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Graph schemes, graph series, and modularity

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To a simple graph we associate a so-called graph series, which can be viewed as the Hilbert–Poincaré series of a certain infinite jet scheme. We study new q-representations and examine modular properties of several examples including Dynkin diagrams of finite and affine type. Notably, we obtain new formulas for graph series of type A_7 and A_8 in terms of "sum of tails" series, and of type D_4 and D_5 in the form of indefinite theta functions of signature $(1, 1)$. We also study examples related to sums of powers of divisors corresponding to 5-cycles. For several examples of graphs, we prove that graph series are mixed quantum modular forms and also mixed mock modular forms.

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1. Introduction and statement of results

The interplay between (multi) q-hypergeometric series and modular forms has been studied extensively over the years. In theoretical physics and representation theory, q-hypergeometric series famously appeared in connection with so-called fermionic formulas of characters of rational conformal field theories. If their summands involve a positive definite quadratic form they are sometimes called Nahm sums [38]. Important examples of this type emerged much earlier as "sum sides" of several classical q-series identities (e.g. Andrews–Gordon identities). More recently, Nahm sums have appeared in several areas including vertex algebras [26,33,37,40], DAHA [18], wall-crossing phenomena and quivers [17,28,34], in connection to colored HOMFLY-PT polynomials [29], and in formulas for the "tail" of colored Jones polynomials [9,22,27].

To define a Nahm sum we choose a positive definite quadratic form $Q : \mathbb{Z}^r \to \mathbb{Q}$ and consider

$$\sum_{\boldsymbol{n}\in\mathbb{N}_0^r} \frac{q^{Q(\boldsymbol{n})+\boldsymbol{b}\cdot\boldsymbol{n}+c}}{(q)_{n_1}\cdots(q)_{n_r}}, \tag{1.1}$$

where $\boldsymbol{n} = (n_1, ..., n_r)$, $(a; q)_n = (a)_n := \prod_{j=0}^{n-1}(1 - aq^j)$, $\boldsymbol{b} \in \mathbb{Q}^r$, $c \in \mathbb{Q}$, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ with $\mathbb{N} := \{n \in \mathbb{Z} : n > 0\}$. One important aspect in the study of these series is to investigate their modularity by considering a suitable constant term in the asymptotic expansion (see [14]). Another interesting question is how to express (1.1) using more familiar q-series, which may or may not be modular. Prominent examples of Nahm sums come from Cartan matrices of classical type and their inverses, as they are all expected to be modular for specific choices of $\boldsymbol{b}$ and c (see [42] for some recent results).

In this paper, we study closely related but different q-hypergeometric series that have origin in algebra and representation theory. Unlike many previous works that link characters of representations of affine Lie algebras with q-series, here we focus on certain q-series coming from commutative algebras and graphs. Our aim is to investigate *graph series* defined as

$$H_\Gamma(q) := \sum_{\boldsymbol{n}\in\mathbb{N}_0^r} \frac{q^{\frac{1}{2}\boldsymbol{n}C_\Gamma\boldsymbol{n}^T+n_1+\cdots+n_r}}{(q)_{n_1}\cdots(q)_{n_r}}, \tag{1.2}$$

where C_Γ is the (symmetric) adjacency matrix of a graph Γ. For many graphs, including all simple graphs, this series is not a Nahm sum (the adjacency matrix is not positive definite!). Another issue is that the matrix C_Γ is often singular, so it is important that we include the linear term in the exponent. The main motivation for studying graph series comes from two sources. As explained in Section 2, for a given graph Γ, there is a graded commutative algebra $J_\infty(R)$, the ring of functions of the infinite jet scheme X_∞, where $X = \mathrm{Spec}(R)$, whose Hilbert series is given by (1.2). This infinitely-generated algebra is closely related to a certain principal subspace vertex algebra constructed from the adjacency matrix of Γ [26,33,37,40], in the sense that its character is the Hilbert series of $J_\infty(R)$.

In this paper we make the first steps in addressing modularity properties of graph series. We focus on examples coming from Dynkin diagrams of finite type, as well as several affine Dynkin diagrams. We show that in many examples their modular properties can be quite interesting in spite of the simplicity of the graph. Since the form of (1.2) does not give any clues about modularity, we first must obtain suitable combinatorial formulas in terms of functions whose modular properties are transparent. For graph series of type A (paths), studied in [25], our first result is two new representations for graph series of type A_7 and A_8 (i.e., paths with seven and eight nodes, respectively).

Theorem 1.1. *We have*

$$H_{A_7}(q) = \frac{q^{-1}}{(1-q)(q)_\infty^4}\left(-1+(q)_\infty D(q)+G(q)+(q)_\infty\right),$$
$$H_{A_8}(q) = \frac{q^{-2}}{(q)_\infty^4}\left(-1+(q)_\infty+3D(q)-2G(q)\right),$$

where $D(q) := \sum_{n\geq 1}\frac{q^n}{1-q^n}$ *and* $G(q) := \sum_{n\geq 0}((q)_n-(q)_\infty)$.

Following Zagier [45], *quantum modular forms* are functions $f:\mathcal{Q}\to\mathbb{C}$ ($\mathcal{Q}\subset\mathbb{Q}$) whose *obstruction to modularity*

$$f(x)-(cx+d)^{-k}f\left(\frac{ax+b}{cx+d}\right),\qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})$$

is “nice” (see Subsection 2.2 for more details). Combined with known q-series representations for $H_{A_j}(q)$, $1\leq j\leq 6$, given in [25, Section 7], we obtain the following result.

Corollary 1.2. *For every* j, $1\leq j\leq 8$, $H_{A_j}(q)\in\mathbb{C}[\frac{1}{1-q},q^{-1},\frac{1}{(q)_\infty},D(q),G(q)]$. *More precisely,* $q(q)_\infty^2 H_{A_4}(q)$ *is a holomorphic quantum modular form of weight one, while* $q(q)_\infty^3 H_{A_5}(q)$ *and* $q(1-q)(q)_\infty^3 H_{A_7}(q)$ *are quantum modular forms of weight* $\frac{3}{2}$.

This indicates the possibility of accommodating all A-type graph series inside a finitely generated ring. We also investigate asymptotic properties of these graph series as $t\to 0^+$, where $q=e^{-t}$ (see Proposition 3.1). Such analysis is important from the geometric viewpoint as we would like to understand the growth of coefficients of the Hilbert series of $J_\infty(R)$.

Graph with cycles are more complicated to analyze. However, for several examples related to 5-cycles we obtain elegant formulas.

Theorem 1.3. *Let C_5 be a 5-cycle graph and Γ_8 the graph in Fig. 1 (see Section 4). Then*

$$H_{C_5}(q) = \frac{q^{-1}}{(q)_\infty^2} \sum_{n\geq 1} \frac{nq^n}{1-q^n}, \tag{1.3}$$

$$H_{\Gamma_8}(q) = \frac{q^{-1}}{(q)_\infty^3} \sum_{n\geq 1} \frac{n^2q^n}{1-q^n}. \tag{1.4}$$

For several examples of graphs of D and E-type, due to a trivalent node, we get graph series with somewhat different combinatorial and modular properties. For this recall that *mixed mock modular forms* are linear combinations of modular forms multiplied by mock modular forms (holomorphic parts of certain non-holomorphic automorphic objects). Here is our main result in this direction.

Theorem 1.4. *We have*

$$H_{D_4}(q) = \frac{1}{(q)_\infty^4} \sum_{n,m\geq 0} (-1)^{n+m}(2n+1)q^{\frac{n^2}{2}+\frac{3m^2}{2}+2nm+\frac{3n}{2}+\frac{5m}{2}}, \tag{1.5}$$

$$H_{D_5}(q) = \frac{1}{(q)_\infty^5} \left(\sum_{n,m\geq 0} - \sum_{n,m<0}\right)(-1)^{n+1}(n+1)^2q^{\frac{n^2+3n}{2}+3nm+3m^2+4m}, \tag{1.6}$$

$$H_{E_6}(q) = \frac{q^{-1}}{(q)_\infty^3} \sum_{n\geq 1} \frac{nq^n}{1-q^n}. \tag{1.7}$$

All three series are mixed mock modular forms.

Although q-series associated to graphs with multiple edges do not directly relate to Hilbert series of jet algebras, they can be viewed as characters of certain principal subspaces (see Section 2). We obtain several q-series identities for graph series of type B_2 and B_3 (ignoring the orientation) and related "coset" series (see Proposition 7.1). For several cases we obtain mixed quantum modular forms (linear combinations of modular forms multiplied by quantum modular forms). This indicates a possible connection with quantum invariants of 3-manifolds and knots where similar series appear [24,30,43]. We also investigate several examples of graph series associated to affine Dynkin diagrams whose modular properties seem more intricate (see Section 8).

The paper is organized as follows. In Section 2 we present the main concepts and definitions, along with preliminary results. In particular, we define the notion of the jet scheme of an affine scheme X, discuss principal subspaces associated to lattices and graphs, and present a few results on mock modular forms. In Section 3, we give two new results on the graph series of type A_7 and A_8 (see Theorem 1.1) and study modular properties and the asymptotic behavior of graph series of type A_n, $1 \leq n \leq 8$ (see Proposition 3.1). In Section 4 we analyze certain graph series coming from 5-cycles.

Section 5 is concerned with the D_4 graph series. The main result here is a new q-representation of the graph series in terms of an indefinite theta function of signature $(1,1)$ (see (1.5)). Moreover, we show that this graph series is a mixed mock modular form. In Section 6, we do a similar analysis for the D_5 graph, but this time we use (mixed) mock modular forms to obtain an indefinite theta function representation (see Theorem 6.2). Section 7 deals with a few examples of graphs with multiple edges of Dynkin type B_2 and B_3. Finally, in Section 8, we investigate graph series from the E_6 Dynkin diagram, 3-cycles (affine type $A_2^{(1)}$), and affine Dynkin graphs of type $D_5^{(1)}$ and $E_6^{(1)}$.

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2. Preliminaries

In this part we introduce the main objects of study and present some preliminary results.

2.1. Graph schemes and graph series

This subsection outlines a construction of the arc space. We define the arc space and the arc algebra (or algebra of infinite jets) of a finitely generated commutative ring R. As usual, let $\mathbb{C}[x_1, x_2, \ldots, x_\ell]$ denote the polynomial algebra in x_j $(1 \leq j \leq \ell)$, let $f_1, f_2, \ldots, f_n$ be a set of polynomials, and define the quotient algebra

$$R := \frac{\mathbb{C}[x_1, x_2, \ldots, x_\ell]}{(f_1, f_2, \ldots, f_n)}.$$

We now introduce new variables $x_{j,(-1-k)}$ for $k \in \{0, \ldots, m\}$. We define a derivation T on

$$\mathbb{C}[x_{j,(-1-k)} : 0 \leq k \leq m,\ 1 \leq j \leq \ell],$$

by letting

$$T\left(x_{j,(-1-k)}\right) := \begin{cases} (-1-k)x_{j,(-k-2)} & \text{for } k \leq m-1, \\ 0 & \text{for } k = m. \end{cases}$$

We also identify x_j with $x_{j,(-1)}$. Set

$$R_m := \frac{\mathbb{C}[x_{j,(-1-k)} : 0 \leq k \leq m,\ 1 \leq j \leq \ell]}{(T^j f_k : 1 \leq k \leq n, j \geq 0)},$$

the *algebra of m-jets* of R. The *arc algebra* of R is defined as the direct limit

$$J_\infty(R) := \lim_{\overrightarrow{m}} R_m = \frac{\mathbb{C}[x_{j,(-1-k)} : 0 \leq k,\ 1 \leq j \leq \ell]}{(T^j f_k : k = 1, \dots n,\ j \geq 0)}.$$

The scheme $X_\infty = \lim_{\overleftarrow{m}} X_m$ (another notation is $J_\infty(X)$), where $X_m = \operatorname{Spec}(R_m)$, is called the *infinite jet scheme* of $X = \operatorname{Spec}(R)$. By construction, the arc algebra $J_\infty(R)$ is a differential commutative algebra. If R is graded, then $J_\infty(R)$ is also graded, and we can define its *Hilbert–(Poincare) series* as:

$$HS_q(J_\infty(R)) := \sum_{m \in \mathbb{Z}} \dim\left(J_\infty(R)_{(m)}\right) q^m.$$

We introduce a grading on R by letting $\deg(x_{j,(-1-k)}) := k + 1$.

Let V be a vertex algebra. Provided that V is strongly finitely generated, we obtain a surjective map from $J_\infty(R_V)$ to $\operatorname{gr}(V)$, the associated graded algebra of V, where R_V is Zhu's commutative algebra of a vertex algebra V. A vertex algebra for which this map is injective is said to be *classically free* [6,7,33–35]. Although this notion is relatively new, non-trivial examples of vertex algebras with this property appeared earlier in the framework of principal subspaces [15,16,20] and Virasoro minimal models [19]. The following theorem was announced in [25, Section 9], it was motivated by [37] and full details were provided in [32,33].

