpha)} \cdots z_n^{(\omega_n, \alpha)}$, where $n = \text{rank}(Q)$. We also let $z_j = e^{2\pi i u_j}$, $u_j \in \mathbb{C}$. Then, for $z = (z_1, \dots, z_n)$, we define

$$\Delta(z) := \prod_{\alpha \in \Delta_-} (1 - \mathbf{z}^\alpha).$$

This function should not to be confused with the modular discriminant!

Let $w \in W$ act on $\mathbb{C}[\mathfrak{h}^*]$, where \mathfrak{h}^* is the dual of the Cartan subalgebra, according to $w \cdot e^\gamma = e^{w(\gamma)}$ and similarly for w acting on \mathbf{z}^γ . Then we have the well known formula

$$w \left(\prod_{\alpha \in \Delta_-} (1 - e^\alpha) \right) = e^{\rho - w(\rho)} (-1)^{\ell(w)} \prod_{\alpha \in \Delta_-} (1 - e^\alpha).$$

For $\lambda \in P^+$ (dominant integral weight) we denote by χ_λ the character of the highest weight module. Recall also the Weyl Character Formula:

$$\chi_\lambda(z) = \sum_{w \in W} (-1)^{\ell(w)} \frac{\mathbf{z}^{w(\lambda + \rho) - \rho}}{\Delta(z)}.$$

4.2. Characters of $W(p)_Q$ -modules. Next, we discuss $W(p)_Q$ -modules (see also Remark 1). As in [18], we choose $\lambda_j = \frac{\omega_j}{\sqrt{p}}$, $j = 1, \dots, n$, to be a basis of L^0 and we also fix representative of Q^0/Q to be $0, \omega_1, \dots, \omega_n$ for type A_n , where ω_j are the fundamental weights. For D_n , we take $0, \omega_1, \omega_{n-1}, \omega_n$; for E_6 we choose $0, \omega_1$ and ω_3 ; for E_7 we take 0 and ω_2 , and for E_8 we choose only 0 .

Again, following [18], each coset L^0/L has a unique representative λ of the form

$$\lambda = \sqrt{p} \widehat{\lambda} + \sum_{j=1}^n (1 - s_j) \lambda_j, \tag{4.1}$$

where $\widehat{\lambda}$ is the representative of Q^0/Q fixed earlier, and $\bar{\lambda} = \sum_{j=1}^n (1 - s_j) \lambda_j$ such that $s_j \in \{1, 2, \dots, p\}$. From now on, elements $\lambda \in L^0/L$ are identified with a unique representative of cosets fixed above.

Example 1. For $Q = A_2$, there are $3p^2$ cosets of L^0/L . For each $\widehat{\lambda} \in \{0, \omega_1, \omega_2\}$ there are p^2 choices of $\bar{\lambda}$, namely:

$$\bar{\lambda} = (1 - s_1) \frac{\omega_1}{\sqrt{p}} + (1 - s_2) \frac{\omega_2}{\sqrt{p}}, \quad s_1, s_2 \in \{1, \dots, p\}.$$

To each coset representative $\lambda \in L^0/L$ as above, we now associate an irreducible $W(p)_Q$ -module denoted by $W(p, \lambda)_Q$; there are $p^n |Q^0/Q|$ such modules in total. As mentioned in the introduction, we omit detailed discussion about modules here. Instead we only focus on their characters (also known as graded dimensions) and refer the reader to [18]. As it is customary in vertex algebra theory, the character of a module M is defined as

$$\text{ch}[M](\tau) := \text{tr}_M q^{L(0) - \frac{c}{24}},$$

where c is the central charge. But for $W(p)_Q$ -modules, it is also convenient to have on disposal complex variables z_j (and u_j). These extra variable should come from yet to be constructed action of the simple Lie algebra \mathfrak{g} on $W(p)_Q$ (for sl_2 this action was constructed in [5]). The resulting expression is called a *full character*. Then we have the following formulas for full characters of atypical irreducible $W(p)_Q$ -modules proposed in [18]:

$$\begin{aligned} \text{ch}[W(p, \lambda)_Q](\tau, z) &= \frac{1}{\eta(\tau)^{\text{rank}(Q)}} \sum_{w \in W} \sum_{\alpha \in Q} (-1)^{\ell(w)} \frac{q^{\frac{1}{2} \|\sqrt{p}\alpha + \lambda + (\sqrt{p} - \frac{1}{\sqrt{p}})\rho\|^2} \mathbf{z}^{w(\alpha + \widehat{\lambda})}}{w(\Delta(z))} \\ &= \frac{1}{\eta(\tau)^{\text{rank}(Q)}} \sum_{\alpha \in Q} q^{\frac{1}{2} \|\sqrt{p}(\alpha + \rho + \widehat{\lambda}) + \bar{\lambda} - \frac{1}{\sqrt{p}}\rho\|^2} \left(\sum_{w \in W} (-1)^{\ell(w)} \frac{\mathbf{z}^{w(\alpha + \rho + \widehat{\lambda}) - \rho}}{\Delta(z)} \right). \end{aligned} \quad (4.2)$$

Observe that the expression in the parentheses of (4.2) does not necessarily equal the Weyl character unless $\alpha + \widehat{\lambda} \in P^+$; rather it can be expressed as the Weyl character of different highest weight, up to a sign. As we shall see momentarily, this formula is very useful for determining modular properties of $\text{ch}[W(p, \lambda)_Q](\tau, z)$ after $z \rightarrow 1$ (i.e., $u \rightarrow 0$).

Next, we rewrite (4.2) by adjusting the Weyl sum (cf. [18]). Observe that we get a non-zero contribution in the sum only if $\alpha + \rho + \widehat{\lambda}$ is in the interior of a Weyl chamber. For such $\alpha + \rho + \widehat{\lambda}$, there is $w \in W$ such that $w(\alpha + \rho + \widehat{\lambda})$ is inside the fundamental Weyl chamber. Thus we can replace the summation over the root lattice as a summation over the $Q \cap P^+$ and an extra summation over the Weyl group. We use this and (4.2) to get

$$\begin{aligned} \text{ch}[W(p, \lambda)_Q](\tau, z) &= \frac{1}{\eta(\tau)^{\text{rank}(Q)}} \sum_{w \in W} \sum_{\alpha \in Q} (-1)^{\ell(w)} \frac{q^{\frac{1}{2} \|\sqrt{p}(\alpha + \rho + \widehat{\lambda}) + \bar{\lambda} - \frac{1}{\sqrt{p}}\rho\|^2} \mathbf{z}^{w(\alpha + \rho + \widehat{\lambda}) - \rho}}{\Delta(z)} \\ &= \frac{1}{\eta(\tau)^{\text{rank}(Q)}} \sum_{w \in W} \sum_{\alpha \in Q \cap P^+} \sum_{w' \in W} (-1)^{\ell(w)} \frac{q^{\frac{1}{2} \|\sqrt{p}w'(\alpha + \rho + \widehat{\lambda}) + \bar{\lambda} - \frac{1}{\sqrt{p}}\rho\|^2} \mathbf{z}^{ww'(\alpha + \rho + \widehat{\lambda}) - \rho}}{\Delta(z)} \\ &= \frac{1}{\eta(\tau)^{\text{rank}(Q)}} \sum_{\alpha \in Q \cap P^+} \sum_{w' \in W} \sum_{t \in W} (-1)^{\ell(t) + \ell(w')} \frac{q^{\frac{1}{2} \|\sqrt{p}w'(\alpha + \rho + \widehat{\lambda}) + \bar{\lambda} - \frac{1}{\sqrt{p}}\rho\|^2} \mathbf{z}^{t(\alpha + \rho + \widehat{\lambda}) - \rho}}{\Delta(z)} \\ &= \frac{1}{\eta(\tau)^{\text{rank}(Q)}} \sum_{\alpha \in Q \cap P^+} \sum_{w \in W} (-1)^{\ell(w)} q^{\frac{1}{2} \|\sqrt{p}w(\alpha + \rho + \widehat{\lambda}) + \bar{\lambda} - \frac{1}{\sqrt{p}}\rho\|^2} \chi_{\widehat{\lambda} + \alpha}(z) \\ &= \sum_{\alpha \in Q \cap P^+} \chi_{\widehat{\lambda} + \alpha}(z) \left(\sum_{w \in W} (-1)^{\ell(w)} \frac{q^{\frac{1}{2} \|\sqrt{p}w(\alpha + \rho + \widehat{\lambda}) + \bar{\lambda} - \frac{1}{\sqrt{p}}\rho\|^2}}{\eta(\tau)^{\text{rank}(Q)}} \right). \end{aligned}$$

The sum in the parenthesis is also the character of an affine W -algebra module [15, 18]. After we take the limit $z \rightarrow 1$, we get

$$\text{ch}[W(p, \lambda)_Q](\tau) = \sum_{\alpha \in Q \cap P^+} \dim \left(V(\widehat{\lambda} + \alpha) \right) \left(\sum_{w \in W} (-1)^{\ell(w)} \frac{q^{\frac{1}{2} \|\sqrt{p}w(\alpha + \rho + \widehat{\lambda}) + \bar{\lambda} - \frac{1}{\sqrt{p}}\rho\|^2}}{\eta(\tau)^{\text{rank}(Q)}} \right), \quad (4.3)$$

where $V(\gamma)$ denote the finite-dimensional \mathfrak{g} -module of highest weight γ . The previous formula is perhaps useful for representation theoretic consideration while equation (4.2) is more suitable for studying modular properties. If $\bar{\lambda} = 0$, the sum over the Weyl group can be further simplified as

$$\text{ch}[W(p, \lambda)_Q](\tau) = \frac{1}{\eta(\tau)^{\text{rank}(Q)}} \sum_{\alpha \in Q \cap P^+} \dim \left(V(\widehat{\lambda} + \alpha) \right) q^{\frac{1}{2p} \|p(\alpha + \rho + \widehat{\lambda}) - \rho\|^2} \prod_{\gamma \in \Delta_+} \left(1 - q^{(\gamma, \alpha + \rho + \widehat{\lambda})} \right),$$

after an application of the Weyl denominator formula.

