HIGHER DEPTH QUANTUM MODULAR FORMS, MULTIPLE EICHLER INTEGRALS, AND $\mathfrak{sl}_3$ FALSE THETA FUNCTIONS

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ABSTRACT. We introduce and study higher depth quantum modular forms. We construct two families of examples coming from rank two false theta functions, whose "companions" in the lower half-plane can be also realized both as double Eichler integrals and as non-holomorphic theta series having values of "double error" functions as coefficients. In particular, we prove that the false theta functions of $\mathfrak{sl}_3$, appearing in the character of the vertex algebra $W^0(p)_{A_2}$, can be written as the sum of two depth two quantum modular forms of positive integral weight.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper, we study higher depth quantum modular forms which occur as rank two false theta functions coming from characters of the vertex algebra $W^0(p)_{A_2}$ for $p \geq 2$. Via asymptotic expansions we relate these to double Eichler integrals which may be viewed as purely non-holomorphic parts of indefinite theta functions.

Let us first recall the classical rank one case. Note that the derivative of a modular form is typically not a modular form (only a so-called quasi-modular form). However, thanks to Bol's identity, differentiating a weight $2-k \in -\mathbb{N}$ modular form $k-1$ times returns a modular form of weight k. Thus it is natural to consider holomorphic Eichler integrals. That is, if $f(\tau) = \sum_{m\geq 1} c_f(m)q^m$ ($q := e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$ throughout) is a cusp form of weight k, then set

$$\widetilde{f}(\tau) := \sum_{m\geq 1} \frac{c_f(m)}{m^{k-1}} q^m. \tag{1.1}$$

It easily follows, by Bol's identity and the modularity of f, that the following function is annihilated by differentiating $k-1$ times

$$R_f(\tau) := \widetilde{f}(\tau) - \tau^{k-2}\widetilde{f}\left(-\frac{1}{\tau}\right). \tag{1.2}$$

This yields that R_f is a polynomial of degree $k-2$ (R_f is the so called *period polynomial* of f). So in particular R_f is much simpler than the starting function $\widetilde{f}$. Note that $\widetilde{f}$ may also be written as an integral, namely, up to constants it equals

$$\int_\tau^{i\infty} f(w)(w-\tau)^{k-2}dw. \tag{1.3}$$

Similarly R_f has an integral representation, namely up to constants it equals

$$\int_0^{i\infty} f(w)(w-\tau)^{k-2}dw.$$

A similar construction works for *weakly holomorphic modular forms*, i.e., those meromorphic modular forms that only have poles at the cusp $i\infty$ and not in $\mathbb{H}$. In this situation, (1.3) needs to be regularized since the integral does not converge. Moreover, there is a "companion integral" (again regularized)

$$\int_{-\overline{\tau}}^{i\infty} g(w)(w+\tau)^{k-2}dw, \tag{1.4}$$

where g is a certain weakly holomorphic modular form related to f in the sense that the corresponding period polynomial, defined analogously to (1.2), basically agrees with R_f.

In contrast, for half-integral weight modular forms there is no half-derivative and thus Bol's identity does not apply. However, one can formally define the analogue of (1.1) for theta functions. This was first investigated by Zagier [26, 27] in connection to Kontsevich's "strange" function

$$K(q) := \sum_{m\geq 0} (q;q)_m,$$

where for $m \in \mathbb{N}_0 \cup \{\infty\}$, $(a;q)_m := \prod_{j=0}^{m-1}(1-aq^j)$ denotes the usual *q-Pochhammer symbol*. The function $K(q)$ does not converge on any open subset of $\mathbb{C}$, but converges as a finite sum for q a root of unity. Zagier's study of K depends on the identity

$$\sum_{m\geq 0}\left(\eta(\tau) - q^{\frac{1}{24}}(q;q)_m\right) = \eta(\tau)D(\tau) + \frac{1}{2}\widetilde{\eta}(\tau), \tag{1.5}$$

with $\eta(\tau) := q^{\frac{1}{24}}(q;q)_\infty = \sum_{m\geq 1}\left(\frac{12}{m}\right)q^{\frac{m^2}{24}}$, $D(\tau) := -\frac{1}{2}+\sum_{m\geq 1}\frac{q^m}{1-q^m}$ and $\widetilde{\eta}(\tau) := \sum_{m\geq 1}\left(\frac{12}{m}\right)mq^{\frac{m^2}{24}}$, where $\left(\frac{\cdot}{\cdot}\right)$ denotes the extended Jacobi symbol. The key observation of Zagier is that in (1.5), the functions $\eta(\tau)$ and $\eta(\tau)D(\tau)$ vanish of infinite order as $\tau \to \frac{h}{k} \in \mathbb{Q}$. So at a root of unity ζ, $K(\zeta)$ is essentially the limiting value of the Eichler integral of η, which Zagier showed has quantum modular properties. Roughly speaking, Zagier defined "quantum modular forms" to be functions $f : \mathcal{Q} \to \mathbb{C}$ ($\mathcal{Q} \subseteq \mathbb{Q}$), such that the error of modularity ($M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$)

$$f(\tau) - (c\tau+d)^{-k}f(M\tau) \tag{1.6}$$

is "nice". The definition is intentionally vague to include many examples; in this paper we require (1.6) to be real-analytic. For example, $\widetilde{f}$ (recall $k \in \mathbb{Z}$ in this case) is a quantum modular form, since R_f is a polynomial and thus real-analytic. Additional examples appear in the study of limits of quantum invariants of 3-manifolds and knots [27], Kashaev invariants of torus knots/links [14, 15], and partial theta functions [11].

Motivated in part by vertex operator algebra theory, further (but similar) examples of quantum modular forms were investigated in the setup of characters of vertex algebra modules in [4] and [9]. These examples are given by characters of $M_{r,s}$, the atypical irreducible modules of the $(1,p)$-singlet algebra for $p \geq 2$ [4, 7]. For $r = 1$ and $1 \leq s \leq p-1$, they take the particularly nice shape

$$\mathrm{ch}_{M_{1,s}}(\tau) = \frac{F_{p-s,p}(p\tau)}{\eta(\tau)},$$

where

$$F_{j,p}(\tau) := \sum_{m\in\mathbb{Z}} \mathrm{sgn}\left(m+\frac{j}{2p}\right)q^{\left(m+\frac{j}{2p}\right)^2}$$

is a *false theta function.* The function $F_{j,p}$ is called "false theta" since getting rid of the sgn-factor yields the theta function $\sum_{m\in\mathbb{Z}} q^{(m+\frac{j}{2p})^2}$, which is a modular form of weight $\frac{1}{2}$. The quantum modularity of $F_{j,p}$ is now given by relating it to a non-holomorphic Eichler integral, as in (1.4). To be more precise, set (correcting a typographical error in [4])

$$F^*_{j,p}(\tau) := -\sqrt{2}i\int_{-\overline{\tau}}^{i\infty} \frac{f_{j,p}(w)}{(-i(w+\tau))^{\frac{1}{2}}}dw,$$

where $f_{j,p}$ is the cuspidal theta function of weight $\frac{3}{2}$

$$f_{j,p}(\tau) := \sum_{m\in\mathbb{Z}} \left(m+\frac{j}{2p}\right) q^{\left(m+\frac{j}{2p}\right)^2}.$$

One can show that $F_{j,p}(\tau)$ agrees for $\tau=\frac{h}{k}$ with $F^*_{j,p}(\tau)$ up to infinite order [4]. Quantum modularity then follows by the (mock) modular transformation of $F^*_{j,p}$ which we recall in Lemma 2.3 below. By "mock-modular", we mean that the occurence of the extra term $r_{f,\frac{d}{c}}$ in Lemma 2.3 prevents the function from being modular. However, there exists a "modular completion" in the sense that after multiplying it with a theta function, $F^*_{j,p}$ is the "purely non-holomorphic part" of a non-holomorphic theta function corresponding to an indefinite quadratic form (of signature $(1,1)$). Its modularity now can be proven by using results of Zwegers [28, Section 2.2]. The functions $\tau\mapsto F_{j,p}(p\tau)$, especially for $p=2$, have appeared in several studies of vertex algebras from different standpoints [3, 7, 12, 16].

In this paper we investigate higher-dimensional analogues. For this we consider certain q-series appearing in representation theory of vertex algebras and W-algebras. They are sometimes called *higher rank false theta functions* and are thoroughly studied in [4, 8]. They appear from extracting the constant term of certain multivariable Jacobi forms [4]. The constant term can be interpreted as the character of the zero weight space of the corresponding Lie algebra representation. In the case of the simple Lie algebra $\mathfrak{sl}_3$, the false theta function takes the following shape ($p\in\mathbb{N}$, $p\geq 2$)

$$F(q) := \sum_{\substack{m_1,m_2\geq 1\\ m_1\equiv m_2 \pmod 3}} \min(m_1,m_2) q^{\frac{p}{3}\left(m_1^2+m_2^2+m_1m_2\right)-m_1-m_2+\frac{1}{p}} \left(1-q^{m_1}\right)\left(1-q^{m_2}\right)\left(1-q^{m_1+m_2}\right). \quad (1.7)$$

Below we decompose this function as $F(q)=\frac{2}{p}F_1(q^p)+2F_2(q^p)$ with F_1 and F_2 defined in (3.1) and (3.2), respectively. The function F_1 and F_2 turn out to have generalized quantum modular properties. This connection goes via an analouge of (1.1). For instance, we show that F_1 asymptotically agrees with an integral of the shape

$$\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{f(w_1,w_2)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1$$

where $f\in S_{\frac{3}{2}}(\chi_1,\Gamma)\otimes S_{\frac{3}{2}}(\chi_2,\Gamma)$ (χ_j are multipliers and $\Gamma\subset \mathrm{SL}_2(\mathbb{Z})$). Modular properties follow from the modularity of f which in turn gives quantum modular properties of F_1. The idea is that here the error of modularity (1.6) is less complicated than the original function. We call the resulting functions higher depth quantum modular forms (see Definition 3 for a precise definition). Roughly speaking (see Definition 3 for a precise definition), depth two quantum modular forms of weight

$k \in \frac{1}{2}\mathbb{Z}$ satisfy, in the simplest case, the modular transformation property ($M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$)

$$f(\tau) - (c\tau + d)^{-k} f(M\tau) \in \mathcal{Q}_\kappa(\Gamma)\mathcal{O}(R) + \mathcal{O}(R)$$

for some $\kappa \in \frac{1}{2}\mathbb{Z}$, where $\mathcal{Q}_\kappa(\Gamma)$ is the space of quantum modular forms of weight κ and $\mathcal{O}(R)$ the space of real analytic functions on $R \subset \mathbb{R}$. Clearly, we can construct examples of depth two simply by multiplying two (depth one) quantum modular forms. Non-trivial examples arise from F (see Theorem 1.1 for precise statement).

Theorem 1.1. *For $p \geq 2$, the higher rank false theta function F can be written as the sum of two depth two quantum modular forms (with quantum set $\mathbb{Q}$) of weight one and two.*

It is worth noting that all of our examples of quantum modular forms, including those studied in [4], have $\mathbb{Q}$ as quantum set. Even though this feature is rare, a possible explanation is that vertex algebra characters are generally better behaved functions and are expected to combine into vector-valued families under the full modular group. Thus in our future work [6] we explore a vector-valued generalization of this theorem and its consequences to representation theory.

Zwegers [28] found an important connection between the error term of the Eichler integral (as in Lemma 2.3) and classical Mordell integrals. This result applied to the case of $F^*_{j,p}$ leads to an elegant expression for the error term as a Mordell integral

$$\int_{\mathbb{R}} \cot\left(\pi i w + \frac{\pi j}{2p}\right) e^{2\pi i p w^2 \tau} dw.$$

In this work we encounter error terms for iterated (double) Eichler integrals, so it is natural to attempt to extend Zwegers' result to two dimensions. In [6] we solve this problem in several special cases. In particular, we find that relevant integrals for the weight one component $\mathcal{E}_1$ (cf. Lemma 5.2) take the form

$$\int_{\mathbb{R}^2} \cot\left(\pi i w_1 + \pi\alpha_1\right) \cot\left(\pi i w_2 + \pi\alpha_2\right) e^{2\pi i\left(3w_1^2 + 3w_1 w_2 + w_2^2\right)\tau} dw_1 dw_2,$$

for some scalars α_1, α_2. This is what we call a *double Mordell* integral. We next turn to the modular completion of these Eichler integrals (see Propostiton 8.1 for a more precise version). For theta functions associated to indefinite quadratic forms, the reader is referred to [1, 17, 20, 23].

Theorem 1.2. *There exists an indefinite theta function, defined via* (8.1)*, of signature* $(2,2)$ *with "purely non-holomorphic" part* $\Theta(\tau)\mathcal{E}_1(\tau)$ *where* Θ *is a theta function of signature* $(2,0)$ *and the Eichler integral* $\mathcal{E}_1$ *is defined in* (5.5)*.*

The paper is organized as follows. In Section 2, we review basic results on special functions, non-holomorphic Eichler integrals, and "double error" functions. We also recall the notion of quantum modular forms and introduce higher depth quantum modular forms. In Section 3, the $\mathfrak{sl}_3$ higher rank false theta function $F(q) = \frac{2}{p}F_1(q^p) + 2F_2(q^p)$ is introduced. In Section 4, we determine the asymptotic behavior of F_1 and F_2 at roots of unity. In Section 5, we introduce multiple Eichler integrals and prove modular transformation formulas for the double Eichler integrals. We also study certain linear combinations of double Eichler integrals associated to F_j. In Section 6, we express special double Eichler integrals as pieces of indefinite theta series. Based on results in this section, in Section 7, we prove the main result, Theorem 1.1, on the quantum modularity of F. Section 8 deals with the completion of certain indefinite theta functions of signature $(2,2)$ associated to the companions of F_j proving Theorem 1.2. We conclude in Section 9 with several questions.

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2. Preliminaries

2.1. **Special functions.** Define, for $u \in \mathbb{R}$,

$$E(u) := 2\int_0^u e^{-\pi w^2} dw.$$

This function is essentially the error function and its derivative is $E'(u) = 2e^{-\pi u^2}$. We have the representation

$$E(u) = \operatorname{sgn}(u)\left(1 - \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}, \pi u^2\right)\right), \tag{2.1}$$

where $\Gamma(\alpha, u) := \int_u^\infty e^{-w} w^{\alpha-1} dw$ is the *incomplete gamma function* and where for $u \in \mathbb{R}$, we set

$$\operatorname{sgn}(u) := \begin{cases} 1 & \text{if } u > 0, \\ -1 & \text{if } u < 0, \\ 0 & \text{if } u = 0. \end{cases}$$

We also require the functional equation of the incomplete Γ-function with $\alpha = \frac{1}{2}$

$$\Gamma\left(\frac{1}{2}, u\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}, u\right) + \frac{1}{\sqrt{u}}e^{-u}. \tag{2.2}$$

Moreover, for $u \neq 0$, set

$$M(u) := \frac{i}{\pi}\int_{\mathbb{R}-iu} \frac{e^{-\pi w^2 - 2\pi i u w}}{w} dw.$$

We have

$$M(u) = E(u) - \operatorname{sgn}(u).$$

Thus, by (2.1)

$$M(u) = -\frac{\operatorname{sgn}(u)}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}, \pi u^2\right). \tag{2.3}$$

This implies that the following bound holds

$$|M(u)| \leq 2e^{-\pi u^2}.$$

We next turn to two-dimensional analogues, following [1] (using slightly different notation). Define $E_2 : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ by (throughout we use bold letters for vectors and denote their components using subscripts)

$$E_2(\kappa;\boldsymbol{u}) := \int_{\mathbb{R}^2} \operatorname{sgn}(w_1)\operatorname{sgn}(w_2+\kappa w_1)\, e^{-\pi\left((w_1-u_1)^2+(w_2-u_2)^2\right)} dw_1 dw_2.$$

Note that

$$E_2(\kappa;-\boldsymbol{u}) = E_2(\kappa;\boldsymbol{u}).$$

Moreover, also following [1], for $u_2, u_1-\kappa u_2 \neq 0$ we set

$$M_2(\kappa;\boldsymbol{u}) := -\frac{1}{\pi^2}\int_{\mathbb{R}-iu_2}\int_{\mathbb{R}-iu_1} \frac{e^{-\pi w_1^2-\pi w_2^2-2\pi i(u_1w_1+u_2w_2)}}{w_2(w_1-\kappa w_2)} dw_1 dw_2.$$

Then we have

$$\begin{aligned} M_2(\kappa;\boldsymbol{u}) &= E_2(\kappa;\boldsymbol{u}) - \operatorname{sgn}(u_2) M(u_1) \\ &\quad - \operatorname{sgn}(u_1-\kappa u_2) M\left(\frac{u_2+\kappa u_1}{\sqrt{1+\kappa^2}}\right) - \operatorname{sgn}(u_1)\operatorname{sgn}(u_2+\kappa u_1). \end{aligned} \tag{2.4}$$

Note that (2.4) extends the definition of M_2 to $u_2=0$ or $u_1=\kappa u_2$. With $x_1 := u_1-\kappa u_2$, $x_2 := u_2$, a direct calculation shows that

$$\begin{aligned} M_2(\kappa;\boldsymbol{u}) &= E_2(\kappa; x_1+\kappa x_2, x_2) + \operatorname{sgn}(x_1)\operatorname{sgn}(x_2) \\ &\quad - \operatorname{sgn}(x_2) E(x_1+\kappa x_2) - \operatorname{sgn}(x_1) E\left(\frac{\kappa x_1}{\sqrt{1+\kappa^2}} + \sqrt{1+\kappa^2}\, x_2\right). \end{aligned}$$

We have the first partial derivatives

$$M_2^{(0,1)}(\kappa;\boldsymbol{u}) = \frac{2}{\sqrt{1+\kappa^2}} e^{-\frac{\pi(u_2+\kappa u_1)^2}{1+\kappa^2}} M\left(\frac{u_1-\kappa u_2}{\sqrt{1+\kappa^2}}\right), \tag{2.5}$$

$$M_2^{(1,0)}(\kappa;\boldsymbol{u}) = 2e^{-\pi u_1^2} M(u_2) + \frac{2\kappa}{\sqrt{1+\kappa^2}} e^{-\frac{\pi(u_2+\kappa u_1)^2}{1+\kappa^2}} M\left(\frac{u_1-\kappa u_2}{\sqrt{1+\kappa^2}}\right), \tag{2.6}$$

and the limiting behavior (cf. [1, Proposition 3.3, iii])

$$M_2(\kappa;\lambda\boldsymbol{u}) \sim -\frac{e^{-\pi\lambda^2\left(u_1^2+u_2^2\right)}}{\lambda^2\pi^2 u_2(u_1-\kappa u_2)} \qquad (\text{as} \quad \lambda\to\infty). \tag{2.7}$$

Lemma 2.1. *For $u_3, u_4+\kappa u_3 \neq 0$, we have the following limits*

$$\lim_{\varepsilon\to 0^+} E_2\left(\varepsilon\kappa; u_1, \varepsilon u_2+\varepsilon^{-1}u_3\right) = \operatorname{sgn}(u_3) E(u_1),$$

$$\lim_{\varepsilon\to 0^+} E_2\left(\kappa; \varepsilon u_1+\varepsilon^{-1}u_3, \varepsilon u_2+\varepsilon^{-1}u_4\right) = \operatorname{sgn}(u_3)\operatorname{sgn}(u_4+\kappa u_3).$$

Proof. We only prove the first statement, the second follows analogously. We may compute the limit inside the integral due to the convergence of the dominating integral $\int_{\mathbb{R}^2} e^{-\pi(w_1^2+w_2^2)} dw = 1$ to obtain

$$\lim_{\varepsilon\to 0^+} E_2\left(\varepsilon\kappa; u_1, \varepsilon u_2+\varepsilon^{-1}u_3\right)$$

$$= \int_{\mathbb{R}^2} e^{-\pi\left(w_1^2+w_2^2\right)} \operatorname{sgn}\left(w_1+u_1\right) \lim_{\varepsilon\to 0^+} \operatorname{sgn}\left(u_3+\varepsilon\left(w_2+\varepsilon\kappa w_1+\varepsilon u_2+\varepsilon\kappa u_1\right)\right) dw_2 dw_1$$
$$= \int_{\mathbb{R}} e^{-\pi w_1^2} \operatorname{sgn}\left(w_1+u_1\right) \int_{\mathbb{R}} e^{-\pi w_2^2} \operatorname{sgn}\left(u_3\right) dw_2 dw_1 = \operatorname{sgn}(u_3) E(u_1).$$

□

2.2. **Euler-Maclaurin summation formula.** We now state a special case of the Euler-Maclaurin summation formula. We only give it in the two-dimensional case; the one-dimensional case can be concluded by viewing the second variable as constant.