Theorem 2.1. *Let Γ be any simple graph with r nodes and without multiple edges. Consider the scheme X_Γ defined by the quadratic equation $x_k x_j = 0$, if k and j are adjacent i.e., $(k, j) \in E(\Gamma)$, and denote by $J_\infty(X_\Gamma)$ the infinite jet algebra of X_Γ. Then the Hilbert series of $J_\infty(X)$, with* $\deg(x_j) = 1$, *is given by*

$$H_\Gamma(q) = \sum_{\boldsymbol{n} \in \mathbb{N}_0^r} \frac{q^{\frac{1}{2}\sum_{(k,j)\in E(\Gamma)} n_k n_j + n_1 + \cdots + n_r}}{(q)_{n_1} \cdots (q)_{n_r}}.$$

Another important result in this context is the following theorem (see [33] and also [37,40]).

Theorem 2.2. *For any Γ (not necessarily simple), the q-series $H_\Gamma(q)$ computes the character of the principal subspace W_Γ [33, Subsection 5.3] of a lattice vertex algebra associated to the adjacency matrix of Γ, equipped with a certain grading.*

We also note that the previous theorem can be used, with slight modifications, for graphs with loops. In that case we need to adjust the linear term of the q-exponent of $H_\Gamma(q)$ (see [34] for some examples). We now give some modular examples.

Examples.
(1) (Affine space $\mathbb{A}^\ell$) For $R = \mathbb{C}[x_1, ..., x_\ell]$, we have

$$J_\infty(R) = \mathbb{C}\left[x_{1,(-1)}, ..., x_{1,(-n)}, ..., x_{\ell,(-1)}, ..., x_{\ell,(-n)}, ...\right].$$

Since $\deg(x_{r,(-j)}) = j$, we immediately get

$$HS_q(J_\infty(R)) = \frac{1}{(q)_\infty^\ell}.$$

(2) (Union of two lines) Let $R = \mathbb{C}[x,y]/(xy)$. Then by [7,12,25] we have

$$HS_q(J_\infty(R)) = \sum_{n_1,n_2\geq 0} \frac{q^{n_1n_2+n_1+n_2}}{(q)_{n_1}(q)_{n_2}} = \frac{1}{(1-q)(q)_\infty}.$$

(3) (“Fat” point) Let $R = \mathbb{C}[x]/(x^2)$. Then by [8,12,15,20] we have

$$HS_q(J_\infty(R)) = \sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty},$$

where the last equality is the first Rogers–Ramanujan identity.

2.2. Modularity results

In this subsection we record some modularity results required for this paper. We start with the transformation laws of the *Dedekind η-function* $\eta(\tau) := q^{\frac{1}{24}}(q)_\infty$:

$$\eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau).$$

We define the *weight two Eisenstein series* ($q := e^{2\pi i\tau}$)

$$E_2(\tau) := 1 - 24\sum_{n\geq 1}\sum_{d|n} dq^n. \tag{2.1}$$

This function is not quite a modular form, but transforms with an additional term under inversion. To be more precise, we have

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2 E_2(\tau) + \frac{6\tau}{\pi i}. \tag{2.2}$$

We also require the following representation of E_2 as a Lerch-type sum. Although the next result is probably known, we include a proof for the sake of completeness.

Lemma 2.3. *We have*

$$\frac{1-E_2(\tau)}{24}=\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}=\sum_{n\geq 1}\frac{(-1)^{n+1}(1+q^n)q^{\frac{n(n+1)}{2}}}{(1-q^n)^2}.$$

Proof. The first equality is just rewriting (2.1). Using L'Hôpital's rule twice the second equality follows from taking the limit $\zeta\to 1$ of the following identity (valid for $0<|\zeta|<|q|<1$):

$$-\frac{\zeta}{(1-\zeta)^2}+\frac{\zeta(q)_\infty^2}{(1-\zeta)\,(\zeta)_\infty(\zeta^{-1}q)_\infty}=\sum_{n\geq 1}\frac{(-1)^{n+1}(1+q^n)q^{\frac{n(n+1)}{2}}}{(1-\zeta q^n)(1-\zeta^{-1}q^n)}. \tag{2.3}$$

Note that (2.3) is implied by the well-known formula (see for instance [31])

$$\frac{(q)_\infty^2}{(\zeta)_\infty(\zeta^{-1}q)_\infty}=\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1-\zeta q^n}. \qquad \square$$

We also require modularity properties of

$$\mathcal{F}(\tau):=\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{(-1)^{n+1}q^{\frac{3n(n+1)}{2}}}{(1-q^n)^2}.$$

For this define ($\tau=u+iv$) the *completion of* $\mathcal{F}$ as

$$\widehat{\mathcal{F}}(\tau):=\mathcal{F}(\tau)-\frac{1}{24}+\frac{E_2(\tau)}{8}-\frac{3\eta(\tau)}{2\sqrt{\pi}}\sum_{n\in\mathbb{Z}-\frac{1}{6}}(-1)^{n+\frac{1}{6}}|n|\Gamma\left(-\frac{1}{2},6\pi n^2v\right)q^{-\frac{3n^2}{2}},$$

where the *incomplete gamma function* is defined as $\Gamma(\alpha,x):=\int_x^\infty e^{-t}t^{\alpha-1}dt$ for $x\in\mathbb{R}^+$ and $\alpha\in\mathbb{R}$. We then have the following modularity result.

Proposition 2.4. *We have for* $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})$,

$$\widehat{\mathcal{F}}\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^2\widehat{\mathcal{F}}(\tau).$$

Proof. Set

$$\begin{aligned} r(\tau)&:=r^+(\tau)+r^-(\tau),\quad\text{with}\\ r^+(\tau)&:=\tfrac{2\pi i}{\eta(\tau)}\left(\mathcal{F}(\tau)-\tfrac{1}{24}+\tfrac{E_2(\tau)}{8}\right),\end{aligned}$$

$$r^-(\tau) := \tfrac{1}{2\pi i}\left[\frac{\partial}{\partial z}\left(\zeta^{-1}q^{-\frac{1}{6}}R(3z+\tau;3\tau)e^{-\frac{\pi^2}{2}E_2(\tau)z^2}\right)\right]_{z=0}.$$

Here $z := x+iy,\ \zeta := e^{2\pi iz}$, and

$$R(z;\tau) := \sum_{n\in\mathbb{Z}+\frac{1}{2}}(-1)^{n-\frac{1}{2}}\left(\operatorname{sgn}(n)-E\left(\left(n+\frac{y}{v}\right)\sqrt{2v}\right)\right)q^{-\frac{n^2}{2}}e^{-2\pi inz},$$

with $E(x) := 2\int_0^x e^{-\pi t^2}dt$. By using [11] we obtain that for $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})$,

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)r\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\eta(\tau)r(\tau). \tag{2.4}$$

We compute

$$\begin{aligned}
&r^-(\tau)\\
&=\frac{1}{2\pi i}q^{-\frac{1}{6}}\sum_{n\in\mathbb{Z}+\frac{1}{2}}(-1)^{n-\frac{1}{2}}\left[\frac{\partial}{\partial z}\left(\left(\operatorname{sgn}(n)-E\left(\left(n+\frac{1}{3}+\frac{y}{v}\right)\sqrt{6v}\right)\right)\zeta^{-3n-1}\right)\right]_{z=0}q^{-\frac{3n^2}{2}-n}\\
&=-\frac{1}{2\pi i}\sum_{n\in\mathbb{Z}-\frac{1}{6}}(-1)^{n+\frac{1}{6}}\left[\frac{\partial}{\partial z}\left(\left(\operatorname{sgn}(n)-E\left(\left(n+\frac{y}{v}\right)\sqrt{6v}\right)\right)\zeta^{-3n}\right)\right]_{z=0}q^{-\frac{3n^2}{2}}.
\end{aligned}$$

We now use the identities

$$E'(x) = 2e^{-\pi x^2},\quad E(x) = \operatorname{sgn}(x)\left(1-\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2},\pi x^2\right)\right), \tag{2.5}$$

$$\Gamma\left(\frac{1}{2},x\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2},x\right)+x^{-\frac{1}{2}}e^{-x} \tag{2.6}$$

to obtain that

$$\left[\frac{\partial}{\partial z}\left(\left(\operatorname{sgn}(n)-E\left(\left(n+\frac{y}{v}\right)\sqrt{6v}\right)\right)\zeta^{-3n}\right)\right]_{z=0} = 3\sqrt{\pi}i|n|\Gamma\left(-\frac{1}{2},6\pi n^2v\right).$$

This gives that

$$r^-(\tau) = -\frac{3}{2\sqrt{\pi}}\sum_{n\in\mathbb{Z}-\frac{1}{6}}(-1)^{n+\frac{1}{6}}|n|\Gamma\left(-\frac{1}{2},6\pi n^2v\right)q^{-\frac{3n^2}{2}}.$$

Thus we have that $\widehat{\mathcal{F}}(\tau) = \eta(\tau)r(\tau)$. The claim then follows from (2.4). $\square$

We also require certain indefinite theta functions, considered by Zwegers in his thesis [47]. We let A be a symmetric $r\times r$ matrix with integral coefficients that is non-degenerate, $Q(\boldsymbol{x}) := \frac{1}{2}\boldsymbol{x}^TA\boldsymbol{x}$ the corresponding quadratic form, and $B(\boldsymbol{x},\boldsymbol{y}) := \boldsymbol{x}^TA\boldsymbol{y}$

the associated bilinear form. We assume that Q has signature $(r-1,1)$. Fix $\boldsymbol{c_0} \in \mathbb{R}^r$ and let

$$C_Q := \{\boldsymbol{c} \in \mathbb{R}^r : Q(\boldsymbol{c}) < 0, B(\boldsymbol{c}, \boldsymbol{c_0}) < 0\}.$$

For $\boldsymbol{c_1}, \boldsymbol{c_2} \in C_Q$, set

$$\varrho(\boldsymbol{n}) = \varrho_A^{\boldsymbol{c_1},\boldsymbol{c_2}}(\boldsymbol{n};\tau) := E\left(\frac{B(\boldsymbol{c_1},\boldsymbol{n})}{\sqrt{-Q(\boldsymbol{c_1})}}\sqrt{v}\right) - E\left(\frac{B(\boldsymbol{c_2},\boldsymbol{n})}{\sqrt{-Q(\boldsymbol{c_2})}}\sqrt{v}\right).$$

Then define

$$\Theta_A(\boldsymbol{z};\tau) = \Theta_A^{\boldsymbol{c_1},\boldsymbol{c_2}}(\boldsymbol{z};\tau) := \sum_{\boldsymbol{n}\in\mathbb{Z}^r} \varrho\left(\boldsymbol{n} + \frac{\mathrm{Im}(\boldsymbol{z})}{v}\right) e^{2\pi i B(\boldsymbol{n},\boldsymbol{z})} q^{Q(\boldsymbol{n})}.$$

We have the following properties (see Proposition 2.7 of [47]).

Proposition 2.5.

(1) *We have*

$$\Theta_A\left(\frac{\boldsymbol{z}}{\tau}; -\frac{1}{\tau}\right) = \frac{i}{\sqrt{-\det(A)}}(-i\tau)^{\frac{r}{2}} \sum_{\boldsymbol{\ell}\in A^{-1}\mathbb{Z}^r/\mathbb{Z}^r} e^{\frac{2\pi i}{\tau}Q(\boldsymbol{z}+\boldsymbol{\ell}\tau)} \Theta_A(\boldsymbol{z}+\boldsymbol{\ell}\tau;\tau).$$

(2) *We have*

$$\Theta_A(-\boldsymbol{z};\tau) = -\Theta_A(\boldsymbol{z};\tau).$$

(3) *For $\boldsymbol{n} \in \mathbb{Z}^2$, $\boldsymbol{m} \in A^{-1}\mathbb{Z}^2$, we have*

$$\Theta_A(\boldsymbol{z} + \boldsymbol{n}\tau + \boldsymbol{m};\tau) = e^{-2\pi i B(\boldsymbol{n},\boldsymbol{z})} q^{-Q(\boldsymbol{n})} \Theta_A(\boldsymbol{z};\tau).$$

We finish this subsection by defining quantum modular forms, following Zagier [45].

Definition. A function $f : \mathcal{Q} \to \mathbb{C}$ (here $\mathcal{Q} \subseteq \mathbb{Q}$) is a *quantum modular form* of weight $k \in \frac{1}{2}\mathbb{Z}$ and multiplier χ for a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ and quantum set $\mathcal{Q}$, if for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ the function

$$f(\tau) - \chi(M)^{-1}(c\tau+d)^{-k} f(M\tau) \tag{2.7}$$

can be extended to an open subset of $\mathbb{R}$ and is real-analytic there.

