GENERALIZED L-FUNCTIONS FOR MEROMORPHIC MODULAR FORMS AND THEIR RELATION TO THE RIEMANN ZETA FUNCTION

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Abstract. In this paper, we construct a family of generalized L-functions, one for each point z in the upper half-plane. We prove that as z approaches $i\infty$, these generalized L-functions converge to an L-function which can be written in terms of the Riemann zeta function.

1. Introduction and statement of results

We begin by recalling some basic properties of L-functions. They encode arithmetic information $c(p^r)$ that comes from local p-adic properties of some global object. This local information is encoded in a series

$$L_p(s) := 1 + \sum_{r=1}^{\infty} \frac{c\left(p^r\right)}{p^{rs}}.$$

In many examples, one may write

$$L_p(s) = \frac{1}{1 - f_p\left(p^{-s}\right) p^{-s}} \tag{1.1}$$

for some polynomials f_p. Extending the definition of $c(p^r)$ multiplicatively, one that then forms the L-series as a Dirichlet series via the *Euler product*

$$L(s) := \prod_p L_p(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}, \tag{1.2}$$

where the product runs over all primes and the series converges for $\mathrm{Re}(s)$ sufficiently large if $c(n)$ grows at most polynomially in n.

By adding archimedean information $L_\infty(s)$ at the real place one may form a complete L-function $\Lambda(s) := L_\infty(s) \prod_p L_p(s)$ which has a meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation

$$\Lambda(k - s) = \Lambda(s) \tag{1.3}$$

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for some $k \in \mathbb{R}$. The original example of an L-function is the *Riemann zeta function*, defined for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}. \tag{1.4}$$

From (1.4), one sees both the Euler product and series representations characterized by (1.2), and the local factors indeed have the shape (1.1). Riemann [15] proved that ζ has a meromorphic continuation to $\mathbb{C}$ with only a simple pole at $s = 1$. He also showed that the function

$$\xi(s) := \frac{1}{2}\pi^{-\frac{s}{2}} s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

where $\Gamma(s)$ is the Gamma function, satisfies the functional equation

$$\xi(s) = \xi(1-s), \tag{1.5}$$

verifying the property (1.3).

The functional equation (1.5) can be explained by viewing the zeta function as the Mellin transform of a modular form (see [17, p. 452]). It is hence natural to ask whether the functional equation (1.3) more generally arises from connections with modular objects. A number of results and important conjectures about the interplay between modular objects and L-functions have originated from this question. For example, Hamburger [9] proved a converse theorem characterizing $\zeta(s)$ by its functional equation which was later extended by Hecke [11] and Weil [16] (see also [2]). Specifically, the version of the converse theorem of Hecke states that if $c(n)$ are slow-growing and (1.3) is satisfied, then the L-function comes from a Mellin transform of a modular form over $\mathrm{SL}_2(\mathbb{Z})$. The numbers $c(n)$ are then realized as the Fourier coefficients of the corresponding modular form. The idea to relate objects in different settings through their corresponding L-functions is one of the cornerstones of the Langlands program [14], which conjectures deep connections between number theory and geometry.

In this paper, we construct a family of functions $L_z(s)$ for $s \in \mathbb{C}$ and $z \in \mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ away from

$$\mathcal{S} := \left\{z \in \mathbb{H} : \exists M \in \mathrm{SL}_2(\mathbb{Z}) \text{ such that } Mz \in i\mathbb{R}^+\right\}.$$

We call these functions generalized L-functions because they resemble L-functions. By the converse theorem of Hecke [11] and Weil [16], if these are not Mellin transforms of modular forms, then either the Euler product and series representations in (1.2) or the functional equation in (1.3) must not hold. We consider the functional equation (1.3) to be the fundamental property of the function in s and then concentrate on the properties as a function of the other variable z. In particular, the (generalized) L-functions L_z satisfy a functional equation, are harmonic and $\mathrm{SL}_2(\mathbb{Z})$-invariant as a function of $z \in \mathbb{H} \setminus \mathcal{S}$, and are related to the Riemann zeta function in the limit $z \to x + i\infty$ with $x \in \mathbb{R} \setminus \mathbb{Z}$.

Theorem 1.1. *Let $s \in \mathbb{C} \setminus \{1\}, z \in \mathbb{H}$. There exists $L_z(s)$ satisfying the following properties:*

(1) *For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H} \setminus \mathcal{S}$ we have*

$$L_{\frac{az+b}{cz+d}}(s) = L_z(s).$$

(2) *The function* $z \mapsto L_z(s)$ *is harmonic on* $\mathbb{H} \setminus \mathcal{S}$.
(3) *We have the functional equation*

$$L_z(2-s) = -L_z(s).$$

(4) *Suppose that* $1 < \mathrm{Re}(s) < 2$. *Then for* $x \in \mathbb{R} \setminus \mathbb{Z}$ *we have*

$$\lim_{y\to\infty}\left(L_{x+iy}(s) - \frac{2\pi i}{s}y^s - \frac{2\pi i}{s-2}y^{2-s} + 2\,\mathrm{Arg}\left(1-e^{2\pi i x}\right)y^{s-1}\right) = -\frac{24i}{(2\pi)^{s-1}}\Gamma(s)\zeta(s)\zeta(s-1),$$

where Arg *denotes the principal value of the argument.*

Remarks.
(1) Although the generalized L-functions $L_z(s)$ do not seem to satisfy the properties of classical L-functions (we expect that one can define a series via the integral in (3.3) below, but we do not study that in this paper), note that the limit in Theorem 1.1 (4) is the L-function associated to the weight two Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$. The series representation for $\zeta(s)\zeta(s-1)$ may be found in (3.2)) and its Euler product follows from the right-hand side of (1.4)).
(2) A more general version of Theorem 1.1 (4) holds for all $s \in \mathbb{C}$ (see Theorem 4.3).

The paper is organized as follows. In Section 2, we recall polar harmonic Maass forms and relate them to the resolvent kernel, as well as introducing some well-known useful functions and their properties. In Section 3, we define the functions $L_z(s)$ and prove Theorem 1.1 (1)–(3). We finally show Theorem 1.1 (4) in Section 4.

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2. Preliminaries

2.1. **Special functions and their properties.** We require certain special functions and their properties. For $s \in \mathbb{C}$ and $y > 0$, we let $\Gamma(s,y) := \int_y^\infty e^{-t}t^{s-1}dt$ be the *incomplete gamma function.* By [6, (8.11.1)–(8.11.3)], as $y \to \infty$ we have

$$\Gamma(s,y) = y^{s-1}e^{-y}\left(\sum_{\ell=0}^{N-1}(s-\ell)_\ell y^{-\ell} + O\left(y^{-N}\right)\right), \tag{2.1}$$

where $(a)_\ell := \prod_{j=0}^{\ell-1}(a+j)$ is the *rising factorial.* For $y_1, y_2 \in \mathbb{R}$ with $y_1y_2 > 0$, we also require the *generalized incomplete gamma function* $\Gamma(s, y_1, y_2) := \int_{y_1}^{y_2} e^{-t}t^{s-1}dt$. For $\mathrm{Re}(s) > 0$, we have the relation $\Gamma(s,y_1,y_2) = \gamma(s,y_2) - \gamma(s,y_1)$ where $\gamma(s,y) := \int_0^y e^{-t}t^{s-1}dt$. We also require the identity (see [6, 8.5.1])

$$\Gamma(s,y_1,y_2) = \frac{y_2^s}{s}{}_1F_1(s;s+1;-y_2) - \frac{y_1^s}{s}{}_1F_1(s;s+1;-y_1). \tag{2.2}$$

