

Massive deformations of Maass forms and Jacobi forms

MARCUS BERG, KATHRIN BRINGMANN^{*}, AND TERRY GANNON[†]

We define one-parameter “massive” deformations of Maass forms and Jacobi forms. This is inspired by descriptions of plane gravitational waves in string theory. Examples include massive Green’s functions (that we write in terms of Kronecker–Eisenstein series) and massive modular graph functions.

AMS 2000 SUBJECT CLASSIFICATIONS: Primary 11F50; Secondary 11F30, 11F37.

1. Introduction and statement of results

This note introduces a class of deformations of some classical objects (Maass forms and real-analytic Jacobi forms), that naturally arise in string theory and in condensed-matter theory. It has been known for a long time that string amplitudes from surfaces (worldsheets) in flat spacetime transform “nicely” with respect to the mapping class group of the surface. For example, torus chiral blocks produce so-called vector-valued modular forms for the full modular group $SL_2(\mathbb{Z})$. The plane (gravitational) wave is a natural one-parameter deformation of flat spacetime. Any spacetime reduces to this type of spacetime in a specific limit [19]. This should lead to one-parameter deformations of e.g. (vector-valued) modular forms, which is of independent interest in mathematics. Similarly, in [22], it was discussed how the statistical mechanics of theories at a critical point can allow interesting one-parameter deformations away from criticality. All the above work in physics raises the question on how special these deformations are and whether one

arXiv: [1910.02745](https://arxiv.org/abs/1910.02745)

^{*}The research of the second author is supported by the Alfried Krupp Prize for Young University Teachers of the Krupp foundation. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 101001179).

[†]The research of the third author is supported in part by an NSERC Discovery Grant.

can develop a mathematical theory of these deformations that is somehow parallel to that of modular or automorphic forms.

The precursor to our classes of functions are some open string amplitudes which Bergman, Gaberdiel, and Green [5] computed in the plane wave background. For example, for $t \in \mathbb{R}^+$ and $m \in \mathbb{R}$, define

$$(1) \quad \mathcal{F}_m(t) := e^{-2\pi c_m t} (1 - e^{-2\pi m t})^{\frac{1}{2}} \prod_{n \geq 1} \left(1 - e^{-2\pi t \sqrt{m^2 + n^2}}\right),$$

where

$$c_m := \frac{1}{(2\pi)^2} \sum_{\ell \geq 1} \int_0^\infty e^{-\ell^2 x - \frac{\pi^2 m^2}{x}} dx.$$

The authors of [5] established in Appendix A the following inversion formula:

$$(2) \quad \mathcal{F}_m(t) = \mathcal{F}_{mt} \left(\frac{1}{t} \right).$$

Moreover, they showed that $\lim_{m \rightarrow 0} c_m = \frac{1}{24}$ and hence we may view $\frac{\mathcal{F}_m(t)}{\sqrt{2\pi t m}}$ as a one-parameter deformation of the *Dedekind eta function* given by $\eta(\tau) := q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n)$ ($q := e^{2\pi i \tau}$), i.e., we obtain $\lim_{m \rightarrow 0} \frac{\mathcal{F}_m(t)}{\sqrt{2\pi t m}} = \eta(it)$. Here, m represents spacetime curvature, thus $m \rightarrow 0$ is the flat-space limit and this is called a *massive deformation*.

It is remarkable that such a complicated-looking deformation of the Dedekind eta function preserves the inversion property of η . According to string theory, this occurs because the space of complex structures (up to equivalence) on an annulus is naturally parametrized by $t \in \mathbb{R}^+$ modulo the inversion $t \mapsto \frac{1}{t}$. This suggests that to get deformations of modular forms which retain modularity with respect to the full modular group $\mathrm{SL}_2(\mathbb{Z})$, one should look at closed string (torus) amplitudes in the plane wave background. Some of these were computed by e.g. Takayanagi [26]; we list those amplitudes, and then repackage them in Section 2 into our prototypical example $\mathcal{E}_{1,\mu}(z; \tau)$. Our paper originated with the challenge of finding a natural mathematical interpretation for these torus amplitudes. In Section 3.1, by studying the special properties of $\mathcal{E}_{1,\mu}(z; \tau)$, we define the new classes of functions.

- A *massive Maass form* is a smooth function $f_\mu(\tau)$ on $(\tau, \mu) \in \mathbb{H} \times \mathbb{R}^+$ which, for each fixed μ , transforms like a modular form, has at most polynomial growth towards the cusps, and is annihilated by some differential operator, given in (11).

• A *massive Maass–Jacobi form* is a smooth function $\phi_\mu(z; \tau)$ on $(z, \tau, \mu) \in \mathbb{C} \times \mathbb{H} \times \mathbb{R}^+$ which, for each fixed μ and z , transforms like a Jacobi form, has at most polynomial growth towards the cusps, and which is annihilated by two differential operators, defined in (12) and (13).

In Section 3.2 we construct families of examples, and study them in Section 4. Recently there has been extensive work in the physics literature on related topics, some of which we review in Section 5.

2. String theory torus amplitudes revisited

2.1. The plane wave torus amplitudes

According to string theory, the functions \mathcal{F}_m given in (1) live on the moduli space of the cylinder, which is why it respects modular inversion but not translation. To find something closer to a modular form, we instead consider closed string amplitudes on a torus.

One of the most basic objects in physics is the one-loop partition function. This can be computed by functional determinants, as reviewed for example in Section 13.4.1 of [10]. We begin with the partition function of a fermion (with twisted boundary conditions) in a plane gravitational wave, as in equation (2.37) of [25]

$$(3) \quad Z_{\alpha, \beta, m}(\tau) := e^{-8\pi c_{\alpha, m} \tau_2} \prod_{n \in \mathbb{Z}} \prod_{\pm} \left(1 - e^{-2\pi \tau_2 \sqrt{m^2 + (n \pm \alpha)^2} + 2\pi i (n \pm \alpha) \tau_1 \pm 2\pi i \beta} \right),$$

with $\alpha, \beta \in \mathbb{R}$ and

$$c_{\alpha, m} := \frac{1}{(2\pi)^2} \sum_{\ell \geq 1} \cos(2\pi \ell \alpha) \int_0^\infty e^{-\ell^2 x - \frac{\pi^2 m^2}{x}} dx = \frac{m}{2\pi} \sum_{\ell \geq 1} \cos(2\pi \ell \alpha) \frac{K_1(2\pi \ell m)}{\ell}.$$

Here and throughout the paper we write $\tau := \tau_1 + i\tau_2$ with $\tau_1 \in \mathbb{R}$ and $\tau_2 \in \mathbb{R}^+$. The *K-Bessel function* enters through (see 10.32.10 of [18]), $\text{Arg}(w) < \frac{\pi}{4}$

$$(4) \quad K_\nu(w) := \frac{1}{2} \left(\frac{w}{2}\right)^\nu \int_0^\infty \exp\left(-x - \frac{w^2}{4x}\right) x^{-\nu-1} dx.$$

In the literature, $Z_{\alpha, \beta, m}(\tau)$ is often written as $Z_{\alpha, \beta, m}(\tau, \bar{\tau})$ to emphasize its non-holomorphicity. Note that $Z_{\alpha, \beta, m}(\tau)$ only depends on α, β modulo one.

In Appendix A of [26], $Z_{\alpha,\beta,m}(\tau)$ was shown to obey

$$(5) \quad Z_{\alpha,\beta,m}(\tau + 1) = Z_{\alpha,\alpha+\beta,m}(\tau), \quad Z_{\alpha,\beta,m}\left(-\frac{1}{\tau}\right) = Z_{\beta,-\alpha,\frac{m}{|\tau|}}(\tau).$$

This modular covariance is quite remarkable, given the unfamiliar square root in the exponent of (3), but it is required by string theory. Computing that $\lim_{m \rightarrow 0} c_{\alpha,m} = \frac{\alpha^2}{4} - \frac{\alpha}{4} + \frac{1}{24}$, we find that

$$(6) \quad \lim_{m \rightarrow 0} Z_{\alpha,\beta,m}(\tau) = e^{-2\pi\alpha^2\tau_2} \left| \frac{\vartheta_1(\alpha\tau + \beta; \tau)}{\eta(\tau)} \right|^2,$$

where for $z \in \mathbb{C}$ the *Jacobi theta function* is defined as

$$\vartheta_1(z; \tau) := -2q^{\frac{1}{8}} \sin(\pi z) \prod_{n=1}^{\infty} (1 - q^n)(1 - e^{2\pi iz} q^n)(1 - e^{-2\pi iz} q^n).$$

In [25], $Z_{\alpha,\beta,m}(\tau)$ was called a massive theta function; we, however, do not use this name since it is not holomorphic even in the limit (6).

2.2. Reformulation of the torus amplitudes

In the inversion formula in (5), it is inconvenient that the parameter m is rescaled; for this reason, the deformation parameter that we use in this paper is the $SL_2(\mathbb{Z})$ -invariant quantity $\mu := m^2|\tau|$. In physics terms, if m^2 is the mass and $|\tau|$ is the area, then μ is dimensionless. The invariant mass parameter μ was not used in the early literature on strings in pp-waves. It was formally introduced in [9] when writing a generating function for modular graph functions, but there was no connection to the plane wave amplitudes.

Just as Jacobi theta functions with characteristics can be recast as Jacobi forms, (6) suggests that we can interpret $Z_{\alpha,\beta,m}(\tau)$ as a function $Z_\mu(z; \tau)$ in $z = \alpha\tau + \beta$ for μ fixed. Then the aforementioned periodicity in α, β , together with (5), implies that for each fixed μ , $Z_\mu(z; \tau)$ transforms under $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ like a Jacobi form of index and weight zero.