Remark. Zagier [46] recently also defined *holomorphic quantum modular forms*. These are holomorphic functions $f : \mathbb{H} \to \mathbb{C}$, such that (2.7) is holomorphic in a larger domain than $\mathbb{H}$.

An example of a holomorphic quantum modular form is the generating function for the number of divisors. We have the following by [10, Theorem 1] (see also [39]).

Lemma 2.6. *The function*

$$D(q) := \sum_{n\geq 1}\sum_{d|n} q^n = \sum_{n\geq 1}\frac{q^n}{1-q^n}$$

is a holomorphic quantum modular form of weight one. Moreover, $\sum_{n\geq 1}\frac{n^2q^n}{1-q^n}$ *is a holomorphic quantum modular form of weight* 3.

2.3. q-series results

We finish this section by recalling several q-series identities needed in the paper. We often use Euler's identity

$$\frac{1}{(\zeta)_\infty} = \sum_{n\geq 0}\frac{\zeta^n}{(q)_n}. \tag{2.8}$$

We also require Bailey's pairs [1, Chapter 3]. Recall that a pair of sequences (α_n, β_n) is called a *Bailey pair* relative to (a, q) if

$$\beta_n = \sum_{0\leq j\leq n}\frac{\alpha_j}{(q)_{n-j}(aq)_{n+j}}.$$

Bailey's Lemma is as follows.

Lemma 2.7. *[1, Theorem 3.4] If (α_n, β_n) is a Bailey pair relative to (a, q), then (assuming convergence conditions) we have*

$$\sum_{n\geq 0}(\varrho_1)_n(\varrho_2)_n\left(\frac{aq}{\varrho_1\varrho_2}\right)^n\beta_n = \frac{\left(\frac{aq}{\varrho_1}\right)_\infty\left(\frac{aq}{\varrho_2}\right)_\infty}{(aq)_\infty\left(\frac{aq}{\varrho_1\varrho_2}\right)_\infty}\sum_{n\geq 0}\frac{(\varrho_1)_n(\varrho_2)_n\left(\frac{aq}{\varrho_1\varrho_2}\right)^n}{\left(\frac{aq}{\varrho_1}\right)_n\left(\frac{aq}{\varrho_2}\right)_n}\alpha_n.$$

Recall an identity by Andrews and Freitas [5, Corollary 4.3]:

$$\frac{1}{(q)_\infty}\sum_{n\geq 0}\zeta^n\left((q)_n-(q)_\infty\right) = \sum_{n\geq 1}\frac{q^n}{(1-\zeta q^n)(q)_n}, \tag{2.9}$$

and another identity [23, formula (5.1)]

$$\sum_{n\geq 0}\zeta^n\left((q)_n-(q)_\infty\right) = \sum_{n\geq 1}q^n\left(1+\zeta+\cdots+\zeta^{n-1}\right)(q)_{n-1}. \tag{2.10}$$

If $\zeta = 1$, this recovers Zagier's identity [43, formula (16)], and for $\zeta = 0$ we get

$$\sum_{n\geq 0} q^{n+1}(q)_n = 1-(q)_\infty. \tag{2.11}$$

We also require an identity of Andrews, Garvan, and Liang [4, Theorem 3.5]

$$\sum_{n\geq 0} \zeta^n \left(\left(q^{n+1}\right)_\infty - 1\right) = \sum_{n\geq 1} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(1-\zeta q^n)(q)_n}. \tag{2.12}$$

Finally, we require two of Fine's identities [21, equations (12.42),(12.45)]

$$\sum_{n\geq 1} \frac{(-1)^{n+1} q^{\frac{n^2+n}{2}}}{(1-q^n)(q)_n} = D(q), \tag{2.13}$$

$$\sum_{n\geq 0} \left(\frac{1}{(q)_\infty} - \frac{1}{(q)_n}\right) = \frac{1}{(q)_\infty} D(q). \tag{2.14}$$

3. *A*-series and the proof of Theorem 1.1

We start our investigation of mock and quantum modular properties of graph series by focusing on the path graphs (i.e., Dynkin diagrams of type A) denoted by A_k, $k \geq 1$ (as usual A_1 is just a single node):

$$1 - 2 - 3 - \cdots - k$$

The corresponding graph series are given by

$$H_{A_k}(q) = \sum_{\boldsymbol{n}\in\mathbb{N}_0^k} \frac{q^{n_1n_2+\cdots+n_{k-1}n_k+n_1+\cdots+n_k}}{(q)_{n_1}\cdots(q)_{n_k}}.$$

Using (2.8) on the sums for n_1, n_2, n_{k+3}, and n_{k+4} and relabeling, it is easy to see that for $k \geq 3$,

$$H_{A_{k+4}}(q) = \frac{1}{(q)_\infty^2}\mathcal{H}_{A_k}(q), \tag{3.1}$$

where for $k \geq 2$

$$\mathcal{H}_{A_k}(q) := \sum_{\boldsymbol{n}\in\mathbb{N}_0^k} \frac{q^{n_1n_2+\cdots+n_{k-1}n_k+n_1+\cdots+n_k}}{(q)_{n_1+1}(q)_{n_2}\cdots(q)_{n_{k-1}}(q)_{n_k+1}}.$$

Further applications of Euler's formula (2.8) give

$$\mathcal{H}_{A_k}(q) = q^{-2} \sum_{n_1,\ldots,n_{k-2}\geq 0} \left(\frac{1}{(q)_\infty} - \frac{1}{(q)_{n_{k-2}}}\right)$$
$$\times \left(\frac{1}{(q)_\infty} - \frac{1}{(q)_{n_1}}\right) \frac{q^{n_1n_2+\ldots+n_{k-3}n_{k-2}+n_2+\cdots+n_{k-3}}}{(q)_{n_2}\cdots(q)_{n_{k-3}}}.$$

The following identities are taken from [25]:

$$H_{A_1}(q) = \frac{1}{(q)_\infty}, \quad H_{A_2}(q) = \frac{1}{(1-q)(q)_\infty},$$
$$H_{A_3}(q) = \frac{q^{-1}(1-(q)_\infty)}{(q)_\infty^2}, \quad H_{A_4}(q) = \frac{q^{-1}}{(q)_\infty^2}D(q),$$
$$H_{A_5}(q) = \frac{q^{-1}}{(q)_\infty^3}\sum_{n\geq 0}((q)_n - (q)_\infty), \quad H_{A_6}(q) = \frac{2q^{-1}}{(q)_\infty^3}D(q) - \frac{q^{-1}}{(q)_\infty^3} + \frac{q^{-1}}{(q)_\infty^2}.$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. For the first identity, we compute

$$H_{A_7}(q) = \frac{1}{(q)_\infty^2}\mathcal{H}_{A_3}(q) = \frac{1}{(q)_\infty^3}\sum_{n\geq 0} q^{-n-2}\left(\frac{1}{(q)_\infty} - \frac{1}{(q)_n}\right)\Big((q)_n - (q)_\infty\Big).$$

To analyze the last sum, we introduce a new parameter ζ and consider

$$F_\zeta(q) := \sum_{n\geq 0}\zeta^n\left(\frac{1}{(q)_\infty} - \frac{1}{(q)_n}\right)\Big((q)_n - (q)_\infty\Big) = \sum_{n\geq 0}\zeta^n\left(\left(q^{n+1}\right)_\infty + \frac{1}{\left(q^{n+1}\right)_\infty} - 2\right).$$

Adding (2.12) and (2.9) results in cancellation of the term $n = 1$, so we have

$$F_\zeta(q) = \sum_{n\geq 2}\frac{(-1)^n q^{\frac{n(n+1)}{2}} + q^n}{(1-\zeta q^n)(q)_n}.$$

Letting $\zeta = q^{-1}$ (which is now allowed) gives

$$F_{q^{-1}}(q) = \sum_{n\geq 0} q^{-n}\left(\frac{1}{(q)_\infty} - \frac{1}{(q)_n}\right)\Big((q)_n - (q)_\infty\Big) = \sum_{n\geq 2}\frac{(-1)^n q^{\frac{n(n+1)}{2}} + q^n}{(1-q^{n-1})(q)_n}.$$

Next we split the right-hand side into two sums. For the first sum we recall (2.13) and also observe

$$\frac{1}{1-q}\sum_{n\geq 1}\frac{(-1)^{n+1}q^{\frac{n^2+n}{2}+1}}{(1-q^n)(q)_n} - \frac{q^2}{(1-q)^2} = \sum_{n\geq 2}\frac{(-1)^n q^{\frac{n^2+n}{2}}}{(1-q^{n-1})(q)_n},$$

which follows from the "finite" identity (here $k \geq 2$)

$$\left(\frac{1}{1-q}\sum_{n=1}^{k-1}\frac{(-1)^{n+1}q^{\frac{n^2+n}{2}+1}}{(1-q^n)\,(q)_n}-\frac{q^2}{(1-q)^2}\right)-\left(\sum_{n=2}^{k}\frac{(-1)^nq^{\frac{n^2+n}{2}}}{(1-q^{n-1})\,(q)_n}\right)=\frac{q^{\frac{k^2}{2}+\frac{k}{2}+1}}{(1-q)(q)_k},$$

after letting $k \to \infty$. Combined this implies that

$$\sum_{n\geq 2}\frac{(-1)^nq^{\frac{n(n+1)}{2}}}{(1-q^{n-1})(q)_n}=\frac{q}{1-q}\sum_{n\geq 2}\frac{q^n}{1-q^n}. \tag{3.2}$$

For the second sum we reintroduce the parameter ζ and use that

$$\sum_{n\geq 2}\frac{q^n}{(1-\zeta q^n)(q)_n}=\sum_{n\geq 1}\frac{q^n}{(1-\zeta q^n)(q)_n}-\frac{q}{(1-q)(1-\zeta q)}.$$

Employing (2.10) and

$$\frac{q}{(1-q)(1-\zeta q)}=\sum_{n\geq 1}\left(1+\cdots+\zeta^{n-1}\right)q^n$$

we get

$$\sum_{n\geq 2}\frac{q^n}{(1-\zeta q^n)(q)_n}=\sum_{n\geq 1}q^n\left(1+\zeta+\cdots+\zeta^{n-1}\right)\left(\frac{1}{(q^n)_\infty}-1\right). \tag{3.3}$$

We let $\zeta = q^{-1}$ in (3.3) to obtain

$$\begin{aligned}&\sum_{n\geq 2}\tfrac{q^n}{(1-q^{n-1})(q)_n}\\ &=\sum_{n\geq 1}q^n\left(1+q^{-1}+\cdots+q^{-n+1}\right)\left(\tfrac{1}{(q^n)_\infty}-1\right)=\tfrac{q}{1-q}\sum_{n\geq 1}(1-q^n)\left(\tfrac{1}{(q^n)_\infty}-1\right)\\ &=-\tfrac{q}{1-q}\sum_{n\geq 1}q^n\left(\tfrac{1}{(q^n)_\infty}-1\right)+\tfrac{q}{1-q}\sum_{n\geq 1}\left(\tfrac{1}{(q^n)_\infty}-1\right).\end{aligned}$$

The first sum evaluates as

$$-\tfrac{q^2}{1-q}\sum_{n\geq 0}q^n\left(\tfrac{1}{(q^{n+1})_\infty}-1\right)=-\tfrac{q^2}{(1-q)(q)_\infty}\sum_{n\geq 0}q^n((q)_n-(q)_\infty)=-\tfrac{q}{(1-q)(q)_\infty}+\tfrac{q}{1-q}+\tfrac{q^2}{(1-q)^2},$$

where we use (2.9) with $\zeta = q$ and (2.8). Thus

$$\sum_{n\geq 2}\frac{q^n}{(1-q^{n-1})\,(q)_n}=-\frac{q}{(1-q)(q)_\infty}+\frac{q}{1-q}+\frac{q^2}{(1-q)^2}+\frac{q}{1-q}\sum_{n\geq 1}\left(\frac{1}{(q^n)_\infty}-1\right).$$

Combined with (3.2), the previous relation gives

$$\sum_{n\geq 2}\frac{(-1)^n q^{\frac{n(n+1)}{2}}+q^n}{(1-q^{n-1})\,(q)_n}$$
$$=\frac{q}{1-q}\sum_{n\geq 2}\frac{q^n}{1-q^n}-\frac{q}{(1-q)(q)_\infty}+\frac{q}{1-q}+\frac{q^2}{(1-q)^2}+\frac{q}{1-q}\sum_{n\geq 1}\left(\frac{1}{(q^n)_\infty}-1\right)$$
$$=\frac{q}{1-q}\left(D(q)-\frac{1}{(q)_\infty}+1+\frac{1}{(q)_\infty}G(q)\right).$$

Finally, after we multiply by $\frac{q^{-2}}{(q)_\infty^3}$ we get the claimed formula.