Let $B_m(x)$ be the m-th Bernoulli polynomial defined by $\frac{te^{xt}}{e^t-1} =: \sum_{m\geq 0} B_m(x)\frac{t^m}{m!}$. We also require

$$B_m(1-x) = (-1)^m B_m(x).$$

The Euler-Maclaurin summation formula implies that, for $\boldsymbol{\alpha}\in\mathbb{R}^2$, $F:\mathbb{R}^2\to\mathbb{R}$ a C^∞-function which has rapid decay, we have (generalizing a result of [25] to include shifts by α)

$$\sum_{\boldsymbol{n}\in\mathbb{N}_0^2} F((\boldsymbol{n}+\boldsymbol{\alpha})t) \sim \frac{\mathcal{I}_F}{t^2} - \sum_{n_2\geq 0} \frac{B_{n_2+1}(\alpha_2)}{(n_2+1)!} \int_0^\infty F^{(0,n_2)}(x_1,0)dx_1 t^{n_2-1}$$
$$- \sum_{n_1\geq 0} \frac{B_{n_1+1}(\alpha_1)}{(n_1+1)!} \int_0^\infty F^{(n_1,0)}(0,x_2)dx_2 t^{n_1-1} + \sum_{n_1,n_2\geq 0} \frac{B_{n_1+1}(\alpha_1)}{(n_1+1)!}\frac{B_{n_2+1}(\alpha_2)}{(n_2+1)!} F^{(n_1,n_2)}(0,0)t^{n_1+n_2}, \tag{2.8}$$

where $\mathcal{I}_F := \int_0^\infty\int_0^\infty F(\boldsymbol{x})dx_1dx_2$. Here by $\sim$ we mean that the difference between the left- and the right-hand side is $O(t^N)$ for any $N\in\mathbb{N}$.

2.3. **Shimura's theta functions.** We require transformation laws of certain theta functions studied, for example, by Shimura [21]. For $\nu\in\{0,1\}$, $h\in\mathbb{Z}$, $N, A\in\mathbb{N}$, with $A|N$, $N|hA$, define

$$\Theta_\nu(A,h,N;\tau) := \sum_{\substack{m\in\mathbb{Z}\\ m\equiv h \pmod N}} m^\nu q^{\frac{Am^2}{2N^2}}. \tag{2.9}$$

Recall the following modular transformation

$$\Theta_\nu(A,h,N;M\tau) = e\left(\frac{abAh^2}{2N^2}\right)\left(\frac{2Ac}{d}\right)\varepsilon_d^{-1}(c\tau+d)^{\frac12+\nu}\Theta_\nu(A,ah,N;\tau) \tag{2.10}$$

for $M=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(2N)$ with $2|b$. Here $e(x):=e^{2\pi i x}$, for odd d, $\varepsilon_d=1$ or i, depending on whether $d\equiv 1 \pmod 4$ or $d\equiv 3 \pmod 4$. Also note that if $h_1\equiv h_2 \pmod N$, then we have

$$\Theta_\nu(A,h_1,N;\tau) = \Theta_\nu(A,h_2,N;\tau), \qquad \Theta_\nu(A,-h,N;\tau) = (-1)^\nu\Theta_\nu(A,h,N;\tau).$$

2.4. **Indefinite theta functions.** We begin by defining (possibly indefinite) theta functions.

Definition 1. Let $A\in M_m(\mathbb{Z})$ be a non-singular symmetric $m\times m$ matrix, $P:\mathbb{R}^m\to\mathbb{C}$ and $\boldsymbol{a}\in\mathbb{Q}^m$. We define the associated theta function by $(\tau=u+iv)$

$$\Theta_{A,P,\boldsymbol{a}}(\tau) := \sum_{\boldsymbol{n}\in\boldsymbol{a}+\mathbb{Z}^m} P\left(\sqrt{v}\boldsymbol{n}\right) q^{\frac12\boldsymbol{n}^TA\boldsymbol{n}}.$$

The following theorem shows that under certain conditions $\Theta_{A,P,\boldsymbol{a}}$ is modular.

Theorem 2.2 (Vignéras, [22]). *Suppose that $A \in M_m(\mathbb{Z})$ is non-singular and that P satisfies the following conditions:*

(1) *For any differential operator D of order two and any polynomial R of degree at most two, we have that $D(\boldsymbol{w})(P(\boldsymbol{w})e^{\pi Q(\boldsymbol{w})})$ and $R(\boldsymbol{w})P(\boldsymbol{w})e^{\pi Q(\boldsymbol{w})}$ belong to $L^2(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$.*

(2) *For some $\lambda \in \mathbb{Z}$ the Vignéras differential equation holds:*

$$\left(\mathcal{D} - \frac{1}{4\pi}\Delta\right) P = \lambda P.$$

Here we define the Euler and Laplace operators ($\boldsymbol{w} := (w_1, \dots, w_m)$, $\partial_{\boldsymbol{w}} := (\frac{\partial}{\partial w_1}, \dots \frac{\partial}{\partial w_m})^T$)

$$\mathcal{D} := \boldsymbol{w}\partial_{\boldsymbol{w}} \qquad \text{and} \qquad \Delta = \Delta_{A^{-1}} := \partial_{\boldsymbol{w}}^T A^{-1} \partial_{\boldsymbol{w}}.$$

Then, assuming that $\Theta_{A,P,\boldsymbol{a}}$ is absolutely locally convergent, $\Theta_{A,P,\boldsymbol{a}}$ is modular of weight $\lambda + \frac{m}{2}$ for some subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

2.5. **Quantum modular forms.** We already motivated quantum modular forms in the introduction. The formal definition is as follows [27].

Definition 2. A function $f : \mathcal{Q} \to \mathbb{C}$ (here $\mathcal{Q} \subseteq \mathbb{Q}$) is called a *quantum modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ and multiplier χ for a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ and quantum set $\mathcal{Q}$* if for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, the function

$$f(\tau) - \chi(M)^{-1}(c\tau + d)^{-k} f(M\tau)$$

can be extended to an open subset of $\mathbb{R}$ and is real-analytic there. We denote the vector space of such forms by $\mathcal{Q}_k(\Gamma, \chi)$.

Remark. Zagier also considered *strong quantum modular forms.* Here one is looking at asymptotic expansions instead of just values.

The introduction already gives examples of quantum modular forms. As mentioned there, the functions $F_{j,p}$ satisfy modular type transformations making them quantum modular forms. More generally, for $f \in S_k(\Gamma, \chi)$, the space of cusp forms of weight k transforming as

$$f(M\tau) = (c\tau + d)^k \chi(M) f(\tau)$$

for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ and χ some multiplier, we set, for $\frac{d}{c} \in \mathbb{Q}$,

$$I_f(\tau) := \int_{-\overline{\tau}}^{i\infty} \frac{f(w)}{(-i(w+\tau))^{2-k}} dw, \qquad r_{f,\frac{d}{c}}(\tau) := \int_{\frac{d}{c}}^{i\infty} \frac{f(w)}{(-i(w+\tau))^{2-k}} dw. \tag{2.11}$$

For weight $k = \frac{1}{2}$, we allow $f \in M_{\frac{1}{2}}(\Gamma, \chi)$, the space of holomorphic modular forms of weight $\frac{1}{2}$. To state the modularity properties of I_f, we let $\Gamma^* := P\Gamma P^{-1}$, where $P := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The proof of the following lemma follows along the same lines as the proof of Theorem 5.1 below.

Lemma 2.3. *We have the transformation, for $M \in \Gamma^*$,*

$$I_f(\tau) - \chi^{-1}(M^*)(c\tau + d)^{k-2} I_f(M\tau) = r_{f,\frac{d}{c}}(\tau).$$

The function I_f is defined on $\mathbb{H} \cup \mathbb{Q}$ whereas $r_{f,\frac{d}{c}}$ exists on all of $\mathbb{R} \setminus \{-\frac{d}{c}\}$ and is real-analytic there. If $f \in S_k(\Gamma, \chi)$, then $r_{f,\frac{d}{c}}$ exists on $\mathbb{R}$.

2.6. **Higher Depth Quantum modular forms.** We next turn to generalizations of quantum modular forms.

Definition 3. A function $f:\mathcal{Q}\to\mathbb{C}$ ($\mathcal{Q}\subset\mathbb{Q}$) is called a *quantum modular form of depth $N\in\mathbb{N}$, weight $k\in\frac{1}{2}\mathbb{Z}$, multiplier χ, and quantum set $\mathcal{Q}$ for* Γ if for $M=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma$

$$f(\tau)-\chi(M)^{-1}(c\tau+d)^{-k}f(M\tau)\in\bigoplus_j \mathcal{Q}_{\kappa_j}^{N_j}(\Gamma,\chi_j)\mathcal{O}(R),$$

where j runs through a finite set, $\kappa_j\in\frac{1}{2}\mathbb{Z}$, $N_j\in\mathbb{N}$ with $\max_j(N_j)=N-1$, the χ_j are characters, $\mathcal{O}(R)$ is the space of real-analytic functions on $R\subset\mathbb{R}$ which contains an open subset of $\mathbb{R}$, $\mathcal{Q}_k^1(\Gamma,\chi):=\mathcal{Q}_k(\Gamma,\chi)$, $\mathcal{Q}_k^0(\Gamma,\chi):=1$, and $\mathcal{Q}_k^N(\Gamma,\chi)$ denotes the space of quantum modular forms of weight k, depth N, multiplier χ for Γ.

Remark. Again one can consider *higher depth strong quantum modular forms* by looking at asymptotic expansions instead of values. The examples of this paper satisfiy this stronger property.

Example. For $f_1\in\mathcal{Q}_{k_1}^1(\Gamma_1,\chi_1)$ and $f_2\in\mathcal{Q}_{k_2}^1(\Gamma_2,\chi_2)$, we have that $f_1f_2\in Q_{k_1+k_2}^2(\Gamma_1\cap\Gamma_2,\chi_1\chi_2)$.

3. A rank two false theta function

We briefly recall a construction from [5, 8, 10]. For $p\in\mathbb{N}_{\geq 2}$, there is a vertex operator algebra $W(p)_{A_2}$ associated to the simple Lie algebra $\mathfrak{sl}_3$ (more precisely, to its root lattice of type A_2). The character formula of $W(p)_Q$, where Q is any ADE root lattice, was proposed in [10] (note that some arguments in [10] are not completely rigorous) and further studied in [5, 8, 10]; see also [2]. Letting $\zeta_j:=e^{2\pi i z_j}$, we have [5, 8]

$$\eta(\tau)^2\mathrm{ch}[W(p)_{A_2}](\tau,\boldsymbol{z})=\sum_{m_1,m_2\in\mathbb{Z}}\frac{q^{p\left(\left(m_1-\frac{1}{p}\right)^2+\left(m_2-\frac{1}{p}\right)^2-\left(m_1-\frac{1}{p}\right)\left(m_2-\frac{1}{p}\right)\right)}}{\left(1-\zeta_1^{-1}\right)\left(1-\zeta_2^{-1}\right)\left(1-\zeta_1^{-1}\zeta_2^{-1}\right)}\Big(\zeta_1^{m_1-1}\zeta_2^{m_2-1}-\zeta_1^{-m_1+m_2-1}$$
$$\times\zeta_2^{m_2-1}-\zeta_1^{m_1-1}\zeta_2^{-m_2+m_1-1}+\zeta_1^{-m_2-1}\zeta_2^{-m_2+m_1-1}+\zeta_1^{-m_1+m_2-1}\zeta_2^{-m_1-1}-\zeta_1^{-m_2-1}\zeta_2^{-m_1-1}\Big).$$

The six term expression in the numerator comes from the summation over the Weyl group W of $\mathfrak{sl}_3$ which is isomorphic to S_3. Thanks to Weyl's character formula, the rational $\boldsymbol{z}$-part is in fact a Laurent polynomial. There are two important operations on this character:

(1) taking the limit $\boldsymbol{z}=(z_1,z_2)\to(0,0)$, yielding a modular form [5];
(2) taking the constant term
$$\mathrm{ch}[W^0(p)_{A_2}](\tau):=\mathrm{CT}_{\zeta_1,\zeta_2}\mathrm{ch}[W(p)_{A_2}](\tau,\boldsymbol{z}),$$
which computes the character of another vertex algebra. It was shown in [5] that
$$\mathrm{ch}[W^0(p)_{A_2}](\tau)=\frac{F(q)}{\eta(\tau)^2}.$$

Note that formulas like $\eta(\tau)^{\mathrm{rank}(Q)}\mathrm{ch}[W^0(p)_Q](\tau)$, where Q is any root lattice, are of interest beyond vertex algebra theory [5, 8]. The coefficients appearing in the q-expansion are essentially dimensions of the zero weight spaces of finite-dimensional irreducible representations of simple Lie algebras (for the recent progress in understanding these numbers see [18]).

Remark. Modular-type properties of regularized (or Jacobi) characters, in particular $\mathrm{ch}[W^0(p)^{\varepsilon}_{A_2}](\tau)$, were investigated in [8] (see also [7]). There are two important differences between the current work and [8]. In this paper, the value of the Jacobi parameter ε is always zero whereas in [8] it is necessarily non-zero. Secondly, there seems to be no clear connection between transformation formulas appearing in [8] and mock modular forms. On the other hand, here we make this connection quite explicit by virtue of generalized Eichler integrals (see Section 5).

Let $n_1 = m_1 - m_2, n_2 = m_2$ in (1.7) and then change $n_1 \mapsto 3n_1$. Then we have, with F given in (1.7),

$$\frac{1}{2}F(q) = f_1(q) + f_2(q) + f_3(q),$$

where, with $Q(\boldsymbol{x}) := 3x_1^2 + 3x_1x_2 + x_2^2$, we define

$$\begin{aligned}
f_1(q) &:= q^{\frac{1}{p}} \sum_{n_1,n_2\geq 0}^{*} n_2 q^{pQ(\boldsymbol{n})}\left(q^{-3n_1-2n_2} - q^{3n_1+2n_2}\right),\\
f_2(q) &:= q^{\frac{1}{p}} \sum_{n_1,n_2\geq 0}^{*} n_2 q^{pQ(\boldsymbol{n})}\left(q^{n_2} - q^{-n_2}\right),\\
f_3(q) &:= q^{\frac{1}{p}} \sum_{n_1,n_2\geq 0}^{*} n_2 q^{pQ(\boldsymbol{n})}\left(q^{3n_1+n_2} - q^{-3n_1-n_2}\right).
\end{aligned}$$

Here $\sum^*$ means that the $n_1 = 0$ term is weighted by $\frac{1}{2}$. We then rewrite

$$\begin{aligned}
f_1(q) = &-\sum_{n_1,n_2\geq 0}\left(n_2+\frac{1}{p}\right) q^{pQ\left(n_1+1,n_2+\frac{1}{p}\right)} + \sum_{n_1,n_2\geq 0}\left(n_2+1-\frac{1}{p}\right) q^{pQ\left(n_1,n_2+1-\frac{1}{p}\right)}\\
&+\frac{1}{p}\sum_{n_1,n_2\geq 0} q^{pQ\left(n_1+1,n_2+\frac{1}{p}\right)} + \frac{1}{p}\sum_{n_1,n_2\geq 0} q^{pQ\left(n_1,n_2+1-\frac{1}{p}\right)} - \frac{1}{2}\sum_{m\geq 0}\left(m+\frac{1}{p}\right) q^{p\left(m+\frac{1}{p}\right)^2}\\
&-\frac{1}{2}\sum_{m\geq 0}\left(m+1-\frac{1}{p}\right) q^{p\left(m+1-\frac{1}{p}\right)^2} + \frac{1}{2p}\sum_{m\geq 0} q^{p\left(m+\frac{1}{p}\right)^2} - \frac{1}{2p}\sum_{m\geq 0} q^{p\left(m+1-\frac{1}{p}\right)^2},\\
f_2(q) = &\sum_{n_1,n_2\geq 0}\left(n_2+\frac{2}{p}\right) q^{pQ\left(n_1+1-\frac{1}{p},n_2+\frac{2}{p}\right)} - \sum_{n_1,n_2\geq 0}\left(n_2+1-\frac{2}{p}\right) q^{pQ\left(n_1+\frac{1}{p},n_2+1-\frac{2}{p}\right)}\\
&-\frac{2}{p}\sum_{n_1,n_2\geq 0} q^{pQ\left(n_1+1-\frac{1}{p},n_2+\frac{2}{p}\right)} - \frac{2}{p}\sum_{n_1,n_2\geq 0} q^{pQ\left(n_1+\frac{1}{p},n_2+1-\frac{2}{p}\right)}\\
&+\frac{q^{\frac{3}{4p}}}{2}\sum_{m\geq 1} m q^{p\left(m-\frac{1}{2p}\right)^2} + \frac{q^{\frac{3}{4p}}}{2}\sum_{m\geq 1} m q^{p\left(m+\frac{1}{2p}\right)^2},\\
f_3(q) = &\sum_{n_1,n_2\geq 0}\left(n_2+1-\frac{1}{p}\right) q^{pQ\left(n_1+\frac{1}{p},n_2+1-\frac{1}{p}\right)} - \sum_{n_1,n_2\geq 0}\left(n_2+\frac{1}{p}\right) q^{pQ\left(n_1+1-\frac{1}{p},n_2+\frac{1}{p}\right)}\\
&+\frac{1}{p}\sum_{n_1,n_2\geq 0} q^{pQ\left(n_1+\frac{1}{p},n_2+1-\frac{1}{p}\right)} + \frac{1}{p}\sum_{n_1,n_2\geq 0} q^{pQ\left(n_1+1-\frac{1}{p},n_2+\frac{1}{p}\right)}
\end{aligned}$$

$$-\frac{q^{\frac{3}{4p}}}{2}\sum_{m\geq 1} mq^{p\left(m+\frac{1}{2p}\right)^2}-\frac{q^{\frac{3}{4p}}}{2}\sum_{m\geq 1} mq^{p\left(m-\frac{1}{2p}\right)^2}.$$

We thus obtain

$$F(q)=\frac{2}{p}F_1\left(q^p\right)+2F_2\left(q^p\right)$$

with

$$F_1(q):=\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\varepsilon(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} q^{Q(\boldsymbol{n})}+\frac{1}{2}\sum_{m\in\mathbb{Z}}\operatorname{sgn}\left(m+\frac{1}{p}\right)q^{\left(m+\frac{1}{p}\right)^2}, \tag{3.1}$$

where

$$\mathscr{S}:=\left\{\left(1-\frac{1}{p},\frac{2}{p}\right),\left(\frac{1}{p},1-\frac{2}{p}\right),\left(1,\frac{1}{p}\right)\left(0,1-\frac{1}{p}\right),\left(\frac{1}{p},1-\frac{1}{p}\right),\left(1-\frac{1}{p},\frac{1}{p}\right)\right\},$$

and for $\boldsymbol{\alpha}\pmod{\mathbb{Z}^2}$, we set

$$\varepsilon(\boldsymbol{\alpha}):=\begin{cases}-2 & \text{if } \boldsymbol{\alpha}\in\left\{\left(1-\frac{1}{p},\frac{2}{p}\right),\left(\frac{1}{p},1-\frac{2}{p}\right)\right\},\\ 1 & \text{otherwise.}\end{cases}$$

Moreover

$$F_2(q):=\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} n_2q^{Q(\boldsymbol{n})}-\frac{1}{2}\sum_{m\in\mathbb{Z}}\left|m+\frac{1}{p}\right|q^{\left(m+\frac{1}{p}\right)^2}, \tag{3.2}$$

where for $\boldsymbol{\alpha}\pmod{\mathbb{Z}^2}$, we let

$$\eta(\boldsymbol{\alpha}):=\begin{cases}1 & \text{if } \boldsymbol{\alpha}\in\left\{\left(1-\frac{1}{p},\frac{2}{p}\right),\left(0,1-\frac{1}{p}\right),\left(\frac{1}{p},1-\frac{1}{p}\right)\right\},\\ -1 & \text{otherwise.}\end{cases}$$

4. Asymptotic behavior of F_1 and F_2

In this section we determine the asymptotic behavior of $F(e^{2\pi i\frac{h}{k}-t})$ ($h,k\in\mathbb{Z}$ with $k>0$ and $\gcd(h,k)=1$) as $t\to 0^+$ and in particular show that the limit exists.