This raises the question what other basic property $Z_\mu(z; \tau)$ possesses. The final ingredient in our repackaging is that the logarithm of a partition function of a free theory can be a Green’s function. The *Green’s functions* $G(x, y)$ of a differential operator L are the solutions to an inhomogeneous (partial) differential equation $L(x)G(x, y) = \delta(x - y)$, where $\delta(x)$ is the Dirac delta distribution; more on this is explained below equation

(9). At least formally, a solution to the differential equation $L(f) = g$ is then $f(x) = \int G(x, y)g(y)dy$. In quantum field theory (or string theory) the Green’s function for the Laplace equation (or Weyl equation, or whatever the equation of motion for the system is) plays a central role e.g. to calculate Feynman diagram integrals. Thus we may hope that $\text{Log}(Z_\mu)$ satisfies a simple differential equation (of course it continues to transform with index and weight zero). Putting all of this together, define

$$\mathcal{E}_{1,\mu}(z; \tau) := -\text{Log} \left(Z_{-\alpha,\beta,\sqrt{\frac{\mu}{|\tau|}}}(\tau) \right).$$

Comparing (6) with (33) then gives that¹

$$(7) \quad \lim_{\mu \rightarrow 0^+} \mathcal{E}_{1,\mu}(z; \tau) = \mathbb{E}_1(z; \tau),$$

with the *Kronecker–Eisenstein series*

$$(8) \quad \mathbb{E}_1(z; \tau) := \frac{\tau_2}{\pi} \sum_{(r,\ell) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{e^{\frac{2\pi i}{\tau_2} \text{Im}((r\tau + \ell)\bar{z})}}{|r\tau + \ell|^2}.$$

Note that $\mathbb{E}_1(z; \tau)$ is up to a factor the Green’s function of the Laplace equation on the torus (see e.g. [21, Chapter 7.2])

$$(9) \quad \partial_z \partial_{\bar{z}}(\mathbb{E}_1(z; \tau)) = -2\pi \delta^{[2]}(z; \tau) + \frac{\pi}{\tau_2},$$

where, for a variable w , we set $\partial_w := \frac{\partial}{\partial w}$. Here, the Dirac delta distribution $\delta^{[2]}(z; \tau)$ is associated to the linear functional on the space of smooth functions on the torus $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ sending such a function $f(z)$ to $f(0) \in \mathbb{C}$. We formally write this linear functional as the integral operator given by $\int \int_P f(z) \delta^{[2]}(z; \tau) d^2z$ where P is a fundamental domain on \mathbb{C} . Since the left-hand side integrates to zero on a compact space, the constant term $\frac{\pi}{\tau_2}$ on the right-hand side of (9) is needed to cancel the integral $\int \int_P \delta^{[2]}(z; \tau) d^2z = 1$. Incidentally, if we allow for quasiperiodicity w as in $\mathbb{E}_1(w, z; \tau)$ in (30), then the constant term $\frac{\pi}{\tau_2}$ in (9) is absent, unless $w \in \mathbb{Q}\tau + \mathbb{Q}$.

To summarize, in this subsection we repackage the string torus amplitudes as $\mathcal{E}_{1,\mu}(z; \tau)$, which can be thought of as a deformed non-holomorphic

¹Here we are starting from the partition function of a fermion. If we would instead start from the partition function of a boson in equation (2.37) of [25], then we would arrive at a different sign.

Jacobi form of weight and index zero, and more precisely as a massive deformation of the torus Green’s function of the Laplace equation. We identify in the next section what differential equation $\mathcal{E}_{1,\mu}(z; \tau)$ satisfies.

Using (33) we rewrite (6) as

$$\lim_{\mu \rightarrow 0^+} \mathcal{E}_{1,\mu}(z; \tau) = \mathbb{E}_1(z; \tau) = -2 \log \left| \frac{\vartheta_1(z; \tau)}{\eta(\tau)} \right| + \frac{2\pi z_2^2}{\tau_2},$$

where $z = z_1 + iz_2$. Since $\vartheta_1(z; \tau)$ vanishes at the lattice points $z \in \mathbb{Z}\tau + \mathbb{Z}$, we see that \mathbb{E}_1 has logarithmic singularities at lattice points. These logarithmic singularities are crucial in mathematical physics, when $\mathbb{E}_1(z; \tau)$ is used as a Green’s function: it is precisely what is required to obtain $\delta^{[2]}$ in (9) above; readers unfamiliar with this may want to consult the textbook Problem 2.1 in [21] with the solution in [11].

3. Massive Jacobi forms

3.1. The definitions

We note that holomorphy is much more restrictive than modularity. To be more precise, it is easy to construct functions which are non-holomorphic but modular-invariant. For example letting $g(z)$ be any smooth function on the sphere and $J(\tau)$ the $\text{SL}_2(\mathbb{Z})$ -Hauptmodul, the composition $g(J(\tau))$ is smooth, modular-invariant, and bounded at the cusps. Of course, the way to proceed is to replace the conditions like $\partial_{\bar{\tau}} = 0$ by higher order differential equations invariant under the appropriate Lie group.

We first consider functions $f_\mu(\tau)$ on $\mathbb{H} \times \mathbb{R}^+$. The group $\text{SL}_2(\mathbb{R})$ acts on \mathbb{H} as usual and fixes \mathbb{R}^+ . The *weight k hyperbolic Laplace operator* for $\text{SL}_2(\mathbb{R})$ is defined as

$$(10) \quad \Delta_{\tau,k} := -\tau_2^2 (\partial_{\tau_1}^2 + \partial_{\tau_2}^2) + ik\tau_2(\partial_{\tau_1} + i\partial_{\tau_2}) = -4\tau_2^2 \partial_\tau \partial_{\bar{\tau}} + 2ik\tau_2 \partial_{\bar{\tau}}.$$

We say that a function g has polynomial growth towards $i\infty$ if

$$g(\tau) = O(\tau_2^a) \quad \text{as } \tau_2 \rightarrow \infty$$

for some $a > 0$. Decay towards the other cusps is defined similarly.

Definition 1. A *massive Maass form* of weight k is a smooth function $f_\mu(\tau)$ on $(\tau, \mu) \in \mathbb{H} \times \mathbb{R}^+$ that transforms like a modular form of weight k for some Fuchsian group, satisfies

$$(11) \quad \Delta_{\tau,k}(f_\mu)(\tau) = (g_2(\mu)\partial_\mu^2 + g_1(\mu)\partial_\mu + g_0(\mu)) f_\mu(\tau)$$

for certain smooth functions $g_j : \mathbb{R}^+ \rightarrow \mathbb{C}$, and for each fixed $\mu \in \mathbb{R}^+$, has polynomial growth towards the cusps. If f_μ is a massive Maass form, and $f(\tau) := \lim_{\mu \rightarrow 0^+} f_\mu(\tau)$ exists for all $\tau \in \mathbb{H}$, then we say f_μ is a *massive deformation* of f .

Any (holomorphic) modular form $f(\tau)$ has massive deformations, e.g. $f(\tau) + a\mu$ for $a \in \mathbb{C}$. In Theorem 3.6 below, we give massive Maass forms $\mathcal{E}_{s,\mu}(0; \tau)$ of weight zero which, for any $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, are massive deformations of the Kronecker–Eisenstein series $\mathbb{E}_s(0; \tau)$ defined in (32).

Note that adding λf_μ to the right-side of (11) is unnecessary as it could be absorbed into g_0 . Also, observe that we can remove the first order term $g_1(\mu)\partial_\mu$ in (3.2) by making the change of variable $\mu = g(\nu)$, provided that we find a solution to the auxiliary ordinary differential equation $g_1(g(\nu))g'(\nu) - g_2(g(\nu))g''(\nu) = 0$.

If $f_\mu(\tau)$ is a massive deformation of a Maass form $f(\tau)$, then for any smooth function $g(\mu)$ on \mathbb{R}^+ with $\lim_{\mu \rightarrow 0^+} g(\mu) = 1$, $g(\mu)f_\mu(\tau)$ is another massive deformation of $f(\tau)$, though with different $g_j(\mu)$. Thus if there is one massive deformation of a given Maass form, then there are many.

Next consider functions $\phi_\mu(z; \tau)$ on $\mathbb{C} \times \mathbb{H} \times \mathbb{R}^+$. The Jacobi group $\text{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ acts on $\mathbb{C} \times \mathbb{H}$ as usual and fixes \mathbb{R}^+ . In particular, restricted to $A = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right] \in \text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, the *weight k index m slash operator* is

$$\begin{aligned} \phi|_{k,m}A(z; \tau) &:= (c\tau + d)^{-k} \exp \left(2\pi i m \left(-c \frac{(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2\tau + 2\lambda z \right) \right) \\ &\quad \times \phi \left(\frac{z + \lambda\tau + \mu}{c\tau + d}; \frac{a\tau + b}{c\tau + d} \right). \end{aligned}$$

The Laplacian here is

$$\Delta_{z,k,m} := 2\tau_2 \partial_z \partial_{\bar{z}} + 8\pi i \alpha \tau_2 m \partial_{\bar{z}} - 2\pi i m.$$

The *Casimir operator* of order three is given by

$$\begin{aligned} C_{k,m} &:= -4\tau_2 z_2 (\partial_z \partial_{\bar{z}}^2 + \partial_{\bar{z}}^2 \partial_z) - 4\tau_2^2 (\partial_{\bar{\tau}} \partial_z^2 + \partial_\tau \partial_{\bar{z}}^2) + 2ik\tau_2 (\partial_z \partial_{\bar{z}} + \partial_{\bar{z}}^2) \\ &\quad + 4\pi i m (8\tau_2^2 \partial_\tau \partial_{\bar{\tau}}^2 - 2z_2^2 \partial_{\bar{z}}^2 + 8\tau_2 z_2 \partial_\tau \partial_{\bar{z}} - 2i(2k - 1)\tau_2 \partial_{\bar{\tau}} + 2kiz_2 \partial_{\bar{z}}). \end{aligned}$$

This operator was introduced into the context of Maass–Jacobi forms in [20]. There are several definitions of those forms in the literature, one example is given in [6].

Definition 2. A *massive Maass–Jacobi form* is a smooth function $\phi_\mu(z; \tau)$ on $(z, \tau, \mu) \in \mathbb{C} \times \mathbb{H} \times \mathbb{R}^+$ that transforms like a Jacobi form of weight k and index m for some discrete subgroup of $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$, satisfies both

$$(12) \quad C_{k,m}(\phi_\mu)(z; \tau) = \lambda \phi_\mu(z; \tau),$$

$$(13) \quad \Delta_{z,k,m}(\phi_\mu)(z; \tau) = - (G_2(\mu)\partial_\mu^2 + G_1(\mu)\partial_\mu + G_0(\mu)) \phi_\mu(z; \tau)$$

for certain smooth functions G_0, G_1 , and $G_2 \in C^\infty(\mathbb{R}^+)$, and has polynomial growth for each fixed z and μ . (The minus sign on the right is to facilitate comparison with Section 5.) If ϕ_μ is a massive Maass–Jacobi form, and $\phi(z; \tau) := \lim_{\mu \rightarrow 0^+} \phi_\mu(z; \tau)$ exists for all $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$, then we say that ϕ_μ is a *massive deformation* of ϕ .

Note that a massive Maass–Jacobi form is, among other things, a one-parameter family of Maass–Jacobi forms. For example, we prove in Corollary 3.5 that $\mathcal{E}_{1,\mu}(z; \tau)$ is a massive Maass–Jacobi form of weight and index zero, for $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, for which $\lambda = G_1(\mu) = G_0(\mu) = 0$ is a possible choice. We generalize this example significantly in Theorem 3.4: e.g. in Corollary 3.5 we give a massive deformation $\mathcal{E}_{s,\mu}(z; \tau)$ of $\mathbb{E}_s(z; \tau)$ for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

Lemma 3.1. Choose any smooth functions $g, \varphi \in C^\infty(\mathbb{R}^+)$ such that both g and φ' never vanish and φ maps \mathbb{R}^+ onto \mathbb{R}^+ .

- (a) Suppose that $f_\mu(\tau)$ is a massive Maass form of weight k , for some discrete subgroup G of $\mathrm{SL}_2(\mathbb{R})$. Then $F_\mu(\tau) := g(\mu)f_{\varphi(\mu)}(\tau)$ is also a massive Maass form of weight k for G .
- (b) Suppose that $\phi_\mu(z; \tau)$ is a massive Maass–Jacobi form of weight k and index m , for some discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$. Then $\Phi_\mu(z; \tau) := g(\mu)\phi_{\varphi(\mu)}(z; \tau)$ is also a massive Maass–Jacobi form of weight k and index m for Γ .