Here ${}_1F_1(a;b;y)$ denotes the confluent hypergeometric function and for $y_j<0$, y_j^s is defined through the principal branch of the logarithm. The asymptotic behaviour of the generalized incomplete gamma function as $y_1\to\infty$ or $y_2\to\infty$ may thus be obtained from the asymptotic behaviour of the ${}_1F_1$-function. Namely, as $y\to\infty$ we have (see [6, 13.7.2], where we note that $M(a,b,y)={}_1F_1(a;b;y)$)

$$ {}_1F_1(a;b;y)=\Gamma(b)\left(\frac{e^y y^{a-b}}{\Gamma(a)}+\frac{e^{-\pi i a}y^{-a}}{\Gamma(b-a)}\right)\left(1+O_{a,b}\left(y^{-1}\right)\right). $$

Assuming that $s\notin\mathbb{Z}$, [6, 13.7.1] furthermore implies that for any $N\in\mathbb{N}_0$

$$ {}_1F_1\left(s;s+1;y\right)\sim se^y y^{-1}\left(\sum_{j=0}^{N}(1-s)_j y^{-j}+O_{s,N}\left(y^{-N-1}\right)\right). \tag{2.3} $$

Recall that for $\ell\in\mathbb{R}$ the *polylogarithm function* is defined for $|Z|<1$ by (see [6, 25.12.10])

$$ \mathrm{Li}_\ell(Z):=\sum_{n=1}^{\infty}\frac{Z^n}{n^\ell}. $$

We frequently use the identity

$$ \lim_{\varepsilon\to0^+}\sum_{n=1}^{\infty}\frac{e^{-2\pi n(ix+\varepsilon y)}}{\left(n(1+\varepsilon)\right)^\ell}=\lim_{\varepsilon\to0^+}\frac{\mathrm{Li}_\ell\left(e^{-2\pi(ix+\varepsilon y)}\right)}{(1+\varepsilon)^\ell}=\mathrm{Li}_\ell\left(e^{-2\pi i x}\right), \tag{2.4} $$

where in the last equality we employ the fact that, since $x\notin\mathbb{Z}$, we avoid the branch cut of $\mathrm{Li}_\ell(z)$ along the positive real axis from 1 to ∞ and hence the limit exists.

2.2. **Modular forms and polar harmonic Maass forms.** As usual, for $k\in\mathbb{Z}$, $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, and $f:\mathbb{H}\to\mathbb{C}$ we define the *weight k slash operator* by

$$ f\big|_k\gamma(\tau):=j(\gamma,\tau)^{-k}f\left(\frac{a\tau+b}{c\tau+d}\right), $$

where $j(\gamma,\tau):=c\tau+d$. To describe certain modular objects, we require the growth of several functions towards points in $\mathbb{H}\cup\{i\infty\}$. For a non-holomorphic modular form f and a point $z\in\mathbb{H}$, we say that f *exhibits the growth g_z at $\tau=z$* if $f(\tau)-g_z(\tau)$ is bounded in an open neighborhood around z. We say that a singularity of f at a point $z\in\mathbb{H}\cup\{i\infty\}$ has *finite order* if the following holds:

(1) If $z\in\mathbb{H}$, then there exists $n\in\mathbb{N}_0$ such that $(\tau-z)^n f(\tau)$ is bounded for τ in a sufficiently small neighborhood of z.
(2) If $z=i\infty$, then there exists $n\in\mathbb{N}_0$ such that $f(\tau)e^{-2\pi n v}$ is bounded for v sufficiently large.

For $\tau=u+iv$, the weight k *hyperbolic Laplace operator* is given by

$$ \Delta_k:=-v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)+ikv\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right). $$

Throughout if we need to specify in which variable an operator $\mathcal{O}_k$ is taken, then we write $\mathcal{O}_{k,\tau}$. We call a function $f : \mathbb{H} \to \mathbb{C}$ a *weight k polar harmonic Maass form* if it satisfies the following properties:

(1) For every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have $f|_k\gamma = f$.
(2) The function f is annihilated by Δ_k except for a discrete set of singularities.
(3) The singularities of f all have finite order.

We require the following relations, which may be found for example in [3, Lemma 5.2], between the hyperbolic Laplace operator and the *Maass raising operator* $R_k := 2i\frac{\partial}{\partial\tau} + \frac{k}{v}$.

Lemma 2.1.

(1) *We have*

$$R_k\left(f\big|_k\gamma\right) = R_k(f)\big|_{k+2}\gamma.$$

In particular, if f satisfies weight k modularity, then $R_k(f)$ satisfies weight $k+2$ modularity.

(2) *If $\Delta_k(f) = \lambda f$, then $\Delta_{k+2}(R_k(f)) = (\lambda + k)R_k(f)$.*

We require properties of some explicit modular functions (i.e., weight zero meromorphic modular forms). To describe these, for even $k \geq 4$ we define the *weight k Eisenstein series* by

$$E_k(\tau) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi i n\tau},$$

where $\sigma_\ell(n) := \sum_{d|n} d^\ell$ and B_k is the k-th Bernoulli number. We denote the unique normalized newform of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$ by

$$\Delta(\tau) := q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

We then set $J(\tau) := j(\tau) - 744$, where $j(\tau) := \frac{E_4(\tau)^3}{\Delta(\tau)}$. We furthermore define the weight two meromorphic modular form

$$H_z(\tau) := \frac{E_4(\tau)^2E_6(\tau)}{\Delta(\tau)(J(\tau)-J(z))}. \tag{2.5}$$

We require the behaviour of $H_z(\tau)$ as $\tau \to i\infty$. To obtain this, we note that by computing the first term of their Fourier expansions, one easily sees that as $v \to \infty$, we have

$$J(\tau) = e^{-2\pi i\tau} + O\left(e^{-2\pi v}\right), \quad \frac{E_4(\tau)^2E_6(\tau)}{\Delta(\tau)} = e^{-2\pi i\tau} + O(1). \tag{2.6}$$

We see in particular that as $\tau \to i\infty$ (resp. $\tau \to 0$), $H_z(\tau)$ exhibits the growth 1 (resp. $\frac{1}{\tau^2}$), applying the weight two modularity in the second case.

2.3. **Real-analytic Eisenstein series.** Setting $\Gamma_\infty := \{\pm\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right) : n \in \mathbb{Z}\}$, the *weight k real-analytic Eisenstein series* is defined for $k \in 2\mathbb{N}_0$ and $w \in \mathbb{C}$ with $\mathrm{Re}(w) > 1$ by

$$E_k(w;\tau) := \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} v^w\big|_k\gamma$$

and then continued meromorphically to the complex w-plane (see [13, Theorem 4.4.2]). Since we make frequent use of results from [7, 8], we note for comparison the definition [8, (2.14) on p. 239] specialized to $N = 1$ (see also the equivalent definition in [7, Section 5]), where the weight $k = 0$ is omitted, the order of the variables is flipped, and a semicolon is used to separate the variables instead of a comma.

We furthermore define the weight two harmonic Eisenstein series as

$$\widehat{E}_2(\tau) := E_2(\tau) - \frac{3}{\pi v} \qquad \text{where } E_2(\tau) := 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)e^{2\pi in\tau}.$$

Using the so-called Hecke trick, Hecke showed that

$$\widehat{E}_2(\tau) = E_2(0;\tau).$$

The weight zero and weight two real-analytic Eisenstein series are related via the raising operator, as a direct calculation shows.

Lemma 2.2. *For $w \in \mathbb{C}$ which is not a pole of $(w-1)E_0(w;z)$ and $k \in 2\mathbb{N}_0$, we have*

$$R_k\left(E_k(w;\tau)\right) = (w+k)E_{k+2}(w-1;\tau).$$

In particular,

$$R_0\left(E_0(1;\tau)\right) = \widehat{E}_2(\tau).$$

Using the fact that $E_2(w;\tau)$ is an eigenfunction under Δ_2, one obtains the following asymptotic behaviour of the Eisenstein series (see [8, (2.17) on p. 240]).