4.1. **The function F_1.** We decompose

$$F_1(q)=F_{1,1}(q)+F_{1,2}(q),$$

where

$$F_{1,1}(q):=\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\varepsilon(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} q^{Q(\boldsymbol{n})},\qquad F_{1,2}(q):=\frac{1}{2}\sum_{m\in\frac{1}{p}+\mathbb{Z}}\operatorname{sgn}(m)q^{m^2}.$$

We first study the asymptotic behavior of $F_{1,1}$, rewriting it in a shape in which we can apply the Euler-Maclaurin formula (2.8). For this, let $\boldsymbol{n}\mapsto\boldsymbol{\ell}+\boldsymbol{n}\frac{kp}{\delta}$ with $\boldsymbol{n}\in\mathbb{N}_0^2$, $0\leq\boldsymbol{\ell}\leq\frac{kp}{\delta}-1$, where

$\delta := \gcd(h,p)$. Here by the inequality we mean that it should hold componentwise. It is not hard to see that, with $\mathcal{F}_1(\boldsymbol{x}) := e^{-Q(\boldsymbol{x})}$,

$$F_{1,1}\left(e^{2\pi i\frac{h}{k}-t}\right) = \sum_{\boldsymbol{\alpha}\in\mathscr{S}} \varepsilon(\boldsymbol{\alpha}) \sum_{0\leq \boldsymbol{\ell}\leq \frac{kp}{\delta}-1} e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})} \sum_{\boldsymbol{n}\in\frac{\delta}{kp}(\boldsymbol{\ell}+\boldsymbol{\alpha})+\mathbb{N}_0^2} \mathcal{F}_1\left(\frac{kp}{\delta}\sqrt{t}\boldsymbol{n}\right).$$

The main term in (2.8) is then

$$\frac{\delta^2}{k^2p^2t}\mathcal{I}_{\mathcal{F}_1} \sum_{\boldsymbol{\alpha}\in\mathscr{S}} \varepsilon(\boldsymbol{\alpha}) \sum_{0\leq \boldsymbol{\ell}\leq \frac{kp}{\delta}-1} e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}. \tag{4.1}$$

It is not hard to see that one may let $\boldsymbol{\ell}$ run modulo $\frac{kp}{\delta}$ (again meant componentwise). We write $\boldsymbol{\ell} = \boldsymbol{N}+k\boldsymbol{\nu}$ with $\boldsymbol{N}$ running modulo k, $\boldsymbol{\nu}$ modulo $\frac{p}{\delta}$, and $\boldsymbol{a}\in\{(-1,2),(1,-2),(0,1),(0,-1),(1,-1),(-1,1)\}$ such that $\boldsymbol{\alpha}-\frac{\boldsymbol{a}}{p}\in\mathbb{Z}^2$. We then compute that the sum over $\boldsymbol{\ell}$ in (4.1) equals (since $Q(\boldsymbol{a})=1$)

$$e^{2\pi i\frac{h}{p^2k}} \sum_{\boldsymbol{N}\pmod k} e^{\frac{2\pi ih}{pk}\left(3\left(pN_1^2+2a_1N_1\right)+3(pN_1N_2+a_2N_1+a_1N_2)+pN_2^2+2a_2N_2\right)} \\ \times \sum_{\boldsymbol{\nu}\ \left(\mathrm{mod}\ \frac{p}{\delta}\right)} e^{\frac{2\pi ih/\delta}{p/\delta}((6a_1+3a_2)\nu_1+(2a_2+3a_1)\nu_2)}.$$

Since $\gcd(\frac{h}{\delta},\frac{p}{\delta})=1$, the inner sum vanishes unless $\frac{p}{\delta}|\,3(2a_1+a_2)$ and $\frac{p}{\delta}|\,(2a_2+3a_1)$. If $3|\frac{p}{\delta}$, then in particular $3\mid a_2$. This is however not satisfied for elements in $\mathscr{S}$. If $3\nmid\frac{p}{\delta}$, then we easily obtain that $a_1\equiv a_2\equiv 0\pmod{\frac{p}{\delta}}$, implying that $\frac{p}{\delta}=1$. We are thus left to show that ($\frac{p}{\delta}=1$)

$$\sum_{\boldsymbol{\alpha}\in\mathscr{S}} \varepsilon(\boldsymbol{\alpha}) \sum_{\boldsymbol{N}\pmod k} e^{\frac{2\pi ih/\delta}{k}\left(3\left(pN_1^2+2a_1N_1\right)+3(pN_1N_2+a_2N_1+a_1N_2)+pN_2^2+2a_2N_2\right)} = 0. \tag{4.2}$$

Changing $\boldsymbol{N}\mapsto\boldsymbol{N}-\boldsymbol{a}\overline{p}$, with $\overline{p}$ the inverse of p modulo k (note that $\frac{p}{\delta}=1$ implies that $\gcd(p,k)=1$), the sum on $\boldsymbol{N}$ equals

$$e^{-\frac{2\pi i\overline{p}h/\delta}{k}} \sum_{\boldsymbol{N}\pmod k} e^{\frac{2\pi ih}{k}Q(\boldsymbol{N})},$$

which is independent of $\boldsymbol{a}$. Thus (4.2) holds.

The second term in (2.8) is

$$-\sum_{\boldsymbol{\alpha}\in\mathscr{S}} \varepsilon(\boldsymbol{\alpha}) \sum_{0\leq\boldsymbol{\ell}\leq\frac{kp}{\delta}-1} e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})} \sum_{n_2\geq 0} \frac{B_{n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(n_2+1)!} \int_0^\infty \mathcal{F}_1^{(0,n_2)}(x_1,0)dx_1 \left(\frac{kp\sqrt{t}}{\delta}\right)^{n_2-1}. \tag{4.3}$$

We claim that the contribution from those n_2 which are even vanishes. This follows, once we show that, for $\boldsymbol{\alpha}\in\mathscr{S}$,

$$\sum_{0\leq\boldsymbol{\ell}\leq\frac{kp}{\delta}-1} \left(e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})} B_{2n_2+1}\left(\frac{\delta\left(\ell_2+\alpha_2\right)}{kp}\right) + e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{1}-\boldsymbol{\alpha})} B_{2n_2+1}\left(\frac{\delta\left(\ell_2+1-\alpha_2\right)}{kp}\right)\right) = 0.$$

This is seen to be true by the change of variables $\boldsymbol{\ell}\mapsto-\boldsymbol{\ell}+(-1+\frac{kp}{\delta})\boldsymbol{1}$ for the second term.

Arguing in the same way for the contribution from n_2 odd, we obtain that (4.3) equals

$$-2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_2\ge0}\frac{B_{2n_2+2}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(2n_2+2)!}\int_0^\infty\mathcal{F}_1^{(0,2n_2+1)}(x_1,0)dx_1\left(\frac{k^2p^2}{\delta^2}t\right)^{n_2},$$

where

$$\mathscr{S}^*:=\left\{\left(1-\frac{1}{p},\frac{2}{p}\right),\left(0,1-\frac{1}{p}\right),\left(\frac{1}{p},1-\frac{1}{p}\right)\right\}.$$

The third term in (2.8) is treated in the same way, yielding the contribution

$$-2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_1\ge0}\frac{B_{2n_1+2}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(2n_1+2)!}\int_0^\infty\mathcal{F}_1^{(2n_1+1,0)}(0,x_2)dx_2\left(\frac{k^2p^2}{\delta^2}t\right)^{n_1}.$$

The final term in (2.8) equals

$$\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}$$
$$\times\sum_{n_1,n_2\ge0}\frac{B_{n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(n_1+1)!}\frac{B_{n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(n_2+1)!}\mathcal{F}_1^{(n_1,n_2)}(0,0)\left(\frac{kp\sqrt{t}}{\delta}\right)^{n_1+n_2}.$$

Arguing in the same way as before this equals

$$2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\ell\le\frac{kp}{\delta}-1}e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}$$
$$\times\sum_{\substack{n_1,n_2\ge0\\ n_1\equiv n_2 \pmod 2}}\frac{B_{n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(n_1+1)!}\frac{B_{n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(n_2+1)!}\mathcal{F}_1^{(n_1,n_2)}(0,0)\left(\frac{kp\sqrt{t}}{\delta}\right)^{n_1+n_2}.$$

The function $F_{1,2}$ is treated similarly, yielding, with $\mathcal{F}_2(x):=e^{-x^2}$,

$$-\sum_{0\le r\le\frac{kp}{\delta}-1}e^{2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2}\sum_{m\ge0}\frac{B_{2m+1}\left(\frac{\delta\left(r+\frac{1}{p}\right)}{kp}\right)}{(2m+1)!}\mathcal{F}_2^{(2m)}(0)\left(\frac{k^2p^2}{\delta^2}t\right)^m.$$

4.2. **The function F_2.** Since the calculations are similar to those for F_1, we skip some of the details. Decompose

$$F_2(q)=F_{2,1}(q)+F_{2,2}(q),$$

with

$$F_{2,1}(q):=\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2}n_2q^{Q(\boldsymbol{n})},\qquad F_{2,2}(q):=-\frac{1}{2}\sum_{m\in\frac{1}{p}+\mathbb{Z}}|m|q^{m^2}.$$

We first study the asymptotic behavior of $F_{2,1}$. Arguing as for $F_{1,1}$, we have

$$F_2\left(e^{2\pi i\frac{h}{k}-t}\right)=\frac{1}{\sqrt{t}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n\in\frac{\delta}{kp}(\boldsymbol{\ell}+\boldsymbol{\alpha})+\mathbb{N}_0^2}\mathcal{G}_1\left(\frac{kp}{\delta}\sqrt{t}\boldsymbol{n}\right),$$

with $\mathcal{G}_1(\boldsymbol{x}):=x_2\mathcal{F}_1(\boldsymbol{x})$. The Euler-Maclaurin main term is

$$\frac{1}{t^{\frac{3}{2}}}\left(\frac{\delta}{kp}\right)^2\mathcal{I}_{\mathcal{G}_1}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{\boldsymbol{\ell}\pmod{\frac{kp}{\delta}}}e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}.$$

As in Subsection 4.1, one can show that this vanishes.

The second term in the Euler-Maclaurin summation formula is

$$-2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_2\ge1}\frac{B_{2n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(2n_2+1)!}\int_0^\infty\mathcal{G}_1^{(0,2n_2)}(x_1,0)dx_1\left(\frac{kp}{\delta}\right)^{2n_2-1}t^{n_2-1},\tag{4.4}$$

again pairing $\boldsymbol{\alpha}$ and $\boldsymbol{1}-\boldsymbol{\alpha}$ and using that $\mathcal{G}_1(x_1,0)=0$.

In the same way we obtain that the third term in the Euler-Maclaurin summation formula is

$$-2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_1\ge0}\frac{B_{2n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(2n_1+1)!}\int_0^\infty\mathcal{G}_1^{(2n_1,0)}(0,x_2)dx_2\left(\frac{kp}{\delta}\right)^{2n_1-1}t^{n_1-1}.\tag{4.5}$$

The final term in Euler-Maclaurin evaluates as

$$2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}$$
$$\times\sum_{\substack{n_1,n_2\ge0\\ n_1\not\equiv n_2\pmod 2}}\frac{B_{n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(n_1+1)!}\frac{B_{n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(n_2+1)!}\mathcal{G}_1^{(n_1,n_2)}(0,0)\left(\frac{kp}{\delta}\right)^{n_1+n_2}t^{\frac{n_1+n_2-1}{2}},$$

again pairing $\boldsymbol{\alpha}$ with $\boldsymbol{1}-\boldsymbol{\alpha}$.

We next determine those terms of $F_{2,1}$ that grow as $t\to0^+$. Inspecting the terms above we see that this comes from the $n_1=0$ term of (4.5) and is given by

$$-\frac{2\delta}{kpt}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}B_1\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\int_0^\infty\mathcal{G}_1(0,x_2)dx_2.\tag{4.6}$$

Using that $\mathcal{G}_1(0,x_2)=x_2e^{-x_2^2}=:\mathcal{G}_2(x_2)$, we obtain that (4.6) equals

$$-\frac{2\delta}{kpt}\mathcal{I}_{\mathcal{G}_2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}B_1\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}.$$

Turning to $F_{2,2}$, its Euler-Maclaurin main term is

$$-\frac{\delta}{kpt}\mathcal{I}_{\mathcal{G}_2}\sum_{r \pmod{\frac{kp}{\delta}}} e^{2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2}. \tag{4.7}$$

Arguing as before, the second term in the Euler-Maclaurin summation formula equals

$$\sum_{0\le r\le \frac{kp}{\delta}-1} e^{2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2}\sum_{m\ge 0}\frac{B_{2m+2}\left(\frac{\delta(\ell+\frac{1}{p})}{kp}\right)}{(2m+2)!}\mathcal{G}_2^{(2m+1)}(0)\left(\frac{kp}{\delta}\right)^{2m+1}t^m.$$

To see that all terms that grow as $t\to 0^+$ cancel, we need to prove that

$$\sum_{r \pmod{\frac{kp}{\delta}}} e^{2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2} = -2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} B_1\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)e^{2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}. \tag{4.8}$$

To show (4.8), we first assume that $\frac{p}{\delta}\not\in\{1,2\}$. Writing $\boldsymbol{\ell}=\boldsymbol{N}+k\boldsymbol{\nu}$, $0\le\boldsymbol{N}<k$, $0\le\boldsymbol{\nu}<\frac{p}{\delta}$ and $\boldsymbol{a}=p\boldsymbol{\alpha}$, we obtain that the sum on $\boldsymbol{\ell}$ equals

$$e^{2\pi i\frac{h}{p^2k}Q(\boldsymbol{a})}\sum_{0\le\boldsymbol{N}<k} e^{\frac{2\pi ih}{pk}\left(3\left(pN_1^2+2a_1N_1\right)+3(pN_1N_2+a_2N_1+a_1N_2)+pN_2^2+2a_2N_2\right)}$$
$$\times\sum_{0\le\boldsymbol{\nu}<\frac{p}{\delta}} B_1\left(\frac{\delta\left(N_1+k\nu_1+\frac{a_1}{p}\right)}{kp}\right)e^{2\pi i\frac{h/\delta}{p/\delta}((6a_1+3a_2)\nu_1+(2a_2+3a_1)\nu_2)}. \tag{4.9}$$

The sum on ν_2 vanishes unless $\frac{p}{\delta}|(2a_2+3a_1)$. It is not hard to see that (under the assumption that $\frac{p}{\delta}\not\in\{1,2\}$) this is not satisfied for elements in $p\mathscr{S}^*$.

We next assume that $\frac{p}{\delta}=1$. It is not hard to see that

$$e^{2\pi i\frac{h}{k}Q\left(k-\ell_1-1+1-\frac{1}{p},\ell_2+3\ell_1+1+\frac{2}{p}\right)} = e^{2\pi i\frac{h}{k}Q\left(\ell_1+\frac{1}{p},\ell_2+1-\frac{1}{p}\right)}. \tag{4.10}$$

This then implies that the contribution of the first and third element in $\mathscr{S}^*$ cancel due to a negative sign from the Bernoulli polynomial and we can shift the sum in ℓ_2 by integers. Thus the right-hand side of (4.8) becomes

$$-2\sum_{0\le\boldsymbol{\ell}<k} B_1\left(\frac{\ell_1}{k}\right)e^{2\pi i\frac{h}{k}Q\left(\ell_1,\ell_2+1-\frac{1}{p}\right)}. \tag{4.11}$$

Now one can show that

$$e^{2\pi i\frac{h}{k}Q\left(k-\ell_1,\ell_2+3\ell_1+1-\frac{1}{p}\right)} = e^{2\pi i\frac{h}{k}Q\left(\ell_1,\ell_2+1-\frac{1}{p}\right)}. \tag{4.12}$$

To finish the claim (4.8), we assume, without loss of generality, that k is odd. We split the sum in (4.11), substitute $(\ell_1,\ell_2)\mapsto(k-\ell_1,\ell_2+3\ell_1)$ in the second part and use (4.12) to obtain

$$-2\sum_{0\le\boldsymbol{\ell}<k} B_1\left(\frac{\ell_1}{k}\right)e^{2\pi i\frac{h}{k}Q\left(\ell_1,\ell_2+1-\frac{1}{p}\right)} = -2\left(\sum_{\substack{0\le\ell_1\le\frac{1}{2}(k-1)\\ \ell_2 \pmod{k}}}+\sum_{\substack{\frac{1}{2}(k+1)\le\ell_1<k\\ \ell_2 \pmod{k}}}\right)B_1\left(\frac{\ell_1}{k}\right)e^{2\pi i\frac{h}{k}Q\left(\ell_1,\ell_2+1-\frac{1}{p}\right)}$$

$$= -2 \sum_{\substack{0\leq \ell_1\leq \frac{1}{2}(k-1)\\ \ell_2 \pmod{k}}} B_1\left(\frac{\ell_1}{k}\right) e^{2\pi i\frac{h}{k}Q\left(\ell_1,\ell_2+1-\frac{1}{p}\right)} - 2 \sum_{\substack{0<\ell_1\leq \frac{1}{2}(k-1)\\ \ell_2 \pmod{k}}} B_1\left(1-\frac{\ell_1}{k}\right) e^{2\pi i\frac{h}{k}Q\left(\ell_1,\ell_2+1-\frac{1}{p}\right)}$$

$$= -2B_1(0) \sum_{\ell_2 \pmod{k}} e^{2\pi i\frac{h}{k}Q\left(0,\ell_2+1-\frac{1}{p}\right)} = \sum_{\ell_2 \pmod{k}} e^{2\pi i\frac{h}{k}\left(\ell_2+1-\frac{1}{p}\right)^2}.$$

The case $\frac{p}{\delta}=2$ is done similarly.

5. Companions in the lower half plane

In this section we investigate multivariable Eichler integrals.

