ON DIVISORS OF MODULAR FORMS

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In celebration of Don Zagier's 65th birthday.

Abstract. The *denominator formula* for the Monster Lie algebra is the product expansion for the modular function $J(z)-J(\tau)$ given in terms of the Hecke system of $\mathrm{SL}_2(\mathbb{Z})$-modular functions $j_n(\tau)$. It is prominent in Zagier's seminal paper on traces of singular moduli, and in the Duncan-Frenkel work on Moonshine. The formula is equivalent to the description of the generating function for the $j_n(z)$ as a weight 2 modular form with a pole at z. Although these results rely on the fact that $X_0(1)$ has genus 0, here we obtain a generalization, framed in terms of polar harmonic Maass forms, for all of the $X_0(N)$ modular curves. We use these functions to study divisors of modular forms.

1. Introduction and statement of results

As usual, let $J(\tau)$ be the $\mathrm{SL}_2(\mathbb{Z})$ Hauptmodul defined by

$$J(\tau)=\sum_{n=-1}^{\infty}c(n)e^{2\pi in\tau}:=\frac{E_4(\tau)^3}{\Delta(\tau)}-744=e^{-2\pi i\tau}+196884e^{2\pi i\tau}+\cdots,$$

where $E_k(\tau):=1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi in\tau}$ is the weight $k\in 2\mathbb{N}$ Eisenstein series, $\sigma_\ell(n):=\sum_{d|\ell}d^\ell$, B_k is the kth Bernoulli number, and $\Delta(\tau):=(E_4(\tau)^3-E_6(\tau)^2)/1728$. By Moonshine (for example, see [14]), $J(\tau)$ is the McKay-Thompson series for the identity (i.e., its coefficients are the graded dimensions of the Monster module $V^\natural$). Moonshine also offers the striking infinite product

$$J(z)-J(\tau)=e^{-2\pi iz}\prod_{m>0,\ n\in\mathbb{Z}}\left(1-e^{2\pi imz}e^{2\pi in\tau}\right)^{c(mn)},$$

the *denominator formula* for the Monster Lie algebra. Here we let $\tau,z\in\mathbb{H}$. This formula is equivalent to the following identity of Asai, Kaneko, and Ninomiya (see Theorem 3 of [2])

$$H_z(\tau):=\sum_{n=0}^{\infty}j_n(z)e^{2\pi in\tau}=\frac{E_4(\tau)^2E_6(\tau)}{\Delta(\tau)}\frac{1}{J(\tau)-J(z)}=-\frac{1}{2\pi i}\frac{J'(\tau)}{J(\tau)-J(z)}. \tag{1.1}$$

The functions $j_n(\tau)$ form a Hecke system. Namely, if we let $j_0(\tau):=1$ and $j_1(\tau):=J(\tau)$, then the others are obtained by applying the normalized Hecke operator $T(n)$

$$j_n(\tau):=j_1(\tau)\mid T(n). \tag{1.2}$$

Remark. The functions $H_z(\tau)$ and $j_n(\tau)$ played central roles in Zagier's [20] seminal paper on traces of singular moduli and the Duncan-Frenkel work [13] on the Moonshine Tower. Carnahan [10] has obtained similar denominator formulas for completely replicable modular functions.

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If $z \in \mathbb{H}$, then $H_z(\tau)$ is a weight 2 meromorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ with a single pole (modulo $\mathrm{SL}_2(\mathbb{Z})$) at the point z. Using these functions, the *divisor modular form* of a normalized weight k meromorphic modular form $f(\tau)$ on $\mathrm{SL}_2(\mathbb{Z})$ was defined in [9] as[1]

$$f^{\mathrm{div}}(\tau) := \sum_{z\in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} e_z \mathrm{ord}_z(f) H_z(\tau), \tag{1.3}$$

where $e_z := 2/\#\mathrm{Stab}_z\left(\mathrm{SL}_2(\mathbb{Z})\right)$ With $\Theta := \frac{1}{2\pi i}\frac{d}{d\tau}$, Theorem 1 of [9] asserts that

$$f^{\mathrm{div}}(\tau) = -\frac{\Theta(f(\tau))}{f(\tau)} + \frac{kE_2(\tau)}{12}. \tag{1.4}$$

Although these results rely on the fact that $X_0(1)$ has genus 0, there is a natural extension for congruence subgroups. This extension requires polar harmonic Maass forms, which are harmonic Maass forms with poles in the upper half-plane (see [5] for details). Here we consider the modular curves $X_0(N)$. For $n \in \mathbb{N}$, we define a Hecke system of $\Gamma_0(N)$ harmonic Maass functions $j_{N,n}(\tau)$ in Section 3 which generalize the $j_n(\tau)$.

In Section 2 we construct weight 2 polar harmonic Maass forms $H^*_{N,z}(\tau)$ which generalize the $H_z(\tau)$. We have two cases for the $H^*_{N,z}(\tau)$, according to whether $z \in \mathbb{H}$ or z is a cusp, which we consider separately. The following theorem summarizes the essential properties of these functions when $z \in \mathbb{H}$.