Lemma 2.3. *Let $w \in \mathbb{C}$ be given such that $\Gamma(w)\zeta(2w-2) \neq 0$. Then, as $t \to \infty$,*

$$E_2(w;it) = t^w - \frac{\sqrt{\pi}w\Gamma\left(w+\frac{1}{2}\right)\zeta(2w+1)}{\Gamma(w+2)\zeta(2w+2)}t^{-w-1} + O_w\left(e^{-t}\right).$$

The dependence on w in the error term is locally uniform.

2.4. **Weight two polar harmonic Maass forms and the resolvent kernel.** In this subsection, we recall certain weight two polar harmonic Maass forms H_z^* on $\mathrm{SL}_2(\mathbb{Z})$. Explicitly, we define (writing $z = x+iy$ throughout)

$$H_z^*(\tau) := -\frac{y}{2\pi}\Psi_2(\tau,z),$$

where $y\Psi_2(\tau,z)$ is the analytic continuation to $w = 0$ of the Poincaré series (see [4, Section 3.1])

$$\mathcal{P}_w(\tau,z) := \sum_{M\in\mathrm{SL}_2(\mathbb{Z})}\frac{\varphi_w(M\tau,z)}{j(M,\tau)^2|j(M,\tau)|^{2w}}.$$

Here $\varphi_w(\tau,z) := y^{w+1}(\tau-z)^{-1}(\tau-\bar{z})^{-1}|\tau-\bar{z}|^{-2w}$. Setting $X_\tau(z) := \frac{z-\tau}{z-\overline{\tau}}$, we use the following properties of $z \mapsto H_z^*$ which follow by [4, Lemma 4.4 and Proposition 5.1], and a direct calculation.

Lemma 2.4. *The function $z \mapsto H_z^*(\tau)$ is a weight zero polar harmonic Maass forms. Moreover, if $\tau \in \mathbb{H}$ is not an elliptic fixed point, then it exhibits the growth $-\frac{1}{4\pi v X_\tau(z)}$ at $z=\tau$ and no other singularities in $(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}) \cup \{i\infty\}$. In particular*

$$2\pi i \lim_{z\to\tau}(z-\tau)H_z^*(\tau) = 1. \tag{2.7}$$

We also require the behaviour of H_z^* as $z \to i\infty$ (see [5, (1.7)] and [5, Theorem 1.2]).

Lemma 2.5. *We have*

$$\lim_{z\to i\infty} H_z^* = -\widehat{E}_2. \tag{2.8}$$

We next relate H_z^* to the resolvent kernel for $\mathrm{SL}_2(\mathbb{Z})$ (see [12] for a full treatment). The *resolvent kernel* is defined by the analytic continuation in w of

$$G_w(z,\tau) := \sum_{M\in\mathrm{PSL}_2(\mathbb{Z})} g_w(Mz,\tau),$$

where

$$g_w(z,\tau) := -\frac{\Gamma(w)^2}{\Gamma(2w)}\left(\frac{2}{1+\cosh(d(z,\tau))}\right)^w {}_2F_1\left(w,w;2w;\frac{2}{1+\cosh(d(z,\tau))}\right).$$

Here ${}_2F_1(a,b;c;Z)$ is Gauss' hypergeometric function. Moreover, $d(z,\tau)$ is the hyperbolic distance between z and τ, which satisfies

$$\cosh(d(z,\tau)) = 1 + \frac{|z-\tau|^2}{2vy}.$$

The function $G_w(z,\tau)$ is invariant under $\mathrm{SL}_2(\mathbb{Z})$ in both variables. Since $H_z^*(\tau)$ satisfies weight zero modularity in z and weight two modularity in τ (see Lemma 2.4 and [5, Theorem 1.1]), it is natural to apply the Maass raising operator in τ. We therefore define

$$\mathcal{G}_w(z,\tau) := \frac{1}{2i} R_{0,\tau}\left(G_w(z,\tau)\right).$$

We next use the invariance in both variables of $G_w(z,\tau)$ under $\mathrm{SL}_2(\mathbb{Z})$ and the well-known facts that it is an eigenfunction under Δ_0 in both variables (see [7, property (b) in Section 5]) and that for τ fixed it has a unique logarithmic singularity in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ at $z=\tau$ (for example, see [7, property (a) in Section 5]). A direct calculation using Lemma 2.1 then yields the following.

Lemma 2.6. *The function $z \mapsto \mathcal{G}_w(z,\tau)$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant and it is an eigenfunction with eigenvalue $w(1-w)$ under $\Delta_{0,z}$. The function $\tau \mapsto \mathcal{G}_w(z,\tau)$ satisfies weight two modularity and has eigenvalue $w(1-w)$ under $\Delta_{2,\tau}$. Moreover, for $\mathrm{Re}(w) \geq 1$ and τ not an elliptic fixed point, $z \mapsto \mathcal{G}_w(z,\tau)$ exhibits the growth $\frac{1}{2i}R_{0,\tau}(g_w(z,\tau))$ at $z=\tau$, which simplifies in the special case $w=1$ to*

$$\lim_{z\to\tau}(z-\tau)\mathcal{G}_1(z,\tau) = -1, \tag{2.9}$$

and does not grow at any point which is $\mathrm{SL}_2(\mathbb{Z})$-inequivalent to τ.

We additionally require the growth of $\mathcal{G}_w(z,\tau)$ as $z\to i\infty$ or $\tau\to i\infty$, which can be obtained from [12, (6.5)].

Lemma 2.7. *Assume that* $\mathrm{Re}(w)\geq 1$.

(1) *For* $y\geq v+\frac{1}{v}+\varepsilon$ *with* $\varepsilon>0$, *we have*

$$\mathcal{G}_w(z,\tau)=\frac{2\pi i}{2w-1}y^{1-w}R_{0,\tau}\left(E_0(w;\tau)\right)+O_{w,\varepsilon}\left(\left(v+\frac{1}{v}\right)^{\frac{1}{2}}e^{\frac{\pi}{2}\left(v+\frac{1}{v}-y\right)}\right),$$

where the error is locally uniform around $w=1$. *In particular, as* $y\to\infty$

$$\mathcal{G}_1(z,\tau)=2\pi i\widehat{E}_2(\tau)+O_v\left(e^{-\frac{\pi y}{2}}\right).$$

(2) *For* $v\geq y+\frac{1}{y}+\varepsilon$ *with* $\varepsilon>0$, *we have*

$$\mathcal{G}_w(z,\tau)=\frac{2\pi i(w-1)}{1-2w}v^{-w}E_0(w;z)+O_{w,\varepsilon}\left(\left(y+\frac{1}{y}\right)^{\frac{1}{2}}e^{\frac{\pi}{4}\left(y+\frac{1}{y}-v\right)}\right),$$

where the bound is again locally uniform around $w=1$.

The function $\mathcal{G}_w(z,\tau)$ is related to H_z^* via the following proposition.