5.1. **Multiple Eichler integrals.** Let $f_j \in S_{k_j}(\Gamma,\chi_j)$; if $k_j=\frac{1}{2}$ we also allow $f_j\in M_{\frac{1}{2}}(\Gamma,\chi_j)$. Define the *double Eichler integral*

$$I_{f_1,f_2}(\tau) := \int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{f_1(w_1)f_2(w_2)}{(-i(w_1+\tau))^{2-k_1}(-i(w_2+\tau))^{2-k_2}}dw_2dw_1,$$

and the *multiple error of modularity*

$$r_{f_1,f_2,\frac{d}{c}}(\tau) := \int_{\frac{d}{c}}^{i\infty}\int_{w_1}^{\frac{d}{c}} \frac{f_1(w_1)f_2(w_2)}{(-i(w_1+\tau))^{2-k_1}(-i(w_2+\tau))^{2-k_2}}dw_2dw_1.$$

Theorem 5.1. *We have, for $M=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma^*$,*

$$I_{f_1,f_2}(\tau) - \chi_1^{-1}(M^*)\chi_2^{-1}(M^*)(c\tau+d)^{k_1+k_2-4}I_{f_1,f_2}(M\tau) = r_{f_1,f_2,\frac{d}{c}}(\tau) + I_{f_1}(\tau)r_{f_2,\frac{d}{c}}(\tau). \tag{5.1}$$

Moreover $r_{f_1,f_2,\frac{d}{c}}\in\mathcal{O}(\mathbb{R}\backslash\{-\frac{d}{c}\})$. If $f_j\in S_{k_j}(\Gamma,\chi_j)$ (for $j=1,2$), then $r_{f_1,f_2,\frac{d}{c}}\in\mathcal{O}(\mathbb{R})$.

Proof of Theorem 5.1. For simplicity, we assume that $\frac{1}{2}\leq k_j\leq 2$ and that f_1,f_2 are cuspidal. The proof in the case that f_1 or f_2 are not cuspidal and of weight $\frac{1}{2}$ is basically the same; we then require the bound

$$f_j\left(iw_j+\frac{d}{c}\right) \ll 1+w_j^{-\frac{1}{2}}.$$

A direct calculation gives that, for $M\in\Gamma^*$,

$$I_{f_1,f_2}(M\tau) = \chi_1(M^*)\chi_2(M^*)(c\tau+d)^{4-k_1-k_2}\int_{-\overline{\tau}}^{\frac{d}{c}}\int_{w_1}^{\frac{d}{c}} \frac{f_1(w_1)f_2(w_2)}{(-i(w_1+\tau))^{2-k_1}(-i(w_2+\tau))^{2-k_2}}dw_2dw_1.$$

The transformation (5.1) now follows by splitting

$$\int_{-\overline{\tau}}^{\frac{d}{c}}\int_{w_1}^{\frac{d}{c}} = \int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} + \int_{\frac{d}{c}}^{i\infty}\int_{\frac{d}{c}}^{w_1} - \int_{-\overline{\tau}}^{i\infty}\int_{\frac{d}{c}}^{i\infty}.$$

Using Lemma 2.3, we are left to show that $r_{f_1,f_2,\frac{d}{c}}$ is real-analytic on $\mathbb{R}$ which follows once we prove that the following function is real-analytic

$$\int_0^\infty\int_0^{w_1} \frac{f_1\left(iw_1+\frac{d}{c}\right)f_2\left(iw_2+\frac{d}{c}\right)}{\left(w_1-i\left(\tau+\frac{d}{c}\right)\right)^{2-k_1}\left(w_2-i\left(\tau+\frac{d}{c}\right)\right)^{2-k_2}}dw_2dw_1. \tag{5.2}$$

We use that for $w_j \geq 1$

$$f_j\left(iw_j+\frac{d}{c}\right) \ll e^{-a_jw_j} \qquad a_j\in\mathbb{R}^+, \tag{5.3}$$

and for $0<w_j\leq 1$ (the implied constant and b_j may depend on c)

$$f_j\left(iw_j+\frac{d}{c}\right) \ll w_j^{-k_j}e^{-\frac{b_j}{w_j}} \qquad b_j\in\mathbb{R}^+. \tag{5.4}$$

To show real-analycity of (5.2) on $\mathbb{R}$, we split it into 3 pieces. Firstly, set

$$I_1 := \int_1^\infty\int_1^{w_1} \frac{f_1\left(iw_1+\frac{d}{c}\right)f_2\left(iw_2+\frac{d}{c}\right)}{\left(w_1-i\left(\tau+\frac{d}{c}\right)\right)^{2-k_1}\left(w_2-i\left(\tau+\frac{d}{c}\right)\right)^{2-k_2}}dw_2dw_1.$$

Using (5.3) and that $w_1\geq 1$ easily gives the locally uniform bound

$$I_1 \ll \int_1^\infty \frac{e^{-a_1w_1}}{w_1^{2-k_1}}dw_1\int_1^\infty \frac{e^{-a_2w_2}}{w_2^{2-k_2}}dw_2 \ll 1.$$

Next consider

$$I_2 := \int_0^1\int_0^{w_1} \frac{f_1\left(iw_1+\frac{d}{c}\right)f_2\left(iw_2+\frac{d}{c}\right)}{\left(w_1-i\left(\tau+\frac{d}{c}\right)\right)^{2-k_1}\left(w_2-i\left(\tau+\frac{d}{c}\right)\right)^{2-k_2}}dw_2dw_1.$$

Using (5.4) gives that

$$I_2 \ll \int_0^1 \frac{e^{-\frac{b_1}{w_1}}}{w_1^2}dw_1\int_0^1 \frac{e^{-\frac{b_2}{w_2}}}{w_2^2}dw_2 \ll 1.$$

Finally, we set

$$I_3 := \int_1^\infty \frac{f_1\left(iw_1+\frac{d}{c}\right)}{\left(w_1-i\left(\tau+\frac{d}{c}\right)\right)^{2-k_1}}dw_1\int_0^1 \frac{f_2\left(iw_2+\frac{d}{c}\right)}{\left(w_2-i\left(\tau+\frac{d}{c}\right)\right)^{2-k_2}}dw_2.$$

Combining the above bounds gives again $I_3\ll 1$.

□

5.2. **Special multiple Eichler integrals of weight one.** Define for $\boldsymbol{\alpha}\in\mathscr{S}^*$

$$\mathcal{E}_{1,\boldsymbol{\alpha}}(\tau) := -\frac{\sqrt{3}}{4}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{\theta_1(\boldsymbol{\alpha};\boldsymbol{w})+\theta_2(\boldsymbol{\alpha};\boldsymbol{w})}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1$$

with

$$\theta_1(\boldsymbol{\alpha};\boldsymbol{w}) := \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2}(2n_1+n_2)n_2e^{\frac{3\pi i}{2}(2n_1+n_2)^2w_1+\frac{\pi i n_2^2w_2}{2}},$$

$$\theta_2(\boldsymbol{\alpha};\boldsymbol{w}) := \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2}(3n_1+2n_2)n_1e^{\frac{\pi i}{2}(3n_1+2n_2)^2w_1+\frac{3\pi i n_1^2w_2}{2}}.$$

Moreover set

$$\mathcal{E}_1(\tau) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\varepsilon(\boldsymbol{\alpha})\mathcal{E}_{1,\boldsymbol{\alpha}}(p\tau), \tag{5.5}$$

$$\Gamma_p := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(12p) : b \equiv 0 \pmod{4p}, d \equiv \pm 1 \pmod{2p} \right\}.$$

Remark. Note that $\Gamma_p^* = \Gamma_p$.

Remark. One can show that

$$\mathcal{E}_1(\tau) = -\frac{\sqrt{3}}{4p} \sum_{\delta \in \{0,1\}} I_{\Theta_1(2p, 1+p\delta, 2p;\, \cdot\,), \Theta_1(6p, 3+3p\delta, 6p;\, \cdot\,)}(\tau).$$

However, as this representation is not required for the remainder of the paper, we do not provide a proof of this identity.

Proposition 5.2. *We have, for* $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_p$,

$$\mathcal{E}_1(\tau) - \left(\frac{-3}{d}\right)(c\tau + d)^{-1} \mathcal{E}_1(M\tau) = \sum_{j=1}^{12} \left(r_{f_j, g_j, \frac{d}{c}}(\tau) + I_{f_j}(\tau) r_{g_j, \frac{d}{c}}(\tau) \right),$$

where f_j, g_j *are cusp forms of weight* $\frac{3}{2}$ *(with some multiplier).*

Proof. To use Theorem 5.1, we write θ_j in terms of Shimura's theta functions (2.9). For θ_1, we set $\nu_1 := 2n_1 + n_2$, $\nu_2 := n_2$. Then $\nu_1 \in 2\alpha_1 + \alpha_2 + \mathbb{Z}$, $\nu_2 \in \alpha_2 + \mathbb{Z}$, and $\nu_1 - \nu_2 \in 2\alpha_1 + 2\mathbb{Z}$ and we obtain

$$\begin{aligned} \theta_1(\boldsymbol{\alpha}; \boldsymbol{w}) &= \sum_{\substack{\boldsymbol{\nu} \in (2\alpha_1 + \alpha_2, \alpha_2) + \mathbb{Z}^2 \\ \nu_1 - \nu_2 \in 2\alpha_1 + 2\mathbb{Z}}} \nu_1 \nu_2 e^{\frac{3\pi i \nu_1^2 w_1}{2} + \frac{\pi i \nu_2^2 w_2}{2}} \\ &= \sum_{\varrho \in \{0,1\}} \sum_{\nu_1 \in 2\alpha_1 + \alpha_2 + \varrho + 2\mathbb{Z}} \nu_1 e^{\frac{3\pi i \nu_1^2 w_1}{2}} \sum_{\nu_2 \in \alpha_2 + \varrho + 2\mathbb{Z}} \nu_2 e^{\frac{\pi i \nu_2^2 w_2}{2}}. \end{aligned}$$

Summing then easily gives

$$\begin{aligned} \sum_{\boldsymbol{\alpha} \in \mathscr{S}^*} \varepsilon(\boldsymbol{\alpha}) \theta_1(\boldsymbol{\alpha}; \boldsymbol{w}) &= \frac{1}{p^2} \sum_{\boldsymbol{A} \in \mathcal{A}} \varepsilon_1(\boldsymbol{A}) \sum_{\nu_1 \equiv A_1 \pmod{2p}} \nu_1 e^{\frac{3\pi i \nu_1^2 w_1}{2p^2}} \sum_{\nu_2 \equiv A_2 \pmod{2p}} \nu_2 e^{\frac{\pi i \nu_2^2 w_2}{2p^2}} \\ &= \frac{1}{p^2} \sum_{\boldsymbol{A} \in \mathcal{A}} \varepsilon_1(\boldsymbol{A}) \Theta_1\left(2p, A_1, 2p; \frac{3w_1}{p}\right) \Theta_1\left(2p, A_2, 2p; \frac{w_2}{p}\right) \end{aligned}$$

with

$$\mathcal{A} := \{(0,2), (p, p+2), (p-1, p-1), (-1,-1), (p+1, p-1), (1,-1)\}, \varepsilon_1(\boldsymbol{A}) := \varepsilon\left(\frac{A_1 - A_2}{2p}, \frac{A_2}{p}\right).$$

For θ_2, we proceed similarly. Set $\nu_1 = 3n_1 + 2n_2$, $\nu_2 = n_1$. Then $\nu_1 \in 3\alpha_1 + 2\alpha_2 + \mathbb{Z}$, $\nu_2 \in \alpha_1 + \mathbb{Z}$, and $\nu_1 - 3\nu_2 \in 2\alpha_2 + 2\mathbb{Z}$ and we obtain

$$\theta_2(\boldsymbol{\alpha}; \boldsymbol{w}) = \sum_{\substack{\boldsymbol{\nu} \in (3\alpha_1 + 2\alpha_2, \alpha_1) + \mathbb{Z}^2 \\ \nu_1 - 3\nu_2 \in 2\alpha_2 + 2\mathbb{Z}}} \nu_1 \nu_2 e^{\frac{\pi i \nu_1^2 w_1}{2} + \frac{3\pi i \nu_2^2 w_2}{2}} = \sum_{\varrho \in \{0,1\}} \sum_{\nu_1 \in 3\alpha_1 + 2\alpha_2 + \varrho + 2\mathbb{Z}} \nu_1 e^{\frac{\pi i \nu_1^2 w_1}{2}} \sum_{\nu_2 \in \alpha_1 + \varrho + 2\mathbb{Z}} \nu_2 e^{\frac{3\pi i \nu_2^2 w_2}{2}}.$$

Summing gives

$$\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*} \varepsilon(\boldsymbol{\alpha})\,\theta_2(\boldsymbol{\alpha};\boldsymbol{w}) = \frac{1}{p^2}\sum_{\boldsymbol{B}\in\mathcal{B}} \varepsilon_2(\boldsymbol{B}) \sum_{\nu_1\equiv B_1 \pmod{2p}} \nu_1 e^{\frac{\pi i\nu_1^2 w_1}{2p^2}} \sum_{\nu_2\equiv B_2 \pmod{2p}} \nu_2 e^{\frac{3\pi i\nu_2^2 w_2}{2p^2}}$$
$$= \frac{1}{p^2}\sum_{\boldsymbol{B}\in\mathcal{B}} \varepsilon_2(\boldsymbol{B})\,\Theta_1\left(2p, B_1, 2p; \frac{w_1}{p}\right)\Theta_1\left(2p, B_2, 2p; \frac{3w_2}{p}\right)$$

with

$$\mathcal{B} := \{(p+1,p-1),(1,-1),(p+2,p),(2,0),(1,1),(p+1,p+1)\},\ \ \varepsilon_2(\boldsymbol{B}) := \varepsilon\left(\frac{B_2-3B_1}{2p}, \frac{B_1}{p}\right).$$

Combining the above yields that

$$\mathcal{E}_1(\tau) = -\frac{\sqrt{3}}{4p}\sum_{\boldsymbol{A}\in\mathcal{A}} \varepsilon_1(\boldsymbol{A}) \int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{\Theta_1(2p,A_1,2p;3w_1)\Theta_1(2p,A_2,2p;w_2)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1$$
$$- \frac{\sqrt{3}}{4p}\sum_{\boldsymbol{B}\in\mathcal{B}} \varepsilon_2(\boldsymbol{B}) \int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{\Theta_1(2p,B_1,2p;w_1)\,\Theta_1(2p,B_2,2p;3w_2)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1.$$

For $M\in\Gamma_p$, we have, using (2.9) and (2.10),

$$\Theta_1(2p,A,2p;\ell M\tau) = \pm\left(\frac{\ell pc}{d}\right)\varepsilon_d^{-1}(c\tau+d)^{\frac{3}{2}}\Theta_1(2p,A,2p;\ell\tau).$$

Theorem 5.1 then finishes the claim using that $\varepsilon_d^2 = (\frac{-1}{d})$.

□

5.3. **Special multiple Eichler integrals of weight two.** Define for $\boldsymbol{\alpha}\in\mathscr{S}^*$

$$\mathcal{E}_{2,\boldsymbol{\alpha}}(\tau) := \frac{\sqrt{3}}{8\pi}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{2\theta_3(\boldsymbol{\alpha};\boldsymbol{w})-\theta_4(\boldsymbol{\alpha};\boldsymbol{w})}{\sqrt{-i(w_1+\tau)}(-i(w_2+\tau))^{\frac{3}{2}}} dw_2 dw_1$$
$$+ \frac{\sqrt{3}}{8\pi}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{\theta_5(\boldsymbol{\alpha};\boldsymbol{w})}{(-i(w_1+\tau))^{\frac{3}{2}}\sqrt{-i(w_2+\tau)}} dw_2 dw_1$$

with

$$\theta_3(\boldsymbol{\alpha};\boldsymbol{w}) := \sum_{n\in\alpha+\mathbb{Z}^2} (2n_1+n_2) e^{\frac{3\pi i}{2}(2n_1+n_2)^2 w_1 + \frac{\pi i n_2^2 w_2}{2}},$$
$$\theta_4(\boldsymbol{\alpha};\boldsymbol{w}) := \sum_{n\in\alpha+\mathbb{Z}^2} (3n_1+2n_2) e^{\frac{\pi i}{2}(3n_1+2n_2)^2 w_1 + \frac{3\pi i n_1^2 w_2}{2}},$$
$$\theta_5(\boldsymbol{\alpha};\boldsymbol{w}) := \sum_{n\in\alpha+\mathbb{Z}^2} n_1 e^{\frac{\pi i}{2}(3n_1+2n_2)^2 w_1 + \frac{3\pi i n_1^2 w_2}{2}}.$$

We then set

$$\mathcal{E}_2(\tau) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}^*} \mathcal{E}_{2,\boldsymbol{\alpha}}(p\tau).$$

Remark. Similarly as for $\mathcal{E}_1$, one can simplify $\mathcal{E}_2$ as

$$\mathcal{E}_2(\tau)=-\frac{\sqrt{3}}{8\pi}\sum_{\boldsymbol{B}\in\mathcal{B}} I_{\Theta_1(2p,B_1,2p;\,\cdot\,),\Theta_0(6p,3B_2,6p;\,\cdot\,)}(\tau).$$

This function again transforms as a depth two quantum modular.

Proposition 5.3. *We have, for $M\in\Gamma_p$,*

$$\mathcal{E}_2(\tau)-\left(\frac{3}{d}\right)(c\tau+d)^{-2}\mathcal{E}_2(M\tau)=\sum_{j=1}^{18}\left(r_{f_j,g_j,\frac{d}{c}}(\tau)+I_{f_j}(\tau)r_{g_j,\frac{d}{c}}(\tau)\right),$$

where f_j and g_j are holomorphic modular forms of weight $\frac{1}{2}$ or cusp forms of weight $\frac{3}{2}$.

Proof. As in the proof of Proposition 5.2, we obtain

$$\sum_{\substack{\boldsymbol{\alpha}\in\mathscr{S}^*\\ \boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2}}(2n_1+n_2)e^{\frac{3\pi i}{2}(2n_1+n_2)^2w_1+\frac{\pi i n_2^2 w_2}{2}}=\frac{1}{p}\sum_{\boldsymbol{A}\in\mathcal{A}}\Theta_1\left(2p,A_1,2p;\frac{3w_1}{p}\right)\Theta_0\left(2p,A_2,2p;\frac{w_2}{p}\right),$$

$$\sum_{\substack{\boldsymbol{\alpha}\in\mathscr{S}^*\\ \boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2}}(3n_1+n_2)e^{\frac{\pi i}{2}(3n_1+2n_2)^2w_1+\frac{3\pi i n_1^2 w_2}{2}}=\frac{1}{p}\sum_{\boldsymbol{B}\in\mathcal{B}}\Theta_1\left(2p,B_1,2p;\frac{w_1}{p}\right)\Theta_0\left(2p,B_2,2p;\frac{3w_2}{p}\right),$$

$$\sum_{\substack{\boldsymbol{\alpha}\in\mathscr{S}^*\\ \boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2}}n_1e^{\frac{\pi i}{2}(3n_1+2n_2)^2w_1+\frac{3\pi i}{2}n_1^2 w_2}=\frac{1}{p}\sum_{\boldsymbol{B}\in\mathcal{B}}\Theta_0\left(2p,B_1,2p;\frac{w_1}{p}\right)\Theta_1\left(2p,B_2,2p;\frac{3w_2}{p}\right).$$

The claim now again follows from Theorem 5.1 using (2.9) and (2.10). □

5.4. **More on double Eichler integrals.** We have an obvious map $S_k(\Gamma,\chi)\to\mathcal{Q}_{2-k}(\Gamma^*,\chi^*)$, where $\chi^*(M):=\chi(M^*)$, which assigns to $f\in S_k(\Gamma,\chi)$ its Eichler integral I_f, defined in (2.11). Clearly, we also have a map from $S_k(\Gamma,\chi)\otimes S_k(\Gamma,\chi)$, actually from its symmetric square, to $(\mathcal{Q}_{2-k}(\Gamma^*,\chi^*))^2$, by mapping $f_1\otimes f_2$ to $I_{f_1}I_{f_2}$. The double Eichler integral construction I_{f_1,f_2} gives rise to a map

$$\Lambda^2\left(S_k(\Gamma,\chi)\right)\longrightarrow\mathcal{Q}^2_{4-2k}\left(\Gamma^*,{\chi^*}^2\right)\Big/\left(\mathcal{Q}_{2-k}\left(\Gamma^*,\chi^*\right)\right)^2,$$

where $\Lambda^2(S_{2-k}(\Gamma,\chi))$ is the second exterior power of $S_{2-k}(\Gamma,\chi)$. To see this, it suffices to observe the simplest *shuffle* relation for iterated integrals

$$I_{f_1,f_2}+I_{f_2,f_1}=I_{f_1}I_{f_2}.$$

Remark. It is now straightforward to consider even more general iterated Eichler integrals ($r\in\mathbb{N}$):

$$I_{f_1,\dots,f_r}:=\int_{-\overline{\tau}}^{i\infty}\int_{w_{r-1}}^{i\infty}\cdots\int_{w_2}^{i\infty}\prod_{j=1}^{r}\frac{f_j(w_j)}{(-i(w_j+\tau))^{2-k_j}}dw_1\cdots dw_r,$$

where the f_j are cusp forms of weight $k_j\geq\frac{1}{2}$ (or possibly holomorphic forms for weight $\frac{1}{2}$). We do not pursue their (mock/quantum) modular properties here – we will address this in our future work [6] (see also Section 9 for related comments).