Remark 3. Observe that, by Weyl's dimension formula, $\dim V(\gamma)$ depends polynomially on the labels m_j of the highest weight $\gamma = m_1\omega_1 + \dots + m_n\omega_n$. The degree of this polynomial is $|\Delta_+|$.

Next, we are interested in modular transformation properties of $W(p)_Q$ -characters. Let us first consider the A_1 case. Here $Q = \mathbb{Z}\alpha$ and $P = \mathbb{Z}\omega$. Two representatives of Q^0/Q are 0 and ω . Then $\widehat{\lambda}$ is either 0 or ω , and the corresponding $2p$ representatives of L^0/L are given by

$$\lambda = \frac{j}{\sqrt{p}}\omega,$$

where $j = -(p-1), \dots, 0$, and $j = 1, \dots, p$, respectively. Clearly, $P^+ \cap Q = \mathbb{N}\alpha = 2\mathbb{N}\omega$ and $\dim(V(2n\omega)) = 2n+1$, $\dim(V((2n-1)\omega)) = 2n$, $\rho = \omega$, and $W = \{\pm 1\}$. There are two series of characters: For $-(p-1) \leq j \leq 0$ ($\widehat{\lambda} = 0$), we have

$$\text{ch} \left[W \left(p, \frac{j\omega}{\sqrt{p}} \right)_{A_1} \right] (\tau) = \frac{1}{\eta(\tau)} \sum_{n=0}^{\infty} (2n+1) \left(q^{p \left(n + \frac{p-j-1}{2p} \right)^2} - q^{p \left(n + \frac{p+j+1}{2p} \right)^2} \right), \quad (4.4)$$

while for $1 \leq j \leq p$ ($\widehat{\lambda} = \omega$), we have

$$\text{ch} \left[W \left(p, \frac{j\omega}{\sqrt{p}} \right)_{A_1} \right] (\tau) = \frac{1}{\eta(\tau)} \sum_{n=0}^{\infty} 2n \left(q^{p \left(n + \frac{2p-j-1}{2p} \right)^2} - q^{p \left(n + \frac{2p+j+1}{2p} \right)^2} \right). \quad (4.5)$$

These formulas are well known character formulas for the triplet vertex algebra modules [19] (see also [2] and references therein).

It is not hard to see that both families of characters can be written as linear combinations of $\frac{\theta_{p,j}(\tau)}{\eta(\tau)}$ and $\frac{\theta'_{p,j}(\tau)}{\eta(\tau)}$, where

$$\theta_{p,j}(\tau) := \sum_{n \in \mathbb{Z}} q^{\frac{1}{4p}(2pn+j)^2}$$

are Jacobi theta functions and $\theta'_{p,j}(\tau) := \sum_{n \in \mathbb{Z}} (2pn+j) q^{(2pn+j)^2/(4p)}$ denote their Jacobi derivatives; see [2, 19] for more details. Consequently, the modular closure of the characters given in (4.4) and (4.5) is finite-dimensional with a basis given by

$$\frac{\theta_{p,\ell}(\tau)}{\eta(\tau)}, \quad 0 \leq \ell \leq p, \quad \frac{\theta'_{p,j}(\tau)}{\eta(\tau)}, \quad 1 \leq j \leq p-1, \quad \tau \frac{\theta'_{p,k}(\tau)}{\eta(\tau)}, \quad 1 \leq k \leq p.$$

Although we do not have a similar explicit description of the modular closure of $W(p)_Q$ -characters, we are able to prove a modularity result for all root lattices Q , except possibly E_8 . For this purpose, we use results from [33]. Let $A \in \text{Pos}(n; \mathbb{Z})$ be even and $P \in \mathbb{C}[v_1, \dots, v_n]$ be a harmonic polynomial

with respect to A^{-1} (as defined in Appendix A, Definition 1) of homogenous degree d . We define a theta-like function associated to A , P , and $\lambda \in A^{-1}\mathbb{Z}^n$ by

$$\theta_{A,P,\lambda}(\tau) := \sum_{r \in \lambda + \mathbb{Z}^n} P(r) q^{\frac{1}{2} r^\top A r}.$$

Then the transformations from [33, Section 2] can be summarized as

$$\theta_{A,P,\lambda}(\tau + 1) = e^{\pi i (\lambda^\top A \lambda)} \theta_{A,P,\lambda}(\tau), \quad (4.6)$$

$$\theta_{A,P,\lambda} \left(-\frac{1}{\tau} \right) = \frac{(-i)^d (-i\tau)^{\frac{n}{2} + d}}{\sqrt{\det(A)}} \sum_{\mu \in A^{-1}\mathbb{Z}^n / \mathbb{Z}^n} e^{2\pi i (\lambda^\top A \mu)} \theta_{A,P,\mu}(\tau). \quad (4.7)$$

Theorem 4.1. *Let Q be an ADE type root lattice, except possibly E_8 (in which case we only have numerical evidence). Then each character $\text{ch}[W(p, \lambda)_Q](\tau)$ is a sum of modular forms of non-negative integral weight. Moreover, the modular closure of $\text{ch}[W(p, \lambda)_Q](\tau)$, $\lambda \in L^0/L$, is finite-dimensional.*

Proof: We first note that $\theta_{A,P,\mu}(\tau)$ transforms the same way as $\theta_{A,1,\mu}(\tau)$, except for the additional multiplicative factor $(-i\tau)^d$ appearing in (4.7). But $\theta_{A,1,\mu}(\tau)$ is an ordinary theta function which is known to be modular (of a certain level and multiplier), and therefore $\theta_{A,P,\mu}(\tau)$ is also modular.

It suffices to show that each irreducible character can be expressed as a linear combination of $\frac{\theta_{A,P,\mu}(\tau)}{\eta(\tau)^n}$, where the matrix A is fixed and where P belongs to a finite set of homogeneous harmonic polynomials. Indeed, provided that this holds, by using relations (4.6) and (4.7) we easily infer that the closure of the given set of characters is contained inside the $\text{SL}_2(\mathbb{Z})$ -invariant space spanned by $\tau^j \frac{\theta_{A,P,\mu}(\tau)}{\eta(\tau)^n}$, where $0 \leq j \leq \deg(P)$, $\mu \in A^{-1}\mathbb{Z}^n / \mathbb{Z}^n$, and P vary through the same finite set. This vector space is clearly finite-dimensional. Now we analyze $\text{ch}[W(p, \lambda)_Q](\tau)$. By Lemma 6.1, combined with (4.2), we see that with $A = pB \in \text{Pos}(n; \mathbb{Z})$, where B is the Gram matrix of Q , the expression $\eta(\tau)^n \text{ch}[W(p, \lambda)_Q](\tau)$ can be written as a linear combination of $\theta_{A,P,\mu}$, $\mu \in A^{-1}\mathbb{Z}^n / \mathbb{Z}^n$, and where P are certain polynomials. Next we show that all P that appears in these linear combinations are harmonic with respect to A^{-1} . For this, it is sufficient to prove that $\prod_{\alpha \in \Delta_+} \langle x_1 \alpha_1 + \cdots + x_n \alpha_n + \tilde{\mu}, \alpha \rangle \in \mathbb{C}[x_1, \dots, x_n]$ is harmonic with respect to A^{-1} for any "shift" $\tilde{\mu}$. This is immediate from Lemma A.3, Theorem A.4, Theorem A.6 and Remark 10 (thus we have to exclude E_8). It remains to show that $\theta_{A,P,\mu}$, where P is harmonic with respect to A^{-1} , can be written as a linear combination of $\theta_{A,P,\mu}$, where P is homogeneous. Clearly, every harmonic polynomial P can be written as a sum of homogeneous harmonic polynomials and this proves the previous claim and the theorem. \square