Theorem 1.1. *If $z \in \mathbb{H}$, then $H^*_{N,z}(\tau)$ is a weight 2 polar harmonic Maass form on $\Gamma_0(N)$ which vanishes at all cusps and has a single simple pole at z. Moreover, the following are true:*

(1) *If $z \in \mathbb{H}$ and $\mathrm{Im}(\tau) > \max\{\mathrm{Im}(z), \frac{1}{\mathrm{Im}(z)}\}$, then we have that*

$$H^*_{N,z}(\tau) = \frac{3}{\pi\left[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)\right]\mathrm{Im}(\tau)} + \sum_{n=1}^{\infty} j_{N,n}(z) e^{2\pi i n\tau}.$$

(2) *For $\gcd(N, n) = 1$, we have $j_{N,n}(\tau) = j_{N,1}(\tau) \mid T(n)$.*
(3) *For $n \mid N$, we have $j_{N,n}(\tau) = j_{\frac{N}{n},1}(n\tau)$.*
(4) *As $n \to \infty$, we have*

$$j_{N,n}(\tau) = \sum_{\substack{\lambda\in\Lambda_\tau \\ \lambda\le n}} \sum_{(c,d)\in S_\lambda} e\left(-\frac{n}{\lambda} r_\tau(c,d)\right) e^{\frac{2\pi n \mathrm{Im}(\tau)}{\lambda}} + O_\tau(n) \tag{1.5}$$

for some real numbers $r_\tau(c, d)$ (see (3.2)), Λ_τ a lattice in $\mathbb{R}$ (see (3.3)), and S_λ the set of solutions to $Q_\tau(c, d) = \lambda$ for a certain positive-definite binary quadratic form Q_τ (see (3.4)).

Four Remarks.
(1) In Theorem 1.1 (1), the inequality on $\mathrm{Im}(\tau)$ is required for convergence.
(2) For $N = 1$, we have that $H^*_{1,z}(\tau) = H_z(\tau) - E^*_2(\tau)$, where $E^*_2(\tau) := -\frac{3}{\pi\mathrm{Im}(\tau)} + E_2(\tau)$ is the usual weight 2 nonholomorphic Eisenstein series, and we have that $j_{1,n}(\tau) = j_n(\tau) + 24\sigma_1(n)$.
(3) The sums (1.5) were introduced by Hardy and Ramanujan [15] (see also [3, 4]) to study the Fourier coefficients of $1/E_6$. Their formulas have been generalized [7, 8] to negative weight meromorphic modular forms. Theorem 1.1 (4) extends these results to weight 0 where the series are not absolutely convergent.

[1] Note that this summation does not include the cusp $i\infty$.

(4) Theorem 1.1 (4) gives asymptotics for $j_{N,n}(z)$ in the n-aspect. If $\mathrm{Im}(z) \geq \mathrm{Im}(Mz)$ for all $M \in \Gamma_0(N)$, then

$$j_{N,n}(z) = e^{-2\pi i n z} + \sum_{\substack{c \geq 1 \\ N|c}} \sum_{\substack{d \in \mathbb{Z} \\ \gcd(c,d)=1 \\ |cz+d|^2=1}} e\left(n\frac{d-a}{c}\right) e^{2\pi i n \overline{z}} + O_z(n) \tag{1.6}$$

as $n \to \infty$.

The second case we consider are those $H^*_{N,\rho}(\tau)$ where ρ is a cusp of $X_0(N)$. These functions are compatible with the $H^*_{N,z}(\tau)$ considered in Theorem 1.1. More precisely, since $z \mapsto H^*_{N,z}(\tau)$ is continuous (even harmonic) and $\Gamma_0(N)$-invariant, it follows that

$$H^*_{N,\rho}(\tau) := \lim_{z \to \rho} H^*_{N,z}(\tau) \tag{1.7}$$

is well-defined and only depends on the equivalence class of ρ. The next result summarizes these functions' properties. We use the Kloosterman sums $K_{i\infty,\rho}(0,-n;c)$ of (2.4) and the weight 2 harmonic Eisenstein series $E^*_{2,N,\rho}(\tau)$ for $\Gamma_0(N)$ defined in Section 2. These have constant term 1 at ρ and vanish at all other cusps.

Theorem 1.2. *We have that $H^*_{N,\rho}(\tau) = -E^*_{2,N,\rho}(\tau)$. Moreover, the following are true:*

(1) *We have*

$$H^*_{N,\rho}(\tau) = \frac{3}{\pi\,[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]\,\mathrm{Im}(\tau)} - \delta_{\rho,\infty} + \sum_{n=1}^{\infty} j_{N,n}(\rho) e^{2\pi i n \tau}, \qquad \textit{with}$$

$$j_{N,n}(\rho) := \lim_{\tau \to \rho} j_{N,n}(\tau) = \frac{4\pi^2 n}{\ell_\rho} \sum_{\substack{c \geq 1 \\ N|c}} \frac{K_{i\infty,\rho}(0,-n;c)}{c^2},$$

where ℓ_ρ denotes the cusp width of ρ and $\delta_{\rho,\infty} := 1$ if $\rho = i\infty$ and 0 otherwise.

(2) *For $\gcd(N,n) = 1$, we have $j_{N,n}(\rho) = \lim_{\tau \to \rho} j_{N,1}(\tau) \mid T(n)$.*

(3) *For $n \mid N$, we have $j_{N,n}(\rho) = \lim_{\tau \to \rho} j_{\frac{N}{n},1}(n\tau)$.*

Two Remarks.