6. Indefinite theta functions

We next realize the double Eichler integrals studied in Section 5 as pieces of indefinite theta functions.

6.1. The function $\mathcal{E}_1$ as an indefinite theta function.

The next lemma rewrites $\mathbb{E}_1(\tau) := \mathcal{E}_1(\frac{\tau}{p})$ in a shape to which one can apply the Euler-Maclaurin summation formula.

Lemma 6.1. *We have*

$$\mathbb{E}_1(\tau) = \frac{1}{2} \sum_{\boldsymbol{\alpha}\in\mathscr{S}^*} \varepsilon(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2} M_2\left(\sqrt{3}; \sqrt{v}\left(2\sqrt{3}n_1 + \sqrt{3}n_2, n_2\right)\right) q^{-Q(\boldsymbol{n})}.$$

Proof. The claim follows, once we prove that

$$\begin{aligned}
&M_2\left(\sqrt{3}; \sqrt{3v}(2n_1+n_2), \sqrt{v}n_2\right)\\
&= -\frac{\sqrt{3}}{2}(2n_1+n_2)n_2 q^{Q(\boldsymbol{n})} \int_{-\overline{\tau}}^{i\infty} \frac{e^{\frac{3\pi i}{2}(2n_1+n_2)^2 w_1}}{\sqrt{-i(w_1+\tau)}} \int_{w_1}^{i\infty} \frac{e^{\frac{\pi i n_2^2 w_2}{2}}}{\sqrt{-i(w_2+\tau)}} dw_2 dw_1\\
&\quad - \frac{\sqrt{3}}{2}(3n_1+2n_2)n_1 q^{Q(\boldsymbol{n})} \int_{-\overline{\tau}}^{i\infty} \frac{e^{\frac{\pi i}{2}(3n_1+2n_2)^2 w_1}}{\sqrt{-i(w_1+\tau)}} \int_{w_1}^{i\infty} \frac{e^{\frac{3\pi i n_1^2 w_2}{2}}}{\sqrt{-i(w_2+\tau)}} dw_2 dw_1.
\end{aligned} \tag{6.1}$$

For simplicity we only show (6.1) for $n_1 \neq 0$. Since, by (2.7),

$$\lim_{\lambda\to\infty} M_2(\kappa; \lambda u_1, \lambda u_2) = 0,$$

we obtain, using (2.5) and (2.6),

$$\begin{aligned}
&M_2(\kappa; u_1, u_2) = -\int_1^\infty \frac{\partial}{\partial w_1} M_2(\kappa; w_1u_1, w_1u_2) dw_1\\
&= -\int_1^\infty \left(u_1 M_2^{(1,0)}(\kappa; w_1u_1, w_1u_2) + u_2 M_2^{(0,1)}(\kappa; w_1u_1, w_1u_2)\right) dw_1\\
&= -2\int_1^\infty \left(u_1 e^{-\pi u_1^2 w_1^2} M(u_2 w_1) + \frac{u_2+\kappa u_1}{\sqrt{1+\kappa^2}} e^{-\frac{\pi(u_2+\kappa u_1)^2 w_1^2}{1+\kappa^2}} M\left(w_1 \frac{u_1-\kappa u_2}{\sqrt{1+\kappa^2}}\right)\right) dw_1\\
&= -\int_1^\infty \left(u_1 e^{-\pi u_1^2 w_1} M\left(u_2\sqrt{w_1}\right) + \frac{u_2+\kappa u_1}{\sqrt{1+\kappa^2}} e^{-\frac{\pi(u_2+\kappa u_1)^2 w_1}{1+\kappa^2}} M\left(\sqrt{w_1}\frac{u_1-\kappa u_2}{\sqrt{1+\kappa^2}}\right)\right) \frac{dw_1}{\sqrt{w_1}} \qquad (6.2)\\
&= \frac{i}{\sqrt{2}} \int_{-\overline{\tau}}^{i\infty} \left(\frac{u_1}{\sqrt{v}} e^{\frac{\pi i u_1^2 w_1}{2v}} q^{\frac{u_1^2}{4v}} M\left(\sqrt{\frac{-i(w_1+\tau)}{2v}} u_2\right)\right.\\
&\qquad \left. + \frac{u_2+\kappa u_1}{\sqrt{(1+\kappa^2)v}} e^{\frac{\pi i(u_2+\kappa u_1)^2 w_1}{2(1+\kappa^2)v}} q^{\frac{(u_2+\kappa u_1)^2}{4(1+\kappa^2)v}} M\left(\sqrt{\frac{-i(w_1+\tau)}{2}} \frac{u_1-\kappa u_2}{\sqrt{(1+\kappa^2)v}}\right)\right) \frac{dw_1}{\sqrt{-i(w_1+\tau)}}.
\end{aligned}$$

Now write for $N \in \mathbb{R}^+$

$$M\left(\sqrt{\frac{-i(w_1+\tau)}{2}} N\right) = \frac{iN}{\sqrt{2}} q^{\frac{N^2}{4}} \int_{w_1}^{i\infty} e^{\frac{\pi i N^2 w_2}{2}} \frac{dw_2}{\sqrt{-i(w_2+\tau)}}.$$

Plugging this into (6.2) easily yields that

$$M_2(\kappa;u_1,u_2)=-\frac{u_1}{2\sqrt{v}}\frac{u_2}{\sqrt{v}}q^{\frac{u_1^2}{4v}+\frac{u_2^2}{4v}}\int_{-\overline{\tau}}^{i\infty}\frac{e^{\frac{\pi i u_1^2 w_1}{2v}}}{\sqrt{-i(w_1+\tau)}}\int_{w_1}^{i\infty}\frac{e^{\frac{\pi i u_2^2 w_2}{2v}}}{\sqrt{-i(w_2+\tau)}}dw_2dw_1$$
$$-\frac{u_2+\kappa u_1}{2\sqrt{(1+\kappa^2)v}}\frac{u_1-\kappa u_2}{\sqrt{(1+\kappa^2)v}}q^{\frac{(u_2+\kappa u_1)^2}{4(1+\kappa^2)v}+\frac{(u_1-\kappa u_2)^2}{4(1+\kappa^2)v}}\int_{-\overline{\tau}}^{i\infty}\frac{e^{\frac{\pi i(u_2+\kappa u_1)^2 w_1}{2(1+\kappa^2)v}}}{\sqrt{-i(w_1+\tau)}}\int_{w_1}^{i\infty}\frac{e^{\frac{\pi i(u_1-\kappa u_2)^2 w_2}{2(1+\kappa^2)v}}}{\sqrt{-i(w_2+\tau)}}dw_2dw_1.$$

From this it is not hard to conclude (6.1). ☐

6.2. **The function $\mathcal{E}_2$ as an indefinite theta function.** We next write $\mathbb{E}_2(\tau):=\mathcal{E}_2(\frac{\tau}{p})$ as a piece of a derivative of an indefinite theta function, having an extra Jacobi variable.

Lemma 6.2. *We have*

$$\mathbb{E}_2(\tau)=\frac{1}{4\pi i}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\times\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2}\left[\frac{\partial}{\partial z}\left(M_2\left(\sqrt{3};\sqrt{3v}(2n_1+n_2),\sqrt{v}\left(n_2-\frac{2\mathrm{Im}(z)}{v}\right)\right)e^{2\pi i n_2 z}\right)\right]_{z=0}q^{-Q(\boldsymbol{n})}.$$

Proof. We first compute

$$\frac{1}{2\pi i}\left[\frac{\partial}{\partial z}\left(M_2\left(\sqrt{3};\sqrt{3v}\left(2n_1+n_2\right),\sqrt{v}\left(n_2-\frac{2\mathrm{Im}(z)}{v}\right)\right)e^{2\pi i n_2 z}\right)\right]_{z=0}$$
$$=n_2M_2\left(\sqrt{3};\sqrt{3v}\left(2n_1+n_2\right),\sqrt{v}n_2\right)+\frac{1}{2\pi\sqrt{v}}e^{-\pi(3n_1+2n_2)^2v}M\left(\sqrt{3v}n_1\right).\tag{6.3}$$

We show below that

$$n_2M_2\left(\sqrt{3};\sqrt{3v}(2n_1+n_2),\sqrt{v}n_2\right)=-\frac{\sqrt{3}}{2\pi}(2n_1+n_2)\int_{2v}^{\infty}\frac{e^{-\frac{3\pi}{2}(2n_1+n_2)^2w_1}}{\sqrt{w_1}}\int_{w_1}^{\infty}\frac{e^{-\frac{\pi n_2^2w_2}{2}}}{w_2^{\frac{3}{2}}}dw_2dw_1$$
$$+\frac{\sqrt{3}}{4\pi}(3n_1+2n_2)\int_{2v}^{\infty}\frac{e^{-\frac{\pi}{2}(3n_1+2n_2)^2w_1}}{\sqrt{w_1}}\int_{w_1}^{\infty}\frac{e^{-\frac{3\pi n_1^2w_2}{2}}}{w_2^{\frac{3}{2}}}dw_2dw_1-\frac{1}{2\pi\sqrt{v}}e^{-\pi(3n_1+2n_2)^2v}M\left(\sqrt{3v}n_1\right)$$
$$-\frac{\sqrt{3}n_1}{4\pi}\int_{2v}^{\infty}\frac{e^{-\frac{\pi}{2}(3n_1+2n_2)^2w_1}}{w_1^{\frac{3}{2}}}\int_{w_1}^{\infty}\frac{e^{-\frac{3\pi n_1^2w_2}{2}}}{w_2^{\frac{1}{2}}}dw_2dw_1.\tag{6.4}$$

Since the third term cancels the second term on the right-hand side of (6.3) this then implies the claim, using that

$$\int_{2v}^{\infty}\frac{e^{-2\pi M^2w_1}}{w_1^{\frac{1}{2}}}\int_{w_1}^{\infty}\frac{e^{-2\pi N^2w_2}}{w_2^{\frac{3}{2}}}dw_2dw_1=-q^{M^2+N^2}\int_{-\overline{\tau}}^{i\infty}\frac{e^{2\pi iM^2w_1}}{(-i(w_1+\tau))^{\frac{1}{2}}}\int_{w_1}^{i\infty}\frac{e^{2\pi iN^2w_2}}{(-i(w_2+\tau))^{\frac{3}{2}}}dw_2dw_1,$$
$$\int_{2v}^{\infty}\frac{e^{-2\pi M^2w_1}}{w_1^{\frac{3}{2}}}\int_{w_1}^{\infty}\frac{e^{-2\pi N^2w_2}}{w_2^{\frac{1}{2}}}dw_2dw_1=-q^{N^2+M^2}\int_{-\overline{\tau}}^{i\infty}\frac{e^{2\pi iM^2w_1}}{(-i(w_1+\tau))^{\frac{3}{2}}}\int_{w_1}^{i\infty}\frac{e^{2\pi iN^2w_2}}{(-i(w_2+\tau))^{\frac{1}{2}}}dw_2dw_1.$$

To prove (6.4), we again, for simplicity, restrict to $n_1 \neq 0$.

Plugging in (6.2) yields

$$M_2\left(\sqrt{3};\sqrt{3v}(2n_1+n_2),\sqrt{v}n_2\right) = -\int_1^\infty \left(\sqrt{3v}(2n_1+n_2)e^{-3\pi v(2n_1+n_2)^2w_1}M\left(\sqrt{vw_1}n_2\right)\right.$$
$$\left.+\sqrt{v}(3n_1+2n_2)e^{-\pi v(3n_1+2n_2)^2w_1}M\left(\sqrt{3vw_1}n_1\right)\right)\frac{dw_1}{\sqrt{w_1}}. \tag{6.5}$$

Using (2.3) and (2.2) the first term in (6.5) multiplied by n_2 gives

$$-\frac{\sqrt{3v}}{2\sqrt{\pi}}|n_2|(2n_1+n_2)\int_1^\infty e^{-3\pi v(2n_1+n_2)^2w_1}\Gamma\left(-\frac{1}{2},\pi vn_2^2w_1\right)\frac{dw_1}{\sqrt{w_1}}$$
$$+\frac{\sqrt{3}}{\pi}(2n_1+n_2)\int_1^\infty e^{-4\pi vQ(n_1,n_2)w_1}\frac{dw_1}{w_1}. \tag{6.6}$$

For the second term in (6.5), we split

$$n_2 = \frac{1}{2}(3n_1+2n_2)-\frac{3}{2}n_1. \tag{6.7}$$

The n_1-term contributes to n_2M_2 as

$$\frac{3}{4}\sqrt{\frac{v}{\pi}}|n_1|(3n_1+2n_2)\int_1^\infty e^{-\pi v(3n_1+2n_2)^2w_1}\Gamma\left(-\frac{1}{2},3\pi vn_1^2w_1\right)\frac{dw_1}{\sqrt{w_1}}$$
$$-\frac{\sqrt{3}}{2\pi}(3n_1+2n_2)\int_1^\infty e^{-4\pi vQ(n_1,n_2)w_1}\frac{dw_1}{w_1}. \tag{6.8}$$

We next use that for $N\in\mathbb{N}_0$, $M\in\mathbb{N}$

$$\int_1^\infty e^{-4\pi N^2vw_1}\Gamma\left(-\frac{1}{2},4\pi vM^2w_1\right)\frac{dw_1}{\sqrt{w_1}} = \frac{1}{2\sqrt{\pi v}|M|}\int_{2v}^\infty \frac{e^{-2\pi N^2w_1}}{\sqrt{w_1}}\int_{w_1}^\infty \frac{e^{-2\pi M^2w_2}}{w_2^{\frac{3}{2}}}dw_2dw_1.$$

We use this to rewrite the first terms in (6.6) and (6.8). The first term in (6.6) is the first term on the right-hand side of (6.4). Similarly, since $n_1\neq 0$, the first term in (6.8) equals the second term in (6.4). Now we combine the second terms in (6.6) and (6.8), to get

$$\frac{\sqrt{3}n_1}{2\pi}\int_1^\infty e^{-4\pi vQ(\boldsymbol{n})w_1}\frac{dw_1}{w_1}. \tag{6.9}$$

Next we compute the contribution from the first term in (6.7),

$$-\frac{\sqrt{v}}{2}\left(3n_1+2n_2\right)^2\int_1^\infty e^{-\pi v(3n_1+2n_2)^2w_1}M\left(\sqrt{3vw_1}n_1\right)\frac{dw_1}{\sqrt{w_1}}$$
$$=\frac{1}{2\pi\sqrt{v}}\int_1^\infty \frac{\partial}{\partial w_1}\left(e^{-\pi v(3n_1+2n_2)^2w_1}\right)\frac{M\left(\sqrt{3vw_1}n_1\right)}{\sqrt{w_1}}dw_1.$$

Using integration by parts, this becomes

$$-\frac{1}{2\pi\sqrt{v}}e^{-\pi v(3n_1+2n_2)^2}M\left(\sqrt{3v}n_1\right)-\frac{\sqrt{3}n_1}{2\pi}\int_1^\infty e^{-4\pi vQ(n_1,n_2)w_1}\frac{dw_1}{w_1}$$

$$+\frac{1}{4\pi\sqrt{v}}\int_1^\infty e^{-\pi v(3n_1+2n_2)^2 w_1}\frac{M\left(\sqrt{3vw_1}n_1\right)}{w_1^{\frac{3}{2}}}dw_1. \quad (6.10)$$

The second term now cancels (6.9) and the first term equals the third term in (6.4).

To rewrite the final term in (6.10), we use that for $M, N \in \mathbb{Z}$ with $N \neq 0$

$$\int_1^\infty e^{-4\pi v M^2 w_1}\frac{M\left(2\sqrt{vw_1}N\right)}{w_1^{\frac{3}{2}}}dw_1 = -2\sqrt{v}N\int_{2v}^\infty \frac{e^{-2\pi M^2 w_1}}{w_1^{\frac{3}{2}}}\int_{w_1}^\infty \frac{e^{-2\pi N^2 w_2}}{w_2^{\frac{1}{2}}}dw_2 dw_1.$$

Thus the last term in (6.10) gives the final term in (6.4).

□

7. Asymptotic behavior of multiple Eichler integrals and proof of Theorem 1.1

In this section, we asymptotically relate F_j and $\mathbb{E}_j$.

7.1. Asymptotic behavior of $\mathbb{E}_1$.

Write

$$F_1\left(e^{2\pi i\frac{h}{k}-t}\right) \sim \sum_{m\geq 0} a_{h,k}(m)t^m \quad \left(t\to 0^+\right).$$

The goal of this subsection is to prove the following.

Theorem 7.1. *We have, for $h, k \in \mathbb{Z}$ with $k > 0$ and $\gcd(h,k) = 1$,*

$$\mathbb{E}_1\left(\frac{h}{k}+\frac{it}{2\pi}\right) \sim \sum_{m\geq 0} a_{-h,k}(m)(-t)^m \quad \left(t\to 0^+\right).$$

Proof. We use Lemma 6.1 and the fact that M_2 is an even function, to rewrite

$$\mathbb{E}_1(\tau) = \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\varepsilon(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} M_2\left(\sqrt{3};\sqrt{v}\left(2\sqrt{3}n_1+\sqrt{3}n_2, n_2\right)\right)q^{-Q(\boldsymbol{n})}$$
$$+\frac{1}{2}\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\varepsilon}(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} M_2\left(\sqrt{3};\sqrt{v}\left(-2\sqrt{3}n_1+\sqrt{3}n_2, n_2\right)\right)q^{-Q(-n_1,n_2)},$$

where

$$\widetilde{\mathscr{S}} := \left\{(1-\alpha_1,\alpha_2) : \boldsymbol{\alpha}\in\mathscr{S}\right\}, \quad \widetilde{\varepsilon}(\boldsymbol{\alpha}) := \varepsilon(1-\alpha_1,\alpha_2).$$

To apply the Euler-Maclaurin summation formula directly, we turn every sgn into sgn*, where $\operatorname{sgn}^*(x) := \operatorname{sgn}(x)$ for $x\neq 0$ and $\operatorname{sgn}^*(0) := 1$. To be more precise, we set

$$M_2^*\left(\sqrt{3};\sqrt{3}(2x_1+x_2),x_2\right) := \operatorname{sgn}^*(x_1)\operatorname{sgn}^*(x_2) + E_2\left(\sqrt{3};\sqrt{3}(2x_1+x_2),x_2\right)$$
$$-\operatorname{sgn}^*(x_2)E\left(\sqrt{3}(2x_1+x_2)\right) - \operatorname{sgn}^*(x_1)E(3x_1+2x_2). \quad (7.1)$$

Using that

$$M_2\left(\sqrt{3};\sqrt{3}x_2,x_2\right) - \lim_{x_1\to 0^+} M_2^*\left(\sqrt{3};\sqrt{3}\left(\pm 2x_1+x_2\right),x_2\right) = \pm M(2x_2), \quad (7.2)$$

we then split

$$\mathbb{E}_1(\tau) = \mathcal{E}_1^*(\tau) + H_1(\tau)$$

with

$$\mathcal{E}_1^*(\tau) := \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\varepsilon(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} M_2^*\left(\sqrt{3};\sqrt{v}\left(2\sqrt{3}n_1+\sqrt{3}n_2,n_2\right)\right)q^{-Q(\boldsymbol{n})}$$
$$+\frac{1}{2}\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\varepsilon}(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} M_2^*\left(\sqrt{3};\sqrt{v}\left(-2\sqrt{3}n_1+\sqrt{3}n_2,n_2\right)\right)q^{-Q(-n_1,n_2)},$$
$$H_1(\tau) := -\frac{1}{2}\sum_{m\in\frac{1}{p}+\mathbb{N}_0} M\left(2\sqrt{v}m\right)q^{-m^2}+\frac{1}{2}\sum_{m\in 1-\frac{1}{p}+\mathbb{N}_0} M\left(2\sqrt{v}m\right)q^{-m^2}.$$

Note that for $n_1=0$ we take the limit $n_1\to 0$ in the M_2^*-functions.