Proposition 2.8. *We have*

$$\mathcal{G}_1(z,\tau)=-2\pi iH_z^*(\tau).$$

Proof. Lemma 2.4 and Lemma 2.6 (with $w=1$) imply that

$$\mathbb{G}_\tau(z):=\lim_{\mathfrak{z}\to z}\left(\mathcal{G}_1(\mathfrak{z},\tau)+2\pi iH_{\mathfrak{z}}^*(\tau)\right),$$

is a polar harmonic Maass form of weight zero on $\mathrm{SL}_2(\mathbb{Z})$. We claim that it vanishes identically. Without loss of generality, it suffices to assume that $\tau\in\mathbb{H}$ is not an elliptic fixed point. For $\mathfrak{z}\in\mathbb{H}$ which is not $\mathrm{SL}_2(\mathbb{Z})$-equivalent to τ, $\mathbb{G}_\tau$ does not have a singularity at $z=\mathfrak{z}$ because neither summand has a singularity by Lemmas 2.4 and 2.6. Moreover, (2.7) and (2.9) yield that $\lim_{z\to\tau}(z-\tau)\mathbb{G}_\tau(z)=0$. We conclude that $\mathbb{G}_\tau$ is an $\mathrm{SL}_2(\mathbb{Z})$-invariant harmonic function that does not have any singularities. Since the only weight zero harmonic functions on $\mathrm{SL}_2(\mathbb{Z})$ without singularities are constant, we conclude that $\mathbb{G}_\tau(z)$ is independent of z and $\lim_{z\to i\infty}\mathbb{G}_\tau(z)=0$ by Lemma 2.5 and Lemma 2.7 (1) implies that $\mathbb{G}_\tau(z)=0$ for all $z\in\mathbb{H}$, yielding the result. □

3. The definition of (generalized) L-functions and the proof of Theorem 1.1 (1)–(3)

In this section we define the relevant (generalized) L-functions. In Section 3.1, we relate an L-function to the weight two Eisenstein series. This L-function plays an important role in the proof of Theorem 1.1 (4) in Section 4. We then define $L_z(s)$ in Section 3.2 and investigate its main properties, proving Theorem 1.1 (1)–(3).

3.1. **An L-function associated to the weight two Eisenstein series.** In order to relate $L_z(s)$ to the Riemann zeta function and prove Theorem 1.1 (4), we first recall the well-known construction of an L-function for the weight two Eisenstein series E_2 and its completion $\widehat{E}_2$. Following a trick of Riemann [15] (used to obtain the zeta function as a regularized Mellin transform), for $t_0 > 0$ we define

$$L\left(\widehat{E}_2, s\right) := \int_{t_0}^{\infty} \left(\widehat{E}_2(it) - 1 + \frac{3}{\pi t}\right) t^{s-1} dt + \int_0^{t_0} \left(\widehat{E}_2(it) + \frac{1}{t^2} - \frac{3}{\pi t}\right) t^{s-1} dt - \frac{t_0^s}{s} - \frac{t_0^{s-2}}{s-2} + \frac{6}{\pi}\frac{t_0^{s-1}}{s-1}. \tag{3.1}$$

We give the main properties of $L(\widehat{E}_2, s)$ and evaluate it in the following lemma, which may be easily proven using the modularity of $\widehat{E}_2$, the growth of $\widehat{E}_2(it)$ as $t \to \infty$, and the identity (see e.g. [10, Theorem 291])

$$\zeta(s)\zeta(s-\ell) = \sum_{n=1}^{\infty} \frac{\sigma_\ell(n)}{n^s}. \tag{3.2}$$

Lemma 3.1.

(1) *The integrals on the right-hand side of* (3.1) *converge absolutely and define meromorphic functions on the complex s-plane with simple poles for $s \in \{0, 1, 2\}$.*
(2) *The definition of $L(\widehat{E}_2, s)$ is independent of the choice of t_0.*
(3) *We have*

$$L\left(\widehat{E}_2, s\right) = -\frac{24}{(2\pi)^s}\Gamma(s)\zeta(s)\zeta(s-1).$$

(4) *We have*

$$L\left(\widehat{E}_2, 2-s\right) = -L\left(\widehat{E}_2, s\right).$$

3.2. **Definition of a generalized L-function for polar harmonic Maass forms.** The goal of this section is to define $L_z(s)$ and to prove Theorem 1.1 (1)–(3).

For $s, s_0, w \in \mathbb{C}$ with $\mathrm{Re}(s), \mathrm{Re}(w)$ sufficiently large, we set

$$L_z(w, s_0; s) := \int_0^{\infty} \mathcal{G}_w(z, it) r_z(it)^{s_0} r_z\left(\frac{i}{t}\right)^{s_0} t^{s-1} dt, \qquad r_z(\tau) := |X_z(\tau)|. \tag{3.3}$$

Lemma 3.2. *Let $s_0, s \in \mathbb{C}$ and $z \in \mathbb{H}$ and if z is equivalent under $\mathrm{SL}_2(\mathbb{Z})$ to a point in $i\mathbb{R}^+$, then assume that $\mathrm{Re}(s_0)$ is sufficiently large. Then the integral defining $L_z(w, s_0; s)$ converges absolutely and locally uniformly for $w \in \mathbb{C}$ with $\mathrm{Re}(w)$ sufficiently large (depending on $\mathrm{Re}(s)$).*

Proof. Using the modularity from Lemma 2.6, one may write (for $\mathrm{Re}(w)$ sufficiently large)

$$L_z(w, s_0; s) = \mathcal{J}_{w,s,s_0,z}(t_0) - \mathcal{J}_{w,2-s,s_0,z}\left(\frac{1}{t_0}\right)$$

with

$$\mathcal{J}_{w,s,s_0,z}(t_0) := \int_{t_0}^{\infty} \mathcal{G}_w(z,it) r_z(it)^{s_0} r_z\left(\frac{i}{t}\right)^{s_0} t^{s-1} dt.$$

The claim then follows by Lemma 2.7 (2). □

We next assume that $z \notin \mathcal{S}$. In this case, we set

$$L_z(w;s) := L_z(w,0;s).$$

Theorem 3.3. *For each $z \in \mathbb{H} \setminus \mathcal{S}$ and $s \in \mathbb{C}$, the function $w \mapsto L_z(w;s)$ has a meromorphic continuation to the whole complex plane. Furthermore, the resulting function in z is an eigenfunction under $\Delta_{0,z}$ with eigenvalue $w(1-w)$ and it is invariant under the action of* $\mathrm{SL}_2(\mathbb{Z})$. *Moreover, $L_z(w;s)$ satisfies the functional equation*

$$L_z(w;s) = -L_z(w;2-s).$$

We prove Theorem 3.3 through a series of lemmas and propositions. For $t_0 > 0$ we define

$$I_{w,s}(z) := \sum_{j=1}^{2} \mathcal{I}_{j,w,s}(t_0;z) - \sum_{j=1}^{2} \mathcal{E}_{j,w,s}(t_0;z), \tag{3.4}$$

where

$$\mathcal{I}_{1,w,s}(t_0;z) := \int_0^{t_0} \left(\mathcal{G}_w(z,it) + \frac{2\pi i(w-1)}{1-2w} E_0(w;z) t^{w-2}\right) t^{s-1} dt,$$
$$\mathcal{I}_{2,w,s}(t_0;z) := \int_{t_0}^{\infty} \left(\mathcal{G}_w(z,it) - \frac{2\pi i(w-1)}{1-2w} E_0(w;z) t^{-w}\right) t^{s-1} dt,$$
$$\mathcal{E}_{1,w,s}(t_0;z) := \frac{2\pi i(w-1) t_0^{s+w-2} E_0(w;z)}{(1-2w)(s+w-2)},$$
$$\mathcal{E}_{2,w,s}(t_0;z) := \frac{2\pi i(w-1) t_0^{s-w} E_0(w;z)}{(1-2w)(s-w)}.$$

We claim that $I_{w,s}(z)$ is independent of the choice of t_0 (see the remark after Lemma 3.4) and we show in Lemma 3.4 below that it agrees with $L_z(w;s)$ for $\mathrm{Re}(w)$ sufficiently large. Following this, we prove in Proposition 3.5 that $I_{w,s}(z)$ gives a meromorphic continuation of $L_z(w;s)$ to the whole complex w-plane. We then show the functional equation of $I_{w,s}$ (and hence also of $L_z(w;s)$ by Lemma 3.4) in Proposition 3.6. The function $L_z(s)$ is then defined by the special value of this function at $w=1$ (see (3.5)). By using the absolute convergence of the integral in Lemma 3.2 for $\mathrm{Re}(w)$ sufficiently large and computing some elementary integrals, one easily sees that the functions $I_{w,s}(z)$ and $L_z(w;s)$ indeed coincide.