Proof. (a) Clearly F_μ satisfies the same transformation law with respect to G as f_μ , and also has the same limiting behavior towards the cusps. To see that (11) holds, with different functions g_0, g_1 , and g_2 , we directly compute

$$\begin{aligned} \Delta_{\tau,k}(F_\mu)(\tau) &= \frac{g(\mu)}{\varphi'(\mu)^2} \left(g_2(\varphi(\mu))\partial_\mu^2 + \left(-\varphi''(\mu)g_2(\varphi(\mu)) + g_1(\varphi(\mu))\varphi'(\mu) \right) \partial_\mu \right. \\ &\quad \left. + g_0(\varphi(\mu))\varphi'(\mu)^2 \right). \end{aligned}$$

The proof of (b) is similar. □

Lemma 3.1 makes the following definitions natural.

Definition 3. For functions $g(\mu), \varphi(\mu), f_\mu(\tau), F_\mu(\tau), \phi_\mu(z; \tau)$, and $\Phi_\mu(z; \tau)$ as in Lemma 3.1, we call $f_\mu(\tau)$ and $F_\mu(\tau)$ *equivalent* as massive Maass forms, and $\phi_\mu(z; \tau)$ and $\Phi_\mu(z; \tau)$ *equivalent* as massive Maass–Jacobi forms.

It is easy to verify that the equivalences defined above indeed yield equivalence relations.

Note that the above objects are truly doubly periodic for $\mu = 0$. It is, however, also of interest to allow for quasiperiodicity, which appears as a parameter w in Section 4.3 below. This can be relevant for vector-valued modular forms.

3.2. Examples

We first note the well-known fact that the Kronecker–Eisenstein series $\mathbb{E}_1(z; \tau)$ is a Maass–Jacobi form of weight and index zero, as we now review.

Proposition 3.2. *For $z \notin \mathbb{Z}\tau + \mathbb{Z}$, we have*

$$C_{0,0}(\mathbb{E}_1(z; \tau)) = 0.$$

Proof. Write $C_{0,0} = C_{0,0;1} + C_{0,0;2}$ with

$$(14) \quad C_{0,0;1} := -4\tau_2 z_2 (\partial_z \partial_{\bar{z}}^2 + \partial_{\bar{z}}^2 \partial_z), \quad C_{0,0;2} := -4\tau_2^2 (\partial_\tau \partial_z^2 + \partial_\tau \partial_{\bar{z}}^2).$$

For $\text{Re}(s)$ sufficiently large, the double sum representation of $\mathbb{E}_s(z; \tau)$ in (32) converges absolutely even after differentiation. We act on each summand, that we denote by $\mathbb{E}_{s,r,\ell}(z; \tau)$, to obtain that

$$\begin{aligned} \tau_2 C_{0,0;1}(\mathbb{E}_{s,r,\ell})(z; \tau) &= 8\pi^3 i z_2 |r\tau + \ell|^2 r \mathbb{E}_{s,r,\ell}(z; \tau), \\ \tau_2 C_{0,0;2}(\mathbb{E}_{s,r,\ell})(z; \tau) &= -8\pi^3 i z_2 |r\tau + \ell|^2 r \mathbb{E}_{s,r,\ell}(z; \tau) \end{aligned}$$

so the contributions from the two pieces of $C_{0,0}$ cancel termwise. The claim then follows via analytic continuation to $s = 1$. □

To determine the action of the operator $C_{0,0}$ on the deformed $\mathcal{E}_{1,\mu}(z; \tau)$, it would be convenient to have a similar double sum representation as for $\mathbb{E}_s(z; \tau)$ in (32). For this purpose, recall the Bessel function defined in (4). Note that $K_\nu(x)$ obeys the differential equation

$$(15) \quad x^2 f''(x) + x f'(x) - (x^2 + \nu^2) f(x).$$

For $\nu \in \mathbb{R}^+$, K_ν has the asymptotic behavior $K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$ as $x \rightarrow \infty$, and $K_\nu(x) \sim \frac{1}{2} \Gamma(\nu) (\frac{x}{2})^{-\nu}$ as $x \rightarrow 0^+$.

Proposition 3.3. *We have*

$$\mathcal{E}_{1,\mu}(z; \tau) = 2\sqrt{\mu\tau_2} \sum_{(r,\ell) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{K_1\left(2\pi\sqrt{\frac{\mu}{\tau_2}}|r\tau + \ell|\right)}{|r\tau + \ell|} e^{\frac{2\pi i}{\tau_2} \text{Im}((r\tau + \ell)\bar{z})}.$$

In particular $\mathcal{E}_{1,0}(z; \tau) = \mathbb{E}_1(z; \tau)$.

We defer the proof of Proposition 3.3 to Section 4 and rather first consider a generalization. Proposition 3.3 suggests the following generalization. To state it, we let \mathcal{S} denote the space of all functions $h \in C^\infty(\mathbb{R}^+)$, which are $O(e^{-ax})$ as $x \rightarrow \infty$, for some $a > 0$, and $O(x^{-b})$ as $x \rightarrow 0$, for some $b > 0$. For example, $K_\nu \in \mathcal{S}$ for each $\nu \in \mathbb{R}^+$. Define

$$(16) \quad \mathcal{E}_{h,\mu,[a,b,c,d,L]}(z; \tau) := \frac{\mu^d}{\tau_2^c} \sum_{(r,\ell) \in \mathbb{Z}^2 \setminus \{(0,0)\}} |r\tau + \ell|^{2c} h\left(\frac{\mu^a}{\tau_2^b} |r\tau + \ell|^{2b}\right) e^{\frac{2\pi i}{\tau_2} L \text{Im}((r\tau + \ell)\bar{z})}.$$

Then for any $h \in \mathcal{S}$, $a \in \mathbb{R}$, $b \in \mathbb{R}^+$, and $L \in \mathbb{Z}$, the double-sum in $\mathcal{E}_{h,\mu,[a,b,c,d,L]}(z; \tau)$ converges absolutely to a smooth function in $\mathbb{C} \times \mathbb{H} \times \mathbb{R}^+$. By Lemma 3.1, $\mathcal{E}_{h,\mu,[a,b,c,d,L]}(z; \tau)$ is a massive Maass–Jacobi form if and only if $\mathcal{E}_{h,\mu,[1,b,c,0,L]}(z; \tau)$ is.

Theorem 3.4. *Choose any $a \in \mathbb{R}^+$, $b \in \mathbb{R}^+$, $L \in \mathbb{Z}$, and any smooth function $h \in \mathcal{S}$ satisfying the differential equation*

$$(17) \quad x^2 h''(x) + \gamma x h'(x) + \left(\kappa - \nu x^{\frac{1}{a}}\right) h(x) = 0$$

for some constants $\gamma, \kappa, \nu \in \mathbb{R}$, $\nu \neq 0$. Then

$$\begin{aligned} \mathcal{E}_{h,\mu,[a,b,L]}(z; \tau) &:= \mathcal{E}_{h,\mu,[1,a,b,0,L]}(z; \tau) \\ &= \sum_{(r,\ell) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{|r\tau + \ell|^{2b}}{\tau_2^b} h\left(\frac{\mu|r\tau + \ell|^{2a}}{\tau_2^a}\right) e^{\frac{2\pi i}{\tau_2} L \text{Im}((r\tau + \ell)\bar{z})} \end{aligned}$$

is a massive Maass–Jacobi form of weight and index zero.

Proof. First note that any $\mathcal{E}_{h,\mu,[a,b,c,d,L]}(z; \tau)$ transforms like a Jacobi form of weight and of index zero. To verify the cusp condition in Definition 2, it suffices to consider $\tau \rightarrow i\infty$ thanks to invariance under $\text{SL}_2(\mathbb{Z})$, and each term in (18) exponentially decays to 0 because $h \in \mathcal{S}$, except the $r = 0$ terms, that have at most polynomial growth.

Because h decays rapidly as $x \rightarrow \infty$, we can verify the partial differential equation (13) term-by-term. Dropping the dependence on a, b, L , and h from the notation, we define

$$f_{\mu,r,\ell}(\tau) := \frac{|r\tau + \ell|^{2b}}{\tau_2^b} h\left(\frac{\mu|r\tau + \ell|^{2a}}{\tau_2^a}\right)$$

and keep $E_{r,\ell,L}(z; \tau) := e^{\frac{2\pi i}{\tau_2} L \operatorname{Im}((r\tau + \ell)\bar{z})}$ as a separate factor. We obtain, with $C_{0,0;1}$ and $C_{0,0;2}$ as defined in (14)

$$\begin{aligned} C_{0,0;1}(f_{\mu,r,\ell}E_{r,\ell,L})(z; \tau) &= 8L^3\pi^3 i\alpha r|r\tau + \ell|^2 f_{\mu,r,\ell}(\tau)E_{r,\ell,L}(z; \tau), \\ C_{0,0;2}(f_{\mu,r,\ell}E_{r,\ell,L})(z; \tau) &= -4L^2\pi^2 ((r\tau + \ell)^2\partial_\tau(f_{\mu,r,\ell})(\tau) \\ &\quad + (r\bar{\tau} + \ell)^2\partial_{\bar{\tau}}(f_{\mu,r,\ell}(\tau))) E_{r,\ell,L}(z; \tau) \\ &\quad - 8\pi^3 r\alpha i L^3|r\tau + \ell|^2 f_{\mu,r,\ell}(\tau)E_{r,\ell,L}(z; \tau). \end{aligned}$$

Since $((r\tau + \ell)^2\partial_\tau + (r\bar{\tau} + \ell)^2\partial_{\bar{\tau}})F(\frac{|r\tau + \ell|^2}{\tau_2})$ vanishes for any smooth function F , we have that $C_{0,0;2}(f_{\mu,r,\ell}E_{r,\ell,L}) = -C_{0,0;1}(f_{\mu,r,\ell}E_{r,\ell,L})$. Hence (12) holds.