(1) Recall that the Fourier expansion in Theorem 1.1 (1) is not valid as $z \to i\infty$.

(2) The $j_{N,n}(\rho)$ are divisor sums, which we leave to the interested reader to verify. From a generalization of the Weil bound (3.9) one can obtain $j_{N,n}(\rho) = O(n^{\frac{3}{2}})$.

We turn to the task of extending (1.4) to generic $\Gamma_0(N)$. Suppose that f is a weight k meromorphic modular form on $\Gamma_0(N)$. In analogy with (1.3), we define the *divisor polar harmonic Maass form*

$$f^{\mathrm{div}}(\tau) := \sum_{z \in X_0(N)} e_{N,z} \mathrm{ord}_z(f) H^*_{N,z}(\tau), \tag{1.8}$$

where $e_{N,z} := 2/\#\mathrm{Stab}_z(\Gamma_0(N))$ and $e_{N,\rho} := 1$ when ρ is a cusp. Generalizing (1.4), we show the following.

Theorem 1.3. *If $S_2(\Gamma_0(N))$ denotes the space of weight 2 cusp forms on $\Gamma_0(N)$, then*

$$f^{\mathrm{div}}(\tau) \equiv \frac{k}{4\pi\,\mathrm{Im}(\tau)} - \frac{\Theta(f(\tau))}{f(\tau)} \pmod{S_2(\Gamma_0(N))}.$$

Three Remarks.
(1) The coefficient of $1/\operatorname{Im}(\tau)$ in $H^*_{N,z}(\tau)$ is independent of z. By the valence formula, summing over every element of $X_0(N)$ in the definition of $f^{\mathrm{div}}(\tau)$ multiplies this constant by $\frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]$, giving the nonholomorphic correction term on the right-hand side of Theorem 1.3.

(2) At first glance, definitions (1.3) and (1.8) might appear different for $N=1$. Indeed, $H^*_{1,z}(\tau)=H_z(\tau)-E^*_2(\tau)$, and the sum in (1.8) includes the cusp $i\infty$ whereas (1.3) omits it. The quasimodular Eisenstein series $E_2(\tau)$ in (1.4) and the valence formula guarantee that they coincide.

(3) The formula in Theorem 1.3 has already been obtained by Choi using a regularized inner product due to Petersson, but without relating the Fourier coefficients of f^{div} to the polar harmonic Maass forms $H^*_{N,z}$ (see Theorem 1.4 of [11]).

Theorem 1.3 can be used to numerically compute divisors of meromorphic modular forms $f(\tau)$, which, in general, is a difficult task (for example, see [12]). The series $-\frac{\Theta(f(\tau))}{f(\tau)}$ is the logarithmic derivative of $f(\tau)$, and this fact converts the points $z\in\mathbb{H}$ in the divisor of $f(\tau)$ into simple poles. These can be identified by the asymptotic properties of the coefficients of $H^*_{N,z}(\tau)$ given in Theorem 1.1. This follows from Theorem 1.3 and the fact that coefficients of cusp forms satisfy Deligne's bound. In the case of the modular functions $j(\tau)-\alpha$, where $\alpha\in\mathbb{C}$, this has been carried out recently by Alwaise [1]. The method is based on the following immediate corollary to Theorems 1.1–1.3.

Corollary 1.4. *Suppose that $f(\tau)$ is a meromorphic modular form of weight k on $\Gamma_0(N)$ whose divisor is not supported at cusps. Let y_1 be the largest imaginary part of any points in the divisor of $f(\tau)$ lying in $\mathbb{H}$. Then if $-\frac{\Theta(f(\tau))}{f(\tau)}=:\sum_{n\gg-\infty}a(n)q^n$ $(q=e^{2\pi i\tau})$, we have that*

$$y_1=\limsup_{n\to\infty}\frac{\log|a(n)|}{2\pi n}.$$

Two Remarks.
(1) We require lim sup in Corollary 1.4 because the $a(n)$ can vanish on arithmetic progressions.

(2) It would be interesting to develop a practical algorithm for numerically computing modular form divisors. The idea would be to carefully peel away poles of $f^{\mathrm{div}}(\tau)$ in descending order until one is left with a linear combination of functions $E^*_{N,\rho}(\tau)$.

Example 1. For the Eisenstein series $E_4(\tau)$, we have

$$-\frac{\Theta(E_4(\tau))}{E_4(\tau)}=-240q+53280q^2-12288960q^3+2835808320q^4-654403831200q^5+\cdots.$$

The sequence $\{b(n)\}_{n\geq1}=\{\log|a(n)|/(2\pi n)\}_{n\geq1}$ converges rapidly. Indeed, $b(2)=0.866066794\ldots$, and $b(10)=0.866025404\ldots$ matches the first 16 digits of the limiting value. The divisor of $E_4(\tau)$ is supported on a zero at $\omega:=(-1+\sqrt{-3})/2$. By (1.6), since ω lies on the unit circle (implying that the second term on the right-hand side of (1.6) appears) for large n, $a(n)$ should very nearly be $\frac{1}{3}\left(e^{-2\pi in\omega}+2e^{2\pi in\overline{\omega}}\right)=e^{-2\pi in\omega}$, which is very easily seen numerically.