Lemma 3.4. *Let $s \in \mathbb{C}$ and $z \in \mathbb{H} \setminus \mathcal{S}$. For* $\mathrm{Re}(w)$ *sufficiently large, $I_{w,s}(z) = L_z(w;s)$.*

Remark. Note that Lemma 3.4 implies that $I_{w,s}$ is independent of the choice of t_0 for $\mathrm{Re}(w)$ sufficiently large because it agrees with $L_z(w;s)$. The Identity Theorem implies that it is independent of t_0 for all w for which its analytic continuation exists.

The next step of the proof of Theorem 3.3 is to prove that $I_{w,s}(z)$ gives a meromorphic continuation of $L_z(w;s)$ to the entire complex w-plane.

Proposition 3.5. *For each $s \in \mathbb{C}$, $z \in \mathbb{H}\setminus\mathcal{S}$, and $t_0 > 0$, the functions $w \mapsto \mathcal{I}_{j,w,s}(t_0; z)$ ($j \in \{1,2\}$) converge absolutely and locally uniformly for $s, w \in \mathbb{C}$ outside of a discrete set of singularities at $w = \frac{1}{2}$ and at the poles of $(w-1)E_0(w; z)$. Moreover, if $w \notin \{\frac{1}{2}, s, 2-s\}$ and w is not a pole of $(w-1)E_0(w; z)$, then the function $z \mapsto I_{w,s}(z)$ is an eigenfunction under Δ_0 with eigenvalue $w(1-w)$ on $\mathbb{H} \setminus \mathcal{S}$ and it is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$.*

Proof. First assume that $w \neq \frac{1}{2}$ and w is not a pole of $(w-1)E_0(w; z)$. By Lemma 2.7 (2), the part of the integral defining $\mathcal{I}_{2,w,s}(t_0; z)$ with $t > y + \frac{1}{y} + \varepsilon$ converges absolutely. Similarly, using the weight two modularity of $\mathcal{G}_w$ from Lemma 2.6, Lemma 2.7 (2) implies that for $t < (y + \frac{1}{y})^{-1}$

$$\mathcal{G}_w(z, it) = \frac{1}{(it)^2}\mathcal{G}_w\left(z, \frac{i}{t}\right) = -\frac{2\pi i(w-1)}{1-2w}E_0(w; z)t^{w-2} + O\left(t^{-2}y^{\frac{1}{2}}e^{c\left(y+\frac{1}{y}-\frac{1}{t}\right)}\right).$$

Hence the integral in $\mathcal{I}_{1,w,s}(t_0; z)$ with $t < (y + \frac{1}{y} + \varepsilon)^{-1}$ also converges absolutely. One sees directly that $w \mapsto \mathcal{E}_{j,w,s}(t_0; z)$ is meromorphic with possible poles for $w \in \{\frac{1}{2}, s, 2-s\}$, and the poles of $(w-1)E_0(w; z)$.

We next show that $I_{w,s}(z)$ is an eigenfunction under $\Delta_{0,z}$. Since the function $E_0(w; z)$ is an eigenfunction with eigenvalue $w(1-w)$ under $\Delta_{0,z}$, we see that $z \mapsto \mathcal{E}_{j,w,s}(t_0; z)$ are eigenfunctions under $\Delta_{0,z}$ with eigenvalue $w(1-w)$. Moreover, since the integrals $I_{j,w,s}(t_0; z)$ are absolutely and locally uniformly convergent, we may take the operator $\Delta_{0,z}$ inside the integrals. Combining the fact that $E_0(w; z)$ is an eigenfunction under $\Delta_{0,z}$ with the fact that $\mathcal{G}_w(z, it)$ is also an eigenfunction with the same eigenvalue by Lemma 2.6, the resulting integrals are also eigenfunctions. Modular invariance in z also follows directly from the modularity of $E_0(z, w)$ and $\mathcal{G}_w(z, it)$, using Lemma 2.6. □

Using the modularity from Lemma 2.6, implies the functional equation of $I_{w,s}(z)$ in the usual way.

Proposition 3.6. *Suppose that $z \in \mathbb{H} \setminus \mathcal{S}$ and $w \in \mathbb{C} \setminus \{\frac{1}{2}\}$ is not a pole of $(w-1)E_0(w; z)$. Then for $s \notin \{w, 2-w\}$ we have*

$$I_{w,2-s}(z) = -I_{w,s}(z).$$

Since $I_{w,s}(z)$ provides the analytic continuation of $L_z(w; s)$ by Lemma 3.4 and Proposition 3.5, we set

$$L_z(s) := I_{1,s}(z). \tag{3.5}$$

Remark. Due to the connection between H_z^* and $\widehat{E}_2$ in (2.8), one may naively consider $L(\widehat{E}_2, s)$ as the generalized L-function at $z = i\infty$. Although one cannot legally interchange the limit with the integrals defining $I_{1,s}(z)$ (and hence $L_z(s)$) to make this connection rigorous, the relationship between these (generalized) L-functions is investigated in Section 4.

Proof of Theorem 3.3. Combining Lemma 3.4 with Propositions 3.5 and 3.6 yields Theorem 3.3. □

Proof of Theorem 1.1 (1)–(3)*.* This follows from (3.5) and Theorem 3.3. □

4. Behavior of the generalized L-functions towards infinity and the proof of Theorem 1.1 (4)

The goal of this section is to investigate the growth of $L_z(s)$ as $z \to i\infty$, ultimately proving Theorem 1.1 (4). In particular, in Theorem 4.3 we obtain an expansion of the type

$$L_z(s) = 2\pi i L\left(\widehat{E}_2, s\right) + \sum_{\ell=0}^{\lfloor \mathrm{Re}(s) \rfloor} c_{\ell,s}(x) y^{s-\ell} + \sum_{\ell=0}^{\lfloor 2-\mathrm{Re}(s) \rfloor} d_{\ell,s}(x) y^{2-s-\ell} + o(1). \tag{4.1}$$

Due to the functional equations for $L_z(s)$ and $L(\widehat{E}_2, s)$, an expansion of the type (4.1), if it exists, has a further restricted shape.

Lemma 4.1. *An expansion of the type* (4.1) *exists if and only if*

$$L_z(s) = 2\pi i L\left(\widehat{E}_2, s\right) + \sum_{\ell=0}^{\lfloor \mathrm{Re}(s) \rfloor} c_{\ell,s}(x) y^{s-\ell} - \sum_{\ell=0}^{\lfloor 2-\mathrm{Re}(s) \rfloor} c_{\ell,2-s}(x) y^{2-s-\ell} + o(1). \tag{4.2}$$

Moreover, such an expansion holds if and only if it holds for $\mathrm{Re}(s) \geq 1$*.*

We next relate $L_z(s)$ to $L(\widehat{E}_2, s)$. For ease of notation, we define (with $0 \leq y_1 \leq y_2 \leq \infty$)

$$\mathbb{J}_{z,s,0}(y_1, y_2) := \int_{y_1}^{y_2} \left(H_z(it) + \frac{1}{t^2}\right) t^{s-1} dt,$$
$$\mathbb{J}_{z,s,i\infty}(y_1, y_2) := \int_{y_1}^{y_2} \left(H_z(it) - 1\right) t^{s-1} dt,$$

where the subscripts 0 and $i\infty$ indicate that we subtract the main growth of $H_z(it)$, which is defined in (2.5), towards 0 or $i\infty$, respectively.