Turning to (13), we obtain that

$$\Delta_{z,0,0}(f_{\mu,r,\ell}E_{r,\ell,L})(z; \tau) = -2\pi^2 L^2 \mu^{-\frac{b+1}{a}} X_\mu(\tau)^{\frac{b+1}{a}} h(X_\mu(\tau))E_{r,\ell,L}(z; \tau),$$

where $X_\mu(\tau) := \frac{\mu|r\tau + \ell|^{2a}}{\tau_2^a}$. We also compute, for any functions $G_j(\mu)$ as above,

$$\begin{aligned} &(G_2(\mu)\partial_\mu^2 + G_1(\mu)\partial_\mu + G_0(\mu)) f_{\mu,r,\ell}(\tau) \\ &= G_2(\mu)\mu^{-2-\frac{b}{a}} X_\mu(\tau)^{\frac{b}{a}+2} h''(X_\mu(\tau)) + G_1(\mu)\mu^{-1-\frac{b}{a}} X_\mu(\tau)^{\frac{b}{a}+1} h'(X_\mu(\tau)) \\ &\quad + G_0(\mu)\mu^{-\frac{b}{a}} X_\mu(\tau)^{\frac{b}{a}} h(X_\mu(\tau)). \end{aligned}$$

Choosing $G_0(\mu) := L^2 \frac{2\pi^2 \kappa}{\nu} \mu^{-\frac{1}{a}}$, $G_1(\mu) := L^2 \frac{2\pi^2 \gamma}{\nu} \mu^{1-\frac{1}{a}}$, $G_2(\mu) := L^2 \frac{2\pi^2}{\nu} \mu^{2-\frac{1}{a}}$ and using (17), we obtain that (13) holds termwise. \square

Remark. We see from the proof of Theorem 3.4 that the parameters $\gamma, \kappa, \nu \in \mathbb{R}, \nu \neq 0$ in (17) can be used to “tune” the ordinary differential equation (17). This means that if applications demand a certain partial differential operator in μ in (13), i.e., certain functions $G_0(\mu), G_1(\mu)$, and $G_2(\mu)$, the ordinary differential equation (17) adjusts accordingly. An explicit example is given in Section 25. By the change of variables $x \mapsto (\frac{w}{2b\sqrt{\nu}})^{2b}$ and $h \mapsto x^{\frac{1-\gamma}{2}} H$, the

differential equation (17) for $h(x)$ transforms to the Bessel equation (15) for H with $\nu^2 = b^2((\gamma - 1)^2 - 4\kappa)$. However, this change of variables may not be admissible in certain applications, for example it generically changes the massless limit, so we treat (17) as a generalization of (15). If one elects to extend Definition 3 to allow further powers of ∂_μ on the right-side of (13), then Theorem 3.4 is extended to include families for which the corresponding (17) is not a Bessel-type equation. For example, if $g_3(\mu)\partial_\mu^3$ is included in (13), then the proof of Theorem 3.4 yields massive Maass–Jacobi forms for certain functions $h(x)$ satisfying

$$x^3 h'''(x) + \varrho x^2 h''(x) + \gamma x h'(x) + \left(\kappa - \nu x^{\frac{1}{a}}\right) h(x) = 0$$

for $\varrho, \gamma, \kappa, \nu \in \mathbb{R}$.

From Theorem 3.4, it is natural to generalize Proposition 3.3 as in the following corollary. For this, we set

$$\mathcal{E}_{s,\mu}(z; \tau) := 2 \sum_{(r,\ell) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left(\frac{\sqrt{\mu\tau_2}}{|r\tau + \ell|}\right)^s K_s\left(2\pi\sqrt{\frac{\mu}{\tau_2}}|r\tau + \ell|\right) e^{2\pi i(r\beta - \ell\alpha)}.$$

Corollary 3.5. *For any $s \in \mathbb{C}$ with $\text{Re}(s) > 0$, the function $\mathcal{E}_{s,\mu}(z; \tau)$ is a massive Maass–Jacobi form for $\text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, of weight and index zero. For $\mu \rightarrow 0^+$ we have $\mathcal{E}_{s,\mu}(z; \tau) \rightarrow \mathbb{E}_s(z; \tau)$.*

Proof. We find that $\mathcal{E}_{s,\mu}(z; \tau) = 2 \mathcal{E}_{2K_s(2\pi x), \mu, [\frac{1}{2}, \frac{1}{2}, \frac{-s}{2}, \frac{s}{2}, 1]}(z; \tau)$, and therefore, by Theorem 3.4, $\mathcal{E}_{s,\mu}(z; \tau)$ is a massive Maass–Jacobi form for $\text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, of weight and index zero. Taking the limit as $\mu \rightarrow 0^+$ and comparing with (32) below, we obtain $\mathbb{E}_s(z; \tau)$ as desired. \square

It is interesting that we get massive deformations for \mathbb{E}_s , even though $\Delta_{z,0,0}(\mathbb{E}_s)$ is proportional to \mathbb{E}_{s-1} , and not to \mathbb{E}_s (see (36)). What makes this possible are the $G_j(\mu)$ in (13).

Theorem 3.6. *For any $h \in \mathcal{S}$, and any $a, c, d \in \mathbb{R}$ and $b \in \mathbb{R}^+$, with $a \neq 0$, the function $\mathcal{E}_{h,\mu[a,b,c,d,0]}(0; \tau)$ is a massive Maass form of weight zero for $\text{SL}_2(\mathbb{Z})$.*

Proof. Theorem 3.4 yields that each $\mathcal{E}_{h,\mu,[a,b,c,d,L]}(0; \tau)$ transforms like a weight zero modular form for $\text{SL}_2(\mathbb{Z})$ and satisfies the cusp condition, so all that remains is to verify (11). By Lemma 3.1, it suffices to take $a = 1$

and $d = 0$. Writing $f_{\mu,r,\ell}(\tau) := \frac{|r\tau + \ell|^{2c}}{\tau_2^c} h\left(\frac{\mu|r\tau + \ell|^{2b}}{\tau_2^b}\right)$, we compute

$$\begin{aligned} \Delta_{\tau,0}(f_{\mu,r,\ell})(\tau) &= -b^2\mu^2 \frac{|r\tau + \ell|^{4b+2c}}{\tau_2^{2b+c}} h''(X_\mu(\tau)) \\ &\quad - (b^2 + 2bc + b)\mu \frac{|r\tau + \ell|^{2b+2c}}{\tau_2^{b+c}} h'(X_\mu(\tau)) - (c^2 + c) \frac{|r\tau + \ell|^{2c}}{\tau_2^c} h(X_\mu(\tau)) \end{aligned}$$

with $X_\mu(\tau) := \frac{\mu}{\tau_2^b} |r\tau + \ell|^{2b}$. Note that this can also be written as

$$\begin{aligned} \Delta_{\tau,0}(f_{\mu,r,\ell})(\tau) &= -\mu^{-\frac{c}{b}} X_\mu(\tau)^{\frac{c}{b}} (b^2 X_\mu(\tau)^2 h''(X_\mu(\tau)) \\ &\quad + b(b + 2c + 1) X_\mu(\tau) h'(X_\mu(\tau)) + c(c + 1) h(X_\mu(\tau))). \end{aligned}$$

Similarly to the proof of Theorem 3.4, we use this explicit expression for $\Delta_{\tau,0}(f_{\mu,r,\ell})(\tau)$ to obtain that (11) is satisfied with $g_2(\mu) := -b^2\mu^2$, $g_1(\mu) := -(b^2 + 2bc + b)\mu$, and $g_0(\mu) := -c^2 - c$; note that there is no condition on h in this case. \square

The examples of massive Maass forms in Theorem 3.6 are often deformations of Maass forms. For example, if $h(x) = 2K_s(2\pi x)$ for $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, then $\mathcal{E}_s(0; \tau) = \mathcal{E}_{h,\mu, [\frac{1}{2}, \frac{1}{2}, \frac{-s}{2}, \frac{s}{2}, 0]}(0; \tau)$ is a massive deformation of $\mathbb{E}_s(0; \tau)$, as in Corollary 3.5.

4. Fourier coefficients of $\mathcal{E}_{1,\mu}(z; \tau)$ and the proof of Proposition 3.3

4.1. Proof of Proposition 3.3

We are now ready to compute the Fourier expansion of $\mathcal{E}_{1,\mu}(z; \tau)$. Note that in the massless case $\mu = 0$, calculating the Fourier expansion of $\mathcal{E}_{1,\mu}(z; \tau)$ is equivalent to proving Kronecker’s second limit formula, as expressed in Appendix A; in physics, this is the Fourier expansion of the Green’s function of the Laplace equation on the torus, see Problem 7.3 in [21].

Proof of Proposition 3.3. Taking the logarithm of equation (3) and recalling that $\mathcal{E}_{1,\mu}(z; \tau)$ is defined with $-\alpha$ gives

$$\mathcal{E}_{1,\mu}(z; \tau) = 8\pi c_{\alpha,\mu} \tau_2 + \sum_{n \in \mathbb{Z}} \sum_{\pm} \text{Log} \left(1 - e^{-2\pi\tau_2 \sqrt{\frac{\mu}{\tau_2} + (n \pm \alpha)^2 + 2\pi i(n \pm \alpha)\tau_1 \mp 2\pi i\beta}} \right).$$

Using Poisson summation, the (ℓ, r) -th $(\ell, r \in \mathbb{Z})$ Fourier coefficient of the second term equals

$$\begin{aligned}
 & \int_0^1 \int_0^1 e^{-2\pi i(\ell\alpha - r\beta)} \sum_{n \in \mathbb{Z}} \sum_{\pm} \text{Log} \left(1 - e^{-2\pi\tau_2 \sqrt{\frac{\mu}{\tau_2} + (n \pm \alpha)^2} + 2\pi i(n \pm \alpha)\tau_1 \mp 2\pi i\beta} \right) d\alpha d\beta \\
 (18) \quad & = \sum_{n \in \mathbb{Z}, j \geq 1, \pm} \frac{1}{j} \int_0^1 e^{2\pi i(-\ell\alpha + (n \pm \alpha)j\tau_1) - 2\pi j\tau_2 \sqrt{\frac{\mu}{\tau_2} + (n \pm \alpha)^2}} d\alpha \int_0^1 e^{2\pi i(-r \mp j)\beta} d\beta,
 \end{aligned}$$

inserting the series expansion of the logarithm. The integral on β now vanishes unless $j = \mp r$ in which case it equals 1. Since $j \geq 1$ we have no solution if $r = 0$. We thus assume for the remaining calculation that $r \neq 0$. We obtain that $\mp = \text{sgn}(r)$ and $j = |r|$, thus (18) equals

$$(19) \quad \frac{1}{|r|} \sum_{n \in \mathbb{Z}} \int_0^1 e^{2\pi i(-\ell\alpha + (n - \text{sgn}(r)\alpha)|r|\tau_1) - 2\pi|r|\tau_2 \sqrt{\frac{\mu}{\tau_2} + (n - \text{sgn}(r)\alpha)^2}} d\alpha.$$

Noting that $e^{-2\pi i\ell\alpha}$ is invariant under $\alpha \mapsto \alpha + \text{sgn}(r)n$, (19) becomes

$$(20) \quad \frac{1}{|r|} \int_{\mathbb{R}} e^{-2\pi i(\ell + r\tau_1)\alpha - 2\pi|r|\tau_2 \sqrt{\frac{\mu}{\tau_2} + \alpha^2}} d\alpha.$$

We next use (26) on page 16 of [2], which states that for $A, B \in \mathbb{C}$ with $\text{Re}(A), \text{Re}(B) > 0$

$$\int_0^\infty e^{-B\sqrt{x^2 + A^2}} \cos(xy) dx = \frac{AB}{\sqrt{y^2 + B^2}} K_1 \left(A\sqrt{y^2 + B^2} \right).$$

Thus (20) becomes

$$2\sqrt{\mu\tau_2} \frac{1}{|r\tau + \ell|} K_1 \left(2\pi\sqrt{\frac{\mu}{\tau_2}} |r\tau + \ell| \right).$$

This yields

$$\mathcal{E}_{1,\mu}(z; \tau) = 8\pi c_{\alpha,\mu}\tau_2 + 2\sqrt{\mu_2\tau_2} \sum_{\substack{r \in \mathbb{Z} \setminus \{0\} \\ \ell \in \mathbb{Z}}} \frac{K_1 \left(2\pi\sqrt{\frac{\mu}{\tau_2}} |r\tau + \ell| \right)}{|r\tau + \ell|} e^{2\pi i(r\beta - \ell\alpha)}.$$

Plugging in the definition of $c_{\alpha,\mu}$ then gives the claimed Fourier expansion. The $c_{\alpha,\mu}$ term furnishes the $r = 0$ terms that exhibit the expected polynomial growth towards the cusp $i\infty$.