Example 2. We consider $f(\tau):=E_4(2\tau)+\frac{\eta^{16}(2\tau)}{\eta^8(\tau)}$, where $\eta(\tau)$ is Dedekind's eta-function. By the valence formula for $\Gamma_0(2)$, it has a single zero, say z_0, in $X_0(2)$. We find that

$$-\frac{\Theta(f(\tau))}{f(\tau)}=-q-495q^2+659q^3+113233q^4-261211q^5+\cdots.$$

After the first 3000 terms the sequence $\log|a(n)|/(2\pi n)$ stabilizes and offers $\operatorname{Im}(z_0)\approx0.4357$. As $f(\tau)$ has real coefficients and there is only one zero, $-\overline{z}_0$ must be $\Gamma_0(2)$-equivalent to z_0. We choose the fundamental domain

$$\left\{z \in \mathbb{H}\colon -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2} \text{ and } \forall M \in \Gamma_0(2)\colon \left(\mathrm{Im}\left(Mz\right) \geq \mathrm{Im}(z) \text{ and } \mathrm{Im}\left(Mz\right) > \mathrm{Im}(z) \text{ if } \mathrm{Re}(z) < 0\right)\right\}.$$

Thus, either $\mathrm{Re}(z) \in \{0, \frac{1}{2}\}$, or z lies on the arc $|2z-1| = 1$. The first two cases are easily excluded by the sign patterns of $a(n)$, and the zero on the arc is easily approximated as $z_0 \approx 0.2547 + 0.4357i$.

This paper is organized as follows. In Section 2 we construct the weight 2 polar harmonic Maass forms $H^*_{N,z}(\tau)$. In Section 3 we relate their Fourier coefficients to the values of the weight 0 weak Maass forms at $\tau = z$, proving Theorems 1.1, 1.2, and 1.3.

2. Weight 2 Polar Harmonic Maass forms

2.1. The $H^*_{N,z}(\tau)$ when $z \in \mathbb{H}$. Define for $z, \tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$

$$P_{N,s}(\tau, z) := \sum_{M \in \Gamma_0(N)} \frac{\varphi_s\left(M\tau, z\right)}{j\left(M, \tau\right)^2 \left|j\left(M, \tau\right)\right|^{2s}} \tag{2.1}$$

with $j(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), \tau) := c\tau + d$ and

$$\varphi_s(\tau, z) := \left(\mathrm{Im}(z)\right)^{1+s} (\tau - z)^{-1} (\tau - \overline{z})^{-1} \left|\tau - \overline{z}\right|^{-2s}.$$

These functions were introduced and investigated in the z-variable in [6], where it was shown that these are *polar harmonic Maass forms*. These functions are allowed to have poles in the upper half plane instead of only at the cusps. In this paper, we are interested in properties of $P_{N,s}(\tau, z)$ as functions of τ. A direct calculation shows that for $L \in \Gamma_0(N)$

$$P_{N,s}\left(L\tau, z\right) = j\left(L, \tau\right)^2 \left|j\left(L, \tau\right)\right|^{2s} P_{N,s}(\tau, z).$$

In [6] it was shown, by a lengthy calculation, that the function $P_{N,s}(\tau, z)$ has an analytic continuation to $s = 0$, which we denote by $\mathrm{Im}(z)\Psi_{2,N}(\tau, z)$. Let $\mathcal{H}_k(\Gamma_0(N))$ be the space of weight k polar harmonic Maass forms with respect to $\Gamma_0(N)$. Lemma 4.4 of [6] then states that $z \mapsto \mathrm{Im}(z)\Psi_{2,N}(\tau, z) \in \mathcal{H}_0(\Gamma_0(N))$. In the τ variable, these functions are also polar harmonic Maass forms, as the next proposition shows. For this, for $w \in \mathbb{C}$, let $e(w) := e^{2\pi i w}$, and

$$K(m, n; c) := \sum_{\substack{a,d \pmod{c} \\ ad \equiv 1 \pmod{c}}} e\left(\frac{md + na}{c}\right).$$

Moreover, I_k and J_k denote the usual I- and J-Bessel functions. The following proposition can be obtained by a careful inspection of the proof of Theorem 3.1 of [6].

Proposition 2.1. *We have that* $\tau \mapsto \mathrm{Im}(z)\Psi_{2,N}(\tau, z) \in \mathcal{H}_2(\Gamma_0(N))$. *For* $\mathrm{Im}(\tau) > \max\{\mathrm{Im}(z), \frac{1}{\mathrm{Im}(z)}\}$, *its Fourier expansion (in τ) has the form*

$$\begin{aligned}
\mathrm{Im}(z)\Psi_{2,N}(\tau, z) = &-\frac{6}{\left[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)\right] \mathrm{Im}(\tau)} - 2\pi \sum_{m \geq 1} \left(e^{-2\pi i m z} - e^{-2\pi i m \overline{z}}\right) e^{2\pi i m \tau} \\
&- 4\pi^2 \sum_{m \geq 1} \sum_{\substack{n, c \geq 1 \\ N | c}} \sqrt{\frac{m}{n}} \frac{K(m, -n; c)}{c} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{2\pi i n z} e^{2\pi i m \tau} \\
&- 4\pi^2 \sum_{m \geq 1} \sum_{\substack{n, c \geq 1 \\ N | c}} \sqrt{\frac{m}{n}} \frac{K(m, n; c)}{c} J_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{-2\pi i n \overline{z}} e^{2\pi i m \tau} - 8\pi^3 \sum_{m \geq 1} m \sum_{\substack{c \geq 1 \\ N | c}} \frac{K(m, 0; c)}{c^2} e^{2\pi i m \tau}.
\end{aligned}$$