For $Q = A_1$, it was shown in [3] that the relation $(L(0) - L_{ss}(0))^2 = 0$ holds on any logarithmic $W(p)_{A_1}$ -module, where $L_{ss}(0)$ denotes the semisimple part of the Virasoro operator $L(0)$. Existence of a non-trivial $L(0)$ Jordan block is closely tied to the fact that $\theta'_{p,j}(\tau)/\eta(\tau)$, modular form of weight one, appears in the modular closure of irreducible characters. Observe that the proof of Theorem 4.1 implies that all irreducible characters contain $\theta_{A,P,\lambda}(\tau)/\eta(\tau)^n$ where $\deg(P) := |\Delta_+|$, where Δ_+ is the set of positive roots (cf. Remark 3). Based on this observation we make the following

Conjecture 2. *We have $(L(0) - L_{ss}(0))^{|\Delta_+|+1} = 0$ on any logarithmic $W(p)_Q$ -module. In other words, $L(0)$ admits Jordan blocks of size at most $|\Delta_+| + 1$.*

4.3. Example: modular transformations for the character of $W(p)_{A_2}$. Here we examine the $Q = A_2$ case, that is $\mathfrak{g} = \mathfrak{sl}(3)$. Let $\{\alpha_1, \alpha_2\}$ be simple roots of A_2 . The entries of the Cartan matrix are given by: $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2$ and $(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1) = -1$. For simplicity, we only

consider the case $\lambda = \widehat{\lambda} = 0$. We have

$$\dim(V(m_1\omega_1 + m_2\omega_2)) = \frac{(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)}{2},$$

and by the Weyl character formula

$$\dim(V(m_1\omega_1 + m_2\omega_2)_0) = \min\{m_1, m_2\} + 1.$$

For $\alpha = m_1\alpha_1 + m_2\alpha_2$, with $m_1, m_2 \in \mathbb{Z}$, we have

$$\lim_{z \rightarrow 1} \left(\sum_{w \in S_3} (-1)^{\ell(w)} \frac{\mathbf{z}^{w(\alpha+\rho)-\rho}}{\Delta(z)} \right) = \frac{(2m_1 - m_2 + 1)(2m_2 - m_1 + 1)(m_1 + m_2 + 2)}{2}.$$

We now present two formulas for $\text{ch}[W(p)A_2](\tau)$. Formula (4.2), after $\mathbf{z} \rightarrow 1$, yields, after shifting $m_j \mapsto m_j - 1$

$$\text{ch}[W(p)A_2](\tau) = \sum_{(m_1, m_2) \in \mathbb{Z}^2} (2m_1 - m_2)(2m_2 - m_1)(m_1 + m_2) \frac{q^p \left(\left(m_1 - \frac{1}{p} \right)^2 + \left(m_2 - \frac{1}{p} \right)^2 - \left(m_1 - \frac{1}{p} \right) \left(m_2 - \frac{1}{p} \right) \right)}{2\eta(\tau)^2}.$$

Similarly, relation (4.3), easily gives another expression for $\text{ch}[W(p)A_2]$:

$$\sum_{\substack{m_1, m_2 = 0 \\ m_1 \equiv m_2 \pmod{3}}}^{\infty} m_1 m_2 (m_1 + m_2) \frac{q^{t_p \left(m_1 - \frac{1}{p}, m_2 - \frac{1}{p} \right)} (1 - q^{m_1}) (1 - q^{m_2}) (1 - q^{m_1 + m_2})}{2\eta(\tau)^2}, \quad \text{where}$$

$$t_p(m_1, m_2) := \frac{p}{3} (m_1^2 + m_2^2 + m_1 m_2). \quad (4.8)$$

Because we are interested modular properties of $\text{ch}[W(p)A_2](\tau)$ we shall focus on the first formula. If we let

$$P(m_1, m_2) := (2m_1 - m_2)(2m_2 - m_1)(m_1 + m_2),$$

then

$$\widetilde{\text{ch}}[W(p)A_2](\tau) := 2\eta(\tau) \text{ch}[W(p)A_2](\tau) = \Theta_{B_p, P_p, \lambda_p}(\tau) \quad \text{with}$$

$$B_p := p \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad P_p(m_1, m_2) := P \left(m_1 + \frac{1}{p}, m_2 + \frac{1}{p} \right), \quad \lambda_p := -\frac{1}{p} (1, 1)^\top.$$

Then we have

$$\begin{aligned} & 2\eta(\tau)^2 \sum_{(m_1, m_2) \in \mathbb{Z}^2} (2m_1 - m_2)(2m_2 - m_1)(m_1 + m_2) q^p \left(\left(m_1 - \frac{1}{p} \right)^2 + \left(m_2 - \frac{1}{p} \right)^2 - \left(m_1 - \frac{1}{p} \right) \left(m_2 - \frac{1}{p} \right) \right) \\ &= 2\eta(\tau)^2 \sum_{(n_1, n_2) \in \left(-\frac{1}{p}, -\frac{1}{p} \right) + \mathbb{Z}^2} \left(2n_1 - n_2 + \frac{1}{p} \right) \left(2n_2 - n_1 + \frac{1}{p} \right) \left(n_1 + n_2 + \frac{2}{p} \right) q^{p(n_1^2 + n_2^2 - n_1 n_2)} \\ &= 2\eta(\tau) \left(\sum_{n = (n_1, n_2) \in \left(-\frac{1}{p}, -\frac{1}{p} \right) + \mathbb{Z}^2} (2n_1 - n_2)(2n_2 - n_1)(n_1 + n_2) q^{\frac{1}{2}n^T B_p n} + \frac{2}{p^3} \sum_{n \in \left(-\frac{1}{p}, -\frac{1}{p} \right) + \mathbb{Z}^2} q^{\frac{1}{2}n^T B_p n} \right. \\ & \quad \left. - \frac{3}{p} \sum_{n \in \left(-\frac{1}{p}, -\frac{1}{p} \right) + \mathbb{Z}^2} (n_1^2 - 4n_1 n_2 + n_2^2) q^{\frac{1}{2}n^T B_p n} + \frac{3}{p^2} \sum_{n \in \left(-\frac{1}{p}, -\frac{1}{p} \right) + \mathbb{Z}^2} (n_1 + n_2) q^{\frac{1}{2}n^T B_p n} \right). \end{aligned}$$

Now each polynomial above is homogeneous and harmonic with respect to B_p^{-1} (use Example 3 and Remark 8) so our character is a linear combination of $\theta_{P, B_p, \lambda}$ and thus formula (4.7) can be applied.

Remark 4. Alternatively, one can use (3) to obtain modular transformation formulas of $W(p)_{A_2}$ -characters.

5. CHARACTERS OF $W^0(p)_Q$ -MODULES

5.1. Characters as constant terms. Next we discuss irreducible $W^0(p)_Q$ -modules. In parallel to [10, 11], we are primarily interested in atypical irreducibles. Typical irreducibles are isomorphic to Fock spaces so their structure and character formulas are well-known (see Section 8). Because the vertex operator algebra $W^0(p)_Q$ is a subalgebra of the Fock space, it automatically admits infinitely many irreducible modules. In particular, infinitely many modules are obtained by restriction from $W(p)_Q$ -modules - those are realized inside the lattice vertex algebra module. This way we obtain $W^0(p)_Q$ -modules parametrized by $\beta \in L^0$ which we denote by $W^0(p, \beta)_Q$. Again, motivated by the singlet case, we assume that all such $W^0(p)_Q$ -modules are irreducible (i.e., we conjecture that $W(p, \lambda)_Q$ is completely reducible as $W^0(p)_Q$ -module) and that all atypical modules arise this way (cf. Remark 1). With these assumptions, we can easily compute characters of all atypical by using the full character $\text{ch}[W(p, \lambda)_Q](\tau, z)$ as follows. Let $\beta \in L^0$ and $\beta \mapsto \lambda$ under the natural map $L^0 \rightarrow L^0/L$, where λ is one of the representatives fixed earlier. Observe that \mathbf{z} -powers in $\text{ch}[W(p, \lambda)_Q](\tau, z)$ are contained in the set $\widehat{\lambda} + Q \subset P$. Moreover, because $\beta = \lambda + \sqrt{p}\gamma$, where $\gamma \in Q$, we extract the coefficient of $\mathbf{z}^{\beta'}$, $\beta' = \widehat{\lambda} + \gamma$, expressed as a constant term:

$$\text{ch}[W^0(p, \beta)_Q](\tau) = \text{CT}_{\mathbf{z}} \left\{ \mathbf{z}^{-\beta'} \text{ch}[W(p, \lambda)_Q](\tau, z) \right\}. \quad (5.1)$$

The most prominent case here is $\beta = 0$, where $\lambda = \widehat{\lambda} = 0$, which gives the character of the vertex algebra $W^0(p)_Q$ (also discussed in the introduction)

$$\text{ch}[W^0(p)_Q](\tau) = \text{CT}_{\mathbf{z}} \{ \text{ch}[W(p)_Q](\tau, z) \} = \sum_{\alpha \in Q \cap P^+} \dim(V(\alpha)_0) \left(\sum_{w \in W} (-1)^{\ell(w)} q^{\frac{1}{2} \|\sqrt{p}w(\alpha + \rho) - \frac{1}{\sqrt{p}}\rho\|^2} \right) \frac{1}{\eta(\tau)^{\text{rank}(Q)}},$$

where $V(\alpha)_0$ denotes the zero weight subspace of $V(\alpha)$. Unfortunately, not much is known about the dimension of $V(\alpha)_0$ for general \mathfrak{g} and α (let alone any explicit formulas!).