The limit in Proposition 3.3 is clear using that $\lim_{x \rightarrow 0} xK_1(x) = 1$. □

4.2. The Mellin transform

The Mellin transform of Jacobi-form-like objects (or, vector-valued modular forms) is interesting to consider since it can produce the corresponding Dirichlet series, or automorphic L-functions. Let us briefly review this, and then apply it to our massive Jacobi forms. The *Mellin transform* (see 2.5.1 of [18]) of a locally integrable function f is a Laplace-like transform

$$\mathcal{M}(f)(s) := \int_0^\infty f(x)x^{s-1}dx$$

that is analytic in some strip $a < \text{Re}(s) < b$, with its inverse (see 2.5.2 of [18]) obtained by integration along a vertical line shifted by any constant c in the strip $a < c < b$,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}(f)(s)x^{-s}ds .$$

Let us consider the Mellin transform of $\mathcal{E}_{1,\mu}(z; \tau)$ with respect to μ and call the Mellin-dual variable s . We have

Proposition 4.1. *For $s \in \mathbb{C}$ with $\text{Re}(s) > 0$, the Mellin transform of $\mu \mapsto \mathcal{E}_{1,\mu}(z; \tau)$ equals*

$$\mathcal{M}(\mathcal{E}_{1,\cdot}(z; \tau))(s) = \frac{\Gamma(s)}{\pi^s} \mathbb{E}_{s+1}(0, z; \tau).$$

Proof. Plugging in Proposition 3.3 gives that the Mellin transform of $\mu \mapsto \mathcal{E}_{1,\mu}(z; \tau)$ is

$$\int_0^\infty \mathcal{E}_{1,\mu}(z; \tau)\mu^{s-1}d\mu = \int_0^\infty 2\sqrt{\mu\tau_2} \sum_{(r,\ell) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{K_1\left(2\pi\sqrt{\frac{\mu}{\tau_2}}|r\tau + \ell|\right)}{|r\tau + \ell|} e^{2\pi i(r\beta - \ell\alpha)} \mu^{s-1}d\mu.$$

Since the sum converges absolutely for $\mu > 0$ (10.25.3 of [18] gives the exponential decay $K_1(x) \sim \sqrt{\frac{\pi}{2x}}e^{-x}$ for $x \rightarrow \infty$), we may interchange summation and integration. We now compute, making the change of variables

$$x = 2\pi\sqrt{\frac{\mu}{\tau_2}}|r\tau + \ell|$$

$$\int_0^\infty K_1\left(2\pi\sqrt{\frac{\mu}{\tau_2}}|r\tau + \ell|\right)\mu^{s-\frac{1}{2}}d\mu = 2\left(\frac{\tau_2}{4\pi^2|r\tau + \ell|^2}\right)^{s+\frac{1}{2}}\int_0^\infty K_1(x)x^{2s}dx.$$

Now 10.43.19 of [18] states that for $a, b \in \mathbb{C}$ with $\operatorname{Re}(a) < \operatorname{Re}(b)$ we have

$$\int_0^\infty K_a(x)x^{b-1}dx = 2^{b-2}\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{a+b}{2}\right).$$

From this it not hard to conclude the claim. □

Note that as long as $\operatorname{Re}(a) < \operatorname{Re}(b)$, it is straightforward to generalize Proposition 4.1 to other $\mathcal{E}_{s,\mu}(z; \tau)$ than $\mathcal{E}_{1,\mu}(z; \tau)$. It is surprising that an integral transform of the massive ($\mu > 0$) Kronecker–Eisenstein series produces the massless ($\mu = 0$) Kronecker–Eisenstein series. This means that we can take the inverse Mellin transform of the classical Kronecker–Eisenstein series to obtain our main example, namely

$$(21) \quad \mathcal{E}_{1,\mu}(z; \tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (\pi\mu)^{-s}\Gamma(s)\mathbb{E}_{s+1}(0; z; \tau)ds,$$

where as in Subsection 4.2, c lies in the region of convergence, here the positive real axis. In Appendix A of [26], it was proven that $\mathcal{E}_{1,\mu}(z; \tau)$ transforms like a Jacobi form of weight and index zero. Equation (21) gives a reproof of this fact using the modular invariance of the undeformed $\mathbb{E}_{s+1}(0; z; \tau)$ (see e.g. Section 4 of [23]), since μ is invariant.

Proposition 4.1 and its inverse equation (21) may be useful in the following sense. There is a vast literature in string theory where integrals over sums of products of Kronecker–Eisenstein series were performed. Particularly relevant examples here include [7, 13, 16, 24]. In the last few years, this long-standing theme in string theory has been put in mathematical terms as modular graph functions [8], we give an example of this in Section 5.1. Proposition 4.1 and equation (21) open up the possibility to generalize some of that vast literature on massless objects ($\mu = 0$, flat space) to mass-deformed objects ($\mu \neq 0$, as in the plane gravitational wave) simply by representing them as inverse Mellin transforms of the well-studied massless objects.

4.3. An alternative formal representation

The goal of this subsection is to introduce an additional parameter $w = A\tau + B$ in $\mathcal{E}_{1,\mu}$, making it quasiperiodic in z in the sense of equation (30).

To be more precise a function $f(w, z; \tau) : \mathbb{C}^2 \times \mathbb{H} \rightarrow \mathbb{C}$ is called *quasiperiodic in z* if

$$(22) \quad f(w, z + 1; \tau) = e^{2\pi i A} f(w, z; \tau), \quad f(w, z + \tau; \tau) = e^{-2\pi i B} f(w, z; \tau).$$

For this, we take (21) as starting point and use (34) to introduce a nonzero first argument $\mathbb{E}_{s+1}(w, z; \tau)$. This turns out to provide a representation of the massive Kronecker–Eisenstein series as a power series in μ . For this define

$$\mathcal{E}_{1,\mu}(w, z; \tau) := \int_{c-i\infty}^{c+i\infty} (\pi\mu)^{-s} \Gamma(s) \mathbb{E}_{s+1}(w, z; \tau) ds.$$

Proposition 4.2. *We have*

$$(23) \quad \mathcal{E}_{1,\mu}(w, z; \tau) = e^{\frac{2\pi i}{\tau_2} \text{Im}(w\bar{z})} \sum_{n \geq 0} \frac{(-\pi\mu)^n}{n!} \mathbb{E}_n(z, w; \tau)$$

with $\mathbb{E}_{s+1}(w, z; \tau)$ defined in (34). In particular $\mathcal{E}_{1,\mu}(w, z; \tau)$ is quasiperiodic in z .

Proof. For $z \notin \mathbb{Z}\tau + \mathbb{Z}$, the integral representation in equation (34) extends $\mathbb{E}_s(w, z; \tau)$ to all s . We integrate along the vertical path of integration in the inverse Mellin transform (21) and close the contour along a semicircle at infinity around $s \rightarrow -\infty$. Using the functional equation (35), we receive contributions from residues at the poles² of $\Gamma(s)$ at $s = -n$ for $n \geq 0$, producing $e^{\frac{2\pi i}{\tau_2} \text{Im}(w\bar{z})} \frac{(-1)^n}{n!} \mu^n \mathbb{E}_n(z, w; \tau)$ as residue. □

Note that if we try to use the double sum representation from equation (8) to perform the sum on r and ℓ before the inverse Mellin transform (21), the double sum representation does not converge for $\text{Re}(s) < 0$, where we pick up the residues of $\Gamma(s)$. That is not a problem as long as we use the analytic continuation (34).

Note that the power series in μ in (23) has no mixed $\mu^n \log(\mu)$ terms, unlike Proposition 3.3 when the Bessel function is expanded in μ .

We can now use this expansion in μ to connect to existing results in the literature, in particular [9]. For this purpose, we assume that $\mu < \frac{1}{2}$, and

²This uses that $\mathbb{E}_s(w, z; \tau)$ for large positive s does not ruin the exponential decay due to the Gamma function for large negative s . For example, for $w = 0$, the author of [23] showed on p. 23, that $\mathbb{E}_n(z, 0; \tau)$ is majorized by $\zeta(2s)$ (with ζ the Riemann zeta-function), and $\zeta(2s)$ decays as 2^{-s} as $s \rightarrow \infty$, whereas it would need to grow to ruin the decay due to the Gamma function. Similarly, for $w \neq 0$ on p. 42 of [23] (set $g = 0$ there), $\mathbb{E}_n(z, w; \tau)$ was majorized by the Hurwitz zeta function.

focus on the special case $w = 0$. We obtain from (23), setting $\mathcal{E}_{1,\mu}(z; \tau) := \mathcal{E}_{1,\mu}(z, 0; \tau)$

$$\begin{aligned}
 \mathcal{E}_{1,\mu}(z; \tau) &= \mathbb{E}_0(z, 0; \tau) - \pi\mu\mathbb{E}_1(z, 0; \tau) + \sum_{n \geq 2} \frac{(-\pi\mu)^n}{n} \left(\frac{\tau_2}{\pi}\right)^n \sum_{r,\ell}^* \frac{1}{|z + r\tau + \ell|^2n} \\
 (24) \quad &= \mathbb{E}_1(0, z; \tau) - \pi\mu\mathbb{E}_1(z, 0; \tau) + \sum_{r,\ell}^* \sum_{n \geq 2} \frac{1}{n} \left(\frac{-\mu\tau_2}{|z + r\tau + \ell|^2}\right)^n,
 \end{aligned}$$

where the summation Σ^* indicates that the sum runs over all $(r, \ell) \in \mathbb{Z}^2$ such that $w + r\tau + \ell \neq 0$. This expression can be further manipulated to a double sum of logarithms, but convergence becomes somewhat more complicated and we therefore do not do so here. This last expression has the nice feature that the massless Green’s function $\mathbb{E}_1(0, z; \tau)$ appears as a separate first term, and the remaining part manifestly vanishes for $\mu = 0$, making the limit $\mu \rightarrow 0^+$ more manifest than in (7), Proposition 3.3, and Proposition 4.1.