We proceed as in Subsection 4.1 to determine the asymptotic behavior of $\mathcal{E}_1^*$ and H_1. Firstly we rewrite

$$\mathcal{E}_1^*\left(\frac{h}{k}+\frac{it}{2\pi}\right)=\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n\in\frac{\delta}{kp}(\boldsymbol{\ell}+\boldsymbol{\alpha})+\mathbb{N}_0^2}\mathcal{F}_3\left(\frac{kp}{\delta}\sqrt{t}\boldsymbol{n}\right)$$
$$+\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\varepsilon}(\alpha)\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(-(\ell_1+\alpha_1),\ell_2+\alpha_2)}\sum_{n\in\frac{\delta}{kp}(\boldsymbol{\ell}+\boldsymbol{\alpha})+\mathbb{N}_0^2}\widetilde{\mathcal{F}}_3\left(\frac{kp}{\delta}\sqrt{t}\boldsymbol{n}\right),$$

where

$$\mathcal{F}_3(\boldsymbol{x}) := \frac{1}{2}M_2^*\left(\sqrt{3};\frac{1}{\sqrt{2\pi}}\left(\sqrt{3}\left(2x_1+x_2\right),x_2\right)\right)e^{Q(\boldsymbol{x})},\qquad \widetilde{\mathcal{F}}_3(\boldsymbol{x}) := \mathcal{F}_3(-x_1,x_2).$$

The contribution from the $\mathcal{F}_3$ term to the first term in (2.8) is

$$\frac{\delta^2}{k^2p^2t}\mathcal{I}_{\mathcal{F}_3}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}=0,$$

conjugating (4.2). In the same way the main term coming from $\widetilde{\mathcal{F}}_3$ is shown to vanish.

The contribution to the second term of Euler-Maclaurin is

$$-2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_2\ge 0}\frac{B_{2n_2+2}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(2n_2+2)!}$$
$$\times\int_0^\infty\left(\mathcal{F}_3^{(0,2n_2+1)}(x_1,0)+\widetilde{\mathcal{F}}_3^{(0,2n_2+1)}(x_1,0)\right)dx_1\left(\frac{k^2p^2t}{\delta^2}\right)^{n_2}.$$

We now claim that

$$\int_0^\infty\left(\mathcal{F}_3^{(0,2n_2+1)}(x_1,0)+\widetilde{\mathcal{F}}_3^{(0,2n_2+1)}(x_1,0)\right)dx_1=(-1)^{n_2}\int_0^\infty\mathcal{F}_1^{(0,2n_2+1)}(x_1,0)dx_1. \tag{7.3}$$

Firstly the right-hand side of (7.3) equals

$$\left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\int_0^\infty\mathcal{F}_1(x_1,x_2)dx_1\right]_{x_2=0}=\left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left(e^{-\frac{x_2^2}{4}}\int_0^\infty e^{-3\left(x_1+\frac{x_2}{2}\right)^2}dx_1\right)\right]_{x_2=0}. \tag{7.4}$$

Now the integral in (7.4) evaluates as

$$\sqrt{\frac{\pi}{3}}\int_{\frac{\sqrt{3}x_2}{2\sqrt{\pi}}}^{\infty} e^{-\pi x_1^2}dx_1 = \frac{\sqrt{\pi}}{2\sqrt{3}}\left(1-E\left(\frac{\sqrt{3}x_2}{2\sqrt{\pi}}\right)\right).$$

Thus (7.4) becomes

$$\frac{\sqrt{\pi}}{2\sqrt{3}}\left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left(e^{-\frac{x_2^2}{4}}\left(1-E\left(\frac{\sqrt{3}x_2}{2\sqrt{\pi}}\right)\right)\right)\right]_{x_2=0} = -\frac{\sqrt{\pi}}{2\sqrt{3}}\left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left(e^{-\frac{x_2^2}{4}}E\left(\frac{\sqrt{3}x_2}{2\sqrt{\pi}}\right)\right)\right]_{x_2=0}. \tag{7.5}$$

To compute the left-hand side of (7.3), we decompose, according to (7.1),

$$M_2^*\left(\sqrt{3},\sqrt{3}(2x_1+x_2),x_2\right) = \operatorname{sgn}^*(x_1)\operatorname{sgn}^*(x_2)+h_1(\boldsymbol{x})-\operatorname{sgn}^*(x_2)h_2(\boldsymbol{x})-\operatorname{sgn}^*(x_1)h_3(\boldsymbol{x}),$$

where

$$h_1(\boldsymbol{x}) := E_2\left(\sqrt{3};\sqrt{3}\left(2x_1+x_2\right),x_2\right),\quad h_2(\boldsymbol{x}) := E\left(\sqrt{3}\left(2x_1+x_2\right)\right),\quad h_3(\boldsymbol{x}) := E\left(3x_1+2x_2\right).$$

Setting

$$a_0(\boldsymbol{x}) := e^{Q(\boldsymbol{x})},\qquad a_j(\boldsymbol{x}) := h_j\left(\frac{1}{\sqrt{2\pi}}(\boldsymbol{x})\right)e^{Q(\boldsymbol{x})},$$

we then obtain

$$\begin{aligned}
&\mathcal{F}_3^{(0,2n_2+1)}(x_1,0)+\widetilde{\mathcal{F}}_3^{(0,2n_2+1)}(x_1,0)\\
&\quad=\frac{1}{2}\left(a_0^{(0,2n_2+1)}(x_1,0)+a_1^{(0,2n_2+1)}(x_1,0)-a_2^{(0,2n_2+1)}(x_1,0)-a_3^{(0,2n_2+1)}(x_1,0)\right)\\
&\qquad+\frac{1}{2}\left(-a_0^{(0,2n_2+1)}(-x_1,0)+a_1^{(0,2n_2+1)}(-x_1,0)-a_2^{(0,2n_2+1)}(-x_1,0)+a_3^{(0,2n_2+1)}(-x_1,0)\right)\\
&\quad=a_0^{(0,2n_2+1)}(x_1,0)-a_2^{(0,2n_2+1)}(x_1,0),
\end{aligned}$$

using that a_0 and a_1 are even and a_2 and a_3 are odd. Plugging in the definition of a_0 and a_2, we need to consider

$$-\left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left(e^{x_2^2}\int_0^{\infty}e^{3x_1^2+3x_1x_2}M\left(\sqrt{\frac{3}{2\pi}}(2x_1+x_2)\right)dx_1\right)\right]_{x_2=0}. \tag{7.6}$$

Changing variables $w := \sqrt{\frac{3}{2\pi}}(2x_1+x_2)$, the function in (7.6) before differentiation is

$$-\sqrt{\frac{\pi}{6}}e^{\frac{x_2^2}{4}}\int_{\sqrt{\frac{3}{2\pi}}x_2}^{\infty}M(w)e^{\frac{\pi w^2}{2}}dw = -\sqrt{\frac{\pi}{6}}e^{\frac{x_2^2}{4}}\left(\int_0^{\infty}M(w)e^{\frac{\pi w^2}{2}}dw-\int_0^{\sqrt{\frac{3}{2\pi}}x_2}M(w)e^{\frac{\pi w^2}{2}}dw\right).$$

The first integral vanishes upon differentiating an odd number of times and then setting $x_2=0$. In the second integral we decompose $M(w)=E(w)-1$. The contribution of the E-function vanishes, since E is an odd function. We are left with

$$-\sqrt{\frac{\pi}{6}}\left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left(e^{\frac{x_2^2}{4}}\int_0^{\sqrt{\frac{3}{2\pi}}x_2}e^{\frac{\pi w^2}{2}}dw\right)\right]_{x_2=0} = -\sqrt{\frac{\pi}{6}}i^{-2n_2-1}\left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left(e^{-\frac{x_2^2}{4}}\int_0^{\sqrt{\frac{3}{2\pi}}x_2i}e^{\frac{\pi w^2}{2}}dw\right)\right]_{x_2=0}.$$

The integral equals

$$i\sqrt{2}\int_0^{\frac{\sqrt{3}x_2}{2\sqrt{\pi}}} e^{-\pi w^2}dw = \frac{i}{\sqrt{2}}E\left(\frac{\sqrt{3}x_2}{2\sqrt{\pi}}\right).$$

Thus we obtain

$$\frac{\sqrt{\pi}}{2\sqrt{3}}(-1)^{n_2+1}\left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left(e^{-\frac{x_2^2}{4}}E\left(\frac{\sqrt{3}x_2}{2\sqrt{\pi}}\right)\right)\right]_{x_2=0},$$

as claimed, by comparing with (7.5).

In the same way one can show that the third term in Euler-Maclaurin equals

$$-2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\varepsilon(\boldsymbol{\alpha})\sum_{0\leq\boldsymbol{\ell}\leq\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_1\geq 0}\frac{B_{2n_1+2}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(2n_1+2)!}\int_0^\infty \mathcal{F}_1^{(2n_1+1,0)}(0,x_2)dx_2\left(-\frac{k^2p^2t}{\delta^2}\right)^{n_1}.$$

The contribution to the final term is, pairing as in Section 4

$$2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\varepsilon(\boldsymbol{\alpha})\sum_{0\leq\boldsymbol{\ell}\leq\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{\substack{n_1,n_2\geq 0\\ n_1\equiv n_2 \pmod 2}}\frac{B_{n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(n_1+1)!}\frac{B_{n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(n_1+1)!}$$
$$\times\left(\mathcal{F}_3^{(\boldsymbol{n})}(\boldsymbol{0})-(-1)^{n_1}\widetilde{\mathcal{F}}_3^{(\boldsymbol{n})}(\boldsymbol{0})\right)\left(\frac{kp\sqrt{t}}{\delta}\right)^{n_1+n_2}.$$

We next show that

$$\mathcal{F}_3^{(n_1,n_2)}(\boldsymbol{0})-(-1)^{n_1}\widetilde{\mathcal{F}}_3^{(n_1,n_2)}(\boldsymbol{0}) = i^{n_1+n_2}\mathcal{F}_1^{(n_1,n_2)}(\boldsymbol{0}).$$

For this, we compute

$$\mathcal{F}_3^{(n_1,n_2)}(\boldsymbol{0})-(-1)^{n_1}\widetilde{\mathcal{F}}_3^{(n_1,n_2)}(\boldsymbol{0}) = a_0^{(n_1,n_2)}(\boldsymbol{0})-a_3^{(n_1,n_2)}(\boldsymbol{0}).$$

Since $a_3(-x_1,-x_2)=-a_3(\boldsymbol{x})$, we obtain

$$a_3^{(n_1,n_2)}(\boldsymbol{0}) = (-1)^{n_1+n_2+1}a_3^{(n_1,n_2)}(\boldsymbol{0}).$$

Because in the sums of interest $n_1\equiv n_2 \pmod 2$, the contribution of a_3 vanishes. As claimed, we are left with

$$a_0^{(n_1,n_2)}(\boldsymbol{0}) = i^{n_1+n_2}\left[\frac{\partial^{n_1}}{\partial x_1^{n_1}}\frac{\partial^{n_2}}{\partial x_2^{n_2}}e^{-Q(\boldsymbol{x})}\right]_{\boldsymbol{x}=0} = i^{n_1+n_2}\mathcal{F}_1^{(\boldsymbol{n})}(\boldsymbol{0}).$$

Finally, the contribution from H_1 gives, observing that the Euler-Maclaurin main term vanishes,

$$\sum_{0\leq r\leq\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2}\sum_{m\geq 0}\frac{B_{2m+1}\left(\frac{\delta\left(r+\frac{1}{p}\right)}{kp}\right)}{(2m+1)!}\mathcal{F}_4^{(2m)}(0)\left(\frac{k^2p^2}{\delta^2}t\right)^m$$

with $\mathcal{F}_4(x):=M(\sqrt{\frac{2}{\pi}}x)e^{x^2}$. The claim then follows, observing that

$$\mathcal{F}_4^{(2m)}(0) = (-1)^{m+1}\left[\frac{\partial^{2m}}{\partial x^{2m}}e^{-x^2}\right]_{x=0} = (-1)^{m+1}\mathcal{F}_2^{(2m)}(0).$$

□

7.2. **Asymptotics of $\mathcal{E}_2$.** We write

$$F_2\left(e^{2\pi i\frac{h}{k}-t}\right)\sim\sum_{m\geq 0}b_{h,k}(m)t^m\qquad\left(t\to 0^+\right),$$

Theorem 7.2. *We have, for $h,k\in\mathbb{Z}$ with $k>0$ and $\gcd(h,k)=1$,*

$$\mathbb{E}_2\left(\frac{h}{k}+\frac{it}{2\pi}\right)\sim\sum_{m\geq 0}b_{-h,k}(m)(-t)^m\qquad\left(t\to 0^+\right).$$

Proof. We write, using Lemma 6.2 and (6.3)

$$\mathbb{E}_2(\tau)=\mathcal{E}_{2,1}(\tau)+\mathcal{E}_{2,2}(\tau),$$

where

$$\begin{aligned}\mathcal{E}_{2,1}(\tau):=&\frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2}n_2M_2\left(\sqrt{3};\sqrt{3v}(2n_1+n_2),\sqrt{v}n_2\right)q^{-Q(\boldsymbol{n})}\\&+\frac{1}{2}\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\eta}(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2}n_2M_2\left(\sqrt{3};\sqrt{3v}(-2n_1+n_2),\sqrt{v}n_2\right)q^{-Q(-n_1,n_2)},\\ \mathcal{E}_{2,2}(\tau):=&\frac{1}{4\pi\sqrt{v}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2}e^{-\pi(3n_1+2n_2)^2v}M\left(\sqrt{3v}n_1\right)q^{-Q(\boldsymbol{n})}\\&+\frac{1}{4\pi\sqrt{v}}\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\eta}(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2}e^{-\pi(-3n_1+2n_2)^2v}M\left(\sqrt{3v}n_1\right)q^{-Q(-n_1,n_2)},\end{aligned}$$

where $\widetilde{\eta}(\boldsymbol{\alpha}):=\eta(1-\alpha_1,\alpha_2)$. We then again use (7.2), to split

$$\mathcal{E}_{2,1}(\tau)=\mathcal{E}_2^*(\tau)+H_2(\tau),$$

where

$$\begin{aligned}\mathcal{E}_2^*(\tau):=&\frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2}n_2M_2^*\left(\sqrt{3};\sqrt{3v}(2n_1+n_2),\sqrt{v}n_2\right)q^{-Q(\boldsymbol{n})}\\&+\frac{1}{2}\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\eta}(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2}n_2M_2^*\left(\sqrt{3};\sqrt{3v}(-2n_1+n_2),\sqrt{v}n_2\right)q^{-Q(-n_1,n_2)},\\ H_2(\tau):=&\frac{1}{2}\sum_{\beta\in\left\{\frac{1}{p},1-\frac{1}{p}\right\}}\sum_{m\in\beta+\mathbb{N}_0}mM\left(2\sqrt{v}m\right)q^{-m^2}.\end{aligned}$$

Using that $\lim_{x\to 0^+}M^*(\pm x)=\mp 1$, where we let $M^*(x):=E(x)-\operatorname{sgn}^*(x)$, we split

$$\mathcal{E}_{2,2}(\tau)=\mathcal{E}_{2,2}^*(\tau)+H_3(\tau),$$

where

$$\mathcal{E}_{2,2}^*(\tau):=\frac{1}{4\pi\sqrt{v}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2}e^{-\pi(3n_1+2n_2)^2v}M^*\left(\sqrt{3v}n_1\right)q^{-Q(\boldsymbol{n})}$$

$$+\frac{1}{4\pi\sqrt{v}}\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\eta}(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2}e^{-\pi(-3n_1+2n_2)^2v}M^*\left(-\sqrt{3v}n_1\right)q^{-Q(-n_1,n_2)},$$

$$H_3(\tau):=\frac{1}{4\pi\sqrt{v}}\sum_{\beta\in\left\{\frac{1}{p},1-\frac{1}{p}\right\}}\sum_{m\in\beta+\mathbb{N}_0}e^{-4\pi m^2v}q^{-m^2}.$$

We first investigate asymptotic properties of $\mathcal{E}_2^*$. Writing $\mathcal{G}_3(\boldsymbol{x}):=x_2\mathcal{F}_3(\boldsymbol{x})$ and $\widetilde{\mathcal{G}}_3(\boldsymbol{x}):=\mathcal{G}_3(-x_1,x_2)$, we have

$$\mathcal{E}_2^*\left(\frac{h}{k}+\frac{it}{2\pi}\right)=\frac{1}{\sqrt{t}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{0\leq\boldsymbol{\ell}\leq\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{\boldsymbol{n}\in\frac{\delta}{kp}(\boldsymbol{\ell}+\boldsymbol{\alpha})+\mathbb{N}_0^2}\mathcal{G}_3\left(\frac{kp}{\delta}\sqrt{t}\boldsymbol{n}\right)$$
$$+\frac{1}{\sqrt{t}}\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\eta}(\boldsymbol{\alpha})\sum_{0\leq\boldsymbol{\ell}\leq\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(-\ell_1-\alpha_1,\ell_2+\alpha_2)}\sum_{\boldsymbol{n}\in\frac{\delta}{kp}(\boldsymbol{\ell}+\boldsymbol{\alpha})+\mathbb{N}_0^2}\widetilde{\mathcal{G}}_3\left(\frac{kp}{\delta}\sqrt{t}\boldsymbol{n}\right).$$

The contribution from $\mathcal{G}_3$ to the Euler-Maclaurin main term is, as in Subsection 4.2,

$$\frac{\delta^2}{k^2p^2t^{\frac{3}{2}}}\mathcal{I}_{\mathcal{G}_3}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{0\leq\boldsymbol{\ell}\leq\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}=0.$$

In the same way we see that the contribution from $\widetilde{\mathcal{G}}_3$ to the main term vanishes.

The contribution to the second term in Euler-Maclaurin is, as in Subsection 4.2,

$$-2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\leq\boldsymbol{\ell}\leq\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_2\geq1}\frac{B_{2n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(2n_2+1)!}$$
$$\times\int_0^\infty\left(\mathcal{G}_3^{(0,2n_2)}(x_1,0)+\widetilde{\mathcal{G}}_3^{(0,2n_2)}(x_1,0)\right)dx_1\left(\frac{kp}{\delta}\right)^{2n_2-1}t^{n_2-1}.$$

We claim that

$$\int_0^\infty\left(\mathcal{G}_3^{(0,2n_2)}(x_1,0)+\widetilde{\mathcal{G}}_3^{(0,2n_2)}(x_1,0)\right)dx_1=(-1)^{n_2+1}\int_0^\infty\mathcal{G}_1^{(0,2n_2)}(x_1,0)dx_1. \tag{7.7}$$

Since we need to differentiate the x_2-factor exactly once, we have

$$\mathcal{G}_3^{(0,2n_2)}(x_1,0)+\widetilde{\mathcal{G}}_3^{(0,2n_2)}(x_1,0)=2n_2\left(\mathcal{F}_3^{(0,2n_2-1)}(x_1,0)+\widetilde{\mathcal{F}}_3^{(0,2n_1-1)}(x_1,0)\right),$$
$$\mathcal{G}_1^{(0,2n_2)}(x_1,0)=2n_2\mathcal{F}_1^{(0,2n_2-1)}(x_1,0).$$

The claim (7.7) then follows from (7.3). This gives the correspondence to (4.4).