We then set

$$H^*_{N,z}(\tau) := -\frac{\operatorname{Im}(z)}{2\pi}\Psi_{2,N}(\tau, z). \tag{2.2}$$

Remark. We have, as $\tau \to z$,

$$H^*_{N,z}(\tau) = \frac{1}{2\pi i e_{N,z}}\frac{1}{\tau - z} + O(1) \tag{2.3}$$

with $e_{N,z}$ as defined after (1.8).

2.2. **The $H^*_{N,z}(\tau)$ for cusps.** We require the Fourier expansion of the functions $H^*_{N,\rho}(\tau)$ defined in (1.7). For any cusp ρ of $\Gamma_0(N)$, denote by ℓ_ρ the cusp width and let M_ρ be a matrix in $\mathrm{SL}_2(\mathbb{Z})$ with $\rho = M_\rho i\infty$. For two cusps $\mathfrak{a}, \mathfrak{b}$ of $\Gamma_0(N)$, the generalized Kloosterman sums are

$$K_{\mathfrak{a},\mathfrak{b}}(m, n; c) := \sum_{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_\infty^{\ell_\mathfrak{a}} \backslash M_\mathfrak{a}^{-1}\Gamma_0(N) M_\mathfrak{b} / \Gamma_\infty^{\ell_\mathfrak{b}}} e\left(\frac{md}{\ell_\mathfrak{b} c} + \frac{na}{\ell_\mathfrak{a} c}\right) \tag{2.4}$$

with $\Gamma_\infty := \{\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1\end{smallmatrix}\right) : n \in \mathbb{Z}\}$. Note that we have $K_{i\infty,i\infty}(m, n; c) = K(m, n; c)$.

Lemma 2.2 (Lemma 5.4 of [6])**.** *We have*

$$H^*_{N,\rho}(\tau) = \frac{3}{\pi\left[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)\right]\operatorname{Im}(\tau)} - \delta_{\rho,\infty} + \frac{4\pi^2}{\ell_\rho}\sum_{n\ge1} n \sum_{c\ge1}\frac{K_{\rho,i\infty}(n,0;c)}{c^2}e^{2\pi i n\tau}.$$

The Fourier expansions in Lemma 2.2 yield a relation with the harmonic weight 2 Eisenstein series $E^*_{2,N,\rho}(\tau)$ for $\Gamma_0(N)$. For $\operatorname{Re}(s) > 0$, define

$$E^*_{2,N,\rho,s}(\tau) := \sum_{M\in\Gamma_\rho\backslash\Gamma_0(N)} j\left(M_\rho M, \tau\right)^{-2}\left|j\left(M_\rho M, \tau\right)\right|^{-2s}. \tag{2.5}$$

Using the Hecke trick, it is well-known (cf. Satz 6 of [16]) that $E^*_{2,N,\rho,s}(\tau)$ has an analytic continuation to $s = 0$, denoted by $E^*_{2,N,\rho}(\tau)$. Applying equations (5.3) and (5.4) in Theorem 1 of [19] with $v = 1$, $A_j = M_\rho$, Γ=$\Gamma_0(N)$, and $\mu = 0$ to obtain the Fourier expansion of $E^*_{2,N,\rho}$, we see that

$$H^*_{N,\rho}(\tau) = -E^*_{2,N,\rho}(\tau). \tag{2.6}$$

3. The $j_{N,n}(z)$ and the proofs of Theorems 1.1 and 1.2

3.1. **The functions $j_{N,n}(z)$.** The functions $j_{N,n}(z)$ are constructed as analytic continuations of Niebur's Poincaré series [18]. To be more precise, set for $n \in \mathbb{N}$ and $\operatorname{Re}(s) > 1$

$$F_{N,-n,s}(z) := \sum_{M\in\Gamma_\infty\backslash\Gamma_0(N)} e\left(-n\operatorname{Re}(Mz)\right)\operatorname{Im}(Mz)^{\frac12} I_{s-\frac12}\left(2\pi n\operatorname{Im}(Mz)\right).$$

These functions are *weak Maass forms* of weight 0; instead of being annihilated by Δ_0, they have eigenvalue $s(1-s)$. To obtain an analytic continuation to $s = 1$, one computes the Fourier expansion of $F_{N,-n,s}(z)$ and uses

$$\lim_{s\to1} y^{\frac12} I_{s-\frac12}\left(2\pi n y\right) = y^{\frac12} I_{\frac12}\left(2\pi n y\right) = \frac{1}{\pi\sqrt{n}}\sinh\left(2\pi n y\right) = \frac{e^{2\pi n y} - e^{-2\pi n y}}{2\pi\sqrt{n}}.$$