In anticipation of the comparison to string theory literature in Section 5 below, note that

$$\begin{aligned}
 -[\partial_\mu^2 (\mathcal{E}_{1,\mu}(0, z; \tau))]_{z=0} &= \left[\sum_{r,\ell}^* \frac{\tau_2^2}{(|z + r\tau + \ell|^2 + \mu\tau_2)^2} \right]_{z=0} \\
 &= \sum_{r,\ell}^* \frac{\tau_2^2}{(|r\tau + \ell|^2 + \mu\tau_2)^2}.
 \end{aligned}$$

This is essentially a simpler version of the function \mathcal{W} occurring in Section 4 of [9], a generating function of (massless) modular graph functions.

5. Comparison to string theory literature

This section is mainly intended for physics readers, or mathematicians who are curious why physicists might be interested in the objects we study here.

In string theory, just as in quantum field theory, it is natural to consider massive worldsheet fields, either in a gravitational wave background, as in [3], or as a technical trick in flat space, as in [15], including as generating function of modular graph functions [9].

5.1. Massive modular graph functions

There is recent interest in modular graph functions [8], that are constructed by integrating various combinations of (massless) Green’s functions over a

fundamental domain for the action of the lattice $\mathbb{Z}\tau + \mathbb{Z}$. The simplest non-trivial example of a modular graph function is just the non-holomorphic Eisenstein series $\mathbb{E}_2(0; \tau)$, arising from the integral over a fundamental domain P of the product of two Kronecker–Eisenstein series: $\mathbb{E}_1(z; \tau)\mathbb{E}_1(-z; \tau)$, viewed as Green’s functions for the Laplace equation on the torus.

A simple corollary of the considerations in this paper is that one can replace $\mathbb{E}_1(z; \tau)\mathbb{E}_1(-z; \tau)$ with $\mathcal{E}_{1,\mu}(z; \tau)\mathcal{E}_{1,\mu}(-z; \tau)$ to create a mass-deformed modular graph function that we might call $\mathcal{E}_{1,1,\mu}(0; \tau)$. We integrate over the region P with corners at the complex numbers $z = 0, z = 1, z = \tau$, and $z = \tau + 1$, a fundamental domain for $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$. The integration is

$$\begin{aligned}
 \mathcal{E}_{1,1,\mu}(0; \tau) &= \frac{1}{\tau_2} \int_P \mathcal{E}_{1,\mu}(z; \tau)\mathcal{E}_{1,\mu}(-z; \tau)d^2z \\
 &= \int_0^1 \int_0^1 \sum_{r_1, \ell_1}^* \frac{\sqrt{\mu\tau_2}K_1(2\pi\sqrt{\frac{\mu}{\tau_2}}|r_1\tau + \ell_1|)}{|r_1\tau + \ell_1|} e^{2\pi i(r_1\beta - \ell_1\alpha)} \\
 (25) \qquad &\qquad \times \sum_{r_2, \ell_2}^* \frac{\sqrt{\mu\tau_2}K_1(2\pi\sqrt{\frac{\mu}{\tau_2}}|r_2\tau + \ell_2|)}{|r_2\tau + \ell_2|} e^{2\pi i(\ell_2\alpha - r_2\beta)} d\alpha d\beta \\
 &= \mu\tau_2 \sum_{r_1, \ell_1}^* \sum_{r_2, \ell_2}^* \frac{K_1\left(2\pi\sqrt{\frac{\mu}{\tau_2}}|r_1\tau + \ell_1|\right)}{|r_1\tau + \ell_1|} \frac{K_1\left(2\pi\sqrt{\frac{\mu}{\tau_2}}|r_2\tau + \ell_2|\right)}{|r_2\tau + \ell_2|} \delta_{r_1, r_2} \delta_{\ell_1, \ell_2} \\
 &= \mu\tau_2 \sum_{r, \ell}^* \frac{K_1\left(2\pi\sqrt{\frac{\mu}{\tau_2}}|r\tau + \ell|\right)^2}{|r\tau + \ell|^2}.
 \end{aligned}$$

For $\mu \rightarrow 0^+$, this clearly reduces to the massless modular graph function $\mathbb{E}_2(0; \tau)$. In the third equality above, the integration over α, β produces factors like $\int_0^1 e^{2\pi i(r_1 - r_2)\alpha} d\alpha = \delta_{r_1, r_2}$ for $r_j \in \mathbb{Z}$, where $\delta_{r_1, r_2} = 0$ unless $r_1 = r_2$ in which case it equals 1. This collapses the two double sums in $\mathcal{E}_{1,\mu}(z; \tau)\mathcal{E}_{1,\mu}(-z; \tau)$ to a single double sum.

Note that in the differential equations in the sections above, we view τ and z as independent variables. Here, τ is considered to be fixed and we integrate over z , and are free to change variables of integration from z to (α, β) as independent real variables, with Jacobian τ_2 .

In this calculation, unlike in its massless counterpart, each double sum converges exponentially. The undeformed $\mu = 0$ eigenvalue equation is $\Delta_{\tau,0}(\mathbb{E}_2(0; \tau)) = -2\mathbb{E}_2(0; \tau)$. For the μ -deformed modular graph function, we have the following:

Proposition 5.1. *We have*

$$(\Delta_{\tau,0} - 2\mu\partial_\mu + \mu^2\partial_\mu^2) \mathcal{E}_{1,1,\mu}(0; \tau) = -2\mathcal{E}_{1,1,\mu}(0; \tau).$$

Proof. The double sum in $\mathcal{E}_{1,1,\mu}(0; \tau)$ in (25) converges absolutely so we can differentiate term by term. Each term has the form $\mathcal{E}_{1,1,\mu,r,\ell}(\tau) = \frac{\mu\tau_2}{\omega} f(X_\mu(\tau))^2$ for some function f , where $X_\mu(\tau) := 2\pi\sqrt{\frac{\mu\omega(\tau)}{\tau_2}}$ with $\omega(\tau) := |r\tau + \ell|^2$. We find that

$$\frac{\Delta_{\tau,0}(\mathcal{E}_{1,1,\mu,r,\ell})(\tau)}{\mathcal{E}_{1,1,\mu,r,\ell}(\tau)} = \frac{\pi\mu\omega(\tau)}{2f(X_\mu(\tau))^2\tau_2} \left(2\sqrt{\frac{\tau_2}{\mu\omega(\tau)}} f(X_\mu(\tau)) f'(X_\mu(\tau)) - 4\pi (f(X_\mu(\tau)) f''(X_\mu(\tau)) + f'(X_\mu(\tau))^2) \right).$$

The action of $\mu^2\partial_\mu^2$ is similar, namely

$$\frac{\mu^2\partial_\mu^2(\mathcal{E}_{1,1,\mu,r,\ell})(\tau)}{\mathcal{E}_{1,1,\mu,r,\ell}(\tau)} = \frac{\pi\mu\omega(\tau)}{2\tau_2 f(X_\mu(\tau))^2} \left(\frac{6\sqrt{\tau_2} f(X_\mu(\tau)) f'(X_\mu(\tau))}{\sqrt{\mu\omega(\tau)}} + 4\pi (f(X_\mu(\tau)) f''(X_\mu(\tau)) + f'(X_\mu(\tau))^2) \right).$$

Finally, we compute

$$\frac{-2\mu\partial_\mu(\mathcal{E}_{1,1,\mu,r,\ell})(\tau)}{\mathcal{E}_{1,1,\mu,r,\ell}(\tau)} = -2 - 4\pi\sqrt{\frac{\mu\omega}{\tau_2}} \frac{f'(X_\mu(\tau))}{f(X_\mu(\tau))}.$$

Combining gives the claim, without using any properties of f . □

We are writing Proposition 5.1 in the form above to make the relation to the undeformed modular graph function explicit. In terms of the massive Maass forms in Theorem 3.6, we move the μ -terms in the differential operator to the right, namely

$$\Delta_{\tau,0}(\mathcal{E}_{1,1,\mu})(0; \tau) = (-\mu^2\partial_\mu^2 + 2\mu\partial_\mu - 2) \mathcal{E}_{1,1,\mu}(0; \tau),$$

so we identify $g_2(\mu) = -\mu^2$, $g_1(\mu) = 2\mu$, and $g_0(\mu) = -2$. In the proof of Theorem 3.6, we have $g_2(\mu) = -b^2\mu^2$, $g_1(\mu) = -(b^2 + 2bc + b)\mu$, and $g_0(\mu) = -c^2 - c$, so this corresponds to $(b, c) = (1, -2)$, since we are requiring $b > 0$. But Theorem 3.6 holds for $f_{\mu,r,\ell}(\tau) = \frac{|r\tau + \ell|^{2c}}{\tau_2^c} h(\frac{\mu|r\tau + \ell|^{2b}}{\tau_2^b})$ for any h , and the proof of Proposition 5.1 holds for any f , so there is the question of equivalence relations as discussed in Section 3.2. Indeed, the example we are discussing below Theorem 3.6 has K_2 in the double sum for $\mathcal{E}_{2,\mu}(0; \tau)$,

not K_1^2 as we have here in $\mathcal{E}_{1,1,\mu}(0; \tau)$ (hence the notation “1,1”). There are many relations between Bessel functions, e.g. recurrence relations, that need to be taken into account in a systematic study of connections between massive Maass forms and massive modular graph functions. The objective here is just to provide one entry point into those connections, and we leave such investigations to the future.

5.2. Helmholtz equation

The ordinary meaning of the “massive” Laplace equation is the Helmholtz equation, i.e., its Green’s function is a solution of

$$(2\partial_z\partial_{\bar{z}} - m^2) G(z; \tau) = -\delta^{[2]}(z; \tau).$$

The differential operator $2\partial_z\partial_{\bar{z}}$ is not invariant under the Jacobi group, since e.g. under an S transformation, $z \mapsto \frac{z}{\tau}$. Multiplying by τ_2 brings the differential operator to our $\Delta_{z,0,0} = 2\tau_2\partial_z\partial_{\bar{z}}$, and m^2 is completed to the invariant $\tau_2 m^2 = \mu$. Finding the solution for $G(z; \tau)$ as a double sum is a standard exercise if $\tau_1 = 0$, see e.g. Appendix E of [15]. Using the basis functions $e^{2\pi i(r\beta - \ell\alpha)}$ as for $\mathbb{E}(z; \tau)$, we find that in coordinates $z = \alpha\tau + \beta$, we have $\Delta_{z,0,0} = 2\tau_2\partial_z\partial_{\bar{z}} = \frac{1}{\tau_2}(\tau_2^2\partial_\beta^2 + \partial_\alpha^2)$, and we multiply the differential operator with τ_2 to find

$$(26) \quad G(z; \tau) := \sum_{(r,\ell) \in \mathbb{Z}^2} \frac{e^{2\pi i(r\beta - \ell\alpha)}}{4\pi^2(r^2\tau_2^2 + \ell^2) + \mu}.$$