Lemma 4.2. *Suppose that $z \in \mathbb{H} \setminus \mathcal{S}$. Then for any $t_0 > 0$ we have*

$$L_z(s) = 2\pi i L\left(\widehat{E}_2, s\right) - 2\pi i \mathbb{J}_{z,s,0}(0, t_0) - 2\pi i \mathbb{J}_{z,s,i\infty}(t_0, \infty) + 2\pi i \frac{t_0^s}{s} + 2\pi i \frac{t_0^{s-2}}{s-2}.$$

Proof. Recalling the definition (3.5) and the absolute and locally uniform convergence of $I_{w,s}(z)$ shown in Proposition 3.5, we may directly plug $w = 1$ into (3.4). It is well-known that (see [8, p. 239, before (2.14)])

$$\lim_{w \to 1} (w-1) E_0(w; z) = \frac{3}{\pi}.$$

Plugging this into the definition following (3.4), we see directly that

$$\mathcal{E}_{1,1,s}(t_0; z) = \mathcal{E}_{2,1,s}(t_0; z) = -\frac{6 i t_0^{s-1}}{s-1}.$$

Moreover, using Proposition 2.8, we obtain

$$\sum_{j=1}^{2} \mathcal{I}_{j,1,s}(t_0; z) = -2\pi i \int_0^{t_0} \left(H_z^*(it) + \frac{3}{\pi t}\right) t^{s-1} dt - 2\pi i \int_{t_0}^{\infty} \left(H_z^*(it) - \frac{3}{\pi t}\right) t^{s-1} dt. \tag{4.3}$$

We then use an identity of Asai, Kaneko, and Ninomiya [1, Theorem 3] and the second remark following [5, Theorem 1.1] to rewrite

$$H_z^*(\tau) = H_z(\tau) - \widehat{E}_2(\tau).$$

Plugging this into (4.3) and recalling the definition (3.1) yields the claim. $\square$

Setting

$$C_{\ell,s}(x) := \begin{cases} \frac{2\pi i}{s} & \text{if } \ell = 0, \\ 4\pi i \frac{(1-s)_{\ell-1}}{(2\pi)^\ell} \operatorname{Re}\left(\operatorname{Li}_\ell\left(e^{2\pi i x}\right)\right) & \text{if } 2 \leq \ell \leq \lfloor \operatorname{Re}(s) \rfloor \text{ is even}, \\ -4\pi \frac{(1-s)_{\ell-1}}{(2\pi)^\ell} \operatorname{Im}\left(\operatorname{Li}_\ell\left(e^{2\pi i x}\right)\right) & \text{if } 1 \leq \ell \leq \lfloor \operatorname{Re}(s) \rfloor \text{ is odd}, \end{cases}$$

the generalized L-function $L_z(s) = I_{1,s}(z)$ is related to the Riemann zeta function via the following theorem.

Theorem 4.3. *An expansion of the type* (4.2) *exists. More precisely, for fixed $x \in \mathbb{R} \setminus \mathbb{Z}$ we have*

$$\lim_{y \to \infty} \left(L_{x+iy}(s) - \sum_{\ell=0}^{\lfloor \operatorname{Re}(s) \rfloor} C_{\ell,s}(x) y^{s-\ell} + \sum_{\ell=0}^{\lfloor 2-\operatorname{Re}(s) \rfloor} C_{\ell,2-s}(x) y^{2-s-\ell} \right) = 2\pi i L\left(\widehat{E}_2, s\right)$$
$$= -\frac{24i}{(2\pi)^{s-1}} \Gamma(s)\zeta(s)\zeta(s-1).$$

Proof. By the Identity Theorem, it suffices to prove the claim for $s \notin \mathbb{Z}$ and $\operatorname{Re}(s) \geq 1$. Lemma 4.2 and Lemma 3.1 (3) then imply that the claim of Theorem 4.3 is equivalent to

$$\lim_{y \to \infty} \left(\mathbb{J}_{z,s,0}(0, t_0) + \mathbb{J}_{z,s,i\infty}(t_0, \infty) - \frac{i}{2\pi} \sum_{\ell=0}^{\lfloor \operatorname{Re}(s) \rfloor} C_{\ell,s}(x) y^{s-\ell} + \frac{i}{2\pi} \sum_{\ell=0}^{\lfloor 2-\operatorname{Re}(s) \rfloor} C_{\ell,2-s}(x) y^{2-s-\ell} \right)$$
$$= \frac{t_0^s}{s} + \frac{t_0^{s-2}}{s-2}. \quad (4.4)$$

We assume without loss of generality that $\frac{2}{y} < t_0 < \frac{y}{2}$ and further split the integrals inside the limit. We claim that, as $y \to \infty$,

$$\mathbb{J}_{z,s,0}\left(0, \frac{1}{y}\right) = -\frac{1}{2\pi} \operatorname{Li}_1\left(e^{-2\pi i x}\right) y^{1-s} + o_{x,s}(1), \quad (4.5)$$

$$\mathbb{J}_{z,s,0}\left(\frac{1}{y}, t_0\right) = \frac{t_0^{s-2} - y^{2-s}}{s-2} + \frac{1}{2\pi} \operatorname{Li}_1\left(e^{2\pi i x}\right) y^{1-s} + o_{x,s}(1), \quad (4.6)$$

$$\mathbb{J}_{z,s,i\infty}(t_0, y) = \frac{t_0^s - y^s}{s} - \sum_{\ell=1}^{\lfloor \operatorname{Re}(s) \rfloor} \frac{(1-s)_{\ell-1}}{(2\pi)^\ell} \operatorname{Li}_\ell\left(e^{2\pi i x}\right) y^{s-\ell} + o_{x,s}(1), \quad (4.7)$$

$$\mathbb{J}_{z,s,i\infty}(y, \infty) = \sum_{\ell=1}^{\lfloor \operatorname{Re}(s) \rfloor} (-1)^{\ell+1} \frac{(1-s)_{\ell-1}}{(2\pi)^\ell} \operatorname{Li}_\ell\left(e^{-2\pi i x}\right) y^{s-\ell} + o_{x,s}(1). \quad (4.8)$$

Before proving these, note that (4.5), (4.6), (4.7), and (4.8) imply (4.4) because

$$-\operatorname{Li}_\ell\left(e^{2\pi ix}\right)+(-1)^{\ell+1}\operatorname{Li}_\ell\left(e^{-2\pi ix}\right)=\begin{cases}-2\operatorname{Re}\left(\operatorname{Li}_\ell\left(e^{2\pi ix}\right)\right) & \text{if } \ell \text{ is even,}\\ -2i\operatorname{Im}\left(\operatorname{Li}_\ell\left(e^{2\pi ix}\right)\right) & \text{if } \ell \text{ is odd.}\end{cases}$$

We next show (4.5). Changing $t\mapsto\frac{1}{t}$ and using the weight two modularity of $H_z(\tau)$, we have

$$\mathbb{J}_{z,s,0}\left(0,\frac{1}{y}\right)=-\mathbb{J}_{z,2-s,i\infty}(y,\infty)=-\lim_{\varepsilon\to0^+}\mathbb{J}_{z,2-s,i\infty}((1+\varepsilon)y,\infty). \tag{4.9}$$