Proposition 3.1 (Theorem 1 of [18])**.** *The function $F_{N,-n,s}(z)$ has an analytic continuation $F_{N,-n}(z)$ to $s = 1$, and $F_{N,-n}(z) \in \mathcal{H}_0(\Gamma_0(N))$. It has the Fourier expansion*

$$F_{N,-n}(z) = \frac{e^{-2\pi i n z} - e^{-2\pi i n\overline{z}}}{2\pi\sqrt{n}} + c_N(n,0) + \sum_{m\ge1}\left(c_N(n,m)e^{2\pi i m z} + c_N(n,-m)e^{-2\pi i m\overline{z}}\right),$$

where the coefficients are given by

$$c_N(n,m) := \sum_{\substack{c\geq 1\\ N|c}} \frac{K(m,-n;c)}{c} \times \begin{cases} \frac{1}{\sqrt{m}} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right) & \text{if } m>0,\\ \frac{2\pi\sqrt{n}}{c} & \text{if } m=0,\\ \frac{1}{\sqrt{|m|}} J_1\left(\frac{4\pi\sqrt{|m|n}}{c}\right) & \text{if } m<0.\end{cases}$$

We then define the functions $j_{N,n}(z)$ by

$$j_{N,n}(z) := 2\pi\sqrt{n} F_{N,-n}(z). \tag{3.1}$$

For $N=1$, we recover the $j_n(z)$ from the introduction up to the constant $2\pi\sqrt{n}c_1(n,0) = 24\sigma_1(n)$.

3.2. **Proofs of Theorems 1.1 and 1.2.** In order to formally state Theorem 1.1 (4), for an arbitrary solution $a,b\in\mathbb{Z}$ to $ad-bc=1$, we define

$$r_z(c,d) := ac|z|^2 + (ad+bc)\operatorname{Re}(z) + bd, \tag{3.2}$$

$$\Lambda_z := \left\{\alpha^2|z|^2 + \beta\operatorname{Re}(z) + \gamma^2 > 0 : \alpha,\beta,\gamma\in\mathbb{Z}\right\}, \tag{3.3}$$

$$Q_z(c,d) := c^2|z|^2 + 2cd\operatorname{Re}(z) + d^2,$$

$$S_\lambda := \left\{(c,d)\in N\mathbb{Z}\times\mathbb{Z} :\ c\geq 0,\ \gcd(c,d)=1,\ \text{and } Q_z(c,d)=\lambda\right\}. \tag{3.4}$$

Note that although $r_z(c,d)$ is not uniquely determined, $e(-nr_z(c,d)/Q_z(c,d))$ is well-defined.

Proof of Theorem 1.1. (1) For $n\in\mathbb{N}$, inspecting the expansions in Propositions 2.1 and 3.1 yields that $2\pi\sqrt{n}F_{N,-n}(z)$ is the coefficient of $e^{2\pi i n\tau}$ in $-\operatorname{Im}(z)\Psi_{2,N}(\tau,z)/(2\pi)$, yielding the claim.
(2) Since $\gcd(N,n)=1$, $T(n)$ commutes with the action of $\Gamma_0(N)$, and so it suffices to show that (by analytic continuation) $f_n(z) = f_1(z) \mid T(n)$, where

$$f_n(z) = f_{n,s}(z) := e\left(-n\operatorname{Re}(z)\right)\left(n\operatorname{Im}(z)\right)^{\frac{1}{2}} I_{s-\frac{1}{2}}\left(2\pi n\operatorname{Im}(z)\right).$$

Let f be a nonholomorphic modular form of weight 0 with Fourier expansion

$$f(z) = \sum_{m\in\mathbb{Z}} a(\operatorname{Im}(z),m)e^{2\pi i m z}.$$

Then for $\gcd(n,N)=1$, the action of $T(n)$ on f is given by

$$f(z) \mid T(n) = n\sum_{m\in\mathbb{Z}}\sum_{d|\gcd(m,n)} \frac{a\left(\frac{d^2}{n}\operatorname{Im}(z), \frac{mn}{d^2}\right)}{d} e^{2\pi i m z}. \tag{3.5}$$

Write $f_n(z) = f_n^*(\operatorname{Im}(z))e^{-2\pi i n z}$ with $f_n^*(y) := (ny)^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi n y)e^{-2\pi n y}$. The mth coefficient in (3.5) vanishes unless $m=-n$. Moreover, only $d=n$ contributes, giving

$$f_1(z) \mid T(n) = f_n^*\left(n\operatorname{Im}(z)\right)e^{-2\pi i n z} = \left(n\operatorname{Im}(z)\right)^{\frac{1}{2}} I_{s-\frac{1}{2}}\left(2\pi n\operatorname{Im}(z)\right)e^{-2\pi i n\operatorname{Re}(z)} = f_n(z).$$

(3) For $n\mid N$, we rewrite

$$\sum_{M\in\Gamma_\infty\backslash\Gamma_0(N)} f_n(Mz) = \sum_{M\in\Gamma_\infty\backslash\Gamma_0(N)} f_1(nMz).$$

Now, with $M = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(N)$, we have $nMz = \frac{anz+bn}{\frac{c}{n}nz+d}$ and $\left(\begin{smallmatrix} a & bn\\ \frac{c}{n} & d\end{smallmatrix}\right)$ runs through $\Gamma_\infty\setminus\Gamma_0(\frac{N}{n})$ if M runs through $\Gamma_\infty\setminus\Gamma_0(N)$, implying the claim for $n\mid N$.

