ON THE EXPLICIT CONSTRUCTION OF HIGHER DEFORMATIONS OF PARTITION STATISTICS

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Abstract. The modularity of the partition generating function has many important consequences, for example asymptotics and congruences for $p(n)$. In a series of papers the author and Ono [12, 13] connected the rank, a partition statistic introduced by Dyson, to weak Maass forms, a new class of functions which are related to modular forms and which were first considered in [11]. Here we do a further step towards understanding how weak Maass forms arise from interesting partition statistics by placing certain 2-marked Durfee symbols introduced by Andrews [1] into the framework of weak Maass forms. To do this we construct a new class of functions which we call quasiweak Maass forms because they have quasimodular forms as components. As an application we prove two conjectures of Andrews. It seems that this new class of functions will play an important role in better understanding weak Maass forms of higher weight themselves, and also their derivatives. As a side product we introduce a new method which enables us to prove transformation laws for generating functions over incomplete lattices.

1. Introduction and Statement of Results

A partition of a nonnegative integer n is a non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n. We have the generating function

$$P(q) = P(z) := \sum_{n=0}^{\infty} p(n)\, q^{24n-1} = \frac{1}{\eta(24z)}, \tag{1.1}$$

where $q := e^{2\pi i z}$ and $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's η-function. Of the many consequences of the modularity of $P(q)$, two of the most striking are Rademacher's exact formula for $p(n)$, and Ramanujan type congruences. In order to state the exact formula, let $I_s(x)$ be the usual I-Bessel function of order s. Furthermore, if k and n are positive integers, then define the Kloosterman sum

$$A_k(n) := \sum_{h \pmod{k}^*} \omega_{h,k}\, e^{-\frac{2\pi i h n}{k}},$$

where the sum only runs over those h modulo k that are coprime to k, and where

$$\omega_{h,k} := \exp\left(\pi i s(h,k)\right),$$

with

$$s(h,k) := \sum_{\mu \pmod{k}} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right).$$

Here

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R}\setminus\mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

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If n is a positive integer, then Rademacher showed that

$$p(n)=\frac{2\pi}{(24n-1)^{\frac{3}{4}}}\sum_{k=1}^{\infty}\frac{A_k(n)}{k}\cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right). \tag{1.2}$$

In particular, (1.2) implies that

$$p(n)\sim\frac{1}{4n\sqrt{3}}\cdot e^{\pi\sqrt{2n/3}} \qquad \text{as } n\to\infty.$$

The partition function also has nice congruence properties. For example, by Ramanujan we have that

$$\begin{aligned}p(5n+4)&\equiv 0 \pmod{5},\\ p(7n+5)&\equiv 0 \pmod{7},\\ p(11n+6)&\equiv 0 \pmod{11},\end{aligned}$$

for every $n\geq 0$ and Ono [21] showed that for any prime $\ell\geq 5$ there exist infinitely many non-nested arithmetic progressions of the form $An+B$ such that

$$p(An+B)\equiv 0 \pmod{\ell}.$$

In this paper we also consider asymptotics and congruences for certain partition statistics.

To explain the Ramanujan congruences with modulus 5 and 7, Dyson [18] introduced the *rank* of a partition, which is defined to be its largest part minus the number of its parts. Dyson conjectured that the partitions of $5n+4$ (resp. $7n+5$) form 5 (resp. 7) groups of equal size when sorted by their ranks modulo 5 (resp. 7), and this conjecture was proven by Atkin and Swinnerton-Dyer [7]. If $N(m,n)$ denotes the number of partitions of n with rank equal to m, then we have the generating function (1.3)

$$R(w;q):=1+\sum_{m\in\mathbb{Z}}\sum_{n=1}^{\infty}N(m,n)w^m\,q^n=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n}=\frac{(1-w)}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{n}{2}(3n+1)}}{1-wq^n},$$

where $(a;q)_n:=\prod_{j=0}^{n-1}\left(1-aq^j\right)$ and $(a;q)_\infty:=\lim_{n\to\infty}(a;q)_n$. In particular,

$$\begin{aligned}R(1;q)&=P(q),\\ R(-1;q)&=f(q):=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q;q)_n^2}.\end{aligned}$$

The function $f(q)$ is one of the so-called *mock theta functions.* Ramanujan listed 17 such functions in his last letter to Hardy and gave two more in his "Lost Notebook" [22], while Watson [25] defined three further functions. Surprisingly, much remained unknown about these series until recently. For example there was even debate concerning the rigorous definition of such a function. Despite these issues, Ramanujan's mock theta functions have been shown to possess many striking properties, and they have been the subject of an astonishing number of important works (for example, see [1, 2, 5, 15, 16, 17, 19, 26]). Much of this activity was foreshadowed by Dyson:

"*The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future.*"

Freeman Dyson, 1987

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Recently much light has been shed on the nature of Ramanujan's mock theta functions. Building on important work of Zwegers [26] the author and Ono ([12, 13]) solved Dyson's "challenge for the future" by placing the rank generating functions, and thus the mock theta function $f(q)$, in the context of weak Maass forms (see Section 6). Loosely speaking it turns out that for each root of unity $w \neq 1$, the function $R(w;q)$ is the holomorphic part of a weight $\frac{1}{2}$ weak Maass form. Since the case $w = 1$ yields a weight $-\frac{1}{2}$ modular form, one can also view the infinite family $R(w;q)$ as a *deformation* of $P(q)$.

Viewing the rank generating functions in the framework of weak Maass forms has found many applications, including an exact formula for the coefficients of $f(q)$ [12], asymptotics for $N(m,n)$ [8], and identities for rank differences [14]. Moreover we obtain congruences for $N(s,t;n)$ [13], the number of partitions of n with rank congruent to s modulo t, which give a combinatorial decomposition of congruences for $p(n)$.

Naturally it is of wide interest to find other explicit examples of Maass forms. Here we construct a new infinite family of such forms, arising in a more complicated way from an interesting partition statistic introduced by Andrews [3]. Define the *symmetrized second moment function*

$$\eta_2(n) := \sum_{m=-\infty}^{\infty} \binom{m}{2} N(m,n).$$

Andrews showed that this function enumerates the 2-marked Durfee symbols (see Section 2). We have the generating function

$$R_2(q) := \sum_{n=0}^{\infty} \eta_2(n)\, q^n = \frac{1}{(q;q)_\infty} \sum_{n \in \mathbb{Z}\setminus\{0\}} (-1)^{n+1} \frac{q^{\frac{3n}{2}(n+1)}}{(1-q^n)^2}. \tag{1.4}$$

One of the main goals of this paper is to place this function into the framework of weak Maass forms and introduce a new class of functions, which we call quasiweak Maass forms, which seem to play an important role in better understanding weak Maass forms themselves, and also their derivatives. It will turn out that the function (1.4) is part of a weight $\frac{3}{2}$ weak Maass forms and occurs from deforming the weight $\frac{1}{2}$ rank generating function by applying a certain differential operator which is responsible for the higher weight. The proof of the modularity of the usual rank generating function (1.3) heavily relies on the fact that the sum runs through the full lattice $\mathbb{Z}$ which allows Poisson summation. In contrast, the sum of the generating function in (1.4) runs only through the incomplete lattice $\mathbb{Z} \setminus \{0\}$. To our knowledge so far known examples which use Poisson summation to prove modularity come from sums over full lattices. The first guess to deal with missing terms in Poisson summation seems to be to add extra summands during the computations, and remove them at the end. However, this does not work in the present case due to the double pole in the $n = 0$ term.

We overcome those problems by introducing a new method which enables us to handle incomplete sums like (1.4). For this purpose, we embed (1.4) into a bigger family which includes $R_2(q)$ as a limiting case of a certain differential operator. It seems very likely that this new method will be helpful in other settings as well, and may lead to a better understanding of functions like $R(w;q)/(1-w)$ (a modified version of Dyson's rank generating function that removes the "artificial" zero). In our situation a very careful analysis of introduced poles is required since they give rise to additional terms in the transformation law, which we managed to identify as quasimodular components. Recall that a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a *quasimodular form* if it can be written as linear combination of derivatives of modular forms. Note that some authors do not include meromorphic functions in this definition and only allow forms that lie in the algebra generated by E_2, E_4, and E_6.

To state our result define

$$\mathcal{R}(z) := R_2(24z)e^{-2\pi i z},$$

$$\mathcal{N}(z) := \frac{i}{4\sqrt{2}\pi}\int_{-\bar{z}}^{i\infty} \frac{\eta(24\tau)}{(-i(\tau+z))^{\frac{3}{2}}}\, d\tau, \tag{1.5}$$

and

$$\mathcal{M}(z) := \mathcal{R}(z) - \mathcal{N}(z) - \frac{1}{24\eta(24z)} + \frac{E_2(24z)}{8\eta(24z)}, \tag{1.6}$$

where as usual

$$E_2(z) := 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)\, q^n$$

with $\sigma_1(n) := \sum_{d|n} d$. The function $E_2(z)$ is "nearly modular", and in particular satisfies $E_2(z+1) = E_2(z)$ and

$$E_2\left(-\frac{1}{z}\right) = z^2 E_2(z) + \frac{6z}{\pi i}. \tag{1.7}$$

Theorem 1.1. *The function $\mathcal{M}(z)$ is a harmonic weak Maass form of weight $\frac{3}{2}$ on $\Gamma_0(576)$ with Nebentypus character $\chi_{12}(\cdot) := \left(\frac{12}{\cdot}\right)$.*

Two remarks.
1) Since $\mathcal{R}(z)$ is, up to quasimodular forms, the holomorphic part of a weak Maass form, we refer to $R_2(q)$ as a *quasimock theta function.*
2) The function $R_2(q)$ is also crucial for understanding k-marked Durfee symbols for $k > 2$ (see [3] for the definition). In upcoming work Mahlburg and the author show [10] that modularity properties of the generating functions $R_k(q)$ can be concluded from modularity properties of $R(q)$ and $R_2(q)$, and that these functions give rise to a new class of functions which we call *quasiweak Maass forms.* It seems that this new class of functions will play an important role in better understanding weak Maass forms of higher weight themselves, and also their derivatives.

The fact that $\mathcal{M}(z)$ is a weak Maass form has some nice applications. Here we address two of them: congruences and asymptotics. These were formulated by Andrews as open problems (problems 11 and 13 page 39 of [3]).

Theorem 1.2. *We have*

$$\eta_2(n) = \sum_{k=1}^{[\sqrt{n}]} A_k(n)\left(-\frac{3}{2(24n-1)^{\frac{1}{4}}} I_{\frac{1}{2}}\left(\frac{\pi}{6k}\sqrt{24n-1}\right) + \frac{\pi(24n-1)^{\frac{1}{4}}}{12k}\right.$$
$$\left. I_{-\frac{1}{2}}\left(\frac{\pi}{6k}\sqrt{24n-1}\right) + \frac{\pi}{12k(24n-1)^{\frac{3}{4}}}\, I_{\frac{3}{2}}\left(\frac{\pi}{6k}\sqrt{24n-1}\right)\right) + O\left(n^{1+\epsilon}\right).$$

In particular the $k=1$ term gives the main term in the asymptotics expansion.

Corollary 1.3. *As $n \to \infty$*

$$\eta_2(n) \sim \frac{1}{4\sqrt{3}}\, e^{\frac{\pi}{6}\sqrt{24n-1}}.$$

Remark. In particular Corollary 1.3 implies that as $n \to \infty$

$$\eta_2(n) \sim np(n).$$

We next turn to congruences. As in the case of partitions one can associate certain ranks to 2-marked Durfee symbols, which we will describe in Section 2. Let $NF_2(r,t;n)$ denote the number of 2-marked Durfee symbols with (full) rank congruent to r modulo t.

Theorem 1.4. *Let $t > 3$ be an integer, $j \in \mathbb{N}$, and $\mathcal{Q} \nmid 6t$ a prime. Then there exist infinitely many arithmetic progressions $An + B$ such that for every $0 \leq r < t$, we have*

$$NF_2(r,t;An+B) \equiv 0 \pmod{\mathcal{Q}^j}.$$

Remark. With similar methods as in [9], one could also obtain congruences for $t = \mathcal{Q}^\ell$ with $\ell \in \mathbb{N}$.

Theorem 1.4 can be viewed as yielding congruences for $\eta_2(n)$ as well as a combinatorial decomposition of these congruences.

Corollary 1.5. *Let j be a positive integer and $\mathcal{Q} > 3$ a prime. Then there exist infinitely many arithmetic progressions $An + B$ such that*

$$\eta_2(An+B) \equiv 0 \pmod{\mathcal{Q}^j}.$$

From Theorem 1.2 and Corollary 1.5 one can obtain congruences and asymptotics for an interesting new statistic introduced by Andrews [4]. Let $spt(n)$ be the total number of appearances of smallest parts in each integer partition of n. Andrews showed that

$$spt(n) = np(n) - \eta_2(n).$$

Since congruences and asymptotics are known for $p(n)$, we should also be able to obtain similar results for $spt(n)$. However, we do not further address this topic here.

We next consider odd moments

$$\eta_2^o(n) := \sum_{m \in \mathbb{Z}} \binom{m+1}{2} N^o(m,n),$$

where $N^o(m,n)$ is the number of partitions related to an odd Durfee symbol with odd rank m (for the definition of odd ranks and odd Durfee symbols see Section 2). We have

$$R_2^o(q) := \sum_{n=1}^{\infty} \eta_2^o(n)\, q^n = \frac{1}{(q^2;q^2)_\infty} \sum_{n \in \mathbb{Z}} (-1)^n \frac{q^{3n^2+5n+2}}{(1-q^{2n+1})^3}.$$

Defining the sum of Kloosterman type

$$A_k^o(n) := e^{\frac{\pi i k}{2}} \sum_{h \pmod{k}^*} \omega_{2h,k}\, e^{\frac{3\pi i hk}{2}}\, e^{\frac{\pi i h}{2k}(1-4n)}$$

we find the following asymptotic expansion for $\eta_2^o(n)$.