Example 2. ($Q = A_1$). We illustrate how this setup recovers characters studied in our earlier work [10]. Following the previous notation adopted for A_1 , for $-(p-1) \leq j \leq 0$, $\widehat{\lambda} = 0$, $k \in \mathbb{Z}$, we get

$$\text{ch} \left[W^0 \left(p, \frac{j\omega}{\sqrt{p}} + k\sqrt{p}\alpha \right)_{A_1} \right] (\tau) = \frac{1}{\eta(\tau)} \sum_{n \geq k}^{\infty} \left(q^{p \left(n + \frac{p-j-1}{2p} \right)^2} - q^{p \left(n + \frac{p+j+1}{2p} \right)^2} \right),$$

while for $1 \leq j \leq p$, $\widehat{\lambda} = \omega$, $k \in \mathbb{Z}$, we obtain

$$\text{ch} \left[W^0 \left(p, \frac{j\omega}{\sqrt{p}} + k\sqrt{p}\alpha \right)_{A_1} \right] (\tau) = \frac{1}{\eta(\tau)} \sum_{n \geq k}^{\infty} \left(q^{p \left(n + \frac{2p-j-1}{2p} \right)^2} - q^{p \left(n + \frac{2p+j+1}{2p} \right)^2} \right).$$

In particular, we have

$$\text{ch}[W^0(p)_{A_1}](\tau) = \frac{1}{\eta(\tau)} \sum_{n=0}^{\infty} \left(q^{p \left(n + \frac{p-1}{2p} \right)^2} - q^{p \left(n + \frac{p+1}{2p} \right)^2} \right).$$

Remark 5. Observe that the numerator of $\text{ch}[W^0(p)_{A_1}](\tau)$, studied extensively in [10, 11], is closely related to Roger's false theta function. Therefore it seems interesting to consider

$$F_{Q,p}(\tau) := \eta(\tau)^{\text{rank}(Q)} \text{ch}[W^0(p)_Q](\tau).$$

From the Lie theoretic point of view, this can be regarded as a ‘‘higher rank’’ generalization of the classical false theta function. It would be especially interesting to understand these objects from number theoretic perspective.

5.2. Explicit example: the character of $W^0(p)_{A_2}$. Now we consider the character of $W^0(p)_{A_2}$. Observe from the definition

$$\begin{aligned} \eta(\tau)^2 \text{ch}[W(p)_{A_2}](\tau, z) &= \sum_{m,n \in \mathbb{Z}} \frac{q^{p(m+1)^2 + p(n+1)^2 - p(m+1)(n+1) - m - n - 2 + \frac{1}{p}}}{(1 - z_1^{-1})(1 - z_2^{-1})(1 - z_1^{-1}z_2^{-1})} \\ &\times (z_1^m z_2^n - z_1^{-m+n-1} z_2^n - z_1^m z_2^{-n+m-1} + z_1^{-n-2} z_2^{-n+m-1} + z_1^{-m+n-1} z_2^{-m-2} - z_1^{-n-2} z_2^{-m-2}) \\ &= \sum_{m,n \in \mathbb{Z}} \frac{q^{p\left(\left(m - \frac{1}{p}\right)^2 + \left(n - \frac{1}{p}\right)^2 - \left(m - \frac{1}{p}\right)\left(n - \frac{1}{p}\right)\right)}}{(1 - z_1^{-1})(1 - z_2^{-1})(1 - z_1^{-1}z_2^{-1})} \\ &\times (z_1^{m-1} z_2^{n-1} - z_1^{-m+n-1} z_2^{n-1} - z_1^{m-1} z_2^{-n+m-1} + z_1^{-n-1} z_2^{-n+m-1} + z_1^{-m+n-1} z_2^{-m-1} - z_1^{-n-1} z_2^{-m-1}). \end{aligned}$$

To obtain a formula for $\text{ch}[W^0(p)_{A_2}]$, we have to extract the constant term. After a straightforward computation, we get

Proposition 5.1. *We have*

$$\text{ch}[W^0(p)_{A_2}](\tau) = \frac{1}{\eta(\tau)^2} \sum_{\substack{m_1, m_2 = 1 \\ m_1 \equiv m_2 \pmod{3}}}^{\infty} \min(m_1, m_2) q^{t_p(m_1 - \frac{1}{p}, m_2 - \frac{1}{p})} (1 - q^{m_1})(1 - q^{m_2})(1 - q^{m_1 + m_2}),$$

where $t_p(m_1, m_2)$ is defined in (4.8).

6. ASYMPTOTIC EXPANSIONS

In this part, we derive asymptotic expansions of characters $\text{ch}[W^0(p, \beta)](\tau)$.

Lemma 6.1. *Let $\lambda \in P^+$. Then*

$$\lim_{z \rightarrow 1} \left(\sum_{w \in W} (-1)^{\ell(w)} \frac{z^{w\left(\frac{\lambda + \rho}{p}\right) - \rho}}{\Delta(z)} \right) = \frac{1}{p^{|\Delta_+|}} \prod_{\alpha \in \Delta_+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} = \frac{1}{p^{|\Delta_+|}} \dim(V(\lambda)).$$

Proof: Since the proof is analogous to the proof of Weyl's dimension formula, we omit it here. \square

Theorem 6.2. *Let $\beta \in L^0$, whose coset representative in L^0/L is λ . Then, as $t \rightarrow 0^+$, we have*

$$\eta(it)^{\dim(Q)} \text{ch}[W^0(p, \beta)_Q](it) \sim \frac{\dim(V(-\sqrt{p}\bar{\lambda}))}{p^{|\Delta_+|}} + O(\sqrt{t}),$$

where $\bar{\lambda} = \sum_{j=1}^n (1 - s_j) \frac{\omega_j}{\sqrt{p}}$ as in Section 4.2. In particular,

$$\eta(it)^{\text{rank}(Q)} \text{ch}[W^0(p)_Q](it) \sim \frac{1}{p^{|\Delta_+|}} + O(\sqrt{t}), \quad \text{where}$$

$$|\Delta_+| := \begin{cases} \frac{n(n+1)}{2} & \text{if } Q = A_n, \\ n(n-1) & \text{if } Q = D_n, \\ 36 & \text{if } Q = E_6, \\ 63 & \text{if } Q = E_7, \\ 120 & \text{if } Q = E_8. \end{cases}$$

Proof: First we choose λ to be a representative in L^0/L , as fixed above. Next we rewrite

$$\begin{aligned}
\eta(\tau)^{\dim(Q)} \text{ch}[W(p, \lambda)_Q](\tau, z) &= \sum_{\beta \in Q} \sum_{w \in W} (-1)^{\ell(w)} q^{\frac{1}{2} \|\sqrt{p}\beta + w(\lambda + \alpha_0 \rho)\|^2} z^{\beta + w(\widehat{\lambda} + \rho) - \rho} \\
&= \sum_{\beta \in Q} \sum_{w \in W} (-1)^{\ell(w)} q^{\frac{\rho}{2} \|\beta\|^2 + \frac{1}{2} \|\lambda + \rho \alpha_0\|^2 + \sqrt{p}(\beta, w(\lambda + \rho \alpha_0))} z^{\beta + w(\widehat{\lambda} + \rho) - \rho} \\
&= \sum_{w \in W} (-1)^{\ell(w)} q^{\frac{1}{2} \|\lambda + \rho \alpha_0\|^2} \sum_{\beta \in Q} e^{\pi i \tau p \|\beta\|^2 + 2\pi i(\beta, \sqrt{p}\tau w(\lambda + \rho \alpha_0) + \mathbf{u})} z^{w(\widehat{\lambda} + \rho) - \rho} \\
&= q^{\frac{1}{2} \|\lambda + \rho \alpha_0\|^2} \sum_{w \in W} (-1)^{\ell(w)} \Theta_{0, \sqrt{p}\tau w(\lambda + \rho \alpha_0) + \mathbf{u}}(p\tau; A_Q) z^{w(\widehat{\lambda} + \rho) - \rho} \\
&= q^{\frac{1}{2} \|\lambda + \rho \alpha_0\|^2} \sum_{w \in W} (-1)^{\ell(w)} \frac{1}{\sqrt{\det(A_Q)} (-pi\tau)^{\frac{n}{2}}} \Theta_{-\sqrt{p}\tau w(\lambda + \rho \alpha_0) - \mathbf{u}, 0} \left(-\frac{1}{p\tau}; A_{Q^0} \right) z^{w(\widehat{\lambda} + \rho) - \rho}, \quad (6.1)
\end{aligned}$$

where $\mathbf{u} = (u_1, \dots, u_n)$, $\alpha_0 = \sqrt{p} - \frac{1}{\sqrt{p}}$, and A_Q denotes the determinant of the Gram matrix of Q . In the last formula we have to exercise caution. As all vectors involved are in \mathbb{C}^n , the vector $w(\lambda + \rho \alpha_0)$ has to be written appropriately depending on whether the summation is over Q or Q^0 . For this purpose we use $\|\cdot\|_0$ if the summation is over Q^0 . Recall (5.1), where we define $\text{ch}[W^0(p, \lambda)_Q](\tau)$ as the constant term of the full characters $\text{ch}[W(p, \lambda)_Q](\tau, z)z^{-\lambda}$. This expression involves integral powers of z , so we can write

$$\eta(\tau)^{\dim(Q)} \text{ch}[W^0(p, \lambda)_Q](\tau) = \eta(\tau)^{\dim(Q)} \int_{[-\frac{1}{2}, \frac{1}{2}]^n} \text{ch}[W(p, \lambda)_Q](\tau, z) z^{-\lambda} d\mathbf{u}.$$