For $H_{A_8}(q)$, using (3.1), we first get

$$H_{A_8}(q)=\frac{q^{-2}}{(q)_\infty^2}\sum_{n_1,n_2\geq 0}\left(\frac{1}{(q)_\infty}-\frac{1}{(q)_{n_1}}\right)\left(\frac{1}{(q)_\infty}-\frac{1}{(q)_{n_2}}\right)q^{n_1n_2}$$
$$=\frac{q^{-2}}{(q)_\infty^2}\sum_{n_1,n_2\geq 0}\left(\frac{q^{n_1n_2}}{(q)_\infty^2}-\frac{q^{n_1n_2}}{(q)_{n_1}(q)_\infty}-\frac{q^{n_1n_2}}{(q)_{n_2}(q)_\infty}+\frac{q^{n_1n_2}}{(q)_{n_1}(q)_{n_2}}\right).$$

We would like to separate this into four sums but there is a convergence issue. For this reason, we first evaluate the terms with $n_1n_2=0$. For $n_1=n_2=0$ we have

$$\frac{q^{-2}}{(q)_\infty^2}\left(\frac{1}{(q)_\infty^2}-\frac{2}{(q)_\infty}+1\right).$$

For $n_2\geq 1$, $n_1=0$ and $n_1\geq 1$, $n_2=0$ (due to symmetry) we get the contribution

$$-\frac{2q^{-2}}{(q)_\infty^2}\sum_{n\geq 1}\left(-1+\frac{1}{(q)_\infty}\right)\left(\frac{1}{(q)_\infty}-\frac{1}{(q)_n}\right).$$

We are left with

$$\sum_{n_1,n_2\geq 1}\left(\frac{q^{n_1n_2}}{(q)_\infty^2}-\frac{q^{n_1n_2}}{(q)_{n_1}(q)_\infty}-\frac{q^{n_1n_2}}{(q)_{n_2}(q)_\infty}+\frac{q^{n_1n_2}}{(q)_{n_1}(q)_{n_2}}\right)$$
$$=\frac{D(q)}{(q)_\infty^2}-\frac{2}{(q)_\infty}\sum_{n\geq 0}\left(\frac{1}{(q^{n+1})_\infty}-1\right)+\sum_{n_1,n_2\geq 1}\frac{q^{n_1n_2}}{(q)_{n_1}(q)_{n_2}}.$$

For the final sum we use an identity from [25, Section 7.3] (which is essentially (2.14)):

$$\sum_{n_1,n_2\geq 1}\frac{q^{n_1n_2}}{(q)_{n_1}(q)_{n_2}}=1+2\frac{D(q)}{(q)_\infty}-\frac{1}{(q)_\infty}.$$

Combining with the above, and using (2.13), yields the claim. □

Remark. As discussed in [25], Theorem 1.1 implies the formula

$$H_{A_7}(q) = \frac{q^{-1}}{(1-q)(q)_\infty^4}\left(\sum_{n\geq 1}(-1)^n(-3n+1)q^{\frac{3n^2+n}{2}} + \sum_{n\leq -1}(-1)^n(3n+2)q^{\frac{3n^2+n}{2}}\right).$$

Note however that the conjecture for $H_{A_8}(q)$ given in [25] does not hold.

The following result describes the asymptotic behaviors and quantum modular properties of these graph series.

Proposition 3.1. *As $t \to 0^+$, we have:*

(1) $(e^{-t})_\infty H_{A_2}(e^{-t}) = \frac{1}{t} + O(1)$,
(2) $(e^{-t})_\infty^2 H_{A_3}(e^{-t}) = 1 + O(t)$,
(3) $(e^{-t})_\infty^2 H_{A_4}(e^{-t}) = \frac{\gamma-\log(t)}{t} + O(1)$, *where γ is the Euler–Mascheroni constant,*
(4) $(e^{-t})_\infty^3 H_{A_5}(e^{-t}) = 1 + O(t)$,
(5) $(e^{-t})_\infty^3 H_{A_6}(e^{-t}) = \frac{2(\gamma-\log(t))}{t} + O(1)$,
(6) $(e^{-t})_\infty^4 H_{A_7}(e^{-t}) = 1 + O(t)$,
(7) $(e^{-t})_\infty^4 H_{A_8}(e^{-t}) = \frac{3(\gamma-\log(t))}{t} + O(1)$.

Moreover, $q(q)_\infty^2 H_{A_4}(q)$ is a holomorphic quantum modular form of weight one, while $q(q)_\infty^3 H_{A_5}(q)$ and $q(1-q)(q)_\infty^4 H_{A_7}(q) + 1$ are quantum modular forms of weight $\frac{3}{2}$.

Proof. (1) and (2) are immediate. The asymptotic behavior in (3) is well-known and can easily be concluded using the Euler–Maclaurin summation formula [44]. Quantum modular properties of $q(q)_\infty^2 H_{A_4}(q) = D(q)$ are given in Lemma 2.6. For (4), we rewrite (see Theorem 2 of [43]):

$$G(q) = -\frac{1}{2}H(q) + (q)_\infty\left(\frac{1}{2} - D(q)\right), \tag{3.4}$$

where

$$G(q) := \sum_{n\geq 0}((q)_n - (q)_\infty), \qquad H(q) := \sum_{n\geq 1} n\left(\frac{12}{n}\right)q^{\frac{n^2-1}{24}}.$$

Since $H(q)$ is a quantum modular of weight $\frac{3}{2}$ and $(q)_\infty$ vanishes at all roots of unity it follows that $q(q)_\infty^3 H_{A_5}(q) = G(q)$ is also a quantum modular of weight $\frac{3}{2}$. The series $H(q)$ satisfies the asymptotic behavior [43, Theorem 3]

$$H\left(e^{-t}\right) = -2 - 2t + O\left(t^2\right),$$

and thus

$$\left(e^{-t}\right)_{\infty}^{3} H_{A_5}\left(e^{-t}\right) = -\frac{1}{2} H\left(e^{-t}\right)(1+O(t)) = 1 + O(t).$$

To see (5), we use part (3). For (6) we recall an identity from [25, Subsection 7.4]

$$\sum_{n\geq 1}(-3n+1)(-1)^n q^{\frac{3n^2+n}{2}} + \sum_{n\leq -1}(3n+2)(-1)^n q^{\frac{3n^2+n}{2}} = -1 + (q)_\infty D(q) + G(q) + (q)_\infty.$$

Using (3.4) we get

$$-1 + G\left(e^{-t}\right) = t + O\left(t^2\right).$$

Since $\frac{1}{1-e^{-t}} = \frac{1}{t} + O(t)$, the asymptotics in (6) follow. To see quantum modularity of $q(1-q)(q)_\infty^4 H_{A_7}(q) + 1$, we recall the first formula in Theorem 1.1:

$$q(1-q)(q)_\infty^4 H_{A_7}(q) = -1 + (q)_\infty D(q) + G(q) + (q)_\infty. \tag{3.5}$$

This formula implies that $q(1-q)(q)_\infty^4 H_{A_7}(q) + 1$ and $G(q)$ agree at all roots unity and it is argued above that $G(q)$ is quantum modular. For (7) we use exactly the same argument as in (5). $\square$

Based on Proposition 3.1, we conjecture that the following is true.

Conjecture 3.2. *For $n \geq 1$, there exist $a_n, b_n, c_n \in \mathbb{R}^+$ such that*

$$\left(e^{-t}\right)_{\infty}^{n} H_{A_{2n}}\left(e^{-t}\right) \sim \frac{a_n + c_n \log(t)}{t}, \qquad \left(e^{-t}\right)_{\infty}^{n} H_{A_{2n-1}}\left(e^{-t}\right) \sim b_n, \qquad \text{as } t \to 0^+.$$

4. 5-cycles, sums of divisors, and the proof of (1.3) and (1.4)

In this part we are concerned with series coming from certain graphs obtained by glueing 5-cycles. Generally, graph series associated to graphs with cycles are more complicated to analyze. We start from an auxiliary result that, quite surprisingly, allows us to perform computations for several interesting examples of graphs. The next lemma can be viewed as a generalization of the A_2-identity discussed in the previous section.

Lemma 4.1. *For $a, b \in \mathbb{N}_0$, we have*

$$A(a,b) := \sum_{n_1,n_2\geq 0} \frac{q^{n_1n_2+(a+1)n_1+(b+1)n_2}}{(q)_{n_1}(q)_{n_2}} = \frac{1}{\left(q^{b+1}\right)_{a+1}\left(q^{a+1}\right)_\infty} = \frac{1}{\left(q^{a+1}\right)_{b+1}\left(q^{b+1}\right)_\infty}.$$

Proof. The second equality follows due to the symmetry $A(a,b) = A(b,a)$. To show the first, we recall a well-known formula (see [21, equation (6.2)])

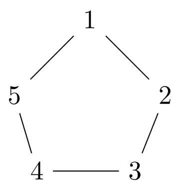
$$\sum_{n\geq 0}\frac{(sq)_n}{(q)_n}t^n=\frac{(stq)_\infty}{(t)_\infty}. \tag{4.1}$$

We compute

$$A(a,b)=\sum_{n_2\geq 0}\frac{q^{(b+1)n_2}}{\left(q^{n_2+a+1}\right)_\infty (q)_{n_2}}=\frac{1}{\left(q^{a+1}\right)_\infty}\sum_{n\geq 0}\frac{\left(q^{a+1}\right)_n q^{(b+1)n}}{(q)_n}=\frac{1}{\left(q^{a+1}\right)_\infty \left(q^{b+1}\right)_{a+1}},$$

where the last equality follows from (4.1), letting $s=q^{a+1}$ and $t=q^{b+1}$. □

Equipped with this result we can now give elegant representations of several (shifted) graph series associated to C_5 (5-cycle)



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in particular proving (1.3).

Proposition 4.2. *We have*

$$H_{C_5}(q)=\sum_{\boldsymbol{n}\in\mathbb{N}_0^5}\frac{q^{n_1n_2+n_1n_5+n_2n_3+n_3n_4+n_4n_5+n_1+n_2+n_3+n_4+n_5}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}}=\frac{q^{-1}}{(q)_\infty^2}\sum_{n\geq 1}\frac{nq^n}{1-q^n},$$

$$\sum_{\boldsymbol{n}\in\mathbb{N}_0^5}\frac{q^{n_1n_2+n_1n_5+n_2n_3+n_3n_4+n_4n_5+2n_1+n_2+n_3+n_4+n_5}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}}=\frac{1}{(1-q)^2(q)_\infty^2},$$

$$\sum_{\boldsymbol{n}\in\mathbb{N}_0^5}\frac{q^{n_1n_2+n_1n_5+n_2n_3+n_3n_4+n_4n_5+n_1+2n_2+n_3+n_4+2n_5}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}}=\frac{q^{-2}}{(q)_\infty^2}\sum_{n\geq 2}\frac{nq^n}{1-q^n}.$$

Proof. Let

$$B_1(q):=\sum_{n\geq 1}\frac{nq^n}{1-q^n},\qquad B_2(q):=\frac{1}{(1-q)^2(q)_\infty^2},\qquad B_3(q):=\frac{q^{-2}}{(q)_\infty^2}\sum_{n\geq 2}\frac{nq^n}{1-q^n}.$$

Note that $B_2(q)+qB_3(q)=B_1(q)$. It is easy to see that the same relation holds for the left-hand sides. Thus it suffices to prove the first two identities. We start with the second identity. Euler's identity (2.8) gives

$$\sum_{\boldsymbol{n}\in\mathbb{N}_0^5}\frac{q^{n_1n_2+n_1n_5+n_2n_3+n_3n_4+n_4n_5+2n_1+n_2+n_3+n_4+n_5}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}}$$

$$= \sum_{n_2,n_3,n_4,n_5\geq 0} \frac{q^{n_2n_3+n_3n_4+n_4n_5+n_2+n_3+n_4+n_5}}{(q^{n_2+n_5+2})_\infty (q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}}$$
$$= \frac{1}{(q)_\infty} \sum_{n_2,n_3,n_4,n_5\geq 0} \frac{q^{n_2n_3+n_3n_4+n_4n_5+n_2+n_3+n_4+n_5}(q)_{n_2+n_5+1}}{(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}}$$
$$= \frac{1}{(q)_\infty} \sum_{n_2,n_5\geq 0} \frac{q^{n_2+n_5}(q)_{n_2+n_5+1}}{(q)_{n_2}(q)_{n_5}} \sum_{n_3,n_4\geq 0} \frac{q^{n_3n_4+(n_2+1)n_3+(n_5+1)n_4}}{(q)_{n_3}(q)_{n_4}}.$$

Using this equals

$$\frac{1}{(q)_\infty} \sum_{n_2,n_5\geq 0} \frac{q^{n_2+n_5}(q)_{n_2+n_5+1}}{(q)_{n_2}(q)_{n_5}} \frac{1}{(q^{n_2+1})_\infty (q^{n_5+1})_{n_2+1}} = \frac{1}{(q)_\infty^2} \sum_{n_2,n_5\geq 0} q^{n_2+n_5}$$
$$= \frac{1}{(1-q)^2(q)_\infty^2}.$$

For the first identity we use a similar argument. We get

$$\sum_{\boldsymbol{n}\in\mathbb{N}_0^5} \frac{q^{n_1n_2+n_1n_5+n_2n_3+n_3n_4+n_4n_5+n_1+n_2+n_3+n_4+n_5}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}}$$
$$= \sum_{n_2,n_3,n_4,n_5\geq 0} \frac{q^{n_2n_3+n_3n_4+n_4n_5+n_2+n_3+n_4+n_5}}{(q^{n_2+n_5+1})_\infty (q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}}$$
$$= \frac{1}{(q)_\infty} \sum_{n_2,n_5\geq 0} \frac{q^{n_2+n_5}}{1-q^{n_2+n_5+1}} = \frac{q^{-1}}{(q)_\infty^2} \sum_{n\geq 1} \frac{nq^n}{1-q^n},$$

as claimed. □

Next we consider the graph series associated to the graph

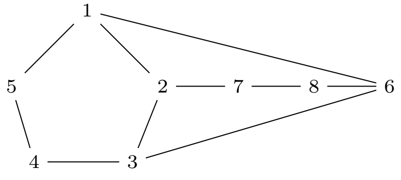


Fig. 1. Graph Γ_8.