Theorem 1.6. *We have*

$$\eta_2^o(n) = -\frac{i}{\sqrt{2}} \sum_{k \text{ odd}}^{[\sqrt{n}]} A_k^o(n) \left(-\frac{3\pi^2(3n-1)^{\frac{1}{4}}}{4} I_{-\frac{1}{2}} \left(\frac{\pi}{3k}\sqrt{3n-1} \right) + \frac{\pi(3n-1)^{\frac{3}{4}}}{16k} I_{-\frac{3}{2}} \left(\frac{\pi}{3k}\sqrt{3n-1} \right) \right) + O(n^{2+\epsilon}).$$

Corollary 1.7. *As $n \to \infty$,*

$$\eta_2^o(n) \sim \frac{3}{8}\sqrt{n}\, e^{\frac{\pi}{3}\sqrt{3n-1}}.$$

The proof of Theorem 1.6 heavily relies on a transformation law for $R_2^o(q)$ which is more complicated than the transformation law for $R_2(q)$ arising from higher order poles of the generating function. For this reason no usual weak Maass forms arise. It would be very interesting to analyze the class of functions $R_2^o(q)$ belong to.

The paper is organized as follows. In Section 2 we recall the connection between higher moments and marked Durfee symbols. In Section 3 we prove a transformation law for $R_2(q)$. To overcome the above mentioned problems, we introduce a new family of functions $\mathcal{R}_2(w;q)$ which have $R_2(q)$ as a limiting case of a certain differential operator L. We first prove a transformation law for $\mathcal{R}_2(w;q)$, and then apply L. Here a careful analysis is required since we introduce summands which have poles when $w = 1$, and it is because of these terms that quasimodular forms show up. In Section 4 we realize the error integral that arises as an integral of theta functions and prove Theorem 1.2. As an application we then prove asymptotics in Section 5 and congruences in Section 6. In Sections 7 and 8 we consider odd rank generating functions.

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2. Some combinatorial results on 2-marked Durfee symbols

Here we recall some results on 2-marked Durfee symbols and their connection to rank moments [3]. Recall that the largest square of nodes in the Ferrers graph of a partition is called the *Durfee square*. The *Durfee symbol* consists of 2 rows and a subscript, where the top row consists of the columns to the right of the Durfee square, the bottom row consists of the rows below the Durfee square, and the subscript denotes the side length of the Durfee square. The number being partitioned is equal to the sum of the rows of the symbol plus the number of nodes in the Durfee square. For example the Durfee symbol

$$\begin{pmatrix} 2 & \\ 3 & 1 \end{pmatrix}_4$$

represents a partition of $2+3+1+4^2 = 22$. To define *2-marked Durfee symbols* one requires two copies of the positive integers denoted by $\{1_1, 2_1, \cdots\}$ and $\{1_2, 2_2, \cdots\}$. We form Durfee symbols as before and use the two copies of the positive integers for parts in both rows. We additionally demand that the following conditions are met:

(1) The sequence of parts and the sequence of subscripts in each row are non-increasing.
(2) The subscript 1 occurs at least once in the top row.
(3) If M_1 is the largest part with subscript 1 in the top row, then all parts in the bottom row with subscript 1 lie in $[1, M_1]$ and with subscript 2 lie in $[M_1, S]$, where S is the side of the Durfee symbol.

We denote by $\mathcal{D}_2(n)$ the number of 2-marked Durfee symbols arising from partitions of n. In [3] it is shown that

$$\mathcal{D}_2(n) = \eta_2(n).$$

As in the case of partitions one can associate ranks to 2-marked Durfee symbols [3]. For a 2-marked Durfee symbol δ we define the *full rank* $FR(\delta)$ by

$$FR(\delta) := \rho_1(\delta) + 2\rho_2(\delta),$$

where

$$\rho_i(\delta) := \begin{cases} \tau_i(\delta) - \beta_i(\delta) - 1 & \text{for } i = 1, \\ \tau_i(\delta) - \beta_i(\delta) & \text{for } i = 2. \end{cases}$$

Here $\tau_i(\delta)$ (resp. $\beta_i(\delta)$) denotes the number of entries in the top (resp. bottom) row of δ with subscript i. We let $NF_2(m,n)$ denote the number of 2-marked Durfee symbols related to n with full rank m and $NF_2(r,t;n)$ the number of 2-marked Durfee symbols related to n with full rank congruent to r modulo t. Let

$$R_2(w;q) := \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} NF_2(m,n)\, w^m\, q^n. \tag{2.1}$$

In particular we have

$$R_2(1;q) = R_2(q).$$

Moreover if $w^3 \neq 1$, then

$$R_2(w;q) = \frac{w^2}{(1-w)(w^3-1)} \left(R(w;q) - R(w^2;q) \right).$$

Therefore in this case properties of R_2, like congruences or asymptotics, can be concluded from properties of the usual rank generating function R using work of the author and Ono.

We next consider odd Durfee symbols. A partition into pairs of consecutive integers (excepting one instance of the largest part) has associated to it the *odd Durfee symbol* $\left(\begin{smallmatrix} a_1 & a_2 & \cdots & a_i \\ b_1 & b_2 & \cdots & b_j \end{smallmatrix}\right)_n$ with the $a's$ and $b's$ being odd numbers $\leq 2n+1$, and the number being partitioned is $2n^2+2n+1+\sum_{m=1}^{i} a_m + \sum_{k=1}^{j} b_k$. The *odd rank* of an odd Durfee symbol is defined as the number of entries in the top minus the number of entries in the bottom row. We denote by $N^o(m,n)$ the number of partitions related to an odd Durfee symbol with odd rank m. We have the generating function

$$R^o(w;q) := \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} N^o(m,n) w^m q^n = \frac{1}{(q^2;q^2)_\infty} \sum_{n \in \mathbb{Z}} (-1)^n \frac{q^{3n^2+3n+1}}{1-wq^{2n+1}}.$$

The definition of an *odd 2-marked Durfee symbol* is almost the same as for a 2-marked Durfee symbol. The only changes are:

(1) All entries in the symbol are odd numbers.
(2) The subscribing rules are the same but instead of adding n^2 to the sum when the Durfee symbol is of side n we now add $2n^2+2n+1$.

If $\mathcal{D}_2^o(n)$ denotes the number of odd 2-marked Durfee symbols of n, then [3]

$$\mathcal{D}_2^o(n) = \eta_2^o(n).$$

3. A transformation law for $R_2(q)$

Due to problems mentioned in the introduction we cannot show a transformation law for $R_2(q)$ directly but we introduce a function of an additional variable w that is related to $R_2(q)$. For this purpose, define for $w \in \mathbb{C}$ with $\mathrm{Re}(w)$ sufficiently small

$$\mathcal{R}_2(w;q) = \mathcal{R}_2(w;\tau) := -\frac{1}{(q;q)_\infty} \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1 - e^{2\pi i w}\, q^n}, \tag{3.1}$$

where $q := e^{2\pi i \tau}$. Defining the operator L for a function g that is differentiable in a neighborhood of 0,

$$L(g(w)) := \frac{1}{2\pi i}\left[\frac{\partial}{\partial w} g\right]_{w=0},$$

we can connect the function $R_2(q;w)$ to $R_2(q)$

$$R_2(q) = L(\mathcal{R}_2(w;q)).$$

We prove a transformation law for $\mathcal{R}_2(w;q)$ from which we conclude a transformation law for $R_2(q)$ by applying L. For this let

$$H_w(x) := \frac{e^x}{1 - e^{2\pi i w}\, e^{2x}}.$$

Then

$$H_w(x + \pi i) = -H_w(x).$$

Moreover for an integer ν and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, define

$$I_{k,\nu}^{\pm}(z;w) := \int_{\mathbb{R}} e^{-\frac{3\pi z x^2}{k}}\, H_w\left(\pm\left(\frac{\pi i \nu}{k} - \frac{\pi i}{6k} - \frac{\pi z x}{k}\right)\right) dx.$$

Theorem 3.1. *For coprime integers h and k with $k > 0$, let $q := e^{\frac{2\pi i}{k}(h+iz)}$ and $q_1 := e^{\frac{2\pi i}{k}\left(h' + \frac{i}{z}\right)}$, where for k odd we let h' be an even solution of $hh' \equiv -1 \pmod{k}$, and for even k we let h' be defined by $hh' \equiv -1 \pmod{2k}$. Then*

$$\begin{aligned}\mathcal{R}_2(w;q) = {} & \frac{\omega_{h,k}\, z^{\frac12}\, e^{\frac{\pi}{12k}\left(z^{-1} - z\right)}}{(q_1;q_1)_\infty}\left(\frac{1}{1 - e^{2\pi i w}} - \frac{i\, e^{\frac{3\pi k w^2}{z} - \pi i w - \frac{\pi w}{z}}}{z\left(1 - e^{-\frac{2\pi w}{z}}\right)}\right) \\ & + i z^{-\frac12} \omega_{h,k} e^{\frac{\pi}{12k}\left(z^{-1} - z\right)} e^{\frac{3\pi k w^2}{z} - \pi i w - \frac{\pi w}{z}}\, \mathcal{R}_2\left(\frac{iw}{z}; q_1\right) \\ & - \frac{z^{\frac12}}{k}\, \omega_{h,k}\, e^{-\frac{\pi z}{12k}} \sum_{\substack{\nu \pmod{k} \\ \pm}} (-1)^\nu\, e^{\frac{\pi i h'}{k}(-3\nu^2 + \nu)} I_{k,\nu}^{\pm}(z;w).\end{aligned}$$

Proof. We first observe that the difference of both sides of the transformation law is a holomorphic function of w in some neighborhood of 0. Indeed this follows immediately if $w \neq 0$ is sufficiently small. Moreover it follows from (3.7) that the point $w = 0$ is a removable singularity. Therefore it is by the identity theorem of holomorphic functions enough to prove the theorem for $w \in \mathbb{C}$ with $\mathrm{Re}(w) \neq 0$ sufficiently small and $z > 0$ real. For those w, the function

$$\widetilde{R}_2(w;q) := \sum_{n \in \mathbb{Z}} \frac{(-1)^{n-1} q^{\frac{n(3n+1)}{2}}}{1 - e^{2\pi i w}\, q^n}$$

is a holomorphic function of z. Writing $n = \nu + km$ with $m \in \mathbb{Z}$ and ν running modulo k, we obtain

$$\widetilde{R}_2(w;q) = -\sum_{\nu \pmod k} (-1)^\nu e^{\frac{3\pi i h \nu^2}{k}} \sum_{m\in\mathbb{Z}} (-1)^m e^{-\frac{3\pi z}{k}(\nu+km)^2} H_w\left(\frac{\pi i \nu h}{k} - \frac{\pi z}{k}(\nu + km)\right). \tag{3.2}$$

Poisson summation gives that the inner sum equals

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{-\frac{3\pi z}{k}(\nu+kx)^2} H_w\left(\frac{\pi i \nu h}{k} - \frac{\pi z}{k}(\nu+kx)\right) e^{\pi i(2n+1)x}\, dx \\
= \;& \frac{1}{k}\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{-\frac{3\pi z x^2}{k}} H_w\left(\frac{\pi i \nu h}{k} - \frac{\pi z x}{k}\right) e^{\frac{\pi i}{k}(2n+1)(x-\nu)}\, dx,
\end{aligned}$$

where for the last equality we made the change of variables $x \mapsto \nu + kx$. Inserting this into (3.2), and changing in the sum over $n \le -1$, $n \mapsto -n-1$, $x \mapsto -x$, and $\nu \mapsto -\nu$, gives

$$\widetilde{R}_2(w;q) = -\frac{1}{k}\sum_{\substack{\nu \pmod k \\ \pm}} (-1)^\nu e^{\frac{3\pi i h \nu^2}{k}} \sum_{n\ge 0} \int_{\mathbb{R}} e^{-\frac{3\pi z x^2}{k}} H_w\left(\pm\left(\frac{\pi i \nu h}{k} - \frac{\pi z x}{k}\right)\right) e^{\frac{\pi i}{k}(2n+1)(x-\nu)}\, dx.$$

Next write

$$H_w\left(\pm\left(\frac{\pi i \nu h}{k} - \frac{\pi z x}{k}\right)\right) = \frac{\pm e^{-\pi i w}}{2\sinh\left(\frac{\pi z x}{k} - \frac{\pi i h \nu}{k} \mp \pi i w\right)}.$$

Using that the function

$$S_w^{\pm}(x) := \frac{\sinh(x \pm k\pi i w)}{\sinh\left(\frac{x}{k} \pm \pi i w\right)} \tag{3.3}$$

is an entire function, we see that the only possible poles of the integrand lie in points

$$x_m^{\pm} := \frac{i}{z}(m \pm kw). \tag{3.4}$$

Moreover for a fixed m this leads at most to a pole if $h\nu \equiv m \pmod k$, and we may choose

$$\nu_m := -h'm. \tag{3.5}$$

Thus poles may at most occur for x_m and ν_m as in (3.4) and (3.5). Using the Residue Theorem we shift the path of integration through

$$\omega_n := \frac{(2n+1)i}{6z}.$$

For this we have to take those $x_m^{\pm}$ into account for which $n \ge 3m \ge \frac{1}{2}(1 \mp 1)$. We denote the associated residues of each summand by $\lambda_{n,m}^{\pm}$. Then

$$\widetilde{R}_2(w;q) = \sum_1 + \sum_2,$$

where

$$\begin{aligned}
\sum_1 &:= -\frac{2\pi i}{k}\left(\sum_{m\ge 0}\sum_{n\ge 3m} \lambda_{n,m}^{+} + \sum_{m>0}\sum_{n\ge 3m} \lambda_{n,m}^{-}\right), \\
\sum_2 &:= -\frac{1}{k}\sum_{\substack{\nu \pmod k \\ \pm}} (-1)^\nu e^{\frac{3\pi i h \nu^2}{k}} \sum_{n\ge 0} \int_{\mathbb{R}+\omega_n} e^{-\frac{3\pi z x^2}{k}} H_w\left(\pm\left(\frac{\pi i \nu h}{k} - \frac{\pi z x}{k}\right)\right) e^{\frac{\pi i}{k}(2n+1)(x-\nu)}\, dx.
\end{aligned}$$