As $t>(1+\varepsilon)y$, for y sufficiently large the asymptotics in (2.6) imply that $|J(it)|>|J(z)|$ and hence, for all $s\in\mathbb{C}$,

$$\begin{aligned}\mathbb{J}_{z,2-s,i\infty}((1+\varepsilon)y,\infty)&=\int_{(1+\varepsilon)y}^{\infty}\left(\sum_{j=1}^{\infty}e^{-2\pi j(iz+t)}+O_x\left(e^{-2\pi t}\right)\left|\sum_{j=0}^{\infty}e^{-2\pi j(iz+t)}\right|\right)t^{1-s}dt\\&=\int_{(1+\varepsilon)y}^{\infty}\sum_{j=1}^{\infty}e^{-2\pi j(iz+t)}t^{1-s}dt+O_{x,s}\left(e^{-\pi y}\right).\end{aligned} \tag{4.10}$$

By the Dominated Convergence Theorem, one can interchange the integral and sum for the main term. Thus the main term of (4.10) gives a contribution to (4.9) of

$$\begin{aligned}-\lim_{\varepsilon\to0^+}\sum_{j=1}^{\infty}e^{-2\pi ijz}\int_{(1+\varepsilon)y}^{\infty}e^{-2\pi jt}t^{1-s}dt&=-\lim_{\varepsilon\to0^+}\sum_{j=1}^{\infty}e^{-2\pi ijz}(2\pi j)^{s-2}\Gamma\left(2-s,2\pi j(1+\varepsilon)y\right) \quad (4.11)\\&=-y^{1-s}\lim_{\varepsilon\to0^+}\sum_{j=1}^{\infty}\frac{e^{-2\pi j(ix+\varepsilon y)}}{2\pi j(1+\varepsilon)}\left(1+O\left(j^{-1}y^{-1}\right)\right),\end{aligned}$$

taking (2.1) with $s\mapsto2-s$, $y\mapsto2\pi(1+\varepsilon)jy$, and $N=1$. It is not hard to see that

$$-y^{1-s}\lim_{\varepsilon\to0^+}\sum_{j=1}^{\infty}\frac{e^{-2\pi j(ix+\varepsilon y)}}{2\pi j(1+\varepsilon)}O\left(j^{-1}y^{-1}\right)\ll y^{-1}\to0$$

as $y\to\infty$. We then obtain (4.5) by (2.4).

We next prove (4.6). We write

$$\mathbb{J}_{z,s,0}\left(\frac{1}{y},t_0\right)=\int_{\frac{1}{y}}^{t_0}H_z(it)t^{s-1}dt+\frac{t_0^{s-2}-y^{2-s}}{s-2}. \tag{4.12}$$

In the integral we make the change of variables $t\mapsto\frac{1}{t}$ and use the modularity of the integrand to obtain that

$$\int_{\frac{1}{y}}^{t_0}H_z(it)t^{s-1}dt=\lim_{\varepsilon\to0^+}\int_{\frac{1}{t_0}}^{(1-\varepsilon)y}H_z(it)t^{1-s}dt.$$

Using (2.6), a straightforward calculation shows that for $\frac{1}{t_0} < t < (1-\varepsilon)y$ and $x \notin \mathbb{Z}$ we have

$$\frac{1}{J(z)-J(it)} = \frac{e^{2\pi iz}}{1-e^{2\pi(t+iz)}}\left(1+O_x\left(e^{-2\pi(t+y)}\right)\right).$$

Combining this with the second equation from (2.6), we obtain that the integral on the right-hand side of (4.12) equals

$$\lim_{\varepsilon\to 0^+}\int_{\frac{1}{t_0}}^{(1-\varepsilon)y}\left(e^{2\pi t}+O(1)\right)\frac{e^{2\pi iz}}{1-e^{2\pi(t+iz)}}\left(1+O_x\left(e^{-2\pi(t+y)}\right)\right)t^{1-s}dt$$
$$= \lim_{\varepsilon\to 0^+}\int_{\frac{1}{t_0}}^{(1-\varepsilon)y}\frac{e^{2\pi(t+iz)}}{1-e^{2\pi(t+iz)}}\left(1+O_x\left(e^{-2\pi t}\right)\right)t^{1-s}dt. \quad (4.13)$$

The O-term in (4.13) vanishes as $y\to\infty$ due to exponential decay of the integrand. To evaluate the main term, we expand $\frac{e^{2\pi(t+iz)}}{1-e^{2\pi(t+iz)}}$ as a geometric series to write the main term as

$$\lim_{\varepsilon\to 0^+}\int_{\frac{1}{t_0}}^{(1-\varepsilon)y}\frac{e^{2\pi(t+iz)}}{1-e^{2\pi(t+iz)}}t^{1-s}dt = \lim_{\varepsilon\to 0^+}\sum_{\ell=1}^{\infty}\int_{\frac{1}{t_0}}^{(1-\varepsilon)y}e^{2\pi\ell(t+iz)}t^{1-s}dt. \quad (4.14)$$

Taking $t\mapsto -\frac{t}{2\pi\ell}$ and then plugging in (2.2) yields that the main term in (4.13) is

$$\lim_{\varepsilon\to 0^+}\sum_{\ell=1}^{\infty}\int_{\frac{1}{t_0}}^{(1-\varepsilon)y}e^{2\pi\ell(t+iz)}t^{1-s}dt = \lim_{\varepsilon\to 0^+}\sum_{\ell=1}^{\infty}e^{2\pi i\ell z}e^{\pi i(s-2)}(2\pi\ell)^{s-2}\Gamma\left(2-s,-\frac{2\pi\ell}{t_0},-2\pi\ell(1-\varepsilon)y\right)$$
$$= \frac{y^{2-s}}{2-s}\lim_{\varepsilon\to 0^+}\sum_{\ell=1}^{\infty}e^{2\pi i\ell z}{}_1F_1\left(2-s;3-s;2\pi\ell(1-\varepsilon)y\right)$$
$$-\frac{t_0^{s-2}}{2-s}\sum_{\ell=1}^{\infty}e^{2\pi i\ell z}{}_1F_1\left(2-s;3-s;\frac{2\pi\ell}{t_0}\right). \quad (4.15)$$

Plugging in (2.3) with $s\mapsto 2-s$ and $N=0$, we obtain that for $|(\ell+1)Z|\to\infty$ (recall that we assume $s\notin\mathbb{Z}$ above)

$${}_1F_1\left(2-s;3-s;2\pi\ell Z\right) = (2-s)e^{2\pi\ell Z}\left(\frac{1}{2\pi\ell Z}+O_s\left(\frac{1}{\ell^2Z^2}\right)\right). \quad (4.16)$$

Taking $Z=t_0^{-1}$, the second term in (4.15) vanishes as $y\to\infty$.

We now take $Z=(1-\varepsilon)y$ in (4.16) and plug into the first term in (4.15) to obtain that (4.15) equals

$$y^{2-s}\lim_{\varepsilon\to 0^+}\sum_{\ell=1}^{\infty}e^{2\pi\ell(iz+(1-\varepsilon)y)}\left(\frac{1}{2\pi(1-\varepsilon)\ell y}+O_s\left(\frac{1}{\ell^2y^2}\right)\right). \quad (4.17)$$

The error term in (4.17) is absolutely convergent uniformly in $\varepsilon\geq 0$ and gives a contribution $\ll y^{-\mathrm{Re}(s)}$, which vanishes because $\mathrm{Re}(s)\geq 1$ by assumption. Plugging (2.4) with $x\mapsto -x$ into the main term of (4.17) implies (4.6).