We need a result for the sum of squares of divisors.

Lemma 4.3. *We have*

$$\sum_{\boldsymbol{n}\in\mathbb{N}_0^3} \frac{q^{n_1+n_2+n_3+1}}{(1-q^{n_1+n_2+1})(1-q^{n_1+n_2+n_3+1})} = \sum_{n\geq 1} \frac{n^2q^n}{1-q^n}.$$

Proof. We can write the left-hand side as

$$\sum_{\substack{0\le\ell\le k\\ k\ge 0}} \frac{(\ell+1)q^{k+1}}{(1-q^{\ell+1})(1-q^{k+1})}$$

$$= \sum_{\substack{1\le\ell\le k\\ k\ge 1}} \frac{\ell q^{k}}{(1-q^{\ell})(1-q^{k})} = \sum_{\substack{1\le\ell\le k\\ k\ge 1}} \frac{\ell q^{k}\left(q^{\ell}+1-q^{\ell}\right)}{(1-q^{k})\,(1-q^{\ell})}$$

$$= \sum_{\substack{1\le\ell\le k\\ k\ge 1}} \frac{\ell q^{k+\ell}}{(1-q^{\ell})(1-q^{k})} + \frac{1}{2}\sum_{k\ge 1}\frac{k(k+1)q^k}{1-q^k}$$

$$= \frac{1}{2}\sum_{k,\ell\ge 1} \frac{\min(k,\ell)q^{k+\ell}}{(1-q^{k})(1-q^{\ell})} + \frac{1}{2}\sum_{k\ge 1}\frac{kq^{2k}}{(1-q^k)^2} + \frac{1}{2}\sum_{k\ge 1}\frac{k(k+1)q^k}{1-q^k}.$$

Finally plugging in [2, equations (5.4) and (6.5)]

$$\sum_{k,\ell\ge 1} \frac{\min(k,\ell)q^{k+\ell}}{(1-q^{k})(1-q^{\ell})} = \sum_{n\ge 1}\frac{n(n-1)q^n}{1-q^n} - \sum_{n\ge 1}\frac{nq^{2n}}{(1-q^n)^2},$$

gives the claim. □

Remark. Recall Bell's identity for the sum of squares of divisors [2, equation (2.3)]

$$\sum_{n\ge 1}\frac{q^n}{(1-q^n)^2}\left(\frac{1}{1-q}+\frac{1}{1-q^2}+\cdots+\frac{1}{1-q^n}\right) = \sum_{n\ge 1}\frac{n^2q^n}{1-q^n}.$$

Curiously, in Lemma 4.3 we prove a slightly different identity

$$\sum_{n\ge 1}\frac{q^n}{1-q^n}\left(\frac{1}{1-q}+\frac{2}{1-q^2}+\cdots+\frac{n}{1-q^n}\right) = \sum_{n\ge 1}\frac{n^2q^n}{1-q^n}.$$

Now we are ready to prove (1.4).

Proof of (1.4). We enumerate the vertices of Γ_8 as on Fig. 1. We use (2.8) for n_1 to obtain

$$\sum_{\boldsymbol{n}\in\mathbb{N}_0^8} \frac{q^{n_1n_2+n_1n_5+n_1n_6+n_2n_3+n_2n_7+n_3n_4+n_3n_6+n_4n_5+n_6n_8+n_7n_8+n_1+n_2+n_3+n_4+n_5+n_6+n_7+n_8}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}(q)_{n_6}(q)_{n_7}(q)_{n_8}}$$

$$= \sum_{n_2,n_3,n_4,n_5,n_6,n_7,n_8\ge 0} \frac{q^{n_2n_3+n_2n_7+n_3n_4+n_3n_6+n_4n_5+n_6n_8+n_7n_8+n_2+n_3+n_4+n_5+n_6+n_7+n_8}}{(q^{n_2+n_5+n_6+1})_\infty(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}(q)_{n_6}(q)_{n_7}(q)_{n_8}}$$

$$= \frac{1}{(q)_\infty}\sum_{n_2,n_5,n_6,n_7,n_8\ge 0} \frac{(q)_{n_2+n_5+n_6}q^{n_2n_7+n_6n_8+n_7n_8+n_2+n_5+n_6+n_7+n_8}}{(q)_{n_2}(q)_{n_5}(q)_{n_6}(q)_{n_7}(q)_{n_8}}$$

$$\times \sum_{n_3,n_4\geq 0} \frac{q^{n_3n_4+(n_2+n_6+1)n_3+(n_5+1)n_4}}{(q)_{n_3}(q)_{n_4}}$$

$$= \frac{1}{(q)_\infty} \sum_{n_2,n_5,n_6,n_7,n_8\geq 0} \frac{(q)_{n_2+n_5+n_6} q^{n_2n_7+n_6n_8+n_7n_8+n_2+n_5+n_6+n_7+n_8}}{(q^{n_5+1})_\infty (q^{n_2+n_6+1})_{n_5+1}(q)_{n_2}(q)_{n_5}(q)_{n_6}(q)_{n_7}(q)_{n_8}}$$

$$= \frac{1}{(q)_\infty^2} \sum_{n_2,n_5,n_6,n_7,n_8\geq 0} \frac{(q)_{n_2+n_5+n_6} q^{n_2n_7+n_6n_8+n_7n_8+n_2+n_5+n_6+n_7+n_8}}{(q^{n_2+n_6+1})_{n_5+1}(q)_{n_2}(q)_{n_6}(q)_{n_7}(q)_{n_8}}$$

using Lemma 4.1 in the penultimate step. Using (2.8), we then rewrite this as

$$\frac{1}{(q)_\infty^2} \sum_{n_2,n_5,n_6\geq 0} \frac{(q)_{n_2+n_5+n_6} q^{n_2+n_5+n_6}}{(q)_{n_2}(q)_{n_6}(q^{n_2+n_6+1})_{n_5+1}(q^{n_2+1})_\infty (q^{n_6+1})_{n_2+1}}$$

$$= \frac{1}{(q)_\infty^3} \sum_{n_2,n_5,n_6\geq 0} \frac{(q)_{n_2+n_5+n_6} q^{n_2+n_5+n_6}}{(q)_{n_6}(q^{n_2+n_6+1})_{n_5+1}(q^{n_6+1})_{n_2+1}}$$

$$= \frac{q^{-1}}{(q)_\infty^3} \sum_{n_2,n_5,n_6\geq 0} \frac{q^{n_2+n_5+n_6+1}}{(1-q^{n_2+n_6+1})(1-q^{n_2+n_5+n_6+1})} = \frac{q^{-1}}{(q)_\infty^3} \sum_{n\geq 1} \frac{n^2q^n}{1-q^n},$$

where the last equality is due to Lemma 4.3. □

Remark. Lemma 2.6 implies that $q(q)_\infty^3 H_{\Gamma_8}(q)$ is a weight 3 holomorphic quantum modular form.

5. D_4 graph series and the proof of (1.5)

In this part we investigate graph series associated to the graph of type D_4:

$$H_{D_4}(q) = \sum_{\boldsymbol{n}\in\mathbb{N}_0^4} \frac{q^{n_1n_2+n_1n_3+n_1n_4+n_1+n_2+n_3+n_4}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}}.$$

We first obtain a representation for $H_{D_4}(q)$ using Appell–Lerch sums. Using this result we then rewrite it as an indefinite theta function of signature $(1,1)$. We also discuss mock modular properties and the asymptotic behavior as $q \to 1^-$. To view modularity properties of H_{D_4}, we use

$$I_1(q) := \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^{n+1}q^{\frac{n(3n+1)}{2}}}{(1-q^n)^2}, \qquad I_2(q) := \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^{n+1}nq^{\frac{n(n+1)}{2}}}{1-q^n},$$

which are mock modular. Moreover, we require the generating function for so-called ranks of strongly unimodal sequences, explicitly given by $U(-\zeta;q)$, where

$$U(\zeta;q) := \sum_{n\geq 0} q^{n+1}(\zeta q)_n \left(\zeta^{-1}q\right)_n.$$

Proposition 5.1. *We have*

$$H_{D_4}(q) = \frac{q^{-1}}{(q)_\infty^4}\left(I_1(q) + \frac{1}{24}(1-E_2(\tau)) + I_2(q)\right) = \frac{q^{-1}}{(q)_\infty^3}U(1;q).$$

Proof. We use Euler's identity (2.8) three times (for n_2, n_3, and n_4) to write

$$H_{D_4}(q) = \sum_{n\geq 0}\frac{q^n}{(q)_n(q^{n+1})_\infty^3} = \frac{1}{(q)_\infty^3}\sum_{n\geq 0} q^n (q)_n^2 = \frac{q^{-1}}{(q)_\infty^3}U(1;q).$$

To see the first identity, we use an identity of Andrews [3, equation (1.2)], giving

$$U(1;q) = \frac{1}{(q)_\infty}\left(\sum_{n\geq 1}\frac{(-1)^{n+1}(1+q^n)q^{\frac{n(3n+1)}{2}}}{(1-q^n)^2} - \sum_{n\geq 1}\frac{q^n}{(1-q^n)^2} + 2\sum_{n\geq 1}\frac{(-1)^{n+1}nq^{\frac{n(n+1)}{2}}}{1-q^n}\right).$$

Rewriting the first and the third sum and using that $\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2} = \frac{1-E_2(\tau)}{24}$ yields the claim. □

We next prove the indefinite theta function representation of H_{D_4} as stated in (1.5). We note that similar formulas already exist in the literature; see for instance [13, Theorem 1.5].

Proof of (1.5). We have

$$(q)_\infty U(1;q) = \sum_{n\geq 1}\frac{(-1)^{n+1}(1+q^n)q^{\frac{n(3n+1)}{2}}}{(1-q^n)^2} - \sum_{n\geq 1}\frac{q^n}{(1-q^n)^2} + 2\sum_{n\geq 1}\frac{(-1)^{n+1}nq^{\frac{n(n+1)}{2}}}{1-q^n}.$$

We first apply Lemma 2.3 to combine the first and second sum in $F(q)$ to obtain

$$(q)_\infty U(1;q) = -\sum_{n\geq 1}\frac{(-1)^{n+1}\left(1+q^n\right)q^{\frac{n(n+1)}{2}}\left(1-q^{n^2}\right)}{(1-q^n)^2} + 2\sum_{n\geq 1}\frac{(-1)^{n+1}nq^{\frac{n(n+1)}{2}}}{1-q^n}.$$

Next we use the identity:

$$-\frac{(1+q^n)\left(1-q^{n^2}\right)}{(1-q^n)^2} = -\sum_{m=0}^{n-1}(2m+1)q^{nm} - 2n\sum_{m\geq n} q^{nm},$$

which follows from expanding the left-hand side as a geometric series. Plugging this in, we have

$$
\begin{aligned}
(q)_\infty U(1;q) &= \sum_{n\geq 1}(-1)^n q^{\frac{n(n+1)}{2}}\left(\sum_{m=0}^{n-1}(2m+1)q^{nm}+2n\sum_{m\geq n}q^{nm}\right)\\
&\quad+\sum_{n\geq 1}\sum_{m\geq 0}(-1)^{n+1}2nq^{\frac{n(n+1)}{2}+nm}\\
&=\sum_{n\geq 1}(-1)^n q^{\frac{n(n+1)}{2}}\sum_{m=0}^{n-1}(2m+1-2n)q^{nm}.
\end{aligned}
$$

The claim now follows by changing $n\mapsto n+m+1$ and using Proposition 5.1. □

Remark. Quantum modular and mock modular properties of $U(1;q)$ are well-understood and therefore follow for $H_{D_4}(q)$. In particular, this gives (as $t\to 0^+$)

$$
\left(e^{-t}\right)_\infty^3 e^{-\frac{23t}{24}} H_{D_4}\left(e^{-t}\right)=\sum_{n\geq 0}\frac{T_n}{n!}\left(\frac{-t}{24}\right)^n,
$$

where T_n are Glaisher's numbers [13, Theorem 1]. Moreover, by [41], $(q)_\infty^3 q^{\frac{1}{24}} H_{D_4}(q)$ is a mixed mock modular form.