We first consider $\sum_1$. We have

$$\lambda_{n,m}^{\pm} = \pm k(-1)^{\nu_m} \frac{e^{-\frac{3\pi z {x_m^{\pm}}^2}{k} + \frac{\pi i}{k}(2n+1)(x_m^{\pm}-\nu_m) + \frac{3\pi i h \nu_m^2}{k} - \pi i w}}{2\cosh\left(\frac{\pi z x_m^{\pm}}{k} - \frac{\pi i h \nu_m}{k} \mp \pi i w\right)\pi z},$$

which yields

$$\lambda_{n+1,m}^{\pm} = e^{\frac{2\pi i}{k}(x_m^{\pm}-\nu_m)} \lambda_{n,m}^{\pm}.$$

Therefore

$$\sum_{n=3m}^{\infty} \lambda_{n,m}^{\pm} = \frac{\lambda_{3m,m}^{\pm}}{1-\exp\left(\frac{2\pi i}{k}(x_m^{\pm}-\nu_m)\right)}.$$

Using this one can prove that

$$\sum_1 = \frac{i}{z}\, e^{\frac{3\pi k w^2}{z} - \pi i w - \frac{\pi w}{z}}\, \widetilde{R}_2\left(q_1; \frac{iw}{z}\right).$$

We next turn to $\sum_2$. With the same argument as before, we change the sum over $n \in \mathbb{N}_0$ back into a sum over all $n \in \mathbb{Z}$. Substituting $x \mapsto x + \omega_n$ and writing $n = 3p + \delta$ with $p \in \mathbb{Z}$ and $\delta \in \{0, \pm 1\}$ gives

$$\sum_2 = -\frac{1}{k}\sum_{\nu,p,\delta} (-1)^{\nu}\, e^{\frac{3\pi i h \nu^2}{k} - \frac{\pi(6p+2\delta+1)^2}{12kz} - \frac{\pi i}{k}(6p+2\delta+1)\nu}$$
$$\int_{\mathbb{R}} e^{-\frac{3\pi z x^2}{k}} H_w\left(\frac{\pi i h \nu}{k} - \frac{\pi z x}{k} - \frac{\pi i(6p+2\delta+1)}{6k}\right) dx.$$

Changing ν into $-h'(\nu+p)$, a lengthy calculation gives

$$\sum_2 = -\frac{1}{k}\sum_{\nu,\delta,p} (-1)^{\nu+p} e^{\frac{\pi i h'}{k}(-3\nu^2+(2\delta+1)\nu) - \frac{\pi}{12kz}(2\delta+1)^2}$$
$$q_1^{\frac{p}{2}(3p+2\delta+1)} \int_{\mathbb{R}} e^{-\frac{3\pi z x^2}{k}} H_w\left(\frac{\pi i \nu}{k} - \frac{\pi i}{6k}(2\delta+1) - \frac{\pi z x}{k}\right) dx.$$

Now the integral is independent of p and the sum over p equals

$$\sum_{p\in\mathbb{Z}} (-1)^p\, q_1^{\frac{p}{2}(3p+2\delta+1)}. \tag{3.6}$$

If $\delta = 1$, then (3.6) vanishes since the pth and the $-(p+1)$th term cancel. If $\delta = 0$ or $\delta = -1$, then (3.6) equals $(q_1; q_1)_\infty$. Changing for $\delta = -1$, $x \mapsto -x$ and $\nu \mapsto -\nu$ gives

$$\sum_2 = -\frac{(q_1;q_1)_\infty}{k} e^{-\frac{\pi}{12kz}} \sum_{\substack{\nu \pmod k \\ \pm}} (-1)^{\nu} e^{\frac{\pi i h'}{k}(-3\nu^2+\nu)} I_{k,\nu}^{\pm}(z; w).$$

From this the theorem follows using

$$(q_1; q_1)_\infty = \omega_{h,k}\, z^{\frac{1}{2}}\, e^{\frac{\pi}{12k}(z^{-1}-z)} (q; q)_\infty.$$

□

From Theorem 3.1 we conclude the desired transformation law for $R_2(q)$.

Corollary 3.2. *For coprime integers h and k with $k>0$, let $q := e^{\frac{2\pi i}{k}(h+iz)}$ and $q_1 := e^{\frac{2\pi i}{k}\left(h'+\frac{i}{z}\right)}$, where for k odd we let h' be an even solution of $hh' \equiv -1 \pmod{k}$, and for even k we let h' be defined by $hh' \equiv -1 \pmod{2k}$. Then*

$$R_2(q) = \omega_{h,k}\, z^{\frac{1}{2}}\, e^{\frac{\pi}{12k}(z^{-1}-z)} \left(\frac{1}{(q_1;q_1)_\infty}\left(-\frac{3k}{4\pi z}+\frac{1}{24z^2}+\frac{1}{24}\right)-\frac{1}{z^2}R_2(q_1)\right) - \frac{z^{\frac{1}{2}}}{2k}\,\omega_{h,k}\, e^{-\frac{\pi z}{12k}} \sum_{\nu \pmod{k}} (-1)^\nu e^{\frac{\pi i h'}{k}(-3\nu^2+\nu)} I_{k,\nu}(z),$$

where

$$I_{k,\nu}(z) := \int_{\mathbb{R}} e^{-\frac{3\pi z x^2}{k}} \frac{\cosh\left(\frac{\pi i\nu}{k}-\frac{\pi i}{6k}-\frac{\pi z x}{k}\right)}{\sinh^2\left(\frac{\pi i\nu}{k}-\frac{\pi i}{6k}-\frac{\pi z x}{k}\right)}\, dx.$$

Proof. One can show that

$$L\left(\frac{1}{1-e^{2\pi i w}} - \frac{i\, e^{3\pi k w^2-\pi i w-\frac{\pi w}{z}}}{z\left(1-e^{-\frac{2\pi w}{z}}\right)}\right) = -\frac{3k}{4\pi z}+\frac{1}{24}+\frac{1}{24z^2}, \tag{3.7}$$

$$L\left(\frac{e^{3\pi k w^2-\pi i w-\frac{\pi w}{z}}}{1-q_1^m\, e^{-\frac{2\pi w}{z}}}\right) = -\frac{1}{2(1-q_1^m)}+\frac{i(1+q_1^m)}{2z(1-q_1^m)^2},$$

$$L\left(H_w\left(\frac{\pi i\nu}{k}-\frac{\pi i}{6k}-\frac{\pi z x}{k}\right)+H_w\left(-\frac{\pi i\nu}{k}+\frac{\pi i}{6k}+\frac{\pi z x}{k}\right)\right) = \frac{\cosh\left(\frac{\pi i\nu}{k}-\frac{\pi i}{6k}-\frac{\pi z x}{k}\right)}{2\sinh^2\left(\frac{\pi i\nu}{k}-\frac{\pi i}{6k}-\frac{\pi z x}{k}\right)}.$$

Moreover

$$\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^m q^{\frac{m}{2}(3m+1)}(1+q^m)}{(1-q^m)^2} = 2\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^m q^{\frac{m}{2}(3m+1)+m}}{(1-q^m)^2} = -2(q;q)_\infty R_2(q),$$

$$\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^m\, q^{\frac{m}{2}(3m+1)}}{1-q^m} = 0,$$

since the mth and $-m$th term cancel. Now the corollary follows from Theorem 3.1. □

4. Proof of Theorem 1.1

In this section we prove that $\mathcal{M}(z)$ is a weak Maass form. If $k \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$, $z = x+iy$ with $x, y \in \mathbb{R}$, then the weight k hyperbolic Laplacian is given by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right). \tag{4.1}$$

If v is odd, then define ϵ_v by

$$\epsilon_v := \begin{cases} 1 & \text{if } v \equiv 1 \pmod{4}, \\ i & \text{if } v \equiv 3 \pmod{4}. \end{cases} \tag{4.2}$$

Moreover we let χ be a Dirichlet character. A *(harmonic) weak Maass form of weight k with Nebentypus χ on a subgroup* $\Gamma \subset \Gamma_0(4)$ is any smooth function $g : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $z \in \mathbb{H}$, we have

$$g(Az) = \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} \chi(d)\,(cz+d)^k\; g(z).$$

(2) We have that $\Delta_k g = 0$.
(3) The function $g(z)$ has at most linear exponential growth at all the cusps of Γ.

Theorem 3.1 implies

Corollary 4.1. *We have*

$$R_2\left(-\frac{1}{z};w\right) e^{\frac{\pi i}{12 z}} = \frac{1}{(-iz)^{\frac{1}{2}} \cdot \eta(z)} \left(\frac{1}{1-e^{2\pi i w}} - z\,\frac{e^{-3\pi i w^2 z - \pi i w + \pi i w z}}{1-e^{2\pi i w z}}\right)$$
$$+\, i(-iz)^{\frac{1}{2}} e^{-\frac{\pi i z}{12} - 3\pi i w^2 z - \pi i w + \pi i w z} R_2\left(z; wz\right) - (-iz)^{-\frac{1}{2}} \left(I_{1,0}^{+}\left(\frac{i}{z};w\right) + I_{1,0}^{-}\left(\frac{i}{z};w\right)\right).$$

Now let

$$J^{\pm}(z;w) := e^{\pm\frac{\pi i}{6}} \int_{\mathbb{R}} \frac{e^{-\frac{3\pi i x^2}{z} \pm \frac{\pi i x}{z}}}{1-e^{2\pi i w \pm \frac{\pi i}{3} \pm \frac{2\pi i x}{z}}} dx.$$

We show that $L\left(J^{+}(z;w) + J^{-}(z;w)\right)$ can be written as a certain Mordell type integral.

Lemma 4.2. *We have*

$$L\left(J^{+}(z;w) + J^{-}(z;w)\right) = \frac{\sqrt{3} z^2}{2\pi} \int_0^{\infty} \frac{\eta(iu)}{(-i(iu+z))^{\frac{3}{2}}}\, du.$$

Proof. Via analytic continuation it is enough to show the claim for $z = it$. The substitution $x \mapsto -\frac{x}{t}$ gives

$$J^{\pm}(it;w) = t \int_{\mathbb{R}} e^{-3\pi t x^2} \frac{e^{\pm\frac{\pi i}{6} \mp \pi x}}{1-e^{2\pi i w \pm \frac{\pi i}{3} \mp 2\pi x}} dx. \tag{4.3}$$

We rewrite the integrand using the Mittag-Leffler theory of partial fraction decompositions.

$$\sum_{\pm} \frac{e^{\pm\frac{\pi i}{6} \mp \pi x}}{1-e^{2\pi i w \pm \frac{\pi i}{3} \mp 2\pi x}} = \mp \frac{e^{-\pi i w}}{2\pi} \sum_{m\in\mathbb{Z}} (-1)^m \sum_{\pm} \frac{1}{x - i\left(m + \frac{1}{6} \pm w\right)}. \tag{4.4}$$

To see this identity, we first observe that both sides of the identity are meromorphic functions of x and it is not hard to see that they have the same poles and residues. Moreover we will need later that the right-hand side is absolutely convergent. Thus in the following we are allowed to interchange summation, integration, and differentiation. Here it is important to consider the terms coming from the $\pm$ terms combined. Since both sides as a function of x have period $i\mathbb{Z}$, it can be easily seen that the difference of both sides is bounded on $\mathbb{C}$ and thus constant. Letting for fixed imaginary part the real part of x go to infinity gives that this constant must be 0. This gives

$$J^{\pm}(it;w) = \mp \frac{e^{-\pi i w}}{2\pi}\, t \sum_{m} (-1)^m \int_{\mathbb{R}} \frac{e^{-3\pi t x^2}}{x - i\left(m + \frac{1}{6} \pm w\right)}\, dx.$$

It is not hard to see that

$$L\left(\frac{e^{-\pi i w}}{iw + x - i\left(m + \frac{1}{6}\right)} - \frac{e^{-\pi i w}}{-iw + x - i\left(m+\frac{1}{6}\right)}\right) = -\frac{1}{\pi\left(x - i\left(m + \frac{1}{6}\right)\right)^2}.$$

Thus

$$L\left(J^{+}(it;w)+J^{-}(it;w)\right)=\frac{t}{2\pi^2}\sum_m(-1)^m\int_{\mathbb{R}}\frac{e^{-3\pi tx^2}}{\left(x-i\left(m+\frac{1}{6}\right)\right)^2}\,dx. \tag{4.5}$$

For $s\neq 0$ we have the identity

$$\int_{\mathbb{R}}\frac{e^{-2\pi tx^2}}{(x-is)^2}dx=-\sqrt{2}\pi t\int_0^\infty\frac{e^{-2\pi us^2}}{(u+t)^{\frac{3}{2}}}du.$$

Thus

$$L\left(J^{+}(it;w)+J^{-}(it;w)\right)=\frac{\sqrt{3}(it)^2}{2\pi}\sum_m(-1)^m\int_0^\infty\frac{e^{-3\pi u\left(m+\frac{1}{6}\right)^2}}{(u+t)^{\frac{3}{2}}}du,$$

which gives the claim of the lemma. □

Combining Corollary 4.1 with Lemma 4.2 gives, as in the proof of Corollary, 3.2

Corollary 4.3.

$$\mathcal{R}\left(-\frac{1}{z}\right)=\frac{2\sqrt{6}}{\eta(z/24)}\left(-\frac{(-iz)^{\frac{1}{2}}}{32\pi}+\frac{(-iz)^{\frac{3}{2}}}{24^3}+\frac{1}{24(-iz)^{\frac{1}{2}}}\right)-\frac{(-iz)^{\frac{3}{2}}}{48\sqrt{6}}\mathcal{R}\left(\frac{z}{576}\right)+\frac{(-iz)^{\frac{3}{2}}}{16\sqrt{3}\pi}\int_0^\infty\frac{\eta(iu/24)}{(-i(iu+z))^{\frac{3}{2}}}\,du.$$

We next study the transformation law of the non-holomorphic part of $\mathcal{M}(z)$ and show that under inversion it introduces the same error integral as $\mathcal{R}(z)$.