In each sum $\Theta_{-\sqrt{p}\tau w(\lambda + \rho \alpha_0) - \mathbf{u}, 0}(-\frac{1}{p\tau}; Q^0)$, the zeroth term dominates so we have

$$\begin{aligned}
&q^{-\frac{1}{2} \|\lambda + \rho \alpha_0\|^2} \eta(\tau)^{\dim(Q)} \text{ch}[W^0(p, \lambda)_Q](\tau) \\
&\sim \sum_{w \in W} e^{-\pi i \tau \|\lambda + \rho \alpha_0\|_0^2} \int_{[-\frac{1}{2}, \frac{1}{2}]^n} e^{-\frac{\pi i}{p\tau} \|\sqrt{p}\tau w(\lambda + \rho \alpha_0) + \mathbf{u}\|_0^2} \frac{z^{w(\widehat{\lambda} + \rho) - \rho - \lambda}}{\Delta(z)} d\mathbf{u} \\
&= \sum_{w \in W} e^{-\pi i \tau \|\lambda + \rho \alpha_0\|_0^2} \int_{[-\frac{1}{2}, \frac{1}{2}]^n} e^{-\frac{\pi i}{p\tau} (2\sqrt{p}\tau(w(\lambda + Q\rho), \mathbf{u}) + \|\mathbf{u}\|_0^2)} \frac{z^{w(\widehat{\lambda} + \rho) - \rho - \lambda}}{\Delta(z)} d\mathbf{u}. \quad (6.2)
\end{aligned}$$

We next turn the integral in (6.2) into an integral over \mathbb{R}^n and show that this only introduces an exponentially small error. More generally, we prove that for $\alpha_j \in \mathbb{R}$ ($j \in \mathbb{N}, j < n$) and $M \in \text{Pos}(n; \mathbb{R})$

$$\int_{(1, \infty)^j \times [-\frac{1}{2}, \frac{1}{2}]^{n-j}} e^{2\pi i \sum_{j=1}^n \alpha_j u_j - \frac{1}{\tau} M[\mathbf{u}]} d\mathbf{u} \quad (6.3)$$

is exponentially small. For this, we use the well-known fact that there exist $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha I_n < M < \beta I_n$, where $A > B$ for $A, B \in \text{Pos}(n; \mathbb{R})$ means there $A - B \in \text{Pos}(n; \mathbb{R})$. Thus (6.3) may be bounded against

$$\ll \int_{(1, \infty)^j \times [-\frac{1}{2}, \frac{1}{2}]^{n-j}} e^{-\frac{\alpha}{\text{Im}(\tau)} \sum_{j=1}^n u_j^2} d\mathbf{u}.$$

The claim follows since for $a \in \mathbb{R}^+$, $\int_{\mathbb{R}} e^{-\frac{a}{\text{Im}(\tau)} u^2} du$ is exponentially small.

Returning to (6.2), we can now write

$$\begin{aligned} q^{-\frac{1}{2}\|\lambda+\rho\alpha_0\|^2} \eta(\tau)^{\dim(Q)} \text{ch}[W^0(p, \lambda)_Q](\tau) &\sim \int_{\mathbb{R}^n} e^{-\frac{2\pi i}{\sqrt{p}}(w(\lambda+\alpha_0\rho), \mathbf{u}) - \frac{\pi i \|\mathbf{u}\|_0^2}{p\tau}} z^{w(\widehat{\lambda}+\rho)-\rho-\lambda} \frac{1}{\Delta(z)} \mathbf{d}\mathbf{u} \\ &= \int_{\mathbb{R}^n} z^{-\frac{\pi i \|\mathbf{u}\|_0^2}{p\tau} - \frac{w(\lambda+\alpha_0\rho)}{\sqrt{p}} + w(\widehat{\lambda}+\rho)-\rho-\lambda} \frac{1}{\Delta(z)} \mathbf{d}\mathbf{u} = \int_{\mathbb{R}^n} z^{-\frac{\pi i \|\mathbf{u}\|_0^2}{p\tau} + w\left(-\frac{\bar{\lambda}}{\sqrt{p}} + \frac{\rho}{p}\right) - \rho - \lambda} \frac{1}{\Delta(z)} \mathbf{d}\mathbf{u}. \end{aligned}$$

Substituting $\tau = it$ and summing the above expression over the Weyl group W , we obtain

$$F(it) := \int_{\mathbb{R}^n} e^{-\frac{\pi \|\mathbf{u}\|_0^2}{pt}} \frac{\sum_{w \in W} (-1)^{\ell(w)} z^{w\left(-\frac{\bar{\lambda}}{\sqrt{p}} + \frac{\rho}{p}\right)}}{z^\rho \Delta(z)} \mathbf{d}\mathbf{u}.$$

Observe that (by Lemma 6.1), as $\mathbf{u} \rightarrow 0$,

$$\frac{\sum_{w \in W} (-1)^{\ell(w)} z^{w\left(-\frac{\bar{\lambda}}{\sqrt{p}} + \frac{\rho}{p}\right)}}{z^\rho \Delta(z)} \rightarrow \frac{\dim(V(-\sqrt{p}\bar{\lambda}))}{p^{|\Delta+|}}.$$

Next, we consider the u -expansion

$$\frac{\sum (-1)^{\ell(w)} z^{w\left(-\frac{\bar{\lambda}}{\sqrt{p}} + \frac{\rho}{p}\right) - \lambda}}{z^\rho \Delta(z)} = \sum_{(m_1, \dots, m_n) \in \mathbb{N}^n} \alpha(m_1, \dots, m_n) u_1^{m_1} \cdots u_n^{m_n}, \quad (6.4)$$

with $\alpha(0, \dots, 0) = \frac{\dim(V(-\sqrt{p}\bar{\lambda}))}{p^{|\Delta+|}}$. Substituting $v_j = u_j \sqrt{\frac{2\pi}{tp}}$ and using (6.4), we get

$$\begin{aligned} F(it) &\sim \sum_{(m_1, \dots, m_n) \in \mathbb{N}^n} \int_{\mathbb{R}^n} e^{-\frac{\|\mathbf{v}\|_0^2}{2}} \left(\sqrt{\frac{tp}{2\pi}}\right)^{n+\sum_{k=1}^n m_k} \alpha(m_1, \dots, m_n) v_1^{m_1} \cdots v_n^{m_n} \mathbf{d}\mathbf{v} \\ &= \frac{\dim(V(-\sqrt{p}\bar{\lambda}))}{p^{|\Delta+|}} \sqrt{\frac{(2\pi)^n}{\det(Q^0)}} \left(\sqrt{\frac{tp}{2\pi}}\right)^n \\ &\quad + \sum_{(m_1, \dots, m_n) \neq 0} \int_{\mathbb{R}^n} e^{-\frac{\|\mathbf{v}\|_0^2}{2}} \left(\sqrt{\frac{tp}{2\pi}}\right)^{n+\sum_{k=1}^n m_k} \alpha(m_1, \dots, m_n) v_1^{m_1} \cdots v_n^{m_n} \mathbf{d}\mathbf{v}, \end{aligned}$$

where we used the generalized Gaussian integral formula

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n C_{j,k} v_j v_k} dv_1 \cdots dv_n = \sqrt{\frac{(2\pi)^n}{\det(C)}},$$

where C is any symmetric positive definite matrix. Taking into account that the prefactor

$$e^{-\pi t \|\lambda + \alpha_0 \rho\|^2 + \pi t \|\lambda + \alpha_0 \rho\|_0^2}$$

does not contribute to the leading asymptotics, along with the formula (6.1), we get

$$\begin{aligned} &\eta(it)^{\text{rank}(Q)} \text{ch}[W^0(p, \lambda)_Q](it) \\ &\sim \frac{\dim(V(-\sqrt{p}\bar{\lambda}))}{p^{|\Delta+|}} \left(\sqrt{\frac{tp}{2\pi}}\right)^n \sqrt{\frac{(2\pi)^n}{\det(A_{Q^0})}} \frac{1}{(\sqrt{tp})^n \sqrt{\det(A_Q)}} + O(\sqrt{t}) = \frac{\dim(V(-\sqrt{p}\bar{\lambda}))}{p^{|\Delta+|}} + O(\sqrt{t}), \end{aligned}$$

as desired. Observe that for general $\lambda \in L^0$, the leading asymptotic only depends on its coset representative, so the proof follows. \square

Remark 6. In general, the coefficients $\alpha(m_1, \dots, m_n)$ are difficult to compute explicitly for all λ . However for $\widehat{\lambda} = 0$ the numerator in the Weyl character formula admits a product form so it is possible to get a more explicit asymptotic expansion. We do this in the next section for $Q = A_2$.