Integrating in α and β gives

$$\int_0^1 \int_0^1 G(z; \tau) d\alpha d\beta = \frac{1}{\mu}.$$

In physics, one may want to normalize this to unity by multiplying $G(z; \tau)$ by μ , but we do not do so here. Note that if we remove the term $(r, \ell) = (0, 0)$ and set $\mu = 0$, then we obtain the Kronecker–Eisenstein series $\mathbb{E}_1(z; i\tau_2)$, up to normalization. Following e.g. Appendix E of [15], we can write the summand in (26) as an integral over a parameter s , then do modular inversion of the sum over ℓ , and evaluate the integral in s . For $\mu = 0$ this gives a logarithm of $|\vartheta_1(z; i\tau_2)|$, as in Kronecker’s second limit formula (33). The result for $\mu \neq 0$ is

$$G(z; \tau) = \sum_{(r,\ell) \in \mathbb{Z}^2} \frac{e^{-2\pi\tau_2\sqrt{4\pi^2r^2\tau_2^2 + \mu}|\ell - \alpha|} e^{2\pi ir\beta}}{2\sqrt{4\pi^2r^2\tau_2^2 + \mu}}.$$

The inversion followed by the integration in s caused r and ℓ to play different roles in the summation. This is not surprising, as we single out the sum on ℓ by hand. In particular, if we now let $\mu \rightarrow 0^+$, then in terms with $r = 0$, the dependence on ℓ and β is trivial which causes the sum on ℓ to diverge, even though that is not the case before those manipulations. One way to regularize is to subtract the massive point-particle Green's function $G_{\text{particle}}(\beta) := \frac{\cosh(m(|\beta| - \pi\tau_2))}{m \sinh(\pi m \tau_2)}$, which is also (quadratically) divergent as $m \rightarrow 0$. Related discussions appear e.g. in [21] Appendix A. The appearance of the divergence is similar to the elementary sum

$$(27) \quad \sum_{\ell \geq 1} \frac{1}{\ell^2 + m^2} = \frac{\pi \coth(\pi m)}{2m} - \frac{1}{2m^2}.$$

The limit of the “coth” term does not exist for $m \rightarrow 0$, but clearly the left-hand side is not divergent, and indeed the term $\frac{1}{2m^2}$ subtracts the principal part. This is close in spirit to (24). There, we “subtract” the entire string Green's function $\mathbb{E}_1(0, z; \tau)$, not just the point-particle Green's function G_{particle} . (“Subtract” in quotation marks since just like in (27) we merely split up an expression in two parts that by themselves would be divergent, we did not subtract anything by hand.)

Going further back in the literature, Sugawara [25] computed the partition function of the gravitational plane wave from the functional integral, in his equation (2.34)

$$Z_m(\tau) = \frac{1}{\prod_{r, \ell \in \mathbb{Z}} \tau_2 (\tau_2^{-2} |r\tau + \ell|^2 + m^2)}$$

which implies that

$$\text{Log}(Z_m(\tau)) = - \sum_{r, \ell \in \mathbb{Z}} \log(\tau_2 (\tau_2^{-2} |r\tau + \ell|^2 + m^2)) .$$

If we regulate this expression by differentiating twice with respect to m , it looks like the double sum representation in (25). But as expected, this formal partition function calculation does not tell us how to traverse the divergence. In (23), we use a twist regularization and analytic continuation.

Finally, we note that there is an extensive literature on related topics. For example, in statistical field theory, similar objects were used to compute renormalization group dependence of the conformal field theory central charge [12, 22]. The Helmholtz equation with periodic boundary conditions has many other applications, e.g. in waveguide physics (see e.g. [17]).

6. Outlook

It would be interesting to continue the discussion of modular graph functions (and forms) from Section 5.1. This should be feasible using the tools given in this paper.

We now give another source for constructing more examples in the future. Note that we can interpret the proof of (2) in [5] as giving the expression

$$(28) \quad \log(\mathcal{F}_m(\tau_2)) = 2\pi\tau_2 c_m + \frac{2\pi}{\tau_2} c_{m\tau_2} - \frac{1}{4} \int_0^\infty e^{-\frac{\pi\tau_2 m^2}{s}} (\vartheta_3(i\tau_2 s) - 1) \left(\vartheta_3\left(\frac{is}{\tau_2}\right) - 1 \right) ds,$$

where $\vartheta_3(\tau) := \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{2}}$. Equation (28) is invariant under the simultaneous transformations $\tau_2 \mapsto \frac{1}{\tau_2}$, $m \mapsto m\tau_2$. An obvious generalisation of (28) is to replace ϑ_3 with functions f, g satisfying, for all $\tau_2 \in \mathbb{R}^+$,

$$f\left(\frac{i}{\tau_2}\right) = \tau_2^k F(i\tau_2), \quad g\left(\frac{i}{\tau_2}\right) = t^k G(i\tau_2)$$

for some $k \in \frac{1}{2}\mathbb{Z}$ and any functions F, G . If f and g are Jacobi theta functions with characteristic, then we recover the special case $\tau = i\tau_2$ of $Z_{\alpha,\beta,m}(\tau)$. It might be possible to construct new massive Maass forms by a clever choice of f, g , but we do not investigate this question in this paper.

Another question is: can we embed the massive deformation in some more general and familiar framework? We believe that there is a sense in which the massive deformations for $SL_2(\mathbb{R})$ “sit” inside automorphic forms for more general Lie groups. We give some remarks on this without any pretense of rigor. The idea is that the results in the main text of this paper may be reproduced and generalized by what is called warped Kaluza–Klein reduction in physics, for example from ordinary (massless) automorphic forms on $SL_3(\mathbb{R})$ or $Sp_2(\mathbb{R})$ to massive automorphic forms on $SL_2(\mathbb{R})$. To illustrate this, let us view the usual Fourier expansion of the non-holomorphic Eisenstein series for $SL_2(\mathbb{R})$ as a warped Kaluza–Klein reduction to massive automorphic form on $SL_2(\mathbb{R})/(\tau_1 \sim \tau_1 + 1)$. This means we make an Ansatz for each nonzero-mode part of its Fourier expansion that $\mathbb{E}_s|_{\tau_1 \text{ piece}} \propto f_m(\tau_2)e^{2\pi im\tau_1}$, so that the Laplacian in τ yields

$$\begin{aligned} \Delta_{\tau,0}(\mathbb{E}_s)|_{\tau_1 \text{ piece}} &\supset -\tau_2^2 (\partial_{\tau_2}^2 + \partial_{\tau_1}^2) f_m(\tau_2)e^{2\pi im\tau_1} \\ &= -\tau_2^2 (\partial_{\tau_2}^2 - 4\pi^2 m^2) f_m(\tau_2)e^{2\pi im\tau_1} \end{aligned}$$

which can be viewed as a “massive” one-dimensional differential operator in τ_2 only. It is not truly the Helmholtz operator, since the “mass” when multiplying out is $4\pi^2 m^2 \tau_2^2$, i.e., it depends on the vertical position τ_2 in the upper half-plane in these coordinates. Demanding that \mathbb{E}_s is an eigenfunction of $\Delta_{\tau,0}$ with eigenvalue $-s(s-1)$ produces a differential equation purely in τ_2 ,

$$\tau_2^2 (\partial_{\tau_2}^2 - 4\pi^2 m^2) f_m(\tau_2) = s(s-1) f_m(\tau_2)$$

which is a Bessel differential equation like (15). Demanding that $f_m(\tau_2)$ falls off as $\tau_2 \rightarrow \infty$ yields specifically that $f_m(\tau_2) = c_m \sqrt{\tau_2} K_{s-\frac{1}{2}}(2\pi m \tau_2)$, with some constant $c_m \in \mathbb{C}$. This is the standard result for the non-zero-mode Fourier expansion of the usual $\mathrm{SL}_2(\mathbb{R})$ Eisenstein series. We call this reduction from $\mathrm{SL}_2(\mathbb{R})$ to $\mathrm{SL}_2(\mathbb{R})/(\tau_1 \sim \tau_1 + 1)$ “warped” essentially because the “mass” $4\pi^2 m^2 \tau_2^2$ depends on τ_2 , unlike in the Helmholtz equation. One can find a coordinate system on the upper half-plane where the mass is constant, but in this (trivial) example it is not necessary to do so to find $f_m(\tau_2)$.

Pursuing this further is beyond the scope of this paper, but a first glimpse can be seen in Kiritsis–Pioline [13], Appendix A, with the Eisenstein series for $\mathrm{SL}_3(\mathbb{R})$. The non-zero-mode terms in their equation (A.4) resemble our Proposition 3.3. It would also be interesting to study the connection to Niebur–Poincaré series [1], where Siegel–Narain theta functions provide multi-parameter families of automorphic forms. The parameters in the Siegel–Narain theta function comprise supersymmetric Calabi–Yau moduli space with zero flux. The gravitational wave background has nonzero flux that produces the worldsheet mass term, so in that sense mass deformation in this paper can be thought of as a more drastic change than moving around in Calabi–Yau moduli space.

Appendix A. Review of the Kronecker–Eisenstein series

$$\mathbb{E}_s(\mathbf{w}, z; \tau)$$

In Appendix A, we review material of Appendix E in [4]. We begin with the question of finding a (non-holomorphic) function $\mathbb{E}_s : \mathbb{C}^2 \times \mathbb{H} \rightarrow \mathbb{C}$ that depends on a parameter $s \in \mathbb{C}$ and is doubly periodic on the torus in the first variable w

$$(29) \quad \mathbb{E}_s(w+1, z; \tau) = \mathbb{E}_s(w, z; \tau), \quad \mathbb{E}_s(w+\tau, z; \tau) = \mathbb{E}_s(w, z; \tau).$$

In the second variable z it should be quasiperiodic, namely

$$(30) \quad \mathbb{E}_s(w, z+1; \tau) = e^{2\pi i A} \mathbb{E}_s(w, z; \tau), \quad \mathbb{E}_s(w, z+\tau; \tau) = e^{-2\pi i B} \mathbb{E}_s(w, z; \tau),$$

where $w = A\tau + B$ ($A, B \in \mathbb{R}$). In physics, expressions corresponding to Feynman graphs are composed of Green’s function of the Laplace equation. For the torus, allowing characteristics as in (30), we have

$$(31) \quad \Delta_{z,0,0}(G(z; \tau)) = -2\pi e^{\frac{2\pi i}{\tau_2} \text{Im}(w\bar{z})} \delta^{[2]}(z; \tau).$$

The factor in front of $\delta^{[2]}(z; \tau)$ ensures compatibility with quasiperiodicity.

We now define the *Kronecker–Eisenstein series*³ ($s \in \mathbb{C}$ with $\text{Re}(s) \geq 1$)

$$(32) \quad \begin{aligned} \mathbb{E}_s(w, z; \tau) &:= \Gamma(s) \left(\frac{\tau_2}{\pi}\right)^s \sum_{r,\ell}^* \frac{e^{\frac{2\pi i}{\tau_2} \text{Im}((w+r\tau+\ell)\bar{z})}}{|w+r\tau+\ell|^{2s}} \\ &= \Gamma(s) \left(\frac{\tau_2}{\pi}\right)^s \sum_{r,\ell}^* \frac{e^{2\pi i(r+A)\beta - 2\pi i(\ell+B)\alpha}}{|(r+A)\tau+\ell+B|^{2s}}, \end{aligned}$$

where the summation Σ^* indicates that the sum runs over all $(r, \ell) \in \mathbb{Z}^2$ such that $w + r\tau + \ell \neq 0$. It is not hard to see that $\mathbb{E}_s(w, z; \tau)$ satisfies the transformations in (29) and (30). For the special case $w = 0$ and $s = 1$, it is also easy to see that $\mathbb{E}_1(0, z; \tau)$ does not quite satisfy (31) but rather (9), with an extra term that is required for the right-hand side to yield zero when integrated over a fundamental domain in z . When $w \neq 0$ there is quasiperiodicity as in (30), and the Laplace operator no longer integrates to zero as it does for a doubly periodic function. That is why the extra term on the right-hand side is not needed in (31).