(4) We first rewrite the claimed asymptotic formula in terms of the corresponding points Mz with $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(N)$. Directly plugging in and simplifying yields $r_z(c,d)/Q_z(c,d) = \mathrm{Re}(Mz)$ and $\mathrm{Im}(z)/Q_z(c,d) = \mathrm{Im}(Mz)$, so the claim in Theorem 1.1 (4) is equivalent to

$$j_{N,n}(z) = \sum_{\substack{M \in \Gamma_\infty \backslash \Gamma_0(N) \\ n\,\mathrm{Im}(Mz) \geq \mathrm{Im}(z)}} e^{-2\pi i n Mz} + O_z(n). \tag{3.6}$$

In order to show (3.6), we only expand the Fourier expansion for large c. That is to say, we write

$$\begin{aligned} j_{N,n}(z) = 2 &\sum_{\substack{1 \leq c \leq \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} \sum_{\substack{d \in \mathbb{Z} \\ \gcd(c,d)=1}} e(-n\,\mathrm{Re}(Mz)) \sinh(2\pi n\,\mathrm{Im}(Mz)) \\ &+ 2\pi\sqrt{n} \sum_{\substack{c > \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} \sum_{m \geq 1} \frac{K(m,-n;c)}{\sqrt{m}c} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{2\pi i m z} + 4\pi^2 n \sum_{\substack{c > \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} \frac{K(0,-n;c)}{c^2} \\ &+ 2\pi\sqrt{n} \sum_{\substack{c > \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} \sum_{m \geq 1} \frac{K(-m,-n;c)}{\sqrt{m}c} J_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{-2\pi i m \overline{z}}. \end{aligned} \tag{3.7}$$

In order to obtain (3.6), we split the main terms with $n\,\mathrm{Im}(Mz) \geq \mathrm{Im}(z)$ off and rewrite

$$2\sinh(2\pi n\,\mathrm{Im}(Mz)) = e^{2\pi n\,\mathrm{Im}(Mz)} - e^{-2\pi n\,\mathrm{Im}(Mz)}. \tag{3.8}$$

The second term above is obviously bounded. Since

$$\mathrm{Im}(z) \leq n\,\mathrm{Im}(Mz) = \frac{n\,\mathrm{Im}(z)}{c^2\,\mathrm{Im}(z)^2 + (d + c\,\mathrm{Re}(z))^2}$$

implies that $c \leq \sqrt{n}/\mathrm{Im}(z) \ll_z \sqrt{n}$ and $|d| \leq |c\,\mathrm{Re}(z)| + \sqrt{n\,\mathrm{Im}(z)} \ll_z \sqrt{n}$, the contribution to the error from the sum of the second terms in (3.8) yields an error of at most $O_z(n)$.

For the second, third, and fourth sums in (3.7), we use the Weil bound for Kloosterman sums

$$|K(m,-n;c)| \leq \sqrt{\gcd(m,n,c)}\sigma_0(c)\sqrt{c} \ll \begin{cases} \sqrt{n}c^{\frac{1}{2}+\varepsilon} & \text{if } m = 0, \\ \sqrt{|m|}c^{\frac{1}{2}+\varepsilon} & \text{if } m \neq 0. \end{cases} \tag{3.9}$$

For the third sum in (3.7), this gives

$$2\pi\sqrt{n} \sum_{\substack{c > \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} \frac{K(0,-n;c)}{c^2} \ll n \sum_{\substack{c > \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} c^{-\frac{3}{2}+\varepsilon} \ll_z n^{\frac{3}{4}+\varepsilon}. \tag{3.10}$$

Next note that for $x \geq 0$ we have $|J_1(x)| \leq I_1(x)$ by their series expansions. Since $x \mapsto \frac{I_1(x)}{x}$ is monotonically increasing and grows at most exponentially, the contribution from the second and fourth terms in (3.7) may be bounded by, using (3.9),

$$\begin{aligned}
&\ll \sum_{\substack{c>\frac{\sqrt{n}}{\mathrm{Im}(z)}\\ N|c}} \sum_{m\geq 1} \frac{|K(\pm m,-n;c)|}{\sqrt{m}c} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{-2\pi m\,\mathrm{Im}(z)}\\
&\ll \sqrt{n} \sum_{\substack{c>\frac{\sqrt{n}}{\mathrm{Im}(z)}\\ N|c}} \sum_{m\geq 1} \frac{|K(\pm m,-n;c)|}{c^2} \frac{I_1\left(4\pi\,\mathrm{Im}(z)\sqrt{m}\right)}{4\pi\,\mathrm{Im}(z)\sqrt{m}} e^{-2\pi m\,\mathrm{Im}(z)} \qquad (3.11)\\
&\ll \sqrt{n} \sum_{m\geq 1} I_1\left(4\pi\,\mathrm{Im}(z)\sqrt{m}\right) e^{-2\pi m\,\mathrm{Im}(z)} \ll \sqrt{n}.
\end{aligned}$$

It remains to bound the terms in the first sum in (3.7) with $|cz+d|^2>n$. Since each term gives a constant contribution, the terms with $|d|<\sqrt{n}+|c\,\mathrm{Re}(z)|$ give an error term of at most $O_z(n)$.