7. EXPLICIT EXAMPLE: THE FULL ASYMPTOTIC EXPANSION OF $\eta(\tau)^2 \text{ch}[W^0(p)_{A_2}](\tau)$

Observe that our method in Section 6 can in theory be used to compute a full asymptotic expansion. In this section we prove Theorem 1.2 from the introduction.

Proof of Theorem 1.2: As in the general case we may write

$$f(\tau) = \eta(\tau)^2 \text{CT}_z \text{ch}[W^0(p)_{A_2}](\tau, z) = \eta(\tau)^2 \int_{[-\frac{1}{2}, \frac{1}{2}]^2} \text{ch}[W^0(p)_{A_2}](\tau, z) du_1 du_2$$

with $z_j := e^{2\pi i u_j}$. Define

$$\mathcal{G}(u, v; \tau) := \sum_{m, n \in \mathbb{Z}} q^{p(m^2 + n^2 - mn) - m - n} e^{2\pi i(mu + nv)}.$$

Then

$$\begin{aligned} f(\tau) = & \int_{[-\frac{1}{2}, \frac{1}{2}]^2} (\mathcal{G}(u_1, u_2; \tau) - \mathcal{G}(-u_1, u_1 + u_2; \tau) - \mathcal{G}(u_1 + u_2, -u_2; \tau) + \mathcal{G}(u_2, -u_1 - u_2; \tau) \\ & + \mathcal{G}(-u_1 - u_2, u_1; \tau) - \mathcal{G}(-u_2, -u_1; \tau)) \frac{z_1^{-1} z_2^{-1} q^{\frac{1}{p}}}{(1 - z_1^{-1})(1 - z_2^{-1})(1 - z_1^{-1} z_2^{-1})} du_1 du_2. \end{aligned} \quad (7.1)$$

An application of (3.1) yields

$$\mathcal{G}(u, v; \tau) = \Theta_{0, \left(\frac{u-\tau}{v-\tau}\right)} \left(2p\tau; \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \right) = \frac{1}{\sqrt{3}p(-i\tau)} \Theta_{\left(\frac{\tau-u}{\tau-v}\right), 0} \left(-\frac{1}{2p\tau}; \frac{4}{3} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \right).$$

Now

$$\Theta_{\left(\frac{\tau-u}{\tau-v}\right), 0} \left(-\frac{1}{2p\tau}; \frac{4}{3} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \right) = e^{-\frac{2\pi i \tau}{p}} \sum_{n, m \in \mathbb{Z}} e^{-\frac{2\pi i}{3p\tau}((n-u)^2 + (m-v)^2 + (n-u)(m-v))} e^{-\frac{2\pi i}{p}((n-u) + m - v)}.$$

We get exponentially small terms unless $n = u$, $m = v$. In particular in all cases that occur only the $n = m = 0$ term contributes. Thus we may approximate the integrand in (7.1) by

$$\begin{aligned} & - \frac{z_1 z_2}{(1 - z_1)(1 - z_2)(1 - z_1 z_2)} e^{-\frac{2\pi i}{3p\tau}(u_1^2 + u_2^2 + u_1 u_2)} \\ & \times \left(e^{\frac{2\pi i}{p}(u_1 + u_2)} - e^{\frac{2\pi i}{p}u_2} - e^{\frac{2\pi i}{p}u_1} + e^{-\frac{2\pi i}{p}u_2} + e^{-\frac{2\pi i}{p}u_1} - e^{-\frac{2\pi i}{p}(u_1 + u_2)} \right) \frac{1}{\sqrt{3}p(-i\tau)}. \end{aligned} \quad (7.2)$$

Moreover, as before we introduce an exponentially small error if in (7.1) we integrate over \mathbb{R}^2 . To compute the remaining integral, we set

$$f(u, v, w) := - \frac{e^{\pi i(u+v+w) + \pi i \frac{(u+v+w)}{p}} \left(1 - e^{-\frac{2\pi i u}{p}}\right) \left(1 - e^{-\frac{2\pi i v}{p}}\right) \left(1 - e^{-\frac{2\pi i w}{p}}\right)}{(1 - e^{2\pi i u})(1 - e^{2\pi i v})(1 - e^{2\pi i w})}.$$

Note that f has only removable singularities in all variables and we may hence write

$$f(u, v, w) =: \sum_{n_1, n_2, n_3 \geq 0} \alpha(n_1, n_2, n_3) u^{n_1} v^{n_2} w^{n_3}. \quad (7.3)$$

Note that in particular $\alpha(0, 0, 0) = 1/p^3$. We then obtain, inserting (7.2) and (7.3) in (7.1),

$$\begin{aligned} f(it) &\sim \frac{1}{\sqrt{3pt}} \int_{\mathbb{R}^2} f(u_1, u_2, u_1 + u_2) e^{-\frac{2\pi}{3pt}(u_1^2 + u_2^2 + u_1 u_2)} du_1 du_2 \\ &= \frac{1}{\sqrt{3pt}} \sum_{n_1, n_2, n_3 \geq 0} \alpha(n_1, n_2, n_3) \int_{\mathbb{R}^2} u_1^{n_1} u_2^{n_2} (u_1 + u_2)^{n_3} e^{-\frac{2\pi}{3pt}(u_1^2 + u_2^2 + u_1 u_2)} du_1 du_2 \\ &= \frac{1}{\sqrt{3pt}} \sum_{n_1, n_2, n_3 \geq 0} \alpha(n_1, n_2, n_3) \left(\sqrt{\frac{3pt}{2\pi}} \right)^{n_1 + n_2 + n_3 + 2} \int_{\mathbb{R}^2} e^{-(u_1^2 + u_2^2 + u_1 u_2)} u_1^{n_1} u_2^{n_2} (u_1 + u_2)^{n_3} du_1 du_2, \end{aligned}$$

where in the last equation we made the change of variables $u_j \mapsto u_j \sqrt{\frac{3pt}{2\pi}}$. To finish the proof we are left to compute $\alpha(n_1, n_2, n_3)$ and the remaining integral.

To determine $\alpha(n_1, n_2, n_3)$, recall that

$$\frac{e^{\pi i u + \frac{\pi i u}{p}} \left(1 - e^{-\frac{2\pi i u}{p}} \right)}{1 - e^{2\pi i u}} = \sum_{n \geq 0} \left(B_n \left(\frac{1}{2} - \frac{1}{2p} \right) - B_n \left(\frac{1}{2} + \frac{1}{2p} \right) \right) \frac{(2\pi i u)^{n-1}}{n!}, \quad (7.4)$$

with $B_n(x)$ the n -th Bernoulli polynomial. Since the left hand side is an even function, it follows that n in the sum is odd. Moreover, noting that $B_n(1-x) = (-1)^n B_n(x)$, gives that (7.4) equals

$$2 \sum_{n \geq 0} \frac{B_{2n+1} \left(\frac{1}{2} - \frac{1}{2p} \right)}{(2n+1)!} (2\pi i u)^{2n}.$$

Thus $\alpha(n_1, n_2, n_3) = 0$ unless n_1, n_2 , and n_3 are even in which case we obtain

$$\alpha(2n_1, 2n_2, 2n_3) = -8 \frac{B_{2n_1+1} \left(\frac{1}{2} - \frac{1}{2p} \right)}{(2n_1+1)!} \frac{B_{2n_2+1} \left(\frac{1}{2} - \frac{1}{2p} \right)}{(2n_2+1)!} \frac{B_{2n_3+1} \left(\frac{1}{2} - \frac{1}{2p} \right)}{(2n_3+1)!} (2\pi i)^{2n_1+2n_2+2n_3}.$$

To finish the proof, define

$$\beta(n_1, n_2, n_3) := \int_{\mathbb{R}^2} u_1^{2n_1} u_2^{2n_2} (u_1 + u_2)^{2n_3} e^{-(u_1^2 + u_2^2 + u_1 u_2)} du_1 du_2. \quad (7.5)$$