Lemma 4.4. *We have*

$$\mathcal{N}(z+1)=\mathcal{N}(z),$$

$$\mathcal{N}\left(-\frac{1}{z}\right)=-\frac{i(-iz)^{\frac{3}{2}}}{16\sqrt{3}\pi}\int_{-\bar{z}}^{i\infty}\frac{\eta(\tau/24)}{(-i(\tau+z))^{\frac{3}{2}}}d\tau+\frac{(-iz)^{\frac{3}{2}}}{16\sqrt{3}\pi}\int_0^\infty\frac{\eta\left(\frac{it}{24}\right)}{(-i(it+z))^{\frac{3}{2}}}\,dt.$$

Proof. The first claim follows directly since $\eta(24\tau)$ is translation-invariant. Moreover

$$\mathcal{N}\left(-\frac{1}{z}\right)=-\frac{i}{4\sqrt{2}\pi}\int_{\frac{1}{\bar{z}}}^{i\infty}\frac{\eta(24\tau)}{\left(-i\left(-\frac{1}{z}+\tau\right)\right)^{\frac{3}{2}}}d\tau.$$

Making the change of variables $\tau\mapsto-\frac{1}{\tau}$ and using the transformation law of the η-function, gives

$$\mathcal{N}\left(-\frac{1}{z}\right)=-\frac{i(-iz)^{\frac{3}{2}}}{16\sqrt{3}\pi}\int_{-\bar{z}}^{i\infty}\frac{\eta(\tau/24)}{(-i(\tau+z))^{\frac{3}{2}}}\,d\tau+\frac{i(-iz)^{\frac{3}{2}}}{16\sqrt{3}\pi}\int_0^{i\infty}\frac{\eta(it/24)}{(-i(z+it))^{\frac{3}{2}}}\,dt,$$

as claimed. □

Combining Corollary 4.3, Lemma 4.4, equation (1.7), and the transformation law of the $\eta-$function gives that $\mathcal{M}(z)$ transforms correctly under $\Gamma_0(576)$ with Nebentypus character χ_{12}. To see that $\mathcal{M}(z)$ is annihilated by $\Delta_{\frac{3}{2}}$, we write $\Delta_{\frac{3}{2}}=-4y^{\frac{1}{2}}\frac{\partial}{\partial z}y^{\frac{3}{2}}\frac{\partial}{\partial\bar{z}}$. Clearly

$$\frac{\partial}{\partial\bar{z}}\left(\mathcal{R}(z)-\frac{1}{24\eta(24z)}+\frac{E_2(24z)}{8\eta(24z)}\right)=0.$$

Moreover

$$-\frac{\partial}{\partial\overline{z}}\mathcal{N}(z)=\frac{i\eta(-24\overline{z})}{16\pi y^{\frac{3}{2}}}.$$

Thus $y^{\frac{3}{2}}\frac{\partial}{\partial\overline{z}}\mathcal{N}(z)$ is antiholomorphic, which implies that $\Delta_{\frac{3}{2}}\left(\mathcal{N}(z)\right)=0$. The claim about the exponential growth in all cusps follows as in [13]

5. Proof of Theorem 1.2

Here we use the circle method and prove Theorem 1.2. First we estimate $I_{k,\nu}(z)$ and certain Kloosterman sums.

Lemma 5.1. *Assume that $n\in\mathbb{N}, \nu\in\mathbb{Z}$, and $z:=\frac{k}{n}-k\Phi i$, $-\frac{1}{k(k+k_1)}\leq\Phi\leq\frac{1}{k(k+k_2)}$, where $\frac{h_1}{k_1}<\frac{h}{k}<\frac{h_2}{k_2}$ are adjacent Farey fractions in the Farey sequence of order N, with $N:=\lfloor n^{\frac{1}{2}}\rfloor$. Then*

$$z^{\frac{1}{2}}\cdot I_{k,\nu}(z)\ll\frac{n^{\frac{1}{4}}}{\left\{\frac{\nu}{k}-\frac{1}{6k}\right\}^2},$$

where for $x\in\mathbb{R}$ we let $\{x\}:=x-\lfloor x\rfloor$.

Proof. We proceed similarly as in [1] and [8]. We write $\frac{\pi z}{k}=Ce^{iA}$. Then $|A|<\frac{\pi}{2}$ since $\mathrm{Re}(z)>0$. Making the substitution $\tau=\frac{\pi z x}{k}$ gives

$$z^{\frac{1}{2}}\cdot I_{k,\nu}(z)=\frac{k}{\pi z^{\frac{1}{2}}}\int_S e^{-\frac{3k\tau^2}{\pi z}}\frac{\cosh\left(\frac{\pi i\nu}{k}-\frac{\pi i}{6k}-\tau\right)}{\sinh^2\left(\frac{\pi i\nu}{k}-\frac{\pi i}{6k}-\tau\right)}\,d\tau,$$

where τ runs on the ray through 0 of elements with argument $\pm A$. One can easily see that for $0\leq t\leq A$

$$\left|e^{-\frac{3kR^2e^{2it}}{\pi z}}\frac{\cosh\left(\frac{\pi i\nu}{k}-\frac{\pi i}{6k}\pm R\,e^{it}\right)}{\sinh^2\left(\frac{\pi i\nu}{k}-\frac{\pi i}{6k}\pm R\,e^{it}\right)}\right|\to 0\qquad\text{as } R\to\infty.$$

Moreover the integrand only has poles at points ir with $r\in\mathbb{R}\setminus\{0\}$. Shifting the path of integration to the real line gives

$$z^{\frac{1}{2}}\cdot I_{k,\nu}(z)=\frac{k}{\pi z^{\frac{1}{2}}}\int_{\mathbb{R}} e^{-\frac{3kt^2}{\pi z}}\cdot\frac{\cosh\left(\frac{\pi i\nu}{k}-\frac{\pi i}{6k}-t\right)}{\sinh^2\left(\frac{\pi i\nu}{k}-\frac{\pi i}{6k}-t\right)}dt.$$

We have

$$\left|e^{-\frac{3kt^2}{\pi z}}\right|=e^{-\frac{3k}{\pi}\mathrm{Re}\left(\frac{1}{z}\right)t^2},$$

$$\left|\cosh\left(\frac{\pi i\nu}{k}-\frac{\pi i}{6k}-t\right)\right|\leq e^t,$$

$$\left|\sinh\left(\frac{\pi i\nu}{k}-\frac{\pi i}{6k}-t\right)\right|^2\gg\begin{cases} e^{2t} & \text{if } t\geq 1,\\ \left\{\frac{\nu}{k}-\frac{1}{6k}\right\}^2 & \text{if } t\leq 1.\end{cases}$$

Thus

$$z^{\frac{1}{2}}\cdot I_{k,\nu}(z)\ll\frac{k}{\left\{\frac{\nu}{k}-\frac{1}{6k}\right\}^2|z|^{\frac{1}{2}}}\int_{\mathbb{R}}e^{-\frac{3k}{\pi}t^2\mathrm{Re}\left(\frac{1}{z}\right)}dt.$$

Making the change of variables $t'=\sqrt{\frac{3k\mathrm{Re}\left(\frac{1}{z}\right)}{\pi}}\,t$ and using that $\mathrm{Re}\left(\frac{1}{z}\right)^{-\frac{1}{2}}\cdot|z|^{-\frac{1}{2}}\ll n^{\frac{1}{4}}\cdot k^{-\frac{1}{2}}$ gives the claim of the lemma. □

We next give estimates for certain sums of Kloosterman type [1, 8].

Lemma 5.2. *Let $n, m \in \mathbb{Z}$, $0 \leq \sigma_1 < \sigma_2 \leq k$, and $D \in \mathbb{Z}$ with $(D,k) = 1$. Then we have*

$$\sum_{\substack{h \pmod{k}^* \\ \sigma_1 \leq Dh' \leq \sigma_2}} \omega_{h,k} \cdot e^{\frac{2\pi i}{k}(hn+h'm)} \ll \gcd(24n+1,k)^{\frac{1}{2}} \cdot k^{\frac{1}{2}+\epsilon}. \tag{5.1}$$

Proof of Theorem 1.2. Using Lemma 5.1, Lemma 5.2, and Corollary 3.2, we now prove Theorem 1.2 using the Hardy-Ramanujan method. By Cauchy's Theorem we have for $n > 0$

$$\eta_2(n) = \frac{1}{2\pi i} \int_C \frac{R_2(q)}{q^{n+1}}\, dq,$$

where C is an arbitrary path inside the unit circle surrounding 0 counterclockwise. Choosing for C the circle with radius $e^{-\frac{2\pi}{n}}$ and as a parametrization $q = e^{-\frac{2\pi}{n}+2\pi i t}$ with $0 \leq t \leq 1$, gives

$$\eta_2(n) = \int_0^1 R_2\left(e^{-\frac{2\pi}{n}+2\pi i t}\right) \cdot e^{2\pi - 2\pi i n t}\, dt.$$

Define

$$\vartheta'_{h,k} := \frac{1}{k(k_1+k)}, \quad \vartheta''_{h,k} := \frac{1}{k(k_2+k)},$$

where $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ are adjacent Farey fractions in the Farey sequence of order $N := \lfloor n^{1/2} \rfloor$. From the theory of Farey fractions it is known that

$$\frac{1}{k+k_j} \leq \frac{1}{N+1} \qquad (j = 1, 2). \tag{5.2}$$

We decompose the path of integration in paths along the Farey arcs $-\vartheta'_{h,k} \leq \Phi \leq \vartheta''_{h,k}$, where $\Phi := t - \frac{h}{k}$. This gives

$$\eta_2(n) = \sum_{h,k} e^{-\frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} R_2\left(e^{\frac{2\pi i}{k}(h+iz)}\right) \cdot e^{\frac{2\pi n z}{k}}\, d\Phi,$$

where $z := \frac{k}{n} - k\Phi i$. Corollary 3.2 gives

$$\eta_2(n) = \sum_{h,k} e^{-\frac{2\pi i h n}{k}} \omega_{h,k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k}} e^{\frac{\pi}{12k}(z^{-1}-z)} z^{\frac{1}{2}} \left(\frac{1}{(q_1;q_1)_\infty}\left(-\frac{3k}{4\pi z} + \frac{1}{24z^2} + \frac{1}{24}\right)\right.$$
$$\left. + \frac{1}{z^2} R_2(q_1)\right) d\Phi - \sum_{h,k} \frac{\omega_{h,k}}{k} e^{-\frac{2\pi i h n}{k}} \sum_{\nu \pmod{k}} (-1)^\nu e^{\frac{\pi i h'}{k}(-3\nu^2+\nu)} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{\frac{1}{2}} I_{k,\nu}(z) e^{\frac{2\pi(n-1/24)z}{k}}\, d\Phi,$$

and we denote the two summands by $\sum_1$ and $\sum_2$. We first treat $\sum_1$ and start with the contribution coming from

$$\frac{1}{24z^2\,(q_1;q_1)_\infty} =: \frac{1}{z^2}\left(\frac{1}{24} + \sum_{r>0} a(r)\, q_1^r\right).$$

We consider the constant term and the term arising from $r \geq 1$ separately since they contribute to the main term and to the error term, respectively. We denote the associated sums by S_1 and S_2, respectively and first estimate S_2. Throughout we need the easily verified facts that $\mathrm{Re}(z) = \frac{k}{n}$, $\mathrm{Re}\left(\frac{1}{z}\right) > \frac{k}{2}$, $|z| \geq \frac{k}{n}$, and $\vartheta'_{h,k} + \vartheta''_{h,k} \leq \frac{2}{k(N+1)}$. Since $k_1, k_2 \leq N$, we can write

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} + \int_{-\frac{1}{k(k_1+k)}}^{-\frac{1}{k(N+k)}} + \int_{\frac{1}{k(N+k)}}^{\frac{1}{k(k_2+k)}} \tag{5.3}$$

and we denote the associated sums by S_{21}, S_{22}, and S_{23}, respectively. We first consider S_{21}. Lemma 5.2 gives

$$S_{21} \ll \left| \sum_{r=1}^{\infty} a(r) \sum_k \sum_h \omega_{h,k} \cdot e^{-\frac{2\pi i h n}{k} + \frac{2\pi i r h'}{k}} \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} z^{-\frac{3}{2}} \cdot e^{-\frac{2\pi}{kz}\left(r - \frac{1}{24}\right) + \frac{2\pi z}{k}\left(n - \frac{1}{24}\right)} \, d\Phi \right|$$
$$\ll \sum_r |a(r)| e^{-\pi r} n \sum_k (24n-1, k)^{\frac{1}{2}} \, k^{-2+\epsilon} \ll n.$$

Since S_{22} and S_{23} are estimated similarly we only consider S_{22}. Writing

$$\int_{-\frac{1}{k(k+k_1)}}^{-\frac{1}{k(N+k)}} = \sum_{l=k_1+k}^{N+k-1} \int_{-\frac{1}{kl}}^{-\frac{1}{k(l+1)}},$$

we see

$$S_{22} \ll \left| \sum_{r=1}^{\infty} a(r) \sum_k \sum_{l=N+1}^{N+k-1} \int_{-\frac{1}{kl}}^{-\frac{1}{k(l+1)}} z^{-\frac{3}{2}} \cdot e^{-\frac{2\pi}{kz}\left(r - \frac{1}{24}\right) + \frac{2\pi z}{k}\left(n - \frac{1}{24}\right)} d\Phi \sum_{\substack{h \\ N < k+k_1 \leq l}} \omega_{h,k} \cdot e^{-\frac{2\pi i h n}{k} + \frac{2\pi i r h'}{k}} \right|. \tag{5.4}$$

It follows from the theory of Farey fractions that

$$k_1 \equiv -h' \pmod{k}, \quad k_2 \equiv h' \pmod{k},$$
$$N - k < k_1 \leq N, \quad N - k < k_2 \leq N.$$

Thus (5.4) can be estimated similarly as S_{21} using Lemma 5.2.