Remark *(ℓ-star graphs).* It is worth noting that for every ℓ-star graph X_ℓ, $\ell\geq 3$, we can write

$$
H_{X_\ell}(q)=\frac{1}{(q)_\infty^\ell}\sum_{n\geq 0}q^n(q)_n^{\ell-1}. \tag{5.1}
$$

For $\ell=2$ we obtain the A_3-graph function discussed in Section 3, via (2.11). However for $\ell>3$, we are not aware of any Appell–Lerch type series representation for the sum in (5.1). It would be interesting to determine their quantum modular properties.

6. D_5 graph series and the proof of (1.6)

In this section we consider the graph series of type D_5:

$$
H_{D_5}(q)=\sum_{\boldsymbol{n}\in\mathbb{N}_0^5}\frac{q^{n_1n_2+n_1n_3+n_1n_4+n_4n_5+n_1+n_2+n_3+n_4+n_5}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}}. \tag{6.1}
$$

Our first result is the following Lerch-type sum representation.

Proposition 6.1. *We have*

$$
H_{D_5}(q)=\frac{q^{-1}}{(q)_\infty^3}\sum_{n\geq 1}\frac{(-1)^{n+1}\left(1+q^n\right)q^{\frac{n(n+1)}{2}}\left(1-q^{n^2}\right)}{\left(1-q^n\right)^2}.
$$

Proof. To begin we use (6.1), and apply (2.8) three times to write

$$H_{D_5}(q) = \frac{1}{(q)_\infty^3} \sum_{n_1,n_4\geq 0} q^{n_1n_4+n_1+n_4}(q)_{n_1} = \frac{1}{(q)_\infty^3} \sum_{n\geq 0} \frac{q^n(q)_n}{1-q^{n+1}}. \tag{6.2}$$

Using [36, Theorem 8], with $a = q^2$, $d = b^2 \to 0$, and $c = q$ and a direct calculation yields that the following are a Bailey pair relative to (q^2, q),

$$\alpha_n := \frac{(-1)^n q^{\frac{n(n+1)}{2}} \left(1+q^{n+1}\right) \left(1-q^{n^2+2n+1}\right)}{1-q^2}, \qquad \beta_n := \frac{1}{(q^2)_n}.$$

Inserting this Bailey pair into Lemma 2.7, with $\varrho_1 = \varrho_2 = q$, yields

$$\sum_{n\geq 0} \frac{q^n(q)_n}{1-q^{n+1}} = \frac{1}{1-q} \sum_{n\geq 0} q^n (q)_n^2 \beta_n = \frac{\left(q^2\right)_\infty^2}{(1-q)\left(q^3,q\right)_\infty} \sum_{n\geq 0} \frac{q^n(q)_n^2 \alpha_n}{(q^2)_n^2}$$
$$= \sum_{n\geq 0} \frac{(-1)^n \left(1+q^{n+1}\right) q^{\frac{n^2+3n}{2}} \left(1-q^{n^2+2n+1}\right)}{\left(1-q^{n+1}\right)^2}$$
$$= -q^{-1} \sum_{n\geq 1} \frac{(-1)^n \left(1+q^n\right) q^{\frac{n(n+1)}{2}} \left(1-q^{n^2}\right)}{\left(1-q^n\right)^2}.$$

Plugging into (6.2) gives the claim. □

We next give an indefinite theta function representation for the series of interest, proving (1.6).

Theorem 6.2. *We have*

$$\sum_{n\geq 1} \frac{(-1)^n \left(1+q^n\right) q^{\frac{n(n+1)}{2}} \left(1-q^{n^2}\right)}{\left(1-q^n\right)^2}$$
$$= -\frac{q}{(q)_\infty^2} \left(\sum_{n,m\geq 0} - \sum_{n,m<0}\right) (-1)^n (n+1)^2 q^{\frac{n^2+3n}{2}+3nm+3m^2+4m}.$$

Proof. Let us first give a sketch of proof. The idea is to view this as an identity between modular forms in trivial spaces. For this, denote the left-hand side by $\mathcal{L}(\tau)$ and the right-hand side by $\mathcal{R}(\tau)$. Then define the completion $\widehat{\mathcal{L}}(\tau)$ and $\widehat{\mathcal{R}}(\tau)$ (see (6.3) and (6.4)). We then show that $\widehat{\mathcal{L}}$ and $\mathcal{L}$ as well as $\widehat{\mathcal{R}}$ and $\mathcal{R}$ differ by the same function (see (6.5)). We finish the proof by showing modulation of $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{R}}$ (see (6.9) and (6.13)).

To find $\widehat{\mathcal{L}}$ we write, using Lemma 2.3,

$$\sum_{n\geq 1} \frac{(-1)^n (1+q^n) q^{\frac{n(n+1)}{2}}}{(1-q^n)^2} = \frac{E_2(\tau)-1}{24},$$
$$-\sum_{n\geq 1} \frac{(-1)^n (1+q^n) q^{\frac{n(3n+1)}{2}}}{(1-q^n)^2} = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^{n+1} q^{\frac{n(3n+1)}{2}}}{(1-q^n)^2} =: \mathcal{F}(\tau),$$

changing $n \mapsto -n$ for the contribution from the term $+q^n$. Thus

$$\mathcal{L}(\tau) = \mathcal{F}(\tau) + \frac{E_2(\tau)-1}{24} = \left(\mathcal{F}(\tau) - \frac{1}{24} + \frac{E_2(\tau)}{8}\right) - \frac{E_2(\tau)}{12}.$$

Define

$$\widehat{\mathcal{L}}(\tau) := \widehat{\mathcal{F}}(\tau) - \frac{E_2(\tau)}{12}. \tag{6.3}$$

To find $\widehat{\mathcal{R}}$, we write

$$\frac{1}{2}\sum_{n,m\in\mathbb{Z}} \left(\operatorname{sgn}\left(n+\frac{1}{2}\right) + \operatorname{sgn}\left(m+\frac{1}{2}\right)\right) (-1)^n (n+1)^2 q^{\frac{n^2+3n}{2}+3nm+3m^2+4m}$$
$$= -\frac{q^{-1}}{2}\sum_{\boldsymbol{n}\in\mathbb{Z}^2} \left(\operatorname{sgn}\left(n_1-\frac{1}{2}\right) + \operatorname{sgn}\left(n_2+\frac{1}{2}\right)\right) (-1)^{n_1} n_1^2 q^{\frac{n_1^2}{2}+\frac{n_1}{2}+3n_1n_2+3n_2^2+n_2}$$
$$= \frac{q^{-1}}{8\pi^2}\left[\frac{\partial^2}{\partial z^2}\sum_{\boldsymbol{n}\in\mathbb{Z}^2} \left(\operatorname{sgn}\left(n_1-\frac{1}{2}\right) + \operatorname{sgn}\left(n_2+\frac{1}{2}\right)\right) e^{2\pi i\left(n_1\left(\frac{1}{2}+z+\frac{\tau}{2}\right)+n_2\tau\right)} q^{Q(\boldsymbol{n})}\right]_{z=0},$$

where $Q(\boldsymbol{n}) := \frac{n_1^2}{2} + 3n_1n_2 + 3n_2^2$. Setting $\boldsymbol{z} := (-2z, \frac{\tau}{6} + \frac{1}{6} + z)$, $\boldsymbol{c_1} := (-2,1)$, $\boldsymbol{c_2} := (-1, \frac{1}{3})$, and choosing $y > 0$ sufficiently small, we may write, using the notation from Section 2,

$$\sum_{\boldsymbol{n}\in\mathbb{Z}^2} \left(\operatorname{sgn}\left(n_1-\frac{1}{2}\right) + \operatorname{sgn}\left(n_2+\frac{1}{2}\right)\right) e^{2\pi i\left(n_1\left(\frac{1}{2}+z+\frac{\tau}{2}\right)+n_2\tau\right)} q^{Q(\boldsymbol{n})}$$
$$= \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \left(\operatorname{sgn}\left(B\left(\boldsymbol{n}+\frac{\operatorname{Im}(\boldsymbol{z})}{v}, \boldsymbol{c_1}\right)\right) - \operatorname{sgn}\left(B\left(\boldsymbol{n}+\frac{\operatorname{Im}(\boldsymbol{z})}{v}, \boldsymbol{c_2}\right)\right)\right) e^{2\pi i B(\boldsymbol{n},\boldsymbol{z})} q^{Q(\boldsymbol{n})}.$$

Define

$$\Phi(z;\tau) := \Theta^{\boldsymbol{c_1},\boldsymbol{c_2}}_{\left(\begin{smallmatrix}1&3\\3&6\end{smallmatrix}\right)}(\boldsymbol{z};\tau), \quad \mathcal{G}(\tau) := \frac{q^{\frac{1}{12}}}{8\pi^2}\left[\frac{\partial^2}{\partial z^2}\Phi(z;\tau)\right]_{z=0}, \quad \widehat{\mathcal{R}}(\tau) := -\frac{\mathcal{G}(\tau)}{\eta(\tau)^2}. \tag{6.4}$$

We next show that

$$\mathcal{L}^-(\tau) := \widehat{\mathcal{L}}(\tau) - \mathcal{L}(\tau) = \widehat{\mathcal{R}}(\tau) - \mathcal{R}(\tau) =: \mathcal{R}^-(\tau). \tag{6.5}$$

For this, we rewrite the right-hand side of this identity. We start by computing

$$\left[\frac{\partial^2}{\partial z^2}\Phi(z;\tau)\right]_{z=0} = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} (-1)^{n_1} q^{\frac{n_1^2}{2}+\frac{n_1}{2}+3n_1n_2+3n_2^2+n_2}$$
$$\times \left[\frac{\partial^2}{\partial z^2}\left(\left(E\left(\left(n_1 - \frac{2y}{v}\right)\sqrt{v}\right) + E\left(\left(n_2 + \frac{1}{6} + \frac{y}{v}\right)\sqrt{6v}\right)\right) e^{2\pi i n_1 z}\right)\right]_{z=0}. \tag{6.6}$$

For the first summand, write

$$\frac{n_1^2}{2} + \frac{n_1}{2} + 3n_1n_2 + 3n_2^2 + n_2 = 3\left(n_2 + \frac{n_1}{2}\right)^2 + \left(n_2 + \frac{n_1}{2}\right) - \frac{n_1^2}{4},$$

make the change of variables $n_1 \mapsto 2n_1 + \delta$, $\delta \in \{0,1\}$, $n_1 \in \mathbb{Z}$, and then let $n_2 \mapsto n_2 - n_1$. Using the second identity in (2.5), the contribution to $\mathcal{R}^-$ is

$$-\frac{1}{\sqrt{\pi}} \sum_{\boldsymbol{n}\in\mathbb{Z}^2} (-1)^{n_1} f(n_1) q^{3\left(n_2+\frac{n_1}{2}\right)^2+\left(n_2+\frac{n_1}{2}\right)-\frac{n_1^2}{4}}$$
$$= -\frac{1}{\sqrt{\pi}} \sum_{\delta\in\{0,1\}} (-1)^\delta \sum_{n_1\in\mathbb{Z}} f(2n_1+\delta) q^{-\frac{1}{4}(2n_1+\delta)^2} \sum_{n_2\in\mathbb{Z}} q^{3\left(n_2+\frac{\delta}{2}\right)^2+n_2+\frac{\delta}{2}},$$

where

$$f(n) := \left[\frac{\partial^2}{\partial z^2}\left(\operatorname{sgn}\left(n - \frac{2y}{v}\right)\Gamma\left(\frac{1}{2}, \pi\left(n - \frac{2y}{v}\right)^2\right) e^{2\pi i n z}\right)\right]_{z=0}.$$

By changing $n_1 \mapsto -n_1 - \delta$ one sees that the sum on n_1 vanishes (since f is an odd function).