We can evaluate (7.5) as follows: we set $u_1 = t_1 + \frac{t_2}{\sqrt{3}}$, $u_2 = -t_1 + \frac{t_2}{\sqrt{3}}$ and

$$\begin{aligned} \beta(n_1, n_2, n_3) &= \left(\frac{2}{\sqrt{3}} \right)^{2n_3+1} \int_{\mathbb{R}^2} \left(t_1 + \frac{t_2}{\sqrt{3}} \right)^{2n_1} \left(t_1 - \frac{t_2}{\sqrt{3}} \right)^{2n_2} t_2^{2n_3} e^{-t_1^2 - t_2^2} dt_1 dt_2 \\ &= \frac{2^{2n_3+1}}{3^{\frac{1}{2}+n_1+n_2+n_3}} \sum_{r=0}^{2n_1} \sum_{j=0}^{2n_2} \binom{2n_1}{r} \binom{2n_2}{j} 3^{r+j} \int_{\mathbb{R}^2} t_1^{r+j} (-1)^j t_2^{2n_1+2n_2+2n_3-r-j} e^{-t_1^2 - t_2^2} dt_1 dt_2 \\ &= \frac{2^{2n_3+1}}{3^{\frac{1}{2}+n_1+n_2+n_3}} \sum_{k=0}^{n_1+n_2} \sum_{j=0}^{2k} \binom{2n_1}{2k-j} \binom{2n_2}{j} (-1)^j 3^k \int_{\mathbb{R}^2} t_1^{2k} t_2^{2n_1+2n_2+2n_3-2k} e^{-t_1^2 - t_2^2} dt_1 dt_2 \\ &= \frac{2^{2n_3+1}}{3^{\frac{1}{2}+n_1+n_2+n_3}} \sum_{k=0}^{n_1+n_2} \sum_{j=0}^{2k} \binom{2n_1}{2k-j} \binom{2n_2}{j} (-1)^j 3^k \Gamma \left(k + \frac{1}{2} \right) \Gamma \left(n_1 + n_2 + n_3 - k + \frac{1}{2} \right), \end{aligned}$$

where we used the substitution $r + j = 2k$ to eliminate summation indices for which the Gaussian integral is vanishing. This gives

$$f(it) \sim -\frac{4}{\pi} \sum_{n_1, n_2, n_3 \geq 0} 2^{2n_3+1} (-2\pi pt)^{n_1+n_2+n_3} \frac{B_{2n_1+1} \left(\frac{1}{2} - \frac{1}{2p}\right)}{(2n_1+1)!} \frac{B_{2n_2+1} \left(\frac{1}{2} - \frac{1}{2p}\right)}{(2n_2+1)!} \frac{B_{2n_3+1} \left(\frac{1}{2} - \frac{1}{2p}\right)}{(2n_3+1)!} \\ \times \sum_{k=0}^{n_1+n_2} \sum_{j=0}^{2k} \binom{2n_1}{2k-j} \binom{2n_2}{j} (-1)^j 3^k \Gamma\left(k + \frac{1}{2}\right) \Gamma\left(n_1 + n_2 + n_3 - k + \frac{1}{2}\right).$$

The claim follows, setting

$$\alpha(N) := -\frac{4}{\pi} \sum_{\substack{n_1, n_2, n_3 \geq 0 \\ N = n_1 + n_2 + n_3}} 2^{2n_3+1} \frac{B_{2n_1+1} \left(\frac{1}{2} - \frac{1}{2p}\right)}{(2n_1+1)!} \frac{B_{2n_2+1} \left(\frac{1}{2} - \frac{1}{2p}\right)}{(2n_2+1)!} \frac{B_{2n_3+1} \left(\frac{1}{2} - \frac{1}{2p}\right)}{(2n_3+1)!} \\ \times \sum_{k=0}^{n_1+n_2} \sum_{j=0}^{2k} \binom{2n_1}{2k-j} \binom{2n_2}{j} (-1)^j 3^k \Gamma\left(k + \frac{1}{2}\right) \Gamma\left(n_1 + n_2 + n_3 - k + \frac{1}{2}\right). \quad (7.6)$$

□

Remark 7. Alternatively one could use a 2-dimensional version of the Euler-McLaurin summation formula as, for example, stated in [35]. However, this requires splitting the function of interest into many summands to which one may apply Euler-McLaurin which makes the calculation much more complicated than the use of Jacobi transformation properties.

8. (ANALYTIC) QUANTUM DIMENSIONS OF $W^0(p)_Q$ -MODULES

As we previously noted (cf. Remark 1), presently there is no complete classification of irreducible $W^0(p)_Q$ -modules for $\text{rank}(Q) \geq 2$. Nevertheless, there are strong indications that irreducible $W^0(p)_Q$ -modules fall into two categories: typical and atypical.

- (i) **Typical** modules are just rank n Fock spaces F_λ , which are irreducible as $W^0(p)_Q$ -modules. Typical characters are therefore given by

$$\text{ch}[F_\lambda](\tau) = \frac{q^{\frac{\|\lambda - \alpha_0 \rho\|^2}{2}}}{\eta(\tau)^n},$$

where $\lambda \notin L^0$ (see [12]).

- (ii) **Atypical** modules, discussed above, form a discrete family parametrized by $\beta \in L^0$, and their characters are

$$\text{ch}[W^0(p, \beta)_Q](\tau).$$

Next we consider asymptotic (or quantum) dimensions of irreducible modules

Theorem 8.1. (i) For β as in Theorem 6.2, we have

$$\text{qdim}(W^0(p, \beta)_Q) = \dim_{\mathbb{C}} V(-\sqrt{p}\lambda).$$

- (ii) For any typical module F_λ , we have

$$\text{qdim}(F_\lambda) = p^{|\Delta_+|}.$$

Proof: It follows directly from Theorem 6.2 and the definition of analytic q -dimension

$$\text{qdim}[M] = \lim_{t \rightarrow 0} \frac{\text{ch}[M](it)}{\text{ch}[V](it)},$$

where M is a module for the vertex operator algebra V . □

Observe that $\dim_{\mathbb{C}}(V((p-1)\rho)) = p^{|\Delta^+|}$. On the other hand, because of $-\bar{\lambda} = (s_1 - 1)\omega_1 + \dots + (s_n - 1)\omega_n$, where $s_j \in \{1, \dots, p\}$, we always have

$$\dim_{\mathbb{C}} V(-\sqrt{p}\bar{\lambda}) \leq p^{|\Delta^+|}.$$

Therefore we conclude that typical modules have the largest quantum dimension among irreducible modules (see also [10]).

Remark 8. Certain regularized characters of $W^0(p)_Q$ -modules including their regularized quantum dimensions are studied in a recent paper [12] (see also [11]).

APPENDIX A. HARMONICITY OF CERTAIN POLYNOMIALS

In this part we prove harmonicity of certain polynomials needed for the proof of modularity of characters in Theorem 4.1.

A.1. Some harmonic polynomials of type A. Consider the rank n root lattice of type A, $Q = \bigoplus_{j=1}^n \mathbb{Z}\alpha_j$, whose Cartan matrix $A_n = (\langle \alpha_j, \alpha_k \rangle)_{j,k=1}^n$ is given by

$$A_n := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}. \quad (\text{A.1})$$

Then A_n^{-1} is the Cartan matrix of the dual (or weight) lattice $P = \bigoplus_{j=1}^n \mathbb{Z}\omega_j$ such that $\langle \omega_j, \alpha_k \rangle = \delta_{j,k}$. Clearly, $A_n^{-1} := (a^{jk})$, where $a^{jk} := \langle \omega_j, \omega_k \rangle$.

Denote by ∇^2 the standard (diagonal) Laplacian on \mathbb{R}^n and let ∇_B^2 be the Laplace-Beltrami operator for $B \in \text{Mat}(n, \mathbb{R})$, that is

$$\nabla_B^2 := \left(\frac{\partial}{\partial x} \right)^T B \frac{\partial}{\partial x} = \sum_{j,k=1}^n b_{jk} \partial_{jk}^2; \quad \partial_{jk}^2 := \frac{\partial^2}{\partial x_j \partial x_k}.$$

Definition 1. We say that polynomial $f(x_1, \dots, x_n)$ is harmonic with respect to B (resp. harmonic) if $\nabla_B^2(f) = 0$ (resp. $\nabla^2(f) = 0$).

Remark 9. Note that f is harmonic with respect to B if and only if it is harmonic with respect to λB for some $\lambda \in \mathbb{R}^\times$.

The following result follows easily, using basic facts from Calculus and Linear Algebra.

Lemma A.1. Consider the linear change of variable $x = By$, where B is an invertible matrix, $C \in \text{Mat}(n, \mathbb{R})$, $x = (x_1, \dots, x_n)$, and $y = (y_1, \dots, y_n)$. Then $\nabla_C^2(f(x)) = \nabla_{(B^{-1})^T B^{-1} C}^2(f^*(y))$, where $f^*(y) = f(B(y))$. In particular, $\nabla^2(f(x)) = \nabla_{(B^{-1})^T B^{-1}}^2(f^*(y))$.

The next result is well-known (see [27, Theorem 5.1.8]).

Proposition A.2. The discriminant $\delta(x_1, \dots, x_n) := \prod_{j < k} (x_j - x_k)$ is harmonic.

Lemma A.3. Suppose that $t \in \mathbb{R}$ and let $f(x_1, \dots, x_n)$ be harmonic with respect to B . Then

$$f(x_1 + a_1 t, \dots, x_j + a_j t, \dots, x_n + a_n t)$$

is also harmonic with respect to B for all $a_j \in \mathbb{R}$. In particular,

$$\prod_{j < k} (x_j - x_k + t(j - k))$$

is harmonic.