In the same way we decompose the terms in $-\frac{3k}{4\pi z (q_1;q_1)_\infty}$, $\frac{1}{24(q_1;q_1)_\infty}$, and $\frac{R_2(q_1)}{z^2}$. One can show that the term with a positive exponent in the Fourier expansions also introduces an error of $O\left(n^{1+\epsilon}\right)$. Thus

$$\sum_1 = \sum_{h,k} e^{-\frac{2\pi i h n}{k}} \omega_{h,k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi\left(n - \frac{1}{24}\right)z}{k} + \frac{\pi}{12kz}} \left(-\frac{3k}{4\pi z^{\frac{1}{2}}} + \frac{1}{24 z^{\frac{3}{2}}} + \frac{z^{\frac{1}{2}}}{24} \right) + O\left(n^{1+\epsilon}\right).$$

We next write the path of integration in a symmetrized way

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{kN}}^{\frac{1}{kN}} - \int_{-\frac{1}{kN}}^{-\frac{1}{k(k+k_1)}} - \int_{\frac{1}{k(k+k_2)}}^{\frac{1}{kN}}$$

and denote the associated sums by S_{11}, S_{12}, and S_{13}, respectively. The sums S_{12} and S_{13} contribute to the error term. Since they have a similar shape, we only consider S_{12}. We again only estimate the error arising from the second summand since it can be shown the other terms lead to an error of at most that size. Writing

$$\int_{-\frac{1}{kN}}^{-\frac{1}{k(k_1+k_1)}} = \sum_{l=N}^{k+k_1-1} \int_{-\frac{1}{kl}}^{-\frac{1}{k(l+1)}}$$

gives

$$S_{12} \ll \left| \sum_k \sum_{l=N}^{N+k-1} \int_{-\frac{1}{kl}}^{-\frac{1}{k(l+1)}} z^{-\frac{3}{2}} e^{\frac{\pi}{12kz}+\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)} d\Phi \sum_{\substack{h \\ l<k+k_1-1\leq N+k-1}} \omega_{h,k} \cdot e^{-\frac{2\pi i h n}{k}} \right|. \tag{5.5}$$

Using that $\mathrm{Re}(z) = \frac{k}{n}$, $\mathrm{Re}\left(\frac{1}{z}\right) < k$, and $|z|^2 \geq \frac{k^2}{n^2}$, (5.5) can by Lemma 5.2 be estimated as before against $O\left(n^{1+\epsilon}\right)$. Thus

$$\sum_1 = \sum_k A_k(n) \int_{-\frac{1}{kN}}^{\frac{1}{kN}} \left(-\frac{3k}{4\pi z^{\frac{1}{2}}} + \frac{1}{24 z^{\frac{3}{2}}} + \frac{z^{\frac{1}{2}}}{24} \right) e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)+\frac{\pi}{12kz}} \, d\Phi + O\left(n^{1+\epsilon}\right). \tag{5.6}$$

To finish the estimation of $\sum_1$, we consider integrals of the form

$$I_{k,r} := \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^r \cdot e^{\frac{2\pi}{k}\left(z\left(n-\frac{1}{24}\right)+\frac{1}{24z}\right)} d\Phi,$$

where $r \in \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}\}$. Substituting $z = \frac{k}{n} - ik\Phi$ gives

$$I_{k,r} = \frac{1}{ki} \int_{\frac{k}{n}-\frac{i}{N}}^{\frac{k}{n}+\frac{i}{N}} z^r \cdot e^{\frac{2\pi}{k}\left(z\left(n-\frac{1}{24}\right)+\frac{1}{24z}\right)} \, dz. \tag{5.7}$$

We denote the circle through $\frac{k}{n} \pm \frac{i}{N}$ and tangent to the imaginary axis at 0 by Γ. If $z = x + iy$, then Γ is given by $x^2 + y^2 = \alpha x$, with $\alpha = \frac{k}{n} + \frac{n}{N^2 k}$. Using the fact that $2 > \alpha > \frac{1}{k}$, $\mathrm{Re}(z) \leq \frac{k}{n}$, and $\mathrm{Re}\left(\frac{1}{z}\right) < k$ on the smaller arc we can show that the integral along the smaller arc is in $O\left(n^{-\frac{1}{4}}\right)$. Moreover the path of integration in (5.7) can be changed by Cauchy's Theorem into the larger arc of Γ. Thus

$$I_{k,r} = \frac{1}{ki} \int_\Gamma z^r \cdot e^{\frac{2\pi}{k}\left(z\left(n-\frac{1}{24}\right)+\frac{1}{24z}\right)} \, dz + O\left(n^{-\frac{1}{4}}\right).$$

Making the substitution $t = \frac{\pi}{12kz}$ gives

$$I_{k,r} = \frac{2\pi}{k} \left(\frac{\pi}{12k}\right)^{1+r} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} t^{-2-r} \cdot e^{t+\frac{\alpha}{t}} \, dt + O\left(n^{-\frac{1}{4}}\right),$$

where $\gamma \in \mathbb{R}^+$ and $\alpha := \frac{\pi^2}{144k^2}(24n-1)$. Now

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} t^{-(2+r)} \, e^{t+\frac{\alpha}{t}} \, dt = \alpha^{-\frac{r+1}{2}} \, I_{r+1}\left(2\sqrt{\alpha}\right).$$

Thus

$$I_{k,r} = \frac{2\pi}{k} (24n-1)^{-\frac{r+1}{2}} I_{r+1}\left(\frac{\pi\sqrt{24n-1}}{6k}\right) + O\left(n^{-\frac{1}{4}+\epsilon}\right).$$

Plugging this into (5.6) gives

$$\sum_1 = \sum_k A_k(n) \Bigg(-\frac{3}{2(24n-1)^{\frac{1}{4}}} I_{\frac{1}{2}}\left(\frac{\pi}{6k}\sqrt{24n-1}\right) + \frac{\pi(24n-1)^{\frac{1}{4}}}{12k}$$

$$I_{-\frac{1}{2}}\left(\frac{\pi}{6k}\sqrt{24n-1}\right) + \frac{\pi}{12k(24n-1)^{\frac{3}{4}}} \, I_{\frac{3}{2}}\left(\frac{\pi}{6k}\sqrt{24n-1}\right) \Bigg) + O\left(n^{1+\epsilon}\right).$$

We next consider $\sum_2$. Using Lemmas 5.1 and 5.2, we obtain

$$\sum_2 \ll n^{-\frac{1}{4}} \sum_k k^{\frac{3}{2}+\epsilon} (24n-1,k)^{\frac{1}{2}} \sum_{\nu \pmod k} \frac{1}{\left(k\left\{\frac{\nu}{k}-\frac{1}{6}\right\}\right)^2} \ll n^{-\frac{1}{4}} \sum_k k^{\frac{3}{2}+\epsilon} \sum_{\nu=1}^{k} \frac{1}{\nu^2} \ll n^{\frac{1}{2}+\epsilon}.$$

Combining the estimates for $\sum_1$ and $\sum_2$ gives Theorem 1.2. ☐

Corollary 1.3 can be concluded using that for $x \to \infty$

$$I_\alpha(x) \sim \frac{1}{\sqrt{2\pi x}}\, e^x.$$

6. Proof of Theorem 1.4

Here we prove Theorem 1.4. First we need to know on which arithmetic progressions the non-holomorphic part of $\mathcal{M}(z)$ is supported. Similarly as in [13] one can show.

Lemma 6.1. *We have*

$$\mathcal{N}(z) = \frac{1}{4\sqrt{\pi}} \sum_{k\in\mathbb{Z}} (-1)^k (6k+1) \Gamma\left(-\frac{1}{2}; 4\pi(6k+1)^2 y\right) q^{-(6k+1)^2},$$

where $\Gamma(\alpha;x) := \int_x^\infty e^{-t}\, t^{\alpha-1}\, dt$ *is the incomplete Gamma-function.*

We next recall results from [13]. For this let $0 < a < c$ and define

$$\begin{aligned} D\left(\frac{a}{c};z\right) &:= -S\left(\frac{a}{c};z\right) + q^{-\frac{\ell_c}{24}} R\left(\zeta_c^a; q^{\ell_c}\right), \\ S\left(\frac{a}{c};z\right) &:= -\frac{i\sin\left(\frac{\pi a}{c}\right)\ell_c^{\frac{1}{2}}}{\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta\left(\frac{a}{c};\ell_c\tau\right)}{\sqrt{i(\tau+z)}} d\tau, \end{aligned}$$

where $\ell_c := \mathrm{lcm}(2c^2, 24)$, $\zeta_c := e^{\frac{2\pi i}{c}}$, and $\Theta\left(\frac{a}{c};\tau\right)$ is a certain weight $\frac{3}{2}$ cuspidal theta function (for the definition see [13]). It turns out (see Theorems 1.1 and 1.2 of [13]) that $D\left(\frac{a}{c};z\right)$ is a harmonic weak Maass form of weight $\frac{1}{2}$.

Next we observe, using orthogonality of roots of unity, that

$$\sum_{n=0}^{\infty} NF_2(r,t;n)\, q^n = \frac{1}{t} R_2(q) + \frac{1}{t} \sum_{j=1}^{t-1} \zeta_t^{-rj} R_2\left(\zeta_t^j; q\right).$$

Thus

$$\begin{aligned} \sum_{n=0}^{\infty} NF_2(r,t;n) q^{\ell_t - \frac{\ell_t}{24}} &= \frac{1}{t}\left(\mathcal{M}\left(\frac{\ell_t z}{24}\right) + \mathcal{N}\left(\frac{\ell_t z}{24}\right) + \frac{1}{24\eta(\ell_t z)} + \frac{E_2(\ell_t z)}{8\eta(\ell_t)}\right) \\ &+ \frac{1}{t}\sum_{j=1}^{t-1} \frac{\zeta_t^{-rj+2j}}{\left(1-\zeta_t^j\right)\left(\zeta_t^{3j}-1\right)} \left(D\left(\frac{j}{t};z\right) - D\left(\frac{2j}{t};z\right) + \left(S\left(\frac{j}{t};z\right) - S\left(\frac{2j}{t};z\right)\right)\right). \end{aligned}$$

Now fix a prime $p > 3$, and let $\mathcal{S}_p := \left\{n \in \mathbb{Z}; \left(\frac{24\ell_t n - \ell_t}{p}\right) = -\left(\frac{\ell_t}{p}\right)\right\}$. From [13], one can conclude that the restriction of $D\left(\frac{js}{t};z\right)$ ($s \in \{1,2\}$) to those coefficients lying in $\mathcal{S}_p$ is a weakly holomorphic modular form of weight $\frac{1}{2}$ on $\Gamma_1(6f_t^2\ell_t p^4)$. In a similar way we see that the restriction of $\mathcal{M}\left(\frac{\ell_t z}{24}\right)$ to those coefficients in $\mathcal{S}_p$ is a weakly holomorphic modular form of weight $\frac{3}{2}$ on $\Gamma_1\left(576\ell_t p^4\right)$. Moreover the restriction of $\frac{1}{24\eta(\ell_t z)}$ to those coefficients in $\mathcal{S}_p$ is a weakly holomorphic modular form of weight $-\frac{1}{2}$

on $\Gamma_1(24\ell_t p^4)$. Moreover by work of Serre it is known that $E_2(z)$ is a p-adic modular form, e.g. we have

$$E_2(z) \equiv E_{p+1}(z) \pmod{p}.$$

From this one can conclude that the restriction of $\frac{E_2(\ell_t z)}{\eta(\ell_t z)}$ to $\mathcal{S}_p$ is congruent to a weakly holomorphic modular form on $\Gamma_1(24\ell_t p^4)$ modulo Q^j. Now the result can be concluded as in [13].

7. A transformation law for $R_2^o(q)$

Throughout we assume the notation of Section 3. Define for $w \in \mathbb{C}$ with $\mathrm{Re}(w)$ sufficiently small

$$\mathcal{R}^o(q;w) = \mathcal{R}^o(z;w) \quad := \quad \frac{1}{(q^2;q^2)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n\, q^{3n^2+n}}{1-e^{2\pi i w}q^{2n+1}}$$

and for a function g that is differentiable in a neighborhood of 0, let

$$L^o(g(w)) \quad := \quad -\frac{1}{8\pi^2}\left[\frac{\partial}{\partial w} e^{-2\pi i w}\frac{\partial}{\partial w} g\right]_{w=0}.$$

Then

$$L^o\left(\mathcal{R}^o(q;w)\right) = R^o(q).$$

We show a transformation law for $\mathcal{R}^o(q;w)$ and then conclude a transformation law for $R^o(q)$ by applying L^o. This transformation law turns out to be more complicated than the transformation law for $R(q;w)$. We distinguish the cases k odd and k even. Let

$$H_w^o(x) := \frac{1}{2\sinh\left(-\pi i w - \frac{x}{2}\right)}.$$

Clearly

$$H_w^o(x+2\pi i) = -H_w^o(x).$$

Let

$$\begin{aligned} h^o(z;w) = h^o(q;w) \quad &:= \quad \frac{1}{(q;q)_\infty}\sum_{m\in\mathbb{Z}}(-1)^m \frac{q^{\frac{1}{2}(3m^2+m)}}{1+e^{\pi i w}q^m},\\ I_{k,\nu}^{o,\pm}(z;w) \quad &:= \quad \int_{\mathbb{R}} e^{-\frac{3\pi z x^2}{2k}+\frac{3\pi z x}{k}}\, H_w^o\left(\pm\frac{2\pi i\nu}{k}-\frac{2\pi z x}{k}\mp\frac{\pi i}{3k}\right)dx. \end{aligned}$$