We finally assume that $|d|\geq\sqrt{n}+|c\,\mathrm{Re}(z)|$. Since $x\mapsto\frac{\sinh(x)}{x}$ is monotonically increasing and $|cz+d|^2>n$, the remaining terms contribute

$$\begin{aligned}
&\left|\sum_{\substack{c\leq\frac{\sqrt{n}}{\mathrm{Im}(z)}\\ N|c}} \sum_{\substack{|d|\geq\sqrt{n}+|c\,\mathrm{Re}(z)|\\ \gcd(c,d)=1,}} e\left(-n\,\mathrm{Re}\left(Mz\right)\right)\sinh\left(2\pi n\,\mathrm{Im}\left(Mz\right)\right)\right| \leq \sum_{c\leq\frac{\sqrt{n}}{\mathrm{Im}(z)}} \sum_{|d|\geq\sqrt{n}+|cx|} \sinh\left(\frac{2\pi n\,\mathrm{Im}(z)}{|cz+d|^2}\right)\\
&\leq \sum_{c\leq\frac{\sqrt{n}}{\mathrm{Im}(z)}} \sum_{|d|\geq\sqrt{n}} \sinh\left(\frac{2\pi n\,\mathrm{Im}(z)}{d^2}\right) \leq 2\pi\sqrt{n} \sum_{d\geq\sqrt{n}} \frac{n}{d^2}\frac{\sinh\left(2\pi\,\mathrm{Im}(z)\right)}{2\pi\,\mathrm{Im}(z)} = O_z(n),
\end{aligned}$$

This implies that the terms in the first sum in (3.7) with $|cz+d|^2>n$ contribute $O_z(n)$. □

Remark.

By replacing $c>\sqrt{n}/\mathrm{Im}(z)$ with $c>C$ in (3.10) and (3.11), one finds that the terms decay like $C^{-\frac{1}{2}+\varepsilon}$ times a power of n. For $c\leq C$, the expansions in Proposition 3.1 decay exponentially in m.

Proof of Theorem 1.2. (1) Let K_s denote the usual K-Bessel function. Expanding $F_{N,-n,s}(z)$ at the cusp ρ as in Section 3.4 of [17], we obtain

$$F_{N,-n,s}\left(M_\rho z\right) = \frac{c_{\rho,s}(n,0)}{2s-1}\left(\mathrm{Im}(z)\right)^{1-s} + \sum_{m\in\mathbb{Z}\setminus\{0\}} c_{\rho,s}(n,m) e^{2\pi i m\frac{\mathrm{Re}(z)}{\ell_\rho}} \left(\mathrm{Im}(z)\right)^{\frac{1}{2}} K_{s-\frac{1}{2}}\left(\frac{2\pi|m|\,\mathrm{Im}(z)}{\ell_\rho}\right),$$

$$\text{with } c_{\rho,s}(n,m) := \sum_{c\geq 1} K_{i\infty,\rho}(m,-n;c) \times \begin{cases} \frac{2}{c\sqrt{\ell_\rho}} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{\ell_\rho c}\right) & \text{if } m>0,\\ \frac{2\pi^s n^{s-\frac{1}{2}}}{\ell_\rho^s c^{2s}\Gamma(s)} & \text{if } m=0,\\ \frac{2}{c\sqrt{\ell_\rho}} J_{2s-1}\left(\frac{4\pi\sqrt{|m|n}}{\ell_\rho c}\right) & \text{if } m<0,\end{cases}$$

The right-hand side is analytic at $s=1$, which gives the expansion of $F_{N,-n}(z)$ at ρ. Plugging in $K_{\frac{1}{2}}(y)=\sqrt{\frac{\pi}{2y}}e^{-y}$ and taking the limit $z\to i\infty$, we obtain

$$j_{N,n}(\rho) = 2\pi\sqrt{n}\lim_{s\to 1^+} c_{\rho,s}(n,0) = \frac{4\pi^2 n}{\ell_\rho}\sum_{c\geq 1}\frac{K_{i\infty,\rho}(0,-n;c)}{c^2}. \qquad (3.12)$$

We have $K_{i\infty,\rho}(0,-n;c) = K_{\rho,i\infty}(n,0;c)$, since $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ runs through $\Gamma_0(N)M_\rho/\Gamma_\infty^{\ell_\rho}$ iff $-M^{-1} = \left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right)$ runs through $\Gamma_\infty^{\ell_\rho}\backslash M_\rho^{-1}\Gamma_0(N)$ in (2.4). Hence (2.6) yields the claim.
Parts (2) and (3) follow by taking limits $\tau \to \rho$ in Theorem 1.1 (2) and (3), respectively. Using the growth in n of $j_{N,n}(\rho)$ from (3.12), these limits may be taken termwise.

□

Proof of Theorem 1.3. We show that the difference of both sides has no poles in $\mathbb{H}$ and decays towards the cusps. We start by considering the points in $\mathbb{H}$. One easily computes that the residue of $-\frac{\Theta(f(\tau))}{f(\tau)}$ at $\tau = z$ equals $\frac{1}{2\pi i}\mathrm{ord}_z(f)$. Using (2.3) gives that the principal part at z agrees. At a cusp ρ one similarly sees that $\frac{\Theta(f(\tau))}{f(\tau)}$ has no pole and its constant term equals $\mathrm{ord}_\rho(f)$. Using that the constant term of $H_{N,z}^*(\tau)$ at ρ is -1 then gives the claim. □

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