Proof: By using Taylor's Theorem, we see that

$$\exp\left(\sum_{j=1}^n ta_j \partial_{x_j}\right) f(x_1, \dots, x_n) = f(x_1 + a_1 t, \dots, x_j + a_j t, \dots, x_n + a_n t).$$

The proof now follows by observing that any pair of partial differential operators with constant coefficients commute with each other, in particular

$$\exp\left(\sum_{i=1}^n ta_i \partial_{x_i}\right) \circ \nabla^2 = \nabla^2 \circ \exp\left(\sum_{j=1}^n ta_j \partial_{x_j}\right).$$

□

Recall the set of positive roots of Q :

$$\Delta^+ = \{\alpha_j, j = 1, \dots, n\} \cup \{\alpha_j + \dots + \alpha_k : 1 \leq j < k \leq n\} \quad \text{and} \quad \rho = \sum_{j=1}^n \omega_j.$$

Next we introduce several families of polynomial with respect to two bases of \mathbb{R}^n . For $\lambda = x_1 \alpha_1 + \dots + x_n \alpha_n$, we let

$$\delta_Q(x_1, \dots, x_n) := \prod_{\alpha \in \Delta^+} (\lambda + \rho, \alpha), \quad \tilde{\delta}_Q(x_1, \dots, x_n) := \prod_{\alpha \in \Delta^+} (\lambda, \alpha).$$

Similarly, for $\mu = x_1 \omega_1 + \dots + x_n \omega_n$, we set

$$\delta_P(x_1, \dots, x_n) := \prod_{\alpha \in \Delta^+} (\mu + \rho, \alpha), \quad \tilde{\delta}_P(x_1, \dots, x_n) := \prod_{\alpha \in \Delta^+} (\mu, \alpha).$$

The subscript Q (resp. P) indicates that we use the α -basis (resp. ω -basis) in the parametrization of λ (resp. μ).

Example 3. With $Q = A_2$, we have

$$\begin{aligned} \delta_Q(x_1, x_2) &= (2x_1 - x_2 + 1)(2x_2 - x_1 + 1)(x_1 + x_2 + 2), \\ \tilde{\delta}_Q(x_1, x_2) &= (2x_1 - x_2)(2x_2 - x_1)(x_1 + x_2), \\ \delta_P(x_1, x_2) &= (x_1 + 1)(x_2 + 1)(x_1 + x_2 + 2), \\ \tilde{\delta}_P(x_1, x_2) &= x_1 x_2 (x_1 + x_2). \end{aligned}$$

From $\mu = x_1 \omega_1 + \dots + x_n \omega_n$, we have

$$\tilde{\delta}_P(x_1, x_2, \dots, x_n) = \prod_{k=1}^n x_k \prod_{1 \leq j < \ell \leq n} (x_j + \dots + x_\ell).$$

Similarly,

$$\delta_P(x_1, x_2, \dots, x_n) = \prod_{k=1}^n (x_k + 1) \prod_{1 \leq j < \ell \leq n} (x_j + \dots + x_\ell + \ell - j + 1).$$

with A_n as in (A.1), we have

Theorem A.4. *The functions $\tilde{\delta}_P$ and δ_P are harmonic with respect to A_n . The functions $\tilde{\delta}_Q$ and δ_Q are harmonic with respect to A_n^{-1} .*

Proof: We apply Proposition A.2 for the discriminant with $n + 1$ variables:

$$\nabla^2 (\delta (x_1, \dots, x_{n+1})) = \sum_{j=1}^{n+1} \partial_{jj}^2 \delta (x_1, \dots, x_{n+1}) = 0.$$

We then make a change of variables:

$$y_1 := x_1 - x_2, \dots, y_n := x_n - x_{n+1}, \quad y_{n+1} := x_{n+1}.$$

With respect to new variables, δ does not depend on y_{n+1} and

$$\delta^*(y_1, \dots, y_{n+1}) = \prod_{j=1}^n y_\ell \prod_{1 \leq j < k \leq n} (y_j + \dots + y_k) = \tilde{\delta}_P(y_1, \dots, y_n).$$

The transition between the y and x variables is controlled by the $(n + 1) \times (n + 1)$ matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -1 & 1 \end{pmatrix}.$$

Note that we have

$$A^T A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 1 \end{pmatrix}.$$

Moreover observe that the principal $n \times n$ minor of $A^T A$ is precisely the Cartan matrix of type A_n . By using Lemma A.1, with $B^{-1} = A$, we get

$$0 = \nabla^2 (\delta(\mathbf{x})) = \nabla_{A^T A}^2 (\delta^*(\mathbf{y})).$$

Although $A^T A$ matrix is of size $n + 1$, $\delta^*(\mathbf{y})$ does not involve the y_{n+1} variable, thus only the summation over the principal minor of A_n is relevant for the laplacian. Consequently,

$$\nabla_{A_n}^2 (\tilde{\delta}_P(\mathbf{y})) = 0$$

as desired. The A_n -harmonicity of δ_P now easily follows from Lemma A.3

Next we consider $\delta_Q(\mathbf{x})$ and $\tilde{\delta}_Q(\mathbf{x})$. Observe that they are related to $\delta_P(\mathbf{x})$ and $\tilde{\delta}_P(\mathbf{x})$, respectively, under a change of basis. More precisely, if we let $\mathbf{y} = A_n \mathbf{x}$ then

$$\tilde{\delta}_P(\mathbf{x}) = \tilde{\delta}_Q(\mathbf{y}).$$

Clearly, $A_n = A_n^T$, so $(A_n^{-1})^T = A_n^{-1}$. Another application of Lemma A.1 results in

$$0 = \nabla_{A_n}^2 (\tilde{\delta}_P) = \nabla_{A_n^{-1} A_n^{-1} A_n}^2 (\tilde{\delta}_Q) = \nabla_{A_n^{-1}}^2 (\tilde{\delta}_Q)$$

as desired. □

A.2. Some harmonic polynomials of D -type. To parametrize the roots of D_n , it is convenient to use the orthogonal basis ε_j , $j = 1, \dots, n$. In this basis positive roots are given by

$$\Delta^+ = \{\varepsilon_j - \varepsilon_k : 1 \leq j < k \leq n\} \cup \{\varepsilon_j + \varepsilon_k : j < k\}.$$

Simple roots are $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$, $j = 1, \dots, n-1$ and $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$. With this ordering the Cartan matrix is

$$D_n = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 & 2 \end{pmatrix}.$$

The next result is taken from [27, Section 5.1.3].

Proposition A.5. *The function*

$$\prod_{j < k} (x_j^2 - x_k^2)$$

is harmonic with respect to D_n .

As before we consider

$$\tilde{\delta}_Q := \prod_{\alpha \in \Delta^+} \langle \lambda, \alpha \rangle, \quad \delta_Q = \prod_{\alpha \in \Delta^+} \langle \lambda + \rho, \alpha \rangle$$

Similarly, we define $\tilde{\delta}_P$ and δ_P .

Example 4. *Let $Q = D_4$, then*

$$\begin{aligned} \delta_Q &= (2x_1 - x_2)(2x_2 - x_1 - x_3 - x_4)(2x_3 - x_2)(2x_4 - x_2)(x_1 + x_2 - x_3 - x_4)(x_2 + x_3 - x_4 - x_1) \\ &\quad \times (x_1 + x_3 - x_4)x_2(x_1 + x_3 + x_4 - x_2)(x_1 + x_4 - x_3)(x_3 + x_4 - x_1)(x_2 + x_4 - x_1 - x_3). \end{aligned}$$

Our next result is analogous to Theorem A.4, now for the D -type.

Theorem A.6. *The function $\tilde{\delta}_Q$ is harmonic with respect to D_n^{-1} and $\tilde{\delta}_P$ is harmonic with respect to D_n .*

Proof: For the proof, we use Proposition A.5 and the change-of-variables matrix

$$D = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & \dots & 1 & 1 \end{pmatrix}.$$

We then easily see that

$$D^T D = D_n.$$

Together with Lemma A.1 this yields that $\tilde{\delta}_P$ is harmonic with respect to D_n . Another change of variables (as in Theorem A.4) easily implies that $\tilde{\delta}_Q$ is harmonic with respect to D_n^{-1} . \square

A.3. Towards harmonic polynomials of type E .

Remark 10. For $Q = E_6, E_7$ and E_8 , we have to consider,

$$\begin{aligned}p_6(x_1, \dots, x_6) &= \nabla_{E_6}^2 \delta_Q(x_1, \dots, x_6), \\p_7(x_1, \dots, x_7) &= \nabla_{E_7}^2 \delta_Q(x_1, \dots, x_7), \\p_8(x_1, \dots, x_8) &= \nabla_{E_8}^2 \delta_Q(x_1, \dots, x_8).\end{aligned}$$

By using MAPLE, we computed p_j , for $j = 6, 7, 8$. We were able to show that $p_6 = p_7 = 0$. But we could not simplify these polynomials to the standard form due to a very high degree for E_8 . Even so, we were able to evaluate p_8 for many randomly chosen x_j -values and all these calculations returned zero. Thus, we can conclude with a high certainty that p_8 is indeed zero.

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