Theorem 7.1. *Assume that k is odd and that $4|h'$. Then we have*

$$\begin{aligned} \mathcal{R}^o(z;w) = &-\frac{1}{\sqrt{2}} z^{-\frac{1}{2}}\omega_{2h,k} e^{\frac{\pi}{24kz}-\frac{2\pi z}{3k}+\frac{\pi i h}{2k}}\, e^{\frac{\pi i}{2}(k+1+3hk)}\, e^{2\pi i w+\frac{3\pi k w^2}{2z}-\frac{\pi w}{2z}}\, h^o\left(q_1^{\frac{1}{2}};\frac{iw}{z}\right)\\ &-\frac{1}{\sqrt{2}\,k} e^{\frac{\pi i}{2}k(3h+1)}\,\omega_{2h,k} z^{\frac{1}{2}} e^{-\pi i w} e^{\frac{\pi i h}{2k}-\frac{2\pi z}{3k}} \sum_{\substack{\nu \pmod{k}\\ \pm}} \pm e^{\frac{\pi i}{2k}\left(h'\left(-3\nu^2+\nu\right)\mp 6\nu\pm 1\right)} I_{k,\nu}^{o,\pm}(z;w). \end{aligned}$$

Proof. We proceed as in the proof of Theorem 3.1. Via analytic continuation it is enough to show the claim for $w \in \mathbb{C}$ with $\mathrm{Re}(w) \neq 0$ sufficiently small and $z > 0$ real. Define

$$\widetilde{R}^o(z;w) := \sum_{n\in\mathbb{Z}} \frac{(-1)^n\, q^{3n^2+n}}{1-e^{2\pi i w}q^{2n+1}}.$$

Then

$$\widetilde{R}^o(z;w) = e^{-\pi i w}\, q^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} (-1)^n e^{\frac{6\pi i(h+iz)n^2}{k}}\, H_w^o\left(\frac{2\pi i}{k}(h+iz)(2n+1)\right). \tag{7.1}$$

We write $n=\nu+km$ with ν running modulo k and $m\in\mathbb{Z}$. This gives that

$$\widetilde{R}^o(z;w) = e^{-\pi i w}\, q^{-\frac{1}{2}} \sum_{\nu \pmod k} (-1)^\nu\, e^{\frac{6\pi i h\nu^2}{k}} \sum_{m\in\mathbb{Z}} (-1)^m e^{-\frac{6\pi z(\nu+km)^2}{k}} H_w^o\left(\frac{2\pi i h}{k}(2\nu+1) - \frac{2\pi z}{k}\left(2\left(\nu+km\right)+1\right)\right). \tag{7.2}$$

Applying Poisson summation to the inner sum and making the change of variables $x\mapsto \nu+kx$ gives that the inner sum equals

$$\frac{1}{k}\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{-\frac{6\pi z x^2}{k}}\, H_w^o\left(\frac{2\pi i h}{k}(2\nu+1) - \frac{2\pi z}{k}(2x+1)\right) e^{\frac{\pi i}{k}(2n+1)(x-\nu)} dx.$$

In (7.2), we change for $n\leq -1$, $n\mapsto -n-1$, $x\mapsto -x$, and $\nu\mapsto -\nu$. This gives

$$\widetilde{R}^o(z;w) = e^{-\pi i w}\, q^{-\frac{1}{2}} \frac{1}{k} \sum_{\substack{\nu \pmod k\\ \pm}} (-1)^\nu\, e^{\frac{6\pi i h\nu^2}{k}} \sum_{n\in\mathbb{N}_0} \int_{\mathbb{R}} e^{-\frac{6\pi z x^2}{k}} H_w^o\left(\frac{2\pi i h}{k}(\pm 2\nu+1) - \frac{2\pi z}{k}(\pm 2x+1)\right) e^{\frac{\pi i}{k}(2n+1)(x-\nu)} dx. \tag{7.3}$$

Making the change of variables $x\mapsto 2x\pm 1$ and $\nu\mapsto \frac{k+1}{2}(\nu\mp 1)$, a lengthy calculation gives

$$\widetilde{R}^o(z;w) = e^{-\pi i w}\, q^{\frac{1}{4}} \frac{1}{2k} \sum_{\substack{\nu \pmod k\\ \pm}} e^{-\frac{\pi i}{2}(\nu\mp 1) + \frac{3\pi i h}{2k}(\nu^2\mp 2\nu) + \frac{\pi i}{2}(k+1+3kh)(1-\nu^2)} \sum_{n\in\mathbb{N}_0} (-1)^{n(\nu+1)} \int_{\mathbb{R}} e^{-\frac{3\pi z}{2k}(x^2\mp 2x)} e^{\frac{\pi i}{2k}(2n+1)(x-\nu)} H_w^o\left(\pm\frac{2\pi i h\nu}{k} \mp \frac{2\pi z x}{k}\right) dx\,.$$

We use the residue theorem to shift the path of integration through

$$\omega_n := \frac{i}{6z}(2n+1).$$

Using again the function $S_w^{\pm}$ defined in (3.3), one sees that poles of the integrand can only come from poles of $\sinh(\pi z x \mp \pi i k w)$, thus lie in

$$x_m^{\pm} := \frac{i}{z}(m \pm kw). \tag{7.4}$$

Moreover for fixed m, there is at most for one ν modulo k a non-trivial residue and this ν can be chosen as

$$\nu_m := -h'm. \tag{7.5}$$

Thus poles may at most occur for x_m and ν_m as in (7.4) and (7.5). If we now shift the path of integration through ω_n, we have to take those $x_m^{\pm}$ into account for which $n\geq 3m>0$, and for x_m^+ we consider additionally $m=0$. Denoting by $\lambda_{n,m}^{\pm}$ the residue of the integrand we get

$$\widetilde{R}^o(z;w) = \sum_1 + \sum_2,$$

$$\sum_1 := \frac{\pi i}{k} e^{-\pi i w}\, q^{\frac14} \left(\sum_{m\geq 0}\sum_{n\geq 3m} \lambda^+_{n,m} + \sum_{m>0}\sum_{n\geq 3m} \lambda^-_{n,m} \right),$$

$$\sum_2 := e^{-\pi i w}\, q^{\frac14} \frac{1}{2k} \sum_{\substack{\nu \pmod k \\ \pm}} e^{-\frac{\pi i}{2}(\nu\mp 1)+\frac{3\pi i h}{2k}(\nu^2\mp 2\nu)+\frac{\pi i}{2}(k+1+3kh)(1-\nu^2)}$$
$$\sum_{n\in\mathbb{N}_0} (-1)^{n(\nu+1)} \int_{\mathbb{R}+\omega_n} e^{-\frac{3\pi z}{2k}(x^2\mp 2x)} e^{\frac{\pi i}{2k}(2n+1)(x-\nu)} H_w^o\left(\pm\frac{2\pi i h\nu}{k} \mp \frac{2\pi z x}{k}\right) dx\,.$$

We first consider $\sum_1$. Since h' is even, we have

$$\lambda^{\pm}_{n,m} = \pm(-1)^n \frac{k\, e^{\frac{\pi i}{2}(k+1+3hk)-\frac{\pi i}{2}(\nu_m\mp 1)+\frac{3\pi i h}{2k}(\nu_m^2\mp 2\nu_m)-\frac{3\pi z}{2k}\left(\left(x_m^{\pm}\right)^2\mp 2x_m^{\pm}\right)+\frac{\pi i}{2k}(2n+1)(x_m^{\pm}-\nu_m)}}{2\pi z\, \cosh(\mp\pi i w - \frac{\pi i h \nu_m}{k} + \frac{\pi z x_m^{\pm}}{k})}.$$

From this we see that

$$\lambda^{\pm}_{n+1,m} = -e^{\frac{\pi i}{k}(x_m^{\pm}-\nu_m)}\lambda^{\pm}_{n,m}.$$

Thus

$$\sum_1 = \frac{\pi i}{k} e^{-\pi i w} q^{\frac14} \left(\sum_{m\geq 0} \frac{\lambda^+_{3m,m}}{1+e^{\frac{\pi i}{k}(x_m^+-\nu_m)}} + \sum_{m>0} \frac{\lambda^-_{3m,m}}{1+e^{\frac{\pi i}{k}(x_m^--\nu_m)}} \right).$$

A lengthy calculation calculation gives that this equals

$$-e^{2\pi i w+\frac{3\pi k w^2}{2z}-\frac{\pi w}{2z}}\, q^{\frac14}\, e^{\frac{\pi i}{2}(k+1+3hk)} \frac{1}{2z} \left(q_1^{1/2}; q_1^{1/2}\right)_\infty h^o\left(q_1^{1/2}; \frac{iw}{z}\right).$$

We next turn to $\sum_2$. As before we can see that

$$\sum_2 = e^{-\pi i w}\, q^{-\frac12} \frac{1}{k} \sum_{\nu \pmod k} (-1)^\nu\, e^{\frac{6\pi i h \nu^2}{k}} \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}+\frac{\omega_n}{2}} e^{-\frac{6\pi z x^2}{k}}$$
$$H_w^o\left(\frac{2\pi i h}{k}(2\nu+1) - \frac{2\pi z}{k}(2x+1)\right) e^{\frac{\pi i}{k}(2n+1)(x-\nu)}\, dx.$$

Substituting $x \mapsto x + \frac{\omega_n}{2}$ and writing $n = 3p+\delta$ with $p\in\mathbb{Z}$ and $\delta\in\{0,\pm1\}$ gives

$$\sum_2 = e^{-\pi i w}\, q^{-\frac12} \frac{1}{k} \sum_{\nu \pmod k} (-1)^\nu\, e^{\frac{6\pi i h \nu^2}{k}} \sum_{\delta,p} e^{-\frac{\pi i}{k}(6p+2\delta+1)\nu-\frac{\pi}{24kz}(6p+2\delta+1)^2}$$
$$\int_{\mathbb{R}} e^{-\frac{6\pi z x^2}{k}} H_w^o\left(\frac{2\pi i h}{k}(2\nu+1) - \frac{2\pi z}{k}(2x+1) - \frac{\pi i}{3k}(6p+2\delta+1)\right) dx.$$

We next change ν into $\frac{1+k}{2}(-h'(\nu+p)-1)$ and $x\mapsto \frac{x-1}{2}$. A lengthy calculation gives

$$\sum_2 = -e^{-\pi i w} q^{\frac14} \frac{1}{2k} e^{\frac{\pi i}{2}k(1+3h)} \sum_{p,\delta,\nu} (-1)^{p+\delta} q_1^{\frac{1}{48}(6p+2\delta+1)^2}\, e^{\frac{\pi i}{2k}((2\delta+1)-6\nu)}$$
$$e^{\frac{\pi i h'}{2k}\left(-3\nu^2+\nu(2\delta+1)-\frac{1}{12}(2\delta+1)^2\right)} \int_{\mathbb{R}} e^{-\frac{3\pi z x^2}{2k}+\frac{3\pi z x}{k}}\, H_w^o\left(\frac{2\pi i\nu}{k} - \frac{2\pi z x}{k} - \frac{\pi i}{3k}(2\delta+1)\right) dx.$$

Now the integral is independent of p. Moreover for $\delta=1$ the sum over p vanishes since the pth and the $-(p+1)$th term cancel. For $\delta=0,-1$ it equals $\eta\left(\frac{\pi i}{k}\left(h'+\frac{i}{z}\right)\right)$. Changing for $\delta=-1$, $\nu\mapsto-\nu$ gives

$$\sum_2 = -e^{-\pi i w} q^{\frac{1}{4}} \frac{1}{2k} e^{\frac{\pi i}{2}k(3h+1)-\frac{\pi i h'}{24k}} \eta\left(\frac{\pi i}{k}\left(h'+\frac{i}{z}\right)\right) \sum_{\substack{\nu \pmod k\\ \pm}} \pm e^{\frac{\pi i}{2k}\left(h'\left(-3\nu^2+\nu\right)\mp 6\nu\pm 1\right)} I_{k,\nu}^{o,\pm}(z;w).$$

Using that

$$\frac{1}{(q^2;q^2)_\infty} = \omega_{2h,k}(2z)^{\frac{1}{2}} \frac{e^{\frac{\pi}{12k}\left(\frac{1}{2z}-2z\right)}}{\left(q_1^{1/2};q_1^{1/2}\right)_\infty}$$

now gives the claim of the theorem. □

We next consider the case that k is even. Define

$$l^o(z;w) := \frac{1}{(q^2;q^2)_\infty} \sum_{m\in\mathbb{Z}} \frac{(-1)^m\, q^{3m^2+5m}}{1-e^{2\pi i w}q^{2m+1}},$$

and

$$J_{k,\nu}^{o,\pm}(z;w) := \int_{\mathbb{R}} e^{-\frac{3\pi z}{2k}\left(x^2-2x\right)} H_w^o\left(\frac{2\pi i}{k}(2\nu+1)-\frac{2\pi z x}{k}\mp\frac{2\pi i}{3k}\right)dx.$$

Theorem 7.2. *Assume that k is even and $hh'\equiv -1 \pmod{4k}$. Then we have*

$$\begin{aligned}R^o(z;w) &= i^{-h'} z^{-\frac{1}{2}} e^{-\frac{2\pi z}{3k}-\frac{10\pi}{3kz}+\frac{\pi i}{2k}(h+7h')}\, e^{2\pi i w+\frac{3\pi w^2 k}{2z}-\frac{2\pi w}{z}} \omega_{h,\frac{k}{2}}\, l^o\left(q_1;\frac{iw}{z}\right)\\ &\quad + \frac{1}{k}(-1)^{\frac{1}{2}(h'+1)}\, e^{-\pi i w}\, \omega_{h,\frac{k}{2}}\, z^{\frac{1}{2}}\, e^{\frac{\pi i}{2k}(h-3h'-6)-\frac{2\pi z}{3k}}\\ &\quad\quad \sum_{\substack{\nu \pmod{k/2}\\ \pm}} (-1)^\nu e^{\frac{\pi i}{2k}\left(h'(-12\nu-12\nu^2\pm 4\nu\pm 2)-12\nu\pm 2\right)} J_{k,\nu}^{o,\pm}(z;w).\end{aligned}$$