For the second term in (6.6), write

$$\frac{n_1^2}{2} + \frac{n_1}{2} + 3n_1n_2 + 3n_2^2 + n_2 = \frac{1}{2}(n_1+3n_2)^2 + \frac{1}{2}(n_1+3n_2) - \frac{3n_2^2}{2} - \frac{n_2}{2}.$$

Then we shift $n_1 \mapsto n_1 - 3n_2$ to obtain that the contribution of the second term to $\mathcal{R}^-$ is

$$-\frac{1}{\sqrt{\pi}} \sum_{\boldsymbol{n}\in\mathbb{Z}^2} (-1)^{n_1} g(n_1,n_2) q^{\frac{1}{2}(n_1+3n_2)^2+\frac{1}{2}(n_1+3n_2)-\frac{3n_2^2}{2}-\frac{n_2}{2}}$$
$$= -\frac{1}{\sqrt{\pi}} \sum_{\boldsymbol{n}\in\mathbb{Z}^2} (-1)^{n_1+n_2} g(n_1-3n_2,n_2) q^{\frac{n_1^2}{2}+\frac{n_1}{2}-\frac{3n_2^2}{2}-\frac{n_2}{2}}, \tag{6.7}$$

where

$$g(\boldsymbol{n}) := \left[\frac{\partial^2}{\partial z^2}\left(\operatorname{sgn}\left(n_2+\frac{1}{6}+\frac{y}{v}\right)\Gamma\left(\frac{1}{2}, 6\pi\left(n_2+\frac{1}{6}+\frac{y}{v}\right)^2 v\right)e^{2\pi i n_1 z}\right)\right]_{z=0}.$$

Making the change of variables $n_1 \mapsto -n_1-1$, we see that

$$\sum_{n_1\in\mathbb{Z}}(-1)^{n_1} h\left(n_1+\frac{1}{2}\right)q^{\frac{n_1(n_1+1)}{2}}=0$$

for any even function h. Thus we obtain that (6.7) equals

$$\begin{aligned}-4\sqrt{\pi}i\sum_{n_1\in\mathbb{Z}}(-1)^{n_1}\left(n_1+\frac{1}{2}\right)q^{\frac{n_1(n_1+1)}{2}}\\ \times\sum_{n_2\in\mathbb{Z}}(-1)^{n_2}\operatorname{sgn}\left(n_2+\frac{1}{6}\right)q^{-\frac{3n_2^2}{2}-\frac{n_2}{2}}\\ \times\left[\frac{\partial}{\partial z}\left(\Gamma\left(\frac{1}{2}, 6\pi\left(n_2+\frac{1}{6}+\frac{y}{v}\right)^2 v\right)e^{-6\pi i\left(n_2+\frac{1}{6}\right)z}\right)\right]_{z=0}.\end{aligned}$$

The sum on n_1 is

$$q^{-\frac{1}{8}}\sum_{n_1\geq 0}(-1)^{n_1}(2n_1+1)q^{\frac{1}{8}(2n_1+1)^2}=q^{-\frac{1}{8}}\eta(\tau)^3.$$

The sum on n_2 is

$$q^{\frac{1}{24}}\sum_{n_2\in\mathbb{Z}+\frac{1}{6}}(-1)^{n_2-\frac{1}{6}}\operatorname{sgn}(n_2)q^{-\frac{3n_2^2}{2}}\left[\frac{\partial}{\partial z}\left(\Gamma\left(\frac{1}{2}, 6\pi\left(n_2+\frac{y}{v}\right)^2 v\right)e^{-6\pi i n_2 z}\right)\right]_{z=0}. \quad (6.8)$$

We then compute, using (2.6),

$$\left[\frac{\partial}{\partial z}\left(\Gamma\left(\frac{1}{2}, 6\pi\left(n_2+\frac{y}{v}\right)^2 v\right)e^{-6\pi i n_2 z}\right)\right]_{z=0}=3\pi i n_2\Gamma\left(-\frac{1}{2}, 6\pi n_2^2 v\right).$$

Thus (6.8) equals (upon changing $n_2 \mapsto -n_2$)

$$\begin{aligned}3\pi i q^{\frac{1}{24}}\sum_{n_2\in\mathbb{Z}+\frac{1}{6}}(-1)^{n_2-\frac{1}{6}}|n_2|\Gamma\left(-\frac{1}{2}, 6\pi n_2^2 v\right)q^{-\frac{3n_2^2}{2}}\\ =3\pi i q^{\frac{1}{24}}\sum_{n_2\in\mathbb{Z}-\frac{1}{6}}(-1)^{n_2+\frac{1}{6}}|n_2|\Gamma\left(-\frac{1}{2}, 6\pi n_2^2 v\right)q^{-\frac{3n_2^2}{2}}.\end{aligned}$$

From this we obtain (6.5).

We next determine the transformation laws of $\widehat{\mathcal{L}}$ and of $\widehat{\mathcal{R}}$. By Proposition 2.4 and (2.2),

$$\widehat{\mathcal{L}}(\tau+1)=\widehat{\mathcal{L}}(\tau), \qquad \widehat{\mathcal{L}}\left(-\frac{1}{\tau}\right)=\tau^2\widehat{\mathcal{L}}(\tau)+\frac{i\tau}{2\pi}. \tag{6.9}$$

We show that $\widehat{\mathcal{R}}$ satisfies the same transformation laws as $\widehat{\mathcal{L}}$. By Proposition 2.5 we obtain

$$\Phi\left(\frac{z}{\tau};-\frac{1}{\tau}\right) = \frac{\tau}{\sqrt{3}}\sum_{\ell \pmod 3} e^{\frac{2\pi i}{\tau}Q\left(-2z,-\frac16+\frac\tau6+z+\frac{\ell\tau}{3}\right)}\Theta_{\left(\begin{smallmatrix}1&3\\3&6\end{smallmatrix}\right)}\left(\left(-2z,-\frac16+\frac\tau6+z+\frac{\ell\tau}{3}\right);\tau\right).$$

Changing $z\mapsto -z$ and using Proposition 2.5 (2), we compute

$$\left[\frac{\partial^2}{\partial z^2}\left(e^{\frac{2\pi i}{\tau}Q\left(-2z,-\frac16+\frac\tau6+z+\frac{\ell\tau}{3}\right)}\Theta_{\left(\begin{smallmatrix}1&3\\3&6\end{smallmatrix}\right)}\left(\left(-2z,-\frac16+\frac\tau6+z+\frac{\ell\tau}{3}\right);\tau\right)\right)\right]_{z=0}$$
$$= -\left[\frac{\partial^2}{\partial z^2}\left(e^{\frac{2\pi i}{\tau}Q\left(-2z,\frac16-\frac\tau6+z-\frac{\ell\tau}{3}\right)}\Theta_{\left(\begin{smallmatrix}1&3\\3&6\end{smallmatrix}\right)}\left(\left(-2z,\frac16-\frac\tau6+z-\frac{\ell\tau}{3}\right);\tau\right)\right)\right]_{z=0}.$$

Thus

$$\mathcal{G}\left(-\frac{1}{\tau}\right) = -\tau^3\frac{e^{-\frac{\pi i}{6\tau}}}{8\sqrt{3}\pi^2}$$
$$\times\sum_{\ell \pmod 3}\left[\frac{\partial^2}{\partial z^2}\left(e^{\frac{2\pi i}{\tau}Q\left(-2z,\frac16-\frac\tau6+z-\frac{\ell\tau}{3}\right)}\Theta_{\left(\begin{smallmatrix}1&3\\3&6\end{smallmatrix}\right)}\left(\left(-2z,\frac16-\frac\tau6+z-\frac{\ell\tau}{3}\right);\tau\right)\right)\right]_{z=0}.$$

Now we choose $\ell\in\{0,-1,-2\}$. Using Proposition 2.5 (2) and (3), we see that the contribution for $\ell=-2$ is an odd function evaluated at zero, and as such vanishes.

Using Proposition 2.5 again to relate the remaining theta functions we obtain

$$\mathcal{G}\left(-\frac{1}{\tau}\right) = -\frac{i\tau^3}{8\pi^2}q^{\frac{1}{12}}\left[\frac{\partial^2}{\partial z^2}\left(e^{-\frac{2\pi i z^2}{\tau}}\Theta_{\left(\begin{smallmatrix}1&3\\3&6\end{smallmatrix}\right)}\left(\left(2z,\frac16+\frac\tau6+z\right);\tau\right)\right)\right]_{z=0}. \tag{6.10}$$

Now

$$\left[\frac{\partial^2}{\partial z^2}\left(e^{-\frac{2\pi i z^2}{\tau}}\Theta_{\left(\begin{smallmatrix}1&3\\3&6\end{smallmatrix}\right)}\left(\left(2z,\frac16+\frac\tau6+z\right);\tau\right)\right)\right]_{z=0}$$
$$=\left[\frac{\partial^2}{\partial z^2}\Theta_{\left(\begin{smallmatrix}1&3\\3&6\end{smallmatrix}\right)}\left(\left(2z,\frac16+\frac\tau6+z\right);\tau\right)\right]_{z=0}-\frac{4\pi i}{\tau}\Theta_{\left(\begin{smallmatrix}1&3\\3&6\end{smallmatrix}\right)}\left(\left(0,\frac16+\frac\tau6\right);\tau\right).$$

The first summand contributes

$$-i\tau^3\mathcal{G}(\tau). \tag{6.11}$$

We next claim that

$$\Theta_{\left(\begin{smallmatrix}1&3\\3&6\end{smallmatrix}\right)}\left(\left(0,\frac{1}{6}+\frac{\tau}{6}\right);\tau\right)=(q)_\infty^2. \tag{6.12}$$

For this, we make the same changes of variables as in the proof of (6.5) to obtain that

$$\begin{aligned}&\Theta_{\left(\begin{smallmatrix}1&3\\3&6\end{smallmatrix}\right)}\left(\left(0,\frac{1}{6}+\frac{\tau}{6}\right);\tau\right)\\&=\sum_{\boldsymbol{n}\in\mathbb{Z}^2}(-1)^{n_1}\left(\operatorname{sgn}(n_1)+\operatorname{sgn}\left(n_2+\frac{1}{2}\right)\right)q^{\frac{n_1^2}{2}+\frac{n_1}{2}+3n_1n_2+3n_2^2+n_2}.\end{aligned}$$

One can now show (as above) that both sides of (6.12) satisfy the same transformation law and lie in a one-dimensional space. Computing one coefficient then gives that they are equal.

Using (6.9), (6.10), and (6.11), we obtain that

$$\widehat{\mathcal{R}}(\tau+1)=\widehat{\mathcal{R}}(\tau),\qquad \widehat{\mathcal{R}}\left(-\frac{1}{\tau}\right)=\tau^2\widehat{\mathcal{R}}(\tau)+\frac{i\tau}{2\pi}. \tag{6.13}$$

This shows that $\mathcal{R}(\tau)-\mathcal{L}(\tau)$ is a weakly holomorphic modular form of weight two for $\mathrm{SL}_2(\mathbb{Z})$. Since one can prove that it does not grow it has to be zero. □

Remark. Propositions 6.1 and 6.2 give that H_{D_5} is a mixed mock modular form, as claimed in Theorem 1.4.

7. Graphs with multiple edges: Kontsevich–Zagier type series

In this part we contemplate graph series with multiple edges. Series of this type do not connect directly with the geometry of jet schemes, but they do naturally appear in vertex algebras (see Theorem 2.2). Here we focus on the two simplest examples coming from the graph

$$\bullet = \bullet$$

of type B_2, and from the graph

$$\bullet - \bullet = \bullet$$

of type B_3. In the setup of principal subspaces, we consider additional q-series arising from cosets in the dual lattices (these compute characters of modules). Thus for B_2 we obtain three q-series

$$F_1(q) := \sum_{n_1,n_2\geq 0} \frac{q^{2n_1n_2+n_1+n_2}}{(q)_{n_1}(q)_{n_2}}, \quad F_2(q) := \sum_{n_1,n_2\geq 0} \frac{q^{2n_1n_2+n_1+2n_2}}{(q)_{n_1}(q)_{n_2}},$$
$$F_3(q) := \sum_{n_1,n_2\geq 0} \frac{q^{2n_1n_2+2n_1+2n_2}}{(q)_{n_1}(q)_{n_2}}.$$

Let us now recall two remarkable false theta functions used to compute unified WRT invariants of the Poincaré 3-sphere [30]. For this we let

$$\chi_+(n) := \begin{cases} (-1)^{\lfloor \frac{n}{30} \rfloor} & \text{if } n^2 \equiv 1 \pmod{120}, \\ 0 & \text{otherwise,} \end{cases}$$
$$\chi_-(n) := \begin{cases} (-1)^{\lfloor \frac{n}{30} \rfloor} & \text{if } n^2 \equiv 49 \pmod{120}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the two q-series

$$\widetilde{\Theta}_+(q) := \sum_{n\geq 1} \chi_+(n) q^{\frac{n^2-1}{120}}, \qquad \widetilde{\Theta}_-(q) := \sum_{n\geq 1} \chi_-(n) q^{\frac{n^2-49}{120}},$$

combine into a vector-valued quantum modular form of weight $\frac{1}{2}$ [30, Section 4]. The following proposition relates the functions F_1, F_2, and F_3 to these false theta functions.