We next show (4.8). Noting that the left-hand side of (4.8) is the negative of the second expression of (4.9) with $s \mapsto 2-s$, we may plug (4.11) with $s \mapsto 2-s$ into (4.10) and then use (2.1) to compute

$$\mathbb{J}_{z,s,i\infty}(y,\infty) = \sum_{\ell=0}^{\lfloor \mathrm{Re}(s)\rfloor -1} (s-\ell)_\ell y^{s-\ell-1} \times \lim_{\varepsilon\to 0^+} \sum_{j=1}^{\infty} e^{-2\pi j(ix+\varepsilon y)} \left((2\pi j(1+\varepsilon))^{-\ell-1} + O\left(j^{-1-\lfloor \mathrm{Re}(s)\rfloor} y^{\ell-\lfloor \mathrm{Re}(s)\rfloor}\right)\right) + o_{x,s}(1).$$

The error term converges absolutely and uniformly in $\varepsilon \geq 0$ and decays as $y \to \infty$. Hence

$$\mathbb{J}_{z,s,i\infty}(y,\infty) = \lim_{\varepsilon\to 0^+} \sum_{\ell=0}^{\lfloor \mathrm{Re}(s)\rfloor -1} (s-\ell)_\ell y^{s-\ell-1} \sum_{j=1}^{\infty} \frac{e^{-2\pi j(ix+\varepsilon y)}}{(2\pi j(1+\varepsilon))^{\ell+1}} + o_{x,s}(1). \tag{4.18}$$

Plugging (2.4) into the main term in (4.18) yields that the main term in (4.18) equals

$$\sum_{\ell=0}^{\lfloor \mathrm{Re}(s)\rfloor -1} (s-\ell)_\ell \frac{1}{(2\pi)^{\ell+1}} \mathrm{Li}_{\ell+1}\left(e^{-2\pi ix}\right) y^{s-\ell-1} = \sum_{\ell=1}^{\lfloor \mathrm{Re}(s)\rfloor} \frac{(s+1-\ell)_{\ell-1}}{(2\pi)^\ell} \mathrm{Li}_\ell\left(e^{-2\pi ix}\right) y^{s-\ell}.$$

To obtain (4.8), we evaluate $(s+1-\ell)_{\ell-1} = (-1)^{\ell+1}(1-s)_{\ell-1}$.

It remains to prove (4.7). By taking $t \mapsto \frac{1}{t}$ we have

$$\mathbb{J}_{z,s,i\infty}(t_0,y) = -\mathbb{J}_{z,2-s,0}\left(\frac{1}{y},\frac{1}{t_0}\right).$$

Plugging (4.13) with $s \mapsto 2-s$ and $t_0 \mapsto \frac{1}{t_0}$ into (4.12) hence yields

$$\mathbb{J}_{z,s,i\infty}(t_0,y) = -\lim_{\varepsilon\to 0^+} \int_{t_0}^{(1-\varepsilon)y} \frac{e^{2\pi(t+iz)}}{1-e^{2\pi(t+iz)}} \left(1+O_x\left(e^{-2\pi t}\right)\right) t^{s-1} dt + \frac{t_0^s - y^s}{s}. \tag{4.19}$$

As in (4.13), the O-term in (4.19) again vanishes as $y \to \infty$. Expanding the main term as done in (4.14), we then plug in (4.15) with $t_0 \mapsto \frac{1}{t_0}$ and $s \mapsto 2-s$ to obtain

$$\lim_{\varepsilon\to 0^+} \int_{t_0}^{(1-\varepsilon)y} \frac{e^{2\pi(t+iz)}}{1-e^{2\pi(t+iz)}} t^{s-1} dt = \lim_{\varepsilon\to 0^+} \sum_{\ell=1}^{\infty} \int_{t_0}^{(1-\varepsilon)y} e^{2\pi\ell(t+iz)} t^{s-1} dt$$

$$= \frac{y^s}{s} \lim_{\varepsilon\to 0^+} \sum_{\ell=1}^{\infty} e^{2\pi i\ell z} {}_1F_1(s;s+1;2\pi\ell(1-\varepsilon)y) - \frac{t_0^s}{s} \sum_{\ell=1}^{\infty} e^{2\pi i\ell z} {}_1F_1(s;s+1;2\pi\ell t_0). \tag{4.20}$$

We then take $s \mapsto 2-s$ and $Z = t_0$ in (4.16) to see that the second term in (4.20) vanishes as $y \to \infty$.

Plugging $Z = 2\pi\ell(1-\varepsilon)y$ and $N = \lfloor \mathrm{Re}(s)\rfloor$ into (2.3), the first term in (4.20) becomes

$$y^s \lim_{\varepsilon\to 0^+} \sum_{\ell=1}^{\infty} e^{2\pi i\ell z} e^{2\pi\ell(1-\varepsilon)y} \left(\sum_{j=0}^{\lfloor \mathrm{Re}(s)\rfloor -1} \frac{(1-s)_j}{(2\pi\ell(1-\varepsilon)y)^{j+1}} + O_s\left((\ell(1-\varepsilon)y)^{-\lfloor \mathrm{Re}(s)\rfloor -1}\right)\right). \tag{4.21}$$

Since $|e^{2\pi i\ell z}e^{2\pi\ell(1-\varepsilon)y}| = e^{-2\pi\ell\varepsilon y} \leq 1$, the contribution from the error term may be bounded against a constant times

$$y^{\mathrm{Re}(s)-\lfloor \mathrm{Re}(s)\rfloor -1}\sum_{\ell=1}^{\infty}\ell^{-\lfloor \mathrm{Re}(s)\rfloor -1}.$$

Since $-\lfloor \mathrm{Re}(s)\rfloor - 1 \leq -2$ (using the assumption $\mathrm{Re}(s) \geq 1$), the sum on ℓ converges absolutely and $\mathrm{Re}(s) - \lfloor \mathrm{Re}(s)\rfloor - 1 < 0$ implies that the error term in (4.21) vanishes as $y \to \infty$.

We then interchange the sum in ℓ and j in the main term of (4.21) (noting the exponential decay in the sum on ℓ) to rewrite (4.21) as

$$\sum_{j=0}^{\lfloor \mathrm{Re}(s)\rfloor -1}(1-s)_j y^{s-j-1}\lim_{\varepsilon\to 0^+}\sum_{\ell=1}^{\infty}\frac{e^{2\pi\ell(ix-\varepsilon y)}}{(2\pi\ell(1-\varepsilon))^{j+1}} + o(1).$$

Plugging in (2.4) with $x \mapsto -x$ yields that the above equals

$$\sum_{j=0}^{\lfloor \mathrm{Re}(s)\rfloor -1}\frac{(1-s)_j}{(2\pi)^{j+1}}\mathrm{Li}_{j+1}\left(e^{2\pi ix}\right)y^{s-1-j} + o(1) = \sum_{j=1}^{\lfloor \mathrm{Re}(s)\rfloor}\frac{(1-s)_{j-1}}{(2\pi)^j}\mathrm{Li}_j\left(e^{2\pi ix}\right)y^{s-j} + o(1),$$

giving (4.7) and completing the proof. □

As a corollary, we obtain Theorem 1.1 (4).

Proof of Theorem 1.1 (4). For $1 < \mathrm{Re}(s) < 2$, the only terms that occur in Theorem 4.3 are the terms $\ell = 0$ and $\ell = 1$ in the first sum and the $\ell = 0$ term in the second sum. Hence in this case Theorem 4.3 states that

$$\lim_{y\to\infty}\left(L_{x+iy}(s) - \frac{2\pi i}{s}y^s - \frac{2\pi i}{s-2}y^{2-s} + 2\,\mathrm{Im}\left(\mathrm{Li}_1\left(e^{2\pi ix}\right)\right)y^{s-1}\right) = -\frac{24i}{(2\pi)^{s-1}}\Gamma(s)\zeta(s)\zeta(s-1).$$

The proof follows noting that

$$\mathrm{Li}_1(z) = -\mathrm{Log}(1-z), \quad \mathrm{Im}\left(\mathrm{Log}(1-z)\right) = \mathrm{Arg}(1-z).$$ □

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