Proof. Via analytic continuation, we may assume that $w\in\mathbb{C}$ with $\mathrm{Re}(w)\neq 0$ sufficiently small and $z>0$ is real. We use the notation from above and write in (7.1) $n=\nu+\frac{k}{2}m$, with ν running modulo $\frac{k}{2}$ and $m\in\mathbb{Z}$. This gives that

$$\widetilde{R}^o(z;w) = e^{-\pi i w}\, q^{-\frac{1}{2}} \sum_{\nu \pmod{k/2}} (-1)^\nu\, e^{\frac{6\pi i h\nu^2}{k}} \sum_{m\in\mathbb{Z}} (-1)^m e^{-\frac{6\pi z}{k}\left(\nu+\frac{km}{2}\right)^2} H_w^o\left(\frac{2\pi i h}{k}(2\nu+1)-\frac{2\pi z}{k}(2\nu+km+1)\right). \tag{7.6}$$

We apply Poisson summation on the inner sum and get, changing x into $2\nu+kx$,

$$\widetilde{R}^o(z;w) = e^{-\pi i w}\, q^{-\frac{1}{2}} \frac{1}{k} \sum_{\nu \pmod{k/2}} (-1)^\nu\, e^{\frac{6\pi i h\nu^2}{k}} \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{-\frac{3\pi z x^2}{2k}} H_w^o\left(\frac{2\pi i h}{k}(2\nu+1)-\frac{2\pi z}{k}(x+1)\right) e^{\frac{\pi i}{k}(2n+1)(x-2\nu)}dx.$$

In the part of the sum on n with $n<0$, we make the change of variables $n\mapsto -n-1$, $x\mapsto -x$, $\nu\mapsto-\nu$, and then $x\mapsto x\pm1$ to get

$$\widetilde{R}^o(z;w)=e^{-\pi iw-\frac{3\pi z}{2k}}\,q^{-\frac12}\frac{1}{k}\sum_{\nu\pmod{k/2}}(-1)^\nu\,e^{\frac{6\pi ih\nu^2}{k}}\sum_{\substack{n\geq0\\ \mp}}\int_{\mathbb{R}}e^{-\frac{3\pi z(x^2\mp2x)}{2k}} H_w^o\left(\frac{2\pi ih}{k}(\pm2\nu+1)\mp\frac{2\pi zx}{k}\right)e^{\frac{\pi i}{k}(2n+1)(x\mp1-2\nu)}\,dx.$$

We next shift the path of integration though

$$\omega_n:=\frac{(2n+1)}{3z}.$$

Using the function $S_w^{\pm}(x)$ defined in (3.3), we see that poles of the integrand can only lie in points

$$x_m^{\pm}:=\frac{i}{z}\left(m\pm wk\right). \tag{7.7}$$

Moreover we see that a non-trivial residue can only occur if m is odd, and for fixed odd m, there is at most for one ν modulo $k/2$ a nontrivial residue. This can be chosen as

$$\nu_m^{\pm}:=\frac12(-h'm\mp1). \tag{7.8}$$

Thus poles may at most occur for x_m and ν_m as in (7.7) and (7.8). Shifting the path of integration through ω_n, we have to take those $x_m^{\pm}$ into account with $m>0$, m odd, and $n\geq\frac{3m\pm1}{2}$. Denoting by $\lambda_{n,m}^{\pm}$ the residue of each summand, we get

$$\widetilde{R}^o(z;w)=\sum_1+\sum_2,$$

where

$$\sum_1=\frac{2\pi i}{k}\,e^{-\pi iw-\frac{3\pi z}{2k}}\,q^{-\frac12}\sum_{\substack{m\geq1,\text{ odd}\\ \pm}}\ \sum_{n\geq\frac12(3m\pm1)}\lambda_{n,m}^{\pm},$$

$$\sum_2=e^{-\pi iw-\frac{3\pi z}{2k}}\,q^{-\frac12}\frac1k\sum_{\nu\pmod{k/2}}(-1)^\nu\,e^{\frac{6\pi ih\nu^2}{k}}\sum_{\substack{n\in\mathbb{N}_0\\ \pm}}\int_{\mathbb{R}+\omega_n}e^{-\frac{3\pi z}{2k}(x^2\mp2x)} H_w^o\left(\frac{2\pi ih}{k}(\pm2\nu+1)\mp\frac{2\pi zx}{k}\right)e^{\frac{\pi i}{k}(2n+1)(x\mp1-2\nu)}\,dx.$$

We first consider $\sum_1$. We have

$$\lambda_{n,m}^{\pm}=\pm\frac{(-1)^{\nu_m^{\pm}}ke^{\frac{6\pi ih\left(\nu_m^{\pm}\right)^2}{k}-\frac{3\pi z}{2k}\left({x_m^{\pm}}^2\mp2x_m^{\pm}\right)+\frac{\pi i}{k}(2n+1)(x_m^{\pm}\mp1-2\nu_m^{\pm})}}{2\pi z\cosh\left(\mp\pi iw-\frac{\pi ih}{k}(2\nu_m^{\pm}\pm1)+\frac{\pi zx_m^{\pm}}{k}\right)}.$$

From this one sees that

$$\lambda_{n+1,m}^{\pm}=e^{\frac{2\pi i}{k}(x_m^{\pm}\mp1-2\nu_m^{\pm})}\,\lambda_{n,m}^{\pm}.$$

A lengthy calculation gives

$$\sum_1 = \frac{1}{z} e^{2\pi i w + \frac{3\pi w^2 k}{2z}} q_1^{\frac{1}{4}} \left(\sum_{\substack{m \geq 1 \text{ odd} \\ \pm}} (-1)^{-\frac{h' m}{2}} \frac{q_1^{\frac{3}{4}m^2 + \frac{1}{2}(1 \pm 1)m} e^{-\frac{\pi w}{z}(1 \pm 1)}}{1 - q_1^m e^{\mp \frac{2\pi w}{z}}} \right).$$

In the sum for the $-$-sign we change $m \mapsto -m$ to obtain a sum over all $m \in \mathbb{Z}$. Then changing $m \mapsto 2m+1$ gives

$$\sum_1 = \frac{i^{-h'}}{z} e^{\frac{3\pi w^2 k}{2z} + 2\pi i w - \frac{2\pi w}{z}} q^{\frac{1}{4}} q_1^{\frac{7}{4}} (q_1^2; q_1^2)_\infty \cdot l^o \left(q_1; \frac{iw}{z} \right).$$

We next consider $\sum_2$. In the same way as before we see that

$$\sum_2 = \frac{2}{k} e^{-\pi i w} q^{-\frac{1}{2}} \sum_{\nu \pmod{k/2}} (-1)^\nu e^{\frac{6\pi i h \nu^2}{k}} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R} + \frac{\omega_n}{2}} e^{-\frac{6\pi z x^2}{k}} H_w^o \left(\frac{2\pi i h}{k}(2\nu+1) - \frac{2\pi z(2x+1)}{k} \right) e^{\frac{2\pi i}{k}(2n+1)(x-\nu)} dx.$$

Substituting $x \mapsto x + \frac{\omega_n}{2}$ and writing $n = 3p + \delta$ with $p \in \mathbb{Z}$ and $\delta \in \{0, \pm 1\}$ gives

$$\sum_2 = \frac{2}{k} e^{-\pi i w} q^{-\frac{1}{2}} \sum_{\nu \pmod{k/2}} (-1)^\nu e^{\frac{6\pi i h \nu^2}{k}} \sum_{p,\delta} e^{-\frac{\pi(6p+2\delta+1)^2}{6kz} - \frac{2\pi i}{k}(6p+2\delta+1)\nu} \int_{\mathbb{R}} e^{-\frac{6\pi z x^2}{k}} H_w^o \left(\frac{2\pi i}{k} \left(h(2\nu+1) - 2p \right) - \frac{2\pi z(2x+1)}{k} - \frac{2\pi i}{3k}(2\delta+1) \right) dx.$$

Changing $\nu \mapsto \frac{1}{2}(-h'(2\nu + 2p + 1) - 1)$ and $x \mapsto 2x + 1$ gives after a lengthy calculation

$$\sum_2 = e^{-\pi i w} q^{\frac{1}{4}} \frac{1}{k} (-1)^{\frac{1}{2}(h'+1)} e^{-\frac{3\pi i h'}{2k} - \frac{3\pi i}{2k}} \sum_{\substack{\nu \pmod{k/2} \\ \pm}} (-1)^\nu e^{\frac{\pi i}{2k} h' \left(-12\nu - 12\nu^2 \pm 4\nu \pm 2 \right)} e^{\frac{\pi i}{2k}(-12\nu \pm 2)} \sum_p (-1)^p q_1^{\frac{1}{12}(6p+2\delta+1)^2} J_{k,\nu}^{o,\pm}(z; w).$$

The sum over p vanishes for $\delta = 1$. If $\delta = 0, -1$, then it equals $\eta \left(\frac{4\pi i}{k} (h' + \frac{i}{z}) \right)$. Using

$$\frac{1}{(q^2; q^2)_\infty} = \frac{\omega_{h, \frac{k}{2}} \, z^{\frac{1}{2}} \, e^{\frac{\pi}{6k}(z^{-1} - z)}}{(q_1^2; q_1^2)_\infty}$$

now easily gives the claim. □

We next conclude from Theorems 7.1 and 7.2 a transformation law for $R_2^o(q)$. For this let

$$\begin{aligned}
g^o(q) &:= \frac{1}{(q;q)_\infty}\sum_{m\in\mathbb{Z}}(-1)^m\frac{q^{\frac12(3m^2+m)}}{(1+q^m)},\\
h^o(q) &:= \frac{1}{(q;q)_\infty}\sum_{m\in\mathbb{Z}}(-1)^m\frac{q^{\frac12(3m^2+m)}(1-3q^m)}{(1+q^m)^3},\\
I_{k,\nu}^{o,\pm}(z) &:= \int_{\mathbb{R}}\frac{e^{-\frac{3\pi zx^2}{2k}+\frac{\pi zx}{k}}}{\sinh^3\left(\mp\frac{\pi i\nu}{k}+\frac{\pi zx}{k}\pm\frac{\pi i}{6k}\right)}\,dx,\\
m^o(q) &:= \frac{1}{(q^2;q^2)_\infty}\sum_{m\in\mathbb{Z}}(-1)^m\frac{q^{3m^2+5m}}{1-q^{2m+1}},\\
J_{k,\nu}^{o,\pm}(z) &:= \int_{\mathbb{R}}\frac{e^{-\frac{3\pi zx^2}{2k}+\frac{\pi zx}{k}}}{\sinh^3\left(-\frac{\pi i}{k}(2\nu+1)+\frac{\pi zx}{k}\pm\frac{\pi i}{3k}\right)}\,dx.
\end{aligned}$$

Corollary 7.3. (1) *If k is odd, then*

$$R_2^o(q)=-\frac{1}{\sqrt2}\omega_{2h,k}\ e^{\frac{\pi}{24kz}-\frac{2\pi z}{3k}+\frac{\pi ih}{2k}+\frac{\pi i}{2}(k+1+3hk)}\left(-\frac{3k}{8\,\pi\,z^{\frac32}}g^o\left(q_1^{\frac12}\right)-\frac{1}{16z^{\frac52}}h^o\left(q_1^{\frac12}\right)\right)$$
$$-\frac{1}{8\sqrt{2k}}\,e^{\frac{\pi i}{2}k(3h+1)+\frac{\pi ih}{2k}-\frac{2\pi z}{3k}}\,\omega_{2h,k}\,z^{\frac12}\sum_{\substack{\nu\pmod k\\ \pm}}\pm e^{\frac{\pi i}{2k}\left(h'\left(-3\nu^2+\nu\right)\pm\frac13\mp2\nu\right)}\,I_{k,\nu}^{o,\pm}(z).$$

(2) *If k is even, then*

$$R_2^o(q)=i^{-h'}e^{-\frac{2\pi z}{3k}-\frac{10\pi}{3kz}+\frac{\pi i}{2k}(h+7h')}\,\omega_{h,\frac k2}\left(-\frac{3k}{8\pi z^{\frac32}}m^o(q_1)-\frac{1}{z^{\frac52}}q_1^{-2}R_2^o(q_1)\right)$$
$$+\frac{1}{8k}(-1)^{\frac12(1+h')}\,\omega_{h,\frac k2}\,z^{\frac12}\,e^{\frac{\pi i}{2k}(h-3h')-\frac{2\pi z}{3k}-\frac{\pi i}{k}}$$
$$\sum_{\substack{\nu\pmod{k/2}\\ \pm}}(-1)^\nu\,e^{\frac{\pi ih'}{k}\left(-6\nu-6\nu^2\pm2\nu\pm1\right)-\frac{2\pi i\nu}{k}\pm\frac{\pi i}{3k}}\,J_{k,\nu}^{o,\pm}(z).$$

Proof. We have

$$L^o\left(\frac{e^{\frac{3\pi kw^2}{2z}+2\pi iw-\frac{\pi w}{2z}}}{1+e^{-\frac{\pi w}{z}}\,q^m}\right)=\frac{i(1-q^m)}{8z(1+q^m)^2}-\frac{3k}{8\pi z(1+q^m)}-\frac{1-6q^m+q^{2m}}{32z^2(1+q^m)^3}.$$

Moreover

$$\sum_{m\in\mathbb{Z}}(-1)^m\frac{q^{\frac12(3m^2+m)}(1-q^m)}{(1+q^m)^2}=0,$$

since the mth and $-m$th term cancel, and

$$\sum_{m\in\mathbb{Z}}(-1)^m\frac{q^{\frac12(3m^2+m)}\left(1-6q^m+q^{2m}\right)}{(1+q^m)^3}=2\sum_{m\in\mathbb{Z}}(-1)^m\frac{q^{\frac12(3m^2+m)}(1-3q^m)}{(1+q^m)^3}.$$

Furthermore

$$L^o\left(e^{-\pi i w}\, H_w^o(x)\right) = \frac{e^x}{8\sinh^3\left(-\frac{x}{2}\right)}. \tag{7.9}$$

Now (i) follows easily using Theorem 7.1.