The third term in Euler-Maclaurin is, in the same way,

$$-2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\leq\boldsymbol{\ell}\leq\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_1\geq0}\frac{B_{2n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(2n_1+1)!}$$
$$\times\int_0^\infty\left(\mathcal{G}_3^{(2n_1,0)}(0,x_2)-\widetilde{\mathcal{G}}_3^{(2n_1,0)}(0,x_2)\right)dx_2\left(\frac{kp}{\delta}\right)^{2n_1-1}t^{n_1-1}.$$

To relate this to (4.5) (skipping the $n_1 = 0$ term in both cases), we compute that

$$\mathcal{G}_3^{(2n_1,0)}(0,x_2) - \widetilde{\mathcal{G}}_3^{(2n_1,0)}(0,x_2) = x_2\left(a_0^{(2n_1,0)}(0,x_2) - a_3^{(2n_1,0)}(0,x_2)\right).$$

Note that

$$a_0(\boldsymbol{x}) - a_3(\boldsymbol{x}) = -e^{Q(\boldsymbol{x})}M^*\left(\sqrt{\frac{1}{2\pi}}(3x_1+2x_2)\right). \tag{7.8}$$

We next show that

$$\begin{aligned}\int_0^\infty & x_2\left(a_0^{(2n_1,0)}(0,x_2) - a_3^{(2n_1,0)}(0,x_2)\right)dx_2\\ &= (-1)^{n_1+1}\int_0^\infty x_2\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}\left(e^{-x_2^2-3x_1x_2-3x_1^2}\right)\right]_{x_1=0}dx_2 + \frac{1}{\sqrt{2}}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}e^{\frac{3x_1^2}{4}}\right]_{x_1=0},\end{aligned} \tag{7.9}$$

where the first terms on the right-hand side corresponds to (4.5). We write it as

$$(-1)^{n_1+1}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}\left(e^{-\frac{3x_1^2}{4}}\int_0^\infty x_2 e^{-\left(x_2+\frac{3x_1}{2}\right)^2}dx_2\right)\right]_{x_1=0}.$$

Now we let

$$f(x_1)e^{\frac{3x_1^2}{4}}(-1)^{n_1+1} := \int_{\frac{3x_1}{2}}^\infty\left(x_2-\frac{3x_1}{2}\right)e^{-x_2^2}dx_2 = \frac{1}{2}e^{-\frac{9x_1^2}{4}} - \frac{3x_1}{2}\frac{\sqrt{\pi}}{2}\left(1-E\left(\frac{3x_1}{2\sqrt{\pi}}\right)\right),$$

using integration by parts. We then compute (using $n_1 > 0$)

$$f^{(2n_1)}(0) = \frac{(-1)^{n_1+1}}{2}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}e^{-3x_1^2}\right]_{x_1=0} + \frac{(-1)^{n_1+1}3\sqrt{\pi}}{4}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}\left(x_1e^{-\frac{3x_1^2}{4}}E\left(\frac{3x_1}{2\sqrt{\pi}}\right)\right)\right]_{x_1=0}. \tag{7.10}$$

For the left-hand side of (7.9) we use (7.8) and consider

$$-\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}\left(\int_0^\infty x_2M^*\left(\frac{2x_2+3x_1}{\sqrt{2\pi}}\right)e^{3x_1^2+3x_1x_2+x_2^2}dx_2\right)\right]_{x_1=0}.$$

Making the change of variables $u = \frac{2x_2+3x_1}{\sqrt{2\pi}}$, the integral before differentiation (including the minus sign) becomes

$$-\sqrt{\frac{\pi}{2}}e^{\frac{3x_1^2}{4}}\frac{1}{2}\int_{\frac{3x_1}{\sqrt{2\pi}}}^\infty\left(\sqrt{2\pi}u-3x_1\right)M^*(u)e^{\frac{\pi u^2}{2}}du. \tag{7.11}$$

Using integration by parts, the contribution from $\sqrt{2\pi}u$ equals

$$\frac{1}{2}e^{3x_1^2}\left(E\left(\frac{3x_1}{\sqrt{2\pi}}\right)-1\right) + \frac{1}{\sqrt{2}}e^{\frac{3x_1^2}{4}}\left(1-E\left(\frac{3x_1}{2\sqrt{\pi}}\right)\right).$$

Thus differentiating $2n_1$ times with respect to x_1 and then setting $x_1 = 0$ gives (using that $z \mapsto E(z)$ is odd)

$$-\frac{1}{2}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}e^{3x_1^2}\right]_{x_1=0} + \frac{1}{\sqrt{2}}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}e^{\frac{3x_1^2}{4}}\right]_{x_1=0} = -\frac{1}{2}(-1)^{n_1}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}e^{-3x_1^2}\right]_{x_1=0} + \frac{1}{\sqrt{2}}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}e^{\frac{3x_1^2}{4}}\right]_{x_1=0}.$$

The first term matches the first term in (7.10), the second term is the second term on the right-hand side of (7.9). For the second term in (7.11), we split

$$\frac{3}{2}\sqrt{\frac{\pi}{2}}e^{\frac{3x_1^2}{4}}x_1\left(\int_0^\infty M^*(u)e^{\frac{\pi u^2}{2}}du-\int_0^{\frac{3x_1}{\sqrt{2\pi}}}E(u)e^{\frac{\pi u^2}{2}}du+\int_0^{\frac{3x_1}{\sqrt{2\pi}}}e^{\frac{\pi u^2}{2}}du\right).$$

Since we take an even number of derivatives only the last term survives, yielding the contribution

$$\frac{3}{2}\sqrt{\frac{\pi}{2}}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}\left(x_1e^{\frac{3x_1^2}{4}}\int_0^{\frac{3x_1}{\sqrt{2\pi}}}e^{\frac{\pi u^2}{2}}du\right)\right]_{x_1=0}=-(-1)^{n_1}\frac{3\sqrt{\pi}}{4}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}\left(x_1e^{-\frac{3x_1^2}{4}}E\left(\frac{3x_1}{2\sqrt{\pi}}\right)\right)\right]_{x_1=0}.$$

This is the second term in (7.10), which implies (7.9).

The left-over term from (7.9) overall contributes as

$$-\sqrt{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_1\ge1}\frac{B_{2n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(2n_1+1)!}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}e^{\frac{3x_1^2}{4}}\right]_{x_1=0}\left(\frac{kp}{\delta}\right)^{2n_1-1}t^{n_1-1}.$$

The final term in Euler-Maclaurin is

$$2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{\substack{n_1,n_2\ge0\\ n_1\not\equiv n_2 \pmod 2}}\frac{B_{n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(n_1+1)!}\frac{B_{n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(n_2+1)!}$$
$$\times\left(\mathcal{G}_3^{(n_1,n_2)}(\mathbf{0})+(-1)^{n_1+1}\widetilde{\mathcal{G}}_3^{(n_1,n_2)}(\mathbf{0})\right)\left(\frac{kp}{\delta}\right)^{n_1+n_2}t^{\frac{n_1+n_2-1}{2}}.$$

Then

$$\mathcal{G}_3^{(n_1,n_2)}(\mathbf{0})+(-1)^{n_1+1}\widetilde{\mathcal{G}}_3^{(n_1,n_2)}(\mathbf{0})=i^{n_1+n_2-1}\mathcal{G}_1^{(n_1,n_2)}(\mathbf{0})$$

gives the relation to (4.7).

We next consider H_2. We have, with $\mathcal{G}_4(x):=x\mathcal{F}_4(x)$,

$$H_2\left(\frac{h}{k}+\frac{it}{2\pi}\right)=\frac{1}{2\sqrt{t}}\sum_{\beta\in\left\{\frac{1}{p},1-\frac{1}{p}\right\}}\sum_{0\le r\le\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}(r+\beta)^2}\sum_{m\in\frac{(r+\beta)\delta}{kp}+\mathbb{N}_0}\mathcal{G}_4\left(\frac{kp}{\delta}\sqrt{t}m\right).$$

The Euler-Maclaurin main term is

$$\frac{1}{2\sqrt{t}}\frac{\delta}{kp\sqrt{t}}\mathcal{I}_{\mathcal{G}_4}\sum_{\beta\in\left\{\frac{1}{p},1-\frac{1}{p}\right\}}\sum_{0\le r\le\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}(r+\beta)^2}=\frac{\delta}{kpt}\mathcal{I}_{\mathcal{G}_4}\sum_{r\pmod{\frac{kp}{\delta}}}e^{-2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2}.$$

The second term becomes

$$-\sum_{0\le r\le\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2}\sum_{m\ge0}\frac{B_{2m+2}\left(\frac{\delta\left(r+\frac{1}{p}\right)}{kp}\right)}{(2m+2)!}\mathcal{G}_4^{(2m+1)}(0)\left(\frac{kp}{\delta}\right)^{2m+1}t^m.$$

Then

$$\mathcal{G}_4^{(2m+1)}(0)=(2m+1)\mathcal{F}_4^{(2m)}(0)=(2m+1)(-1)^{m+1}\mathcal{F}_2^{(2m)}(0)=(-1)^{m+1}\mathcal{G}_2^{(2m+1)}(0).$$

gives the relation to (4.9).

Finally, we consider $\mathcal{E}_{2,2}$. We first study $\mathcal{E}_{2,2}^*$ and write

$$\mathcal{E}_{2,2}^*\left(\frac{h}{k}+\frac{it}{2\pi}\right) = \frac{1}{\sqrt{\pi t}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}}\eta(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{\boldsymbol{n}\in\frac{\delta}{kp}(\boldsymbol{\ell}+\boldsymbol{\alpha})+\mathbb{N}_0^2}\mathcal{G}_5\left(\frac{kp}{\delta}\sqrt{t}\boldsymbol{n}\right)$$
$$+\frac{1}{\sqrt{\pi t}}\sum_{\boldsymbol{\alpha}\in\widetilde{\mathscr{S}}}\widetilde{\eta}(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(-\ell_1-\alpha_1,\ell_2+\alpha_2)}\sum_{\boldsymbol{n}\in\frac{\delta}{kp}(\boldsymbol{\ell}+\boldsymbol{\alpha})+\mathbb{N}_0^2}\widetilde{\mathcal{G}}_5\left(\frac{kp}{\delta}\sqrt{t}\boldsymbol{n}\right),$$

where

$$\mathcal{G}_5(\boldsymbol{x}) := \frac{1}{2\sqrt{2}}e^{-\frac{3x_1^2}{2}-3x_1x_2-x_2^2}M^*\left(\sqrt{\frac{3}{2\pi}}x_1\right),\quad \widetilde{\mathcal{G}}_5(\boldsymbol{x}) := \mathcal{G}_5(-x_1,x_2).$$

As before the main term in Euler-Maclaurin vanishes. The second term equals

$$-\frac{2}{\sqrt{\pi t}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_2\ge0}\frac{B_{2n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(2n_2+1)!}$$
$$\times\int_0^\infty\left(\mathcal{G}_5^{(0,2n_2)}(x_1,0)+\widetilde{\mathcal{G}}_5^{(0,2n_2)}(x_1,0)\right)dx_1\left(\frac{kp}{\delta}\right)^{2n_2-1}t^{n_2-\frac{1}{2}}.$$

It is however not hard to see that

$$\mathcal{G}_5^{(0,2n_2)}(x_1,0)+\widetilde{\mathcal{G}}_5^{(0,2n_2)}(x_1,0)=0.$$

The third term in Euler-Maclaurin is

$$-\frac{2}{\sqrt{\pi t}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_1\ge0}\frac{B_{2n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(2n_1+1)!}$$
$$\times\int_0^\infty\left(\mathcal{G}_5^{(2n_1,0)}(0,x_2)-\widetilde{\mathcal{G}}_5^{(2n_1,0)}(0,x_2)\right)dx_2\left(\frac{kp}{\delta}\right)^{2n_1-1}t^{n_1-\frac{1}{2}}.$$

Now

$$\mathcal{G}_5^{(2n_1,0)}(0,x_2)-\widetilde{\mathcal{G}}_5^{(2n_1,0)}(0,x_2)=2\mathcal{G}_{5,1}^{(2n_1,0)}(0,x_2),$$

where $\mathcal{G}_{5,1}(\boldsymbol{x}) := -\frac{1}{2\sqrt{2}}e^{-\frac{3x_1^2}{2}-3x_1x_2-x_2^2}$. We thus need to compute

$$2\int_0^\infty\mathcal{G}_{5,1}^{(2n_1,0)}(0,x_2)\,dx_2 = -\frac{1}{\sqrt{2}}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}e^{\frac{3x_1^2}{4}}\int_0^\infty e^{-\left(x_2+\frac{3}{2}x_1\right)^2}dx_2\right]_{x_1=0}$$
$$= -\frac{\sqrt{\pi}}{2\sqrt{2}}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}\left(e^{\frac{3x_1^2}{4}}\left(1-E\left(\frac{3x_1}{2\sqrt{\pi}}\right)\right)\right)\right]_{x_1=0} = -\frac{\sqrt{\pi}}{2\sqrt{2}}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}e^{\frac{3x_1^2}{4}}\right]_{x_1=0}.$$

This term then contributes as

$$\frac{1}{\sqrt{2}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_1\ge0}\frac{B_{2n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(2n_1+1)!}\left[\frac{\partial^{2n_1}}{\partial x_1^{2n_1}}e^{\frac{3x_1^2}{4}}\right]_{x_1=0}\left(\frac{kp}{\delta}\right)^{2n_1-1}t^{n_1-1}. \tag{7.12}$$

The final term in the Euler-Maclaurin summation formula is

$$\frac{2}{\sqrt{\pi t}}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*}\eta(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{\substack{n_1,n_2\ge0\\ n_1\not\equiv n_2\pmod 2}}\frac{B_{n+1}\left(\frac{\delta(\ell_1+\alpha_1)}{kp}\right)}{(n_1+1)!}\frac{B_{n+2}\left(\frac{\delta(\ell_2+\alpha_2)}{kp}\right)}{(n_2+1)!}$$
$$\times\left(\mathcal{G}_5^{(n_1,n_2)}(\mathbf{0})+(-1)^{n_1+1}\widetilde{\mathcal{G}}_5^{(n_1,n_2)}(\mathbf{0})\right)\left(\frac{kp}{\delta}\sqrt{t}\right)^{n_1+n_2}.$$

It is easy to see that under the condition $n_1\not\equiv n_2\pmod 2$ we have

$$\mathcal{G}_5^{(n_1,n_2)}(\mathbf{0})+(-1)^{n_1+1}\widetilde{\mathcal{G}}_5^{(n_1,n_2)}(\mathbf{0})=0.$$

Next, we consider

$$H_3\left(\frac{h}{k}+\frac{it}{2\pi}\right)=\frac{1}{2\sqrt{2\pi t}}\sum_{\beta\in\left\{\frac{1}{p},1-\frac{1}{p}\right\}}\sum_{0\le r\le\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}(r+\beta)^2}\sum_{m\in\frac{\delta(r+\beta)}{kp}+\mathbb{N}_0}\mathcal{F}_2\left(\frac{kp}{\delta}\sqrt{t}m\right).$$

The Euler-Maclaurin main term is

$$\frac{\delta}{2kpt\sqrt{2\pi}}\mathcal{I}_{\mathcal{F}_2}\sum_{\beta\in\left\{\frac{1}{p},1-\frac{1}{p}\right\}}\sum_{r\pmod{\frac{kp}{\delta}}}e^{-2\pi i\frac{h}{k}(r+\beta)^2}=\frac{\delta}{kpt\sqrt{2\pi}}\mathcal{I}_{\mathcal{F}_2}\sum_{r\pmod{\frac{kp}{\delta}}}e^{-2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2}.$$

The final term is

$$\frac{1}{\sqrt{2\pi t}}\sum_{0\le r\le\frac{kp}{\delta}-1}e^{-2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2}\sum_{m\ge0}\frac{B_{2m+2}\left(\frac{\delta\left(r+\frac{1}{p}\right)}{kp}\right)}{(2m+2)!}\mathcal{F}_2^{(2m+1)}(0)=0$$

since $\mathcal{F}_2$ is an even function.

Collecting all growing terms gives

$$\frac{\delta}{kpt}\sum_{r\pmod{\frac{kp}{\delta}}}e^{-2\pi i\frac{h}{k}\left(r+\frac{1}{p}\right)^2}\left(\mathcal{I}_{\mathcal{G}_3(0,\cdot)-\widetilde{\mathcal{G}}_3(0,\cdot)}+\mathcal{I}_{\mathcal{G}_4}+\frac{1}{\sqrt{\pi}}\mathcal{I}_{\mathcal{G}_5(0,\cdot)-\widetilde{\mathcal{G}}_5(0,\cdot)}+\frac{\mathcal{I}_{\mathcal{F}_2}}{\sqrt{2\pi}}\right). \tag{7.13}$$

We compute $\mathcal{I}_{\mathcal{F}_2}=\frac{\sqrt{\pi}}{2}$ and, using integration by parts,

$$\mathcal{I}_{\mathcal{G}_4}=\int_0^\infty xe^{x^2}M^*\left(\sqrt{\frac{2}{\pi}}x\right)dx=-\frac{M^*(0)}{2}-\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}dx=\frac{1}{2}-\frac{1}{\sqrt{2}}$$

by conjugating (4.8). Moreover, (7.9) gives

$$\mathcal{I}_{\mathcal{G}_3(0,\cdot)-\widetilde{\mathcal{G}}_3(0,\cdot)}-\int_0^\infty x_2e^{-x_2^2}dx_2+\frac{1}{\sqrt{2}}=\frac{1}{2}\left[e^{-x_2^2}\right]_0^\infty+\frac{1}{\sqrt{2}}=-\frac{1}{2}+\frac{1}{\sqrt{2}},$$

$$\mathcal{I}_{\mathcal{G}_5(0,\cdot)-\widetilde{\mathcal{G}}_5(0,\cdot)} = -\frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Thus the term inside the paranthesis in (7.13) vanishes.

We are left to show that the contributions from (7.8) and (7.12) vanish. For this it suffices to show that, for all $n \in \mathbb{N}$,

$$\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*} \sum_{0\le\boldsymbol{\ell}\le\frac{kp}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(\boldsymbol{\ell}+\boldsymbol{\alpha})} B_{2n+1}\left(\frac{\delta\left(\ell_1+\alpha_1\right)}{kp}\right) = 0.$$

As in (4.9) we get that this sum is zero for $\frac{p}{\delta} \notin \{1,2\}$. Next we consider $\frac{p}{\delta} = 1$. We first combine the first and third element in $\mathscr{S}^*$. Using (4.10) and

$$B_{2m+1}(1-x) = -B_{2m+1}(x) \tag{7.14}$$

gives that these cancel. Thus we need to show that

$$\sum_{0\le\ell<k} B_{2n+1}\left(\frac{\ell_1}{k}\right) e^{-2\pi i\frac{h}{k}Q\left(\ell_1,\ell_2+1-\frac{1}{p}\right)} = 0. \tag{7.15}$$

We use (4.11) and distinguish again whether k is even or odd. If k is odd we do the same change of variables and use (7.14) to obtain that (7.15) equals

$$B_{2n+1}(0) \sum_{\ell_2 \pmod k} e^{-2\pi i\frac{h}{k}\left(\ell_2+1-\frac{1}{p}\right)^2} = 0$$

since for $m \ge 3$ odd, $B_m(0) = 0$.