Note that, by Kronecker’s second limit formula, $\mathbb{E}_1(0, z; \tau)$ can alternatively be represented as (see e.g. Chapter 20 of [14] or Section 5 of [23])

$$(33) \quad \mathbb{E}_1(0, z; \tau) = -\log \left(\left| \frac{\vartheta_1(z; \tau)}{\eta(\tau)} \right|^2 \right) + \frac{2\pi z_2^2}{\tau_2}.$$

The double sum (32) is only absolutely convergent for $\text{Re}(s) > 1$, but an analytic continuation to all complex s can be found as an integral representation. If either z or w are lattice points there are additional pole terms at $s = 0$ or $s = 1$ in this integral representation, that are written out in Appendix E of [4], but in this calculation we for simplicity stay away from

³This has an additional $\Gamma(s)$ as compared to [4], which makes it “completed” in the sense of the reflection formula (35) below.

lattice points. The integral (Mellin) representation is then found from

$$\mathbb{E}_s(w, z; \tau) = \int_0^\infty x^{s-1} \sum_{r, \ell}^* e^{-\frac{\pi x}{\tau_2} |w+r\tau+\ell|^2 + \frac{2\pi i}{\tau_2} \operatorname{Im}((w+r\tau+\ell)\bar{z})} dx,$$

which is valid for $\operatorname{Re}(s) > 1$. The sum is exponentially decaying for $x \rightarrow \infty$ but for $s \leq 1$ it is potentially divergent towards $x \rightarrow 0$. We follow the approach of Riemann, namely to split the integral into one from 0 to 1 and one from 1 to ∞ , then use the modular transformation of a theta function on the first piece and change variables $x \mapsto \frac{1}{x}$ to obtain

$$(34) \quad \mathbb{E}_s(w, z; \tau) = e^{\frac{2\pi i}{\tau_2} \operatorname{Im}(w\bar{z})} \int_1^\infty x^{-s} \sum_{r, \ell}^* e^{-\frac{\pi x}{\tau_2} |z+r\tau+\ell|^2 + \frac{2\pi i}{\tau_2} \operatorname{Im}((z+r\tau+\ell)\bar{w})} dx \\ + \int_1^\infty x^{s-1} \sum_{r, \ell}^* e^{-\frac{\pi x}{\tau_2} |w+r\tau+\ell|^2 + \frac{2\pi i}{\tau_2} \operatorname{Im}((w+r\tau+\ell)\bar{z})} dx.$$

Although we are originally assuming that $\operatorname{Re}(s) > 1$, this integral representation gives an analytic continuation (in s) of $\mathbb{E}_s(w, z; \tau)$. Moreover it directly implies a symmetry under $s \mapsto 1 - s$,

$$(35) \quad \mathbb{E}_s(w, z; \tau) = e^{\frac{2\pi i}{\tau_2} \operatorname{Im}(w\bar{z})} \mathbb{E}_{1-s}(z, w; \tau)$$

which is the functional relation (reflection formula) for $\mathbb{E}_s(w, z; \tau)$. Note that the variables w and z are switched, which gives a motivation to allow $w \neq 0$ in the first place. Note that (as is familiar from discussions of L-functions, but here $\mathbb{E}_s(w, z; \tau)$ depends on τ) the two sides of (35) never simultaneously have convergent double sum representations, and that the reflection formula does not give any information on the behavior of the double sums in the strip $0 < \operatorname{Re}(s) < 1$, which is instead provided by the integral representation.

If $z \notin \mathbb{Z}\tau + \mathbb{Z}$, $\mathbb{E}_s(w, z; \tau)$ satisfies the partial differential equation where we view z and τ as independent variables,

$$(36) \quad \Delta_{z,0,0}(\mathbb{E}_s(w, z; \tau)) = -2\pi(s-1)\mathbb{E}_{s-1}(w, z; \tau).$$

The “twisted” (quasiperiodic) Kronecker–Eisenstein series has a factor in front of the delta function that allows for quasiperiodicity:

$$(37) \quad \Delta_{z,0,0}(\mathbb{E}_1(w, z; \tau)) = -2\pi e^{\frac{2\pi i}{\tau_2} \operatorname{Im}(w\bar{z})} \delta^{[2]}(z; \tau).$$

A formal power series representation of the lattice delta function is

$$(38) \quad \delta^{[2]}(z; \tau) = \sum_{r, \ell \in \mathbb{Z}} e^{2\pi i(r\beta - \ell\alpha)}.$$

The factor in front of the lattice delta function in (37) may seem inconsequential, since the right-hand side is zero away from lattice points and at $z = 0$ the factor is one, but it can be nontrivial at lattice points away from the origin. We do not attempt to write nonzero-index distributions in detail here.

Note that $\mathbb{E}_s(z, w; \tau)$ has weight zero, but it can be used to generate objects with non-zero weight, e.g. those called $\mathbb{E}_{s,k}(z, w; \tau)$ in Appendix E of [4]. We do not discuss them here.

Acknowledgements

The authors thank the referees for their useful comments.

References

- [1] C. Angelantonj, I. Florakis, and B. Pioline, *Threshold corrections, generalised prepotentials and Eichler integrals*, Nucl. Phys. B **897** (2015), 781. [MR3365316](#)
- [2] H. Bateman, *Tables of Integral Transforms*, McGraw-Hill, 1954. Available at authors.library.caltech.edu/43489/.
- [3] D. Berenstein, J. Maldacena, and H. Nastase, *Strings in flat space and pp waves from $N=4$ superYang-Mills*, J. High Energy Phys. **4** (2002). [MR1911426](#)
- [4] M. Berg, M. Haack, J. Kang, and S. Sjörs, *Towards the one-loop Kähler metric of Calabi–Yau orientifolds*, J. High Energy Phys. **12** (2014). [MR3303516](#)
- [5] O. Bergman, M. Gaberdiel, and M. Green, *D-brane interactions in type II B plane-wave background*, J. High Energy Phys. **3** (2003). [MR1976014](#)
- [6] K. Bringmann and O. Richter, *Zagier-type dualities and lifting maps for harmonic Maass–Jacobi forms*, Adv. Math. **225** (2010), 2298–2315. [MR2680205](#)
- [7] J. Broedel, C. Mafra, N. Matthes, and O. Schlotterer, *Elliptic multiple zeta values and one-loop superstring amplitudes* J. High Energy Phys. **7** (2015). [MR3383100](#)

- [8] E. D'Hoker, M. Green, Ö. Gürdogan, and P. Vanhove, *Modular Graph Functions*, Commun. Number Theory Phys. **11** (2017), 165–218. [MR3668971](#)
- [9] E. D'Hoker, M. Green, and P. Vanhove, *On the modular structure of the genus-one Type II superstring low energy expansion*, J. High Energy Phys. **8** (2015). [MR3402124](#)
- [10] P. Fleig, H. Gustafsson, A. Kleinschmidt, and D. Persson, *Eisenstein Series and Automorphic Representations: With Applications in String Theory*, Cambridge University Press, Cambridge, 2018, 567. [MR3793195](#)
- [11] M. Headrick, *A solution manual for Polchinski's 'String Theory'*, [arXiv:0812.4408 \[hep-th\]](#).
- [12] C. Itzykson and J. Drouffe, *Statistical Field Theory. Vol. 2: Strong Coupling, Monte Carlo Methods, Conformal Field Theory, And Random Systems*, Cambridge Monographs on Mathematical Physics, 1989, 405–810. [MR1175177](#)
- [13] E. Kiritsis and B. Pioline, *On R^4 threshold corrections in IIB string theory and (p, q) string instantons*, Nuclear Phys. B **508** (1997), 509–534. [MR1600063](#)
- [14] S. Lang, *Elliptic functions*, Graduate Texts in Mathematics **112**, Springer-Verlag, New York, 1987. [MR0890960](#)
- [15] K. Lee and W. Siegel, *Simpler superstring scattering*, J. High Energy Phys. **6** (2006). [MR2233809](#)
- [16] W. Lerche, B. Nilsson, A. Schellekens, and N. Warner, *Anomaly Cancelling Terms From the Elliptic Genus*, Nuclear Phys. B **299** (1988), 91–116. [MR0928107](#)
- [17] C. Linton, *The Green's Function for the Two-Dimensional Helmholtz Equation in Periodic Domains*, J. Engineering Math. **33** (1998), 377–402. [MR1638860](#)
- [18] NIST Digital Library of Mathematical functions, <http://dlmf.nist.gov>.
- [19] R. Penrose, *Any space-time has a plane wave as a limit* in M. Cahen, M. Flato, Differential Geometry and Relativity. Mathematical Physics and Applied Mathematics, vol 3. Springer, Dordrecht, 1976. [MR0446354](#)
- [20] A. Pitale, *Jacobi Maaß forms*, Abh. Math. Semin. Univ. Hambg. **79** (2009), 87–111. [MR2541345](#)

- [21] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*, Cambridge Monographs on Mathematical Physics, 1998. [MR1648555](#)
- [22] H. Saleur and C. Itzykson, *Two-dimensional field theories close to criticality*, J. Stat. Phys. **48** (1987), 449–475. [MR0914892](#)
- [23] C. Siegel, *Lectures on advanced analytic number theory*, Tata institute, 1961. [MR0262150](#)
- [24] S. Stieberger and T. Taylor, *Non-abelian Born-Infeld action and type I-heterotic duality. II. Nonrenormalization theorems*, Nuclear Phys. B **648** (2003), 3–34. [MR1950178](#)
- [25] Y. Sugawara, *Thermal amplitudes in DLCQ superstrings on PP waves*, Nuclear Phys. B **650** (2003), 75–113. [MR1954580](#)
- [26] T. Takayanagi, *Modular invariance of strings on PP-waves with RR-flux*, J. High Energy Phys. **22** (2002). [MR1960315](#)

MARCUS BERG
DEPARTMENT OF PHYSICS
KARLSTAD UNIVERSITY
651 88 KARLSTAD
SWEDEN
E-mail address: marcus.berg@kau.se

KATHRIN BRINGMANN
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY OF COLOGNE
WEYERTAL 86-90, 50931 COLOGNE
GERMANY
E-mail address: kbringma@math.uni-koeln.de

TERRY GANNON
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON
CANADA T6G 2G1
E-mail address: tjgannon@ualberta.ca

RECEIVED OCTOBER 24, 2019

ACCEPTED MARCH 28, 2021