To see (ii), we observe that

$$L^o\left(\frac{e^{\frac{3\pi k w^2}{2z}+2\pi i w-\frac{2\pi w}{z}}}{1-e^{-\frac{2\pi w}{z}}\, q^{2m+1}}\right) = \frac{i}{2z(1-q^{2m+1})^2} - \frac{3k}{8\pi z(1-q^{2m+1})} - \frac{1+q^{2m+1}}{2z^2(1-q^{2m+1})^3},$$

use again (7.9), and

$$\sum_{m\in\mathbb{Z}} \frac{(-1)^m\, q^{3m^2+5m}}{(1-q^{2m+1})^2} = 0,$$

since the mth and $-(m+1)$th term cancel. Then (ii) follows from Theorem 7.2. □

8. Proof of Theorem 1.6

Here we show asymptotics for $\eta_2^o(n)$. As in Section 3, we estimate the Mordell type integrals occurring in the transformation law of $R^o(q)$ and Kloosterman sums. We assume the notation from above.

Lemma 8.1. (1) *If k is odd, then*

$$z^{\frac{1}{2}} I_{k,\nu}^{o,\pm}(z) \ll \frac{n^{\frac{1}{4}}}{\left\{\frac{\nu}{k}-\frac{1}{6k}\right\}^3}.$$

(2) *If k is even, then*

$$z^{\frac{1}{2}} J_{k,\nu}^{o,\pm}(z) \ll \frac{n^{\frac{1}{4}}}{\left\{\frac{2\nu+1}{k}\mp\frac{1}{3k}\right\}^3}.$$

Lemma 8.2. *Let $l, n \in \mathbb{Z}$.*

(1) *If k is odd, and $hh' \equiv -1 \pmod{k}$ with h' even, then*

$$\sum_{\substack{h \pmod{k}^* \\ \sigma_1\le Dh'\le\sigma_2}} \omega_{2h,k}\, e^{\frac{3\pi i h k}{2}}\, e^{\frac{\pi i}{2k}(h(1+4n)+2lh')} \ll (3n+1,k)^{\frac{1}{2}+\epsilon}\, k^{\frac{1}{2}+\epsilon}. \tag{8.1}$$

(2) *If k is even and $hh' \equiv -1 \pmod{4k}$, then*

$$\sum_{\substack{h \pmod{k}^* \\ \sigma_1\le Dh'\le\sigma_2}} i^{-h'}\, \omega_{h,\frac{k}{2}}\, e^{\frac{\pi i}{2k}((1+4n)h+(7+4l)h')} \ll (96n+25,k)^{\frac{1}{2}}\, k^{\frac{1}{2}+\epsilon}, \tag{8.2}$$

$$\sum_{\substack{h \pmod{k}^* \\ \sigma_1\le Dh'\le\sigma_2}} i^{h'}\, \omega_{h,\frac{k}{2}}\, e^{\frac{\pi i}{2k}((1+4n)h+(-1+4l)h')} \ll (96n+25,k)^{\frac{1}{2}}\, k^{\frac{1}{2}+\epsilon}. \tag{8.3}$$

Proof. (i) Using that h' is even, one can see that the left-hand side of (8.1) is well-defined. We only show the estimate for the full modulus k, the restriction to $\sigma_1 \leq Dh' \leq \sigma_2$ can be concluded as in [20]. We change h into $\bar{2}h$ and h' into $2h'$. This leads to the sum

$$\sum_{h \pmod{k}^*} \omega_{h,k}\, e^{\frac{2\pi i}{k}\left(\left(\bar{2}n+\frac{1}{4}(3k^2+1)\bar{2}\right)h+lh'\right)}. \tag{8.4}$$

By [1], (8.4) can be estimated against

$$\left(24\left(\bar{2}n+\frac{1}{4}\left(3k^2+1\right)\bar{2}\right)+1,k\right)^{\frac{1}{2}} k^{\frac{1}{2}+\epsilon} = (3n+1,k)^{\frac{1}{2}}\, k^{\frac{1}{2}+\epsilon}.$$

(ii) We only show (8.2), since (8.3) can be proven similarly. We first observe that (8.2) is well-defined since changing $h \mapsto h+k$ implies that $h' \mapsto h'+rk$ with $r \equiv k+1 \pmod 4$. We then change the sum into a sum $\pmod{4k}$. This gives

$$\frac{1}{4}\sum_{h \pmod{4k}^*} \omega_{h,\frac{k}{2}}\, e^{\frac{2\pi i}{4k}(h(4n+1)+(4l-7-k)h')} \ll (96n+25,k)^{\frac{1}{2}} k^{\frac{1}{2}+\epsilon}$$

by [1] as claimed . □

Proof of Theorem 1.6. We use the circle method and proceed as in the proof of Theorem 1.2. This gives

$$\eta_2^o(n) = \sum_{h,k} e^{-\frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} R_2^o\left(e^{\frac{2\pi i}{k}(h+iz)}\right)\cdot e^{\frac{2\pi n z}{k}}\, d\Phi.$$

Corollary 7.3 gives

$$\eta_2^o(n) = -\frac{i}{\sqrt{2}}\sum_{\substack{k \text{ odd}\\ h}} e^{-\frac{2\pi i h n}{k}+\frac{\pi i h}{2k}+\frac{\pi i}{2}(k+3hk)}\omega_{2h,k}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi z}{k}\left(n-\frac{1}{3}\right)+\frac{\pi}{24kz}}\left(-\frac{3\pi k}{8z^{\frac{3}{2}}}g^o\left(q_1^{\frac{1}{2}}\right)\right.$$
$$\left.-\frac{1}{16z^{\frac{5}{2}}}h^o\left(q_1^{\frac{1}{2}}\right)\right)d\Phi + \sum_{\substack{k \text{ even}\\ h}} i^{-h'} e^{-\frac{2\pi i h n}{k}+\frac{\pi i}{2k}(h+7h')}\omega_{h,\frac{k}{2}}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi z}{k}\left(n-\frac{1}{3}\right)-\frac{10\pi}{3kz}}\left(-\frac{3k}{8\pi z^{\frac{3}{2}}}m^o(q_1)\right.$$
$$\left.-\frac{1}{z^{\frac{5}{2}}}q_1^{-2}R_2^o(q_1)\right) - \frac{1}{8\sqrt{2}}\sum_{\substack{k \text{ odd}\\ h}}\frac{\omega_{2h,k}}{k}\, e^{-\frac{2\pi i h n}{k}}\, e^{\frac{\pi i}{2}k(3h+1)+\frac{\pi i h}{2k}} \sum_{\substack{\nu \pmod{k}\\ \pm}} \pm e^{\frac{\pi i}{2k}\left(h'\left(-3\nu^2+\nu\right)\pm\frac{1}{3}\mp 2\nu\right)}$$
$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{\frac{1}{2}} e^{\frac{2\pi z}{k}\left(n-\frac{1}{3}\right)} I_{k,\nu}^{o,\pm}(z)d\Phi + \frac{1}{8k}\sum_{\substack{k \text{ even}\\ h}}(-1)^{\frac{1}{2}(1+h')}\omega_{h,\frac{k}{2}}\, e^{\frac{\pi i}{2k}(h-3h')-\frac{\pi i}{k}}\, e^{-\frac{2\pi i h n}{k}}\sum_{\substack{\nu \pmod{k/2}\\ \pm}}(-1)^\nu$$
$$e^{\frac{\pi i}{k}(h'(-6\nu-6\nu^2\pm 2\nu\pm 1))-\frac{2\pi i\nu}{k}\pm\frac{\pi i}{3k}}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{\frac{1}{2}} e^{\frac{2\pi z}{k}\left(n-\frac{1}{3}\right)} J_{k,\nu}^{o,\pm}(z)\, d\Phi.$$

We denote the occurring sums by $\sum_1, \sum_2, \sum_3$, and $\sum_4$. As in the proof of Theorem 1.2 we obtain, using Lemma 8.2,

$$\sum_1 + \sum_2 = -\frac{i}{\sqrt{2}}\sum_{k \text{ odd}} A_k^o(n)\int_{-\frac{1}{kN}}^{\frac{1}{kN}} e^{\frac{2\pi z}{k}\left(n-\frac{1}{3}\right)+\frac{\pi}{24kz}}\left(-\frac{3\pi k}{16z^{\frac{3}{2}}}+\frac{1}{64z^{\frac{5}{2}}}\right)d\Phi + O(n^{2+\epsilon}).$$

We consider integrals of the form

$$I_{k,r} := \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^r e^{\frac{2\pi}{k}\left(z\left(n-\frac{1}{3}\right)+\frac{1}{48z}\right)} d\Phi$$

with $r \in \{-\frac{5}{2}, -\frac{3}{2}\}$. As in the proof of Theorem 1.2 this equals

$$I_{k,r} = \frac{1}{ki}\int_{\Gamma} z^r e^{\frac{2\pi}{k}\left(z\left(n-\frac{1}{3}\right)+\frac{1}{48z}\right)} dz + O(n^{\frac{3}{4}}).$$

Making the substitution $t = \frac{\pi}{24kz}$ gives

$$I_{k,r} = \frac{2\pi}{k}\left(\frac{\pi}{24k}\right)^{1+r} \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} t^{-2-r} e^{t+\frac{\alpha}{t}} dt + O\left(n^{\frac{3}{4}}\right),$$

where $\gamma \in \mathbb{R}$ and $\alpha = \frac{\pi^2}{36k^2}(3n-1)$. Now the integrals can be computed as before and lead to Bessel functions of order $-\frac{1}{2}$ and $-\frac{3}{2}$. More precisely, we obtain that $\sum_1 + \sum_2$ equals

$$-\frac{i}{\sqrt{2}}\sum_{k \text{ odd}} A_k^o(n)\left(-\frac{3\pi^2(3n-1)^{\frac{1}{4}}}{4} I_{-\frac{1}{2}}\left(\frac{\pi}{3k}\sqrt{3n-1}\right) + \frac{\pi(3n-1)^{\frac{3}{4}}}{4k} I_{-\frac{3}{2}}\left(\frac{\pi}{3k}\sqrt{3n-1}\right)\right) + O(n^{2+\epsilon}).$$

$\sum_3$ and $\sum_4$ can be estimated against $O\left(n^{2+\epsilon}\right)$ as in the proof of Theorem 1.2 using Lemma 8.2 and Lemma 8.1. ☐

References

[1] G. E. Andrews, *On the theorems of Watson and Dragonette for Ramanujan's mock theta functions*, Amer. J. Math. **88** (1966), pages 454-490.
[2] G. E. Andrews, *Mock theta functions,* Theta functions - Bowdoin 1987, Part 2 (Brunswick, ME., 1987), pages 283-297, Proc. Sympos. Pure Math. **49**, Part 2, Amer. Math. Soc., Providence, RI., 1989.
[3] G. E. Andrews, *Partitions, Durfee symbols, and the Atkin-Garvan moments of ranks*, Invent. Math., accepted for publication.
[4] G. Andrews, *The number of smallest parts in the partitions of n*, preprint.
[5] G. E. Andrews, F. Dyson, and D. Hickerson, *Partitions and indefinite quadratic forms*, Invent. Math. **91** (1988), pages 391-407.
[6] A.O.L. Atkin and F. Garvan, *Relations between the ranks and the cranks of partitions*, Ramanujan Journal **7** (2003), pages 243-366.
[7] A. O. L. Atkin and H. P. F. Swinnerton-Dyer, *Some properties of partitions*, Proc. London Math. Soc. **4** (1954), pages 84-106.
[8] K. Bringmann, *Asymptotics for rank partition functions*, Transaction of the AMS, accepted for publication.
[9] K. Bringmann, *On certain congruences for Dyson's ranks*, submitted for publication.
[10] K. Bringmann, F. Garvan, and K. Mahlburg, *Higher rank moments*, in preparation.
[11] J. H. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. **125** (2004), pages 45-90.
[12] K. Bringmann and K. Ono, *The $f(q)$ mock theta function conjecture and partition ranks*, Invent. Math. **165** (2006), pages 243-266.
[13] K. Bringmann and K. Ono, *Dysons ranks and Maass forms*, Ann. Math., accepted for publication.
[14] K. Bringmann, K. Ono, and R. Rhoades, *Eulerian series as modular form*, Journal of the American Mathematical Society, accepted for publication.
[15] Y.-S. Choi, *Tenth order mock theta functions in Ramanujan's lost notebook*, Invent. Math. **136** (1999), pages 497-569.
[16] H. Cohen, *q-identities for Maass waveforms*, Invent. Math. **91** (1988), pages 409-422.
[17] L. Dragonette, *Some asymptotic formulae for the mock theta series of Ramanujan*, Trans. Amer. Math. Soc. **72** (1952), pages 474-500.
[18] F. Dyson, *Some guesses in the theory of partitions*, Eureka (Cambridge) **8** (1944), pages 10-15.
[19] D. Hickerson, *On the seventh order mock theta functions*, Invent. Math. **94** (1988), pages 661-677.
[20] J. Lehner, *A partition function connected with the modulus five.* Duke Math. J. **8** (1941), pages 631–655.
[21] K. Ono, *Distribution of the partition function modulo m*, Ann. of Math. **151** (2000), pages 293-307.

[22] S. Ramanujan, *The lost notebook and other unpublished papers*, Narosa, New Delhi, 1988.
[23] G. Shimura, *On modular forms of half integral weight*, Ann. Math. **97** (1973), pages 440-481.
[24] S. Treneer, *Congruences for the coefficients of weakly holomorphic modular forms*, Proc. London Math. Soc. **93** (2006), pages 304–324.
[25] G. N. Watson, *The final problem: An account of the mock theta functions*, J. London Math. Soc. **2** (2) (1936), pages 55-80.
[26] S. P. Zwegers, *Mock theta-functions and real analytic modular forms,* q-series with applications to combinatorics, number theory, and physics (Urbana, IL, 2000), 269–277, Contemp. Math., **291**, Amer. Math. Soc., Providence, RI, 2001.

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