If k is even, then we obtain

$$B_{2n+1}(0) \sum_{\ell_2 \pmod k} e^{-2\pi i\frac{h}{k}\left(\ell_2+1-\frac{1}{p}\right)^2} + B_{2n+1}\left(\frac{1}{2}\right) \sum_{\ell_2 \pmod k} e^{-2\pi i\frac{h}{k}\left(\ell_2+1-\frac{1}{p}\right)^2} = 0$$

since for m odd $B_m(\frac{1}{2}) = 0$.

We next turn to the case $\frac{p}{\delta} = 2$. Then only the second element survives and we want

$$\sum_{0\le\boldsymbol{\ell}\le 2k-1} B_{2n+1}(0) e^{-2\pi i\frac{h}{k}\left(\ell_1,\ell_2+1-\frac{1}{p}\right)} = 0. \tag{7.16}$$

We obtain for the left-hand side of (7.16)

$$\left(B_{2n+1}(0) + B_{2n+1}\left(\frac{1}{2}\right)\right) \sum_{\ell_2 \pmod k} e^{-2\pi i\frac{h}{k}\left(\ell_2+1-\frac{1}{p}\right)^2} = 0.$$

This finally proves the theorem.

□

7.3. **Proof of Theorem 1.1.** We are now ready to prove a refined version of Theorem 1.1.

Theorem 7.3. (1) *The function* $\widehat{F}_1 : \mathbb{Q} \to \mathbb{C}$ *defined by* $\widehat{F}_1(\frac{h}{k}) := F_1(e^{2\pi i \frac{ph}{k}})$ *is a depth two quantum modular form of weight one for* Γ_p *with multiplier* $(\frac{-3}{d})$.

(2) *The function* $\widehat{F}_2 : \mathbb{Q} \to \mathbb{C}$ *defined by* $\widehat{F}_2(\frac{h}{k}) := F_2(e^{2\pi i \frac{ph}{k}})$ *is a depth two quantum modular form of weight two for* Γ_p *with multiplier* $(\frac{3}{d})$.

Proof. (1) We have, by Theorem 7.1,

$$\widehat{F}_1\left(\frac{h}{k}\right) = \lim_{t\to 0^+} F_1\left(e^{2\pi i \frac{ph}{k} - t}\right) = a_{hp_1, \frac{k}{p_2}}(0) = \lim_{t\to 0^+} \mathbb{E}_1\left(-\frac{h}{k} + \frac{it}{2\pi}\right),$$

where $p_1 := p/\gcd(k,p)$, $p_2 := \gcd(k,p)$. Proposition 5.2 then gives the claim.

(2) Theorem 7.2 gives

$$\widehat{F}_2\left(\frac{h}{k}\right) = \lim_{t\to 0^+} F_2\left(e^{2\pi i \frac{ph}{k} - t}\right) = b_{hp_1, \frac{k}{p_2}}(0) = \lim_{t\to 0^+} \mathbb{E}_2\left(-\frac{h}{k} + \frac{it}{2\pi}\right).$$

Proposition 5.3 then gives the claim. □

Remark. For odd d, we have that $(\frac{3}{d}) = (\frac{-3}{d}) = 1$ if and only if $d \equiv 1 \pmod{12}$ so that both F_1 and F_2 can be viewed as quantum modular forms with the trivial character under a suitable subgroup of Γ_p (e.g. the principal congruence subgroup $\Gamma(12p)$).

8. Completed indefinite theta functions

In this section, we embed the double Eichler integrals in a modular context by viewing them as "purely non-holomorphic" parts of indefinite theta series.

8.1. **Weight one.** The functions E_2 and M_2 were introduced in [1], where they played a crucial role in understanding modular indefinite theta functions of signature $(j, 2)$ $(j \in \mathbb{N}_0)$. We consider the quadratic form $Q_1(\boldsymbol{n}) := \frac{1}{2}\boldsymbol{n}^T A_1 \boldsymbol{n}$ and the bilinear form $B_1(\boldsymbol{n}, \boldsymbol{m}) := \boldsymbol{n}^T A_1 \boldsymbol{m}$ given by $A_1 := \left(\begin{smallmatrix} 6&3&6&3\\ 3&2&3&2\\ 6&3&0&0\\ 3&2&0&0 \end{smallmatrix}\right)$, and define $A_0 := \left(\begin{smallmatrix} 6&3\\ 3&2 \end{smallmatrix}\right)$, $P_0(\boldsymbol{n}) := M_2(\sqrt{3}; \sqrt{3}\,(2n_1+n_2), n_2)$ and, for $\boldsymbol{n} \in \mathbb{R}^4$, set

$$P(\boldsymbol{n}) := M_2\left(\sqrt{3}; \sqrt{3}(2n_3+n_4), n_4\right) + (\operatorname{sgn}(2n_3+n_4) + \operatorname{sgn}(n_1))\,(\operatorname{sgn}(3n_3+2n_4) + \operatorname{sgn}(n_2)) \\ + (\operatorname{sgn}(n_4) + \operatorname{sgn}(n_2))\, M_1\left(\sqrt{3}(2n_3+n_4)\right) + (\operatorname{sgn}(n_3) + \operatorname{sgn}(n_1))\, M_1(3n_3+2n_4).$$

Note that, for $\boldsymbol{\alpha} \in \mathscr{S}^*$,

$$2\mathcal{E}_{1,\boldsymbol{\alpha}}(\tau) = \Theta_{-A_0, P_0, \boldsymbol{\alpha}}(\tau).$$

We view this function as "purely non-holomorphic" part of the indefinite theta function

$$\Theta_{A_1, P, \boldsymbol{a}}(\tau) = \sum_{\boldsymbol{n} \in \boldsymbol{a} + \mathbb{Z}^4} P\left(\sqrt{v}\boldsymbol{n}\right) q^{Q_1(\boldsymbol{n})}, \tag{8.1}$$

where $\boldsymbol{a} \in \frac{1}{p}A_1^{-1}\mathbb{Z}^4$ with $(a_3, a_4) = (\alpha_1, \alpha_2)$. One can either employ Section 4.3 of [1] or proceed directly (as we do here) to prove the following proposition.

Proposition 8.1. *Assume that* $\boldsymbol{a} \in \frac{1}{p}A_1^{-1}\mathbb{Z}^4$ *with* $a_1, a_2 \notin \mathbb{Z}$.

(1) *We have*

$$\Theta_{A_1,P^-,\boldsymbol{a}}(\tau) = 2\mathcal{E}_{1,(a_3,a_4)}(\tau)\Theta_{A_0,1,(a_1-a_3,a_2-a_4)}(\tau),$$

where

$$P^-(\boldsymbol{n}) := M_2\left(\sqrt{3}; \sqrt{3}\left(2n_3+n_4\right), n_4\right).$$

(2) *The functions* $\Theta_{A_1,P,\boldsymbol{a}}$ *and* $\Theta_{-A_0,P_0,(a_3,a_4)}$ *converge absolutely and locally uniformly.*
(3) *The function* $\tau \mapsto \Theta_{A_1,P,a}(p\tau)$ *transforms like a modular form of weight two for some subgroup of* $\mathrm{SL}_2(\mathbb{Z})$ *and some character.*

Remark. When considering indefinite theta functions of signature $(j,2)$, one usually obtains four M_2-terms as the purely "non-holomorphic" part. The arguments of these four M_2-functions are dictated by the holomorphic part. The fact that $(1,0,0,0)^T$ and $(0,1,0,0)^T$ (which correspond to n_1 and n_2 occuring in P) have norm zero with respect to A_1^{-1} causes the "missing" M_2-terms to vanish. Therefore we refer to this situation as a *double null limit* (see [1]).

Proof of Proposition 8.1. (1) Shifting $(n_1,n_2,n_3,n_4) \mapsto (n_1-n_3, n_2-n_4, n_3, n_4)$ on the left hand side of the identity gives the claim.

(2) For $\Theta_{-A_0,P_0,(a_3,a_4)}$ we employ the asymptotic given in (2.7), to obtain

$$\left|M_2\left(\sqrt{3}; \sqrt{3v}\left(2n_1+n_2\right), \sqrt{v}n_2\right) q^{-\frac{1}{2}\boldsymbol{n}^TA_0\boldsymbol{n}}\right| \leq \frac{e^{-\pi\left(3(2n_1+n_2)^2+n_2^2\right)v}}{\pi^2 n_1 n_2} e^{\pi\boldsymbol{n}^TA_0\boldsymbol{n}v}$$
$$\leq c_1 e^{-2\pi\boldsymbol{n}^TA_0\boldsymbol{n}v} e^{\pi\boldsymbol{n}^TA_0\boldsymbol{n}v} = c_1 e^{-\pi\boldsymbol{n}^TA_0\boldsymbol{n}v}$$

for some $c_1 \in \mathbb{R}^+$ and $(n_1,n_2) \in (a_3,a_4)+\mathbb{Z}^2$ with $n_1, n_2 \neq 0$. By plugging in the definition, one can show that for some $c_2 \in \mathbb{R}^+$ and $n = (0,n_2) \in (a_3,a_4)+\mathbb{Z}^2$

$$\left|M_2\left(\sqrt{3}; \sqrt{3v}n_2, \sqrt{v}n_2\right) q^{-\frac{1}{2}\boldsymbol{n}^TA_0\boldsymbol{n}}\right| \leq c_2 e^{-\pi\boldsymbol{n}^TA_0\boldsymbol{n}v}$$

(and similarly for the case $n_2 = 0$). Using that A_0 is positive definite, we obtain, for some $c_3 \in \mathbb{R}^+$

$$\sum_{\boldsymbol{n}\in(a_3,a_4)+\mathbb{Z}^2} \left|M_2\left(\sqrt{3}; \sqrt{3v}\left(2n_1+n_2\right), \sqrt{v}n_2\right) q^{-\frac{1}{2}\boldsymbol{n}^TA_0\boldsymbol{n}}\right| \leq c_3 \sum_{\boldsymbol{n}\in(a_3,a_4)+\mathbb{Z}^2} e^{-\pi\boldsymbol{n}^TA_0\boldsymbol{n}v} < \infty,$$

implying the absolute and locally uniform convergence of $\Theta_{-A_0,P_0,(a_3,a_4)}$. Combining this with (1) and the convergence of the positive definite theta series $\Theta_{A_1,1,(a_1-a_3,a_2-a_4)}$, we obtain absolute and locally uniform convergence of the M_2-part of $\Theta_{A_1,P,\boldsymbol{a}}$.

For the part containing only sign-terms

$$\sum_{\boldsymbol{n}\in\boldsymbol{a}+\mathbb{Z}^4} \left(\mathrm{sgn}(2n_3+n_4)+\mathrm{sgn}(n_1)\right)\left(\mathrm{sgn}(3n_3+2n_4)+\mathrm{sgn}(n_2)\right) q^{Q_1(\boldsymbol{n})}, \tag{8.2}$$

we consider the determinant of $\Delta_{A_1}(\boldsymbol{n},\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3,\boldsymbol{b}_4)$, where $\left(\Delta_M(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_5)\right)_{j,\ell} := \boldsymbol{v}_j^T M\boldsymbol{v}_\ell$ and

$$(\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3,\boldsymbol{b}_4) := \frac{1}{3}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 2 & -3 & -1 & 0 \\ -3 & 6 & 0 & -3 \end{pmatrix}.$$

We compute the determinant $\det(\Delta_{A_1}(\boldsymbol{n},\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3,\boldsymbol{b}_4))$ via Laplace expansion to obtain

$$\begin{aligned} e^{-\pi v Q_1(\boldsymbol{n})} &\leq e^{-\pi\left(\frac{15}{16}B_1(\boldsymbol{b}_1,\boldsymbol{n})^2+\frac{2}{9}B_1(\boldsymbol{b}_2,\boldsymbol{n})^2+B_1(\boldsymbol{b}_1,\boldsymbol{n})B_1(\boldsymbol{b}_3,\boldsymbol{n})+2B_1(\boldsymbol{b}_2,\boldsymbol{n})B_1(\boldsymbol{b}_4,\boldsymbol{n})\right)v} \\ &\leq e^{-c_4\left(B_1(\boldsymbol{b}_1,\boldsymbol{n})^2+B_1(\boldsymbol{b}_2,\boldsymbol{n})^2+|B_1(\boldsymbol{b}_3,\boldsymbol{n})|+|B_1(\boldsymbol{b}_4,\boldsymbol{n})|\right)v} \end{aligned}$$

with some $c_4\in\mathbb{R}^+$ for all $\boldsymbol{n}\in\boldsymbol{a}+\mathbb{Z}^4$ which satisfy the condition

$$\left(\operatorname{sgn}(2n_3+n_4)+\operatorname{sgn}(n_1)\right)\left(\operatorname{sgn}(3n_3+2n_4)+\operatorname{sgn}(n_2)\right)\neq 0.$$

Thus (8.2) is dominated by

$$\begin{aligned} &\sum_{\boldsymbol{n}\in\boldsymbol{a}+\mathbb{Z}^4}\left|\left(\operatorname{sgn}(2n_3+n_4)+\operatorname{sgn}(n_1)\right)\left(\operatorname{sgn}(3n_3+2n_4)+\operatorname{sgn}(n_2)\right)e^{-\pi Q_1(\boldsymbol{n})v}\right| \\ &\qquad\leq 4\sum_{\boldsymbol{n}\in\boldsymbol{a}+\mathbb{Z}^4} e^{-c_4\left(B_1(\boldsymbol{b}_1,\boldsymbol{n})^2+B_1(\boldsymbol{b}_2,\boldsymbol{n})^2+|B_1(\boldsymbol{b}_3,\boldsymbol{n})|+|B_1(\boldsymbol{b}_4,\boldsymbol{n})|\right)v}<\infty. \end{aligned}$$

To deal with the contribution of the third and fourth summand of P one combines the approaches of the two previous terms.

(3) We use Lemma 2.1 to rewrite P as a limit of E_2-functions, namely

$$P(\boldsymbol{n})=\lim_{\varepsilon\to 0}\widehat{P}_\varepsilon(\boldsymbol{n}),$$

where

$$\begin{aligned} \widehat{P}_\varepsilon(\boldsymbol{n}) :=& \Bigg(E_2\left(\frac{\varepsilon}{3};\sqrt{3}\left(2n_3+n_4\right),-\varepsilon\left(n_1+n_3+\frac{n_4}{\sqrt{3}}\right)+\frac{3n_2}{\varepsilon\left(2\sqrt{3}-3\right)}\right) \\ &+E_2\left(\frac{\varepsilon}{2};\left(3n_3+2n_4\right),\frac{3n_1}{\varepsilon\left(2\sqrt{3}-3\right)}-\varepsilon\left(n_2+\sqrt{3}n_3+n_4\right)\right)+E_2\left(\sqrt{3};\sqrt{3}\left(2n_3+n_4\right),n_4\right) \\ &+E_2\left(-\sqrt{3};\frac{n_2}{2\varepsilon}-\frac{\varepsilon}{2}\left(n_2+2n_4\right),\frac{\sqrt{3}}{2\varepsilon}\left(2n_1+n_2\right)-\frac{\sqrt{3}}{2}\varepsilon\left(2n_1+n_2+4n_3+2n_4\right)\right)\Bigg). \end{aligned}$$

One can then verify that each occuring term $E_2(\kappa;b^Tn,c^Tn)$ satisfies the Vignéras differential equation given in Theorem 2.2 with $\lambda=0$ and $A=A_1$. A straightforward calculation shows that the Vignéras differential equation is satisfied for $\widehat{P}_\varepsilon$ with respect to A_1 if and only if it is satisfied for $\widehat{P}_{\varepsilon,p}(\boldsymbol{n}):=\widehat{P}_\varepsilon(\sqrt{p}\boldsymbol{n})$ with respect to pA_1. Furthermore, we have

$$\Theta_{A_1,P,\boldsymbol{a}}(p\tau)=\Theta_{pA_1,P_p,\boldsymbol{a}}(\tau)=\lim_{\varepsilon\to 0}\Theta_{pA_1,\widehat{P}_{\varepsilon,p},\boldsymbol{a}}(\tau)$$

where $P_p(\boldsymbol{n}):=P(\sqrt{p}\boldsymbol{n})$. We can apply Theorem 2.2 to obtain weight 2 modularity of $\Theta_{pA_1,\widehat{P}_{\varepsilon,p},\boldsymbol{a}}$ since $\boldsymbol{a}\in(pA_1)^{-1}\mathbb{Z}^4$. Now, taking the limit $\varepsilon\to 0$ proves the claim.

□

8.2. **Completion: weight two.** Similarly as in the previous Section 8.1, the function $\mathbb{E}_2$ may be related to a modular object of weight three. This connection becomes evident when writing $\mathbb{E}_2$ as a Jacobi derivative as in Lemma 6.2. We leave the details to the reader.

8.3. **Lowering.** The indefinite theta series considered in Subsection 8.1 are higher depth harmonic Maass forms following Zagier-Zwegers. Roughly speaking, by this we mean that applying the *Maass lowering operator* $L := -2iv^2\frac{\partial}{\partial\overline{\tau}}$ makes the function simpler. In particular, for the iterated Eichler integral, we have

$$L\left(I_{f_1,f_2}(\tau)\right) = 2^{k_1}v^{k_1}f_1\left(-\overline{\tau}\right)I_{f_2}(\tau).$$

Now $v^{k_1}f_1\left(-\overline{\tau}\right)$ is v^{k_1} times a conjugated modular form of weight k_1 (so transforming of weight $-k_1$) and I_{f_2}, defined in (2.11), is the non-holomorphic part of an harmonic Maass form of weight $2-k_2$.

9. Conclusion and further questions

We conclude here with several comments and research directions

(1) We plan to more systematically study higher depth quantum modular forms and to describe explicitly the quantum S-modular matrix of $F(q)$. This requires a modification of several arguments used here for $F_2(q)$ (note that we restricted ourselves to Γ_p out of necessity). This result would allow us to make a more precise connection between $W(p)_{A_2}$ and its irreducible modules. For one, we should be able to associate an S-matrix to the set of atypical irreducible $W(p)_{A_2}$-characters, in parallel to [8].
(2) Iterated (or multiple) Eichler integrals studied in Section 5 are of independent interest. As in other theories dealing with iterated integrals (e.g. non-commutative modular symbols, Chen's integrals and multiple zeta-values) shuffle relations are expected to play an important role. Another goal worth pursuing is to connect iterated Eichler integrals of half-integral weights to Manin's work [19].
(3) We plan to investigate the asymptotic of $F(q)$ in terms of finite q-series evaluated at root of unity. This requires certain hypergeometric type formulas for double rank two false theta functions.
(4) In recent work [6] we found a new expression for the error of modularity appearing in Propositions 5.2 and 5.3, at least if $M\tau = -\frac{1}{\tau}$. Our formulae involve what we end up calling, "double Mordell" integrals. In the rank one case this connection is well-understood [28, Theorem 1.16].
(5) Very recently, W. Yuasa [24] gave an explicit formula for the *tail* of $(2,2p)$-torus links associated to the sequence of colored Jones polynomials: $J_{n\omega_j}(K,q)$, $n\in\mathbb{N}$, where ω_j, $j=1,2$ are the fundamental weights. We were able to identify the same tail as a summand of $F(q)$, up to the factor $1-q$ (viz. extract the "diagonal" $m_1=m_2$ in formula (1.7)). This raises the following question: Is it true that $F(q)$ is the tail of $J_{n\rho}(K,q)$, $(n\in\mathbb{N})$ (here $\rho=\omega_1+\omega_2$), up to a rational function of q? For related computations of tails colored with $\mathfrak{sl}_3$ representations see [13].

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