Proposition 7.1. *We have*

$$F_1(q) = \frac{\widetilde{\Theta}_-(q)}{(q)_\infty}, \quad F_2(q) = \frac{q^{-1}\left(\widetilde{\Theta}_+(q) - 1\right)}{(q)_\infty}, \quad F_3(q) = \frac{q^{-2}\left(\widetilde{\Theta}_-(q) - \widetilde{\Theta}_+(q)\right)}{(q)_\infty}.$$

Proof. Using (2.8), we obtain

$$F_2(q) = \sum_{n_1,n_2\geq 0} \frac{q^{n_1+(2n_1+2)n_2}}{(q)_{n_1}(q)_{n_2}} = \sum_{n_1\geq 0} \frac{q^{n_1}}{(q)_{n_1}\left(q^{2n_1+2}\right)_\infty} = \frac{1}{(q)_\infty} \sum_{n\geq 0} q^n \left(q^{n+1}\right)_{n+1}.$$

The sum is known to be $q^{-1}(\widetilde{\Theta}_+(q) - 1)$ by (3.14) of [24]. For the first identity, we write

$$F_1(q) = \sum_{n_1,n_2\geq 0} \frac{q^{n_1+(2n_1+1)n_2}}{(q)_{n_1}(q)_{n_2}} = \sum_{n_1\geq 0} \frac{q^{n_1}}{(q)_{n_1}\left(q^{2n_1+1}\right)_\infty} = \frac{1}{(q)_\infty} \sum_{n\geq 0} q^n \left(q^{n+1}\right)_n,$$

and use (3.13) of [24]. Easy manipulations then yield

$$F_3(q) = q^{-2}F_1(q) - q^{-1}F_2(q) - \frac{q^{-2}}{(q)_\infty},$$

which implies the formula for $F_3(q)$. □

For a graph Γ of type B_3 we record similar identities. We first consider a slightly shifted version of $H_\Gamma(q)$ given by

$$H_1(q) := \sum_{\boldsymbol{n}\in\mathbb{N}_0^3} \frac{q^{2n_1n_2+n_2n_3+n_1+2n_2+n_3}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}}.$$

As before, using Euler's identity (2.8) gives

$$H_1(q) = \frac{q^{-1}}{(q)_\infty}\sum_{n_1\geq 0}\frac{q^{n_1+1}}{\left(1-q^{2n_1+2}\right)(q)_{n_1}} = \frac{q^{-1}}{(q)_\infty}\sum_{n\geq 0}\frac{q^{n+1}}{\left(1+q^{n+1}\right)(q)_{n+1}}$$
$$= \frac{q^{-1}}{(q)_\infty}\sum_{n\geq 1}\frac{q^{n}}{\left(1+q^{n}\right)(q)_{n}}.$$

We can rewrite the right-hand side using a sum of tails identity [5, Theorem 4.1] as

$$H_1(q) = \frac{q^{-1}}{(q)_\infty^2}\sum_{n\geq 1}(-1)^n\left((q)_n-(q)_\infty\right).$$

By (2.10) with $\zeta=-1$, we have

$$\sum_{n\geq 0}(-1)^n\left((q)_n-(q)_\infty\right) = \sum_{n\geq 1} q^{2n-1}(q)_{2n-2}.$$

This can be further expressed, using (2.11), as

$$\frac{1}{2}\left(-(q)_\infty+1+\sum_{n\geq 0}(-1)^n q^{n+1}(q)_n\right).$$

The last sum, $\sigma(q) = 1+\sum_{n\geq 0}(-1)^n q^{n+1}(q)_n$, is a quantum modular of weight zero (see the examples in [45]).

For the graph series

$$H_2(q) := \sum_{\boldsymbol{n}\in\mathbb{N}_0^3} \frac{q^{2n_1n_2+n_2n_3+n_1+n_2+n_3}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}},$$

we first deduce that

$$H_2(q) = \frac{1}{(q)_\infty}\sum_{n_1\geq 0}\frac{q^{n_1}}{(1-q^{2n_1+1})(q)_{n_1}} = \frac{1}{(1-q)(q)_\infty} + \frac{1}{(q)_\infty^2}\sum_{n\geq 0} q^n((q)_{2n}-(q)_\infty),$$

by [5, Theorem 4.1] and [43, Theorem 2]. Recall formula (2.10):

$$\sum_{n\geq 0}\zeta^n\left((q)_n-(q)_\infty\right)=\sum_{n\geq 1}q^n\left(1+\zeta+\cdots+\zeta^{n-1}\right)(q)_{n-1}.$$

We extract all even powers of ζ on both sides of this equation:

$$\sum_{n\geq 0}\zeta^{2n}\left((q)_{2n}-(q)_\infty\right)$$
$$=\sum_{n\geq 1}\left(1+\zeta^2+\cdots+\zeta^{2n-2}\right)q^{2n-1}(q)_{2n-2}+\sum_{n\geq 1}\left(1+\zeta^2+\cdots+\zeta^{2n-2}\right)q^{2n}(q)_{2n-1}.$$

Letting $\zeta=q^{\frac{1}{2}}$, the right-hand side can be written as

$$\frac{1}{1-q}\sum_{n\geq 1}\left((1-q^n)q^{2n-1}(q)_{2n-2}+(1-q^n)q^{2n}(q)_{2n-1}\right).$$

Next we use (2.11) to get

$$\sum_{n\geq 1}\left(q^{2n-1}(q)_{2n-2}+q^{2n}(q)_{2n-1}\right)=\sum_{n\geq 0}q^{n+1}(q)_n=1-(q)_\infty.$$

We rewrite the remaining sum

$$\sum_{n\geq 1}\left(-q^nq^{2n-1}(q)_{2n-2}-q^nq^{2n}(q)_{2n-1}\right)=-\sum_{n\geq 0}q^{3n+2}(q)_{2n}-\sum_{n\geq 0}q^{3n+3}(q)_{2n+1}.$$

Hence we have

$$H_2(q)=\frac{1}{(1-q)(q)_\infty^2}\left(1-\sum_{n\geq 0}\left(q^{3n+2}(q)_{2n}+q^{3n+3}(q)_{2n+1}\right)\right).$$

8. Further examples and the proof of (1.7)

Here we consider a few more complicated graphs.

8.1. 3-cycle

As shown in [25], for the three cycle graph Γ we have

$$\sum_{\boldsymbol{n}\in\mathbb{N}_0^3}\frac{q^{n_1n_2+n_1n_3+n_2n_3+n_1+n_2+n_3}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}}=\frac{1}{(q)_\infty}\sum_{n\geq 0}\frac{q^n}{(q^{n+1})_{n+1}},$$

where the sum on the right-hand side is $\chi_1(q)$, a fifth order mock theta function of Ramanujan. Such mock theta functions were introduced in Ramanujan's last letter to Hardy.

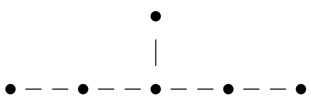
Since $\det(C_\Gamma) = 2$ for $C_\Gamma = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, it is natural to consider an additional q-series coming from the nontrivial coset. We obtain a related identity for another fifth order mock theta function χ_0

$$\sum_{\boldsymbol{n}\in\mathbb{N}_0^3} \frac{q^{n_1n_2+n_1n_3+n_2n_3+2n_1+n_2+n_3}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}} = \frac{1}{(q)_\infty}\sum_{n\geq 0}\frac{q^n}{(q^{n+2})_{n+1}} = \frac{q^{-1}}{(q)_\infty}\left(\chi_0(q)-1\right)\right).$$

Interestingly, $\chi_0(q)$ and $\chi_1(q)$ combine into a vector-valued quantum modular form [24].

8.2. Graph series of E_6

Now Γ is



and the graph series is

$$H_{E_6}(q) = \sum_{\boldsymbol{n}\in\mathbb{N}_0^6} \frac{q^{n_1n_2+n_1n_3+n_1n_4+n_2n_5+n_3n_6+n_1+n_2+n_3+n_4+n_5+n_6}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}(q)_{n_6}}.$$

We next prove (1.7).

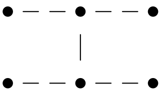
Proof of (1.7). We compute

$$\begin{aligned} H_{E_6}(q) &= \sum_{\boldsymbol{n}\in\mathbb{N}_0^6} \frac{q^{n_1(n_2+n_3+n_4)+n_2n_5+n_3n_6+n_1+n_2+n_3+n_4+n_5+n_6}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}(q)_{n_6}} \\ &= \frac{1}{(q)_\infty^2}\sum_{n_1,n_2,n_3,n_4\geq 0}\frac{q^{n_1(n_2+n_3+n_4)+n_1+n_2+n_3+n_4}}{(q)_{n_1}(q)_{n_4}} \\ &= \frac{1}{(q)_\infty^3}\sum_{n_1,n_2,n_3\geq 0} q^{n_1(n_2+n_3)+n_1+n_2+n_3} \\ &= \frac{1}{(q)_\infty^3}\sum_{n\geq 0}\frac{q^n}{(1-q^{n+1})^2} = \frac{q^{-1}}{(q)_\infty^3}\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2} = \frac{q^{-1}}{(q)_\infty^3}\sum_{n\geq 1}\frac{nq^n}{1-q^n}. \qquad \square \end{aligned}$$

We finish with two examples coming from affine Dynkin diagrams.

8.3. H*-graph (or* $D_5^{(1)}$*)*

Here we consider Γ to be an H graph as in the picture:



This graph series is given by

$$H_\Gamma(q) = \sum_{\boldsymbol{n}\in\mathbb{N}_0^6} \frac{q^{n_1n_2+n_1n_3+n_1n_4+n_4n_5+n_4n_6+n_1+n_2+n_3+n_4+n_5+n_6}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}(q)_{n_6}}.$$

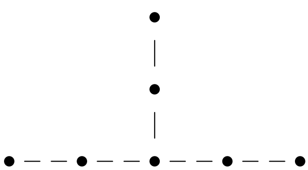
From this we easily find that

$$H_\Gamma(q) = \frac{1}{(q)_\infty^4} \sum_{n,m\geq 0} q^{mn+m+n}(q)_m(q)_n.$$

It would be interesting to explore modular properties of this double series. We believe that it is a (higher depth) quantum modular form.

8.4. T_2*-graph (or* $E_6^{(1)}$*)*

The next example is obtained by adding an extra node to an E_6 graph.



Here we get

$$\begin{aligned} H_{T_2}(q) &= \sum_{\boldsymbol{n}\in\mathbb{N}_0^7} \frac{q^{n_1n_6+n_2n_4+n_2n_5+n_2n_6+n_3n_4+n_3n_5+n_1+n_2+n_3+n_4+n_5+n_6+n_7}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}(q)_{n_6}(q)_{n_7}} \\ &= \frac{1}{(q)_\infty^4} \sum_{n_1,n_2,n_3\geq 0} q^{n_1+n_2+n_3}(q)_{n_1+n_2+n_3} = \frac{1}{2(q)_\infty^4} \sum_{n\geq 0} \left(n^2+3n+2\right) q^n (q)_n. \end{aligned}$$

The last sum can be expressed as a sum of tails by differentiating (2.10) with respect to ζ and then letting $\zeta = 1$. This immediately gives

$$H_{T_2}(q) = \frac{q^{-1}}{(q)_\infty^4} \sum_{n\geq 0} (n+1)\left((q)_n - (q)_\infty\right).$$

Since $\sum_{n\geq 0}((q)_n - (q)_\infty)$ is a quantum modular form, it would be interesting to investigate modular properties of the sum

$$\sum_{n\geq 0} n\left((q)_n - (q)_\infty\right).$$

9. Conclusion and open questions

We hope that this paper generates interest in graph series and their modular properties. However, unlike Nahm sums, they seem not to give rise to usual modular forms even for very simple graphs. Instead we obtain interesting combinations of mixed quantum and mock modular forms. This raises two natural questions:

Is there a simple graph Γ, not totally disconnected, and $a \in \mathbb{Q}$, such that $q^a H_\Gamma(q)$ is a modular form? Can we characterize which graphs have which types of modular properties?

We point out that for many examples we are not aware of any modular properties. This is the case for the following graphs:

$$\frac{q^{-1}}{(q)_\infty^4}\sum_{n\geq 0}(n+1)\left((q)_n - (q)_\infty\right), \quad \frac{1}{(q)_\infty^\ell}\sum_{n\geq 0} q^n (q)_n^{\ell-1}, \quad \ell \geq 4$$
$$\frac{1}{(q)_\infty^4}\sum_{n,m\geq 0} q^{nm+m+n}(q)_n(q)_m, \quad \frac{1}{(q)_\infty}\sum_{n,m\geq 1}\frac{q^{nm}}{(q)_{n+m-1}},$$

corresponding to T_2, ℓ-star graphs $X_\ell, \ell \geq 4$, H, and 4-cycle graphs, respectively. We hope to return to these examples in future work.

Data availability

No data was used for the research described in the article.

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