HIGHER DEPTH QUANTUM MODULAR FORMS AND PLUMBED 3-MANIFOLDS

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ABSTRACT. In this paper we study new invariants $\widehat{Z}_{\boldsymbol{a}}(q)$ attached to plumbed 3-manifolds that were introduced by Gukov, Pei, Putrov, and Vafa. These remarkable q-series at radial limits conjecturally compute WRT invariants of the corresponding plumbed 3-manifold. Here we investigate the series $\widehat{Z}_0(q)$ for unimodular plumbing H-graphs with six vertices. We prove that for every positive definite unimodular plumbing matrix, $\widehat{Z}_0(q)$ is a depth two quantum modular form on $\mathbb{Q}$.

1. INTRODUCTION AND STATEMENT OF RESULTS

A *quantum modular form* is a complex-valued function defined on $\mathbb{Q}$ or a subset thereof, called the *quantum set*, that exhibits modular-like transformation properties up to an obstruction term with "nice" analytic properties (for instance, it can be extended to a real-analytic function on some open subset of $\mathbb{R}$). Quantum modular forms were introduced by Zagier in [23], where he described several non-trivial examples. They have appeared in several areas including quantum invariants of knots and 3-manifolds [16, 17, 18, 19], mock modular forms [13], meromorphic Jacobi forms [8], mathematical physics [12], partial and false theta functions [7], and representation theory [7, 11].

Motivated on the one hand by the concept of higher depth mock modular forms and on the other hand by the appearance of higher rank false theta functions in representation theory, Kaszian and two of the authors [4] defined so-called higher depth quantum modular forms, and gave an infinite family of examples coming from characters of representations of vertex algebras. If the depth is two, these functions satisfy

$$f(\tau) - (c\tau + d)^{-k} f(\gamma\tau) \in \mathcal{Q}^1\mathcal{O}(R), \qquad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}),$$

where $\mathcal{Q}^1$ is the space of quantum modular forms and $\mathcal{O}(R)$ is the space of real-analytic functions on some subset R of $\mathbb{R}$. All known examples of depth two quantum modular come from rank two

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partial theta functions ($q := e^{2\pi i\tau}, \tau \in \mathbb{H}$)

$$\sum_{\mathbf{n}\in\mathbb{N}_0^2+\boldsymbol{\beta}} q^{an_1^2+bn_2^2+cn_1n_2},$$

where $\boldsymbol{\beta} \in \mathbb{Q}^2$ (throughout we write vectors in bold letters and their components with subscripts) and $a, 4ab - c^2 > 0$. Further examples of this kind were studied in [3, 20]. Depth two quantum modular forms also appear as the coefficients of meromorphic Jacobi forms of negative matrix index [5].

In [15], as a part of the construction of homological invariants for closed 3-manifolds, Gukov, Pei, Putrov, and Vafa proposed a new approach to WRT invariants for a large class of 3-manifolds. For any plumbed 3-manifold, homeomorphically represented by a plumbing graph and positive definite linking matrix M [1], they [15] defined a certain family of q-series (called *homological blocks*)

$$\widehat{Z}_{\boldsymbol{a}}(q) := \frac{q^{\frac{-3N+\mathrm{tr}(M)}{4}}}{(2\pi i)^N}\mathrm{PV}\int_{|w_j|=1}\prod_{j=1}^{N} g(w_j)\prod_{(k,\ell)\in E} f(w_k,w_\ell)\Theta_{-M,\boldsymbol{a}}(q;\boldsymbol{w})\frac{dw_j}{w_j}, \tag{1.1}$$

where PV denotes the Cauchy principal value, integrals are oriented counterclockwise throughout, and $\int_{|w_j|=1}$ indicates the integration $\int_{|w_1|=1}\cdots\int_{|w_N|=1}$. Moreover $g(w_j)$ and $f(w_k, w_\ell)$ are certain simple rational functions defined in (2.7) and (2.8), respectively and

$$\Theta_{-M,\boldsymbol{a}}(q;\boldsymbol{w}) := \sum_{\boldsymbol{\ell}\in 2M\mathbb{Z}^N+\boldsymbol{a}} q^{\frac{1}{4}\boldsymbol{\ell}^T M^{-1}\boldsymbol{\ell}}\boldsymbol{w}^{\boldsymbol{\ell}}, \quad \boldsymbol{a} \in 2\mathrm{coker}(M) + \boldsymbol{\delta},$$

where $\boldsymbol{\delta} := (\delta_j)$ such that $\delta_j \equiv \deg(v_j) \pmod 2$ with δ_j denoting the *degree* (or valency) of the j-th node. Conjecturally, a suitable (explicit) linear combination of $\widehat{Z}_{\boldsymbol{a}}(q)$, denoted by $\widehat{Z}(q)$ in [15], is the universal WRT invariant, that is, as $q \to e^{\frac{2\pi i}{k}}$ its limit coincides with the SU(2) WRT invariant of M at level k. This, in particular, leads to another conjecture (attributed in [6] to Gukov) that $\widehat{Z}_{\boldsymbol{a}}(q)$ and $\widehat{Z}(q)$ are quantum modular forms. This conjecture can be verified for specific 3-manifolds obtained from unimodular 3-star plumbing graphs (e.g. the E_8 graph) [6, 9] due to the fact that $\widehat{Z}_{\boldsymbol{a}}(q)$ can be expressed via one-dimensional unary false theta functions

$$\sum_{n\in\mathbb{Z}} \mathrm{sgn}(n)q^{an^2+bn},$$

whose quantum modularity properties are well-understood [7, 17, 18, 19].

In this paper we investigate $\widehat{Z}_{\boldsymbol{a}}(q)$ for a family of non-Seifert plumbed 3-manifolds. We consider the simplest plumbing graph of this kind obtained by splicing two 3-star graphs. This way we obtain the so-called H-graph with six vertices (Figure 1), with the linking matrix

[1] In [15], M is negative definite, which we account for by replacing it with $-M$ when referring to their work.

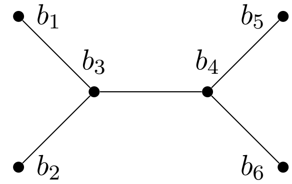

FIGURE 1. The H-graph

$$M = \begin{pmatrix} b_1 & 0 & -1 & 0 & 0 & 0 \\ 0 & b_2 & -1 & 0 & 0 & 0 \\ -1 & -1 & b_3 & -1 & 0 & 0 \\ 0 & 0 & -1 & b_4 & -1 & -1 \\ 0 & 0 & 0 & -1 & b_5 & 0 \\ 0 & 0 & 0 & -1 & 0 & b_6 \end{pmatrix}. \tag{1.2}$$

We only consider positive definite unimodular matrices whose 3-manifolds are integral homology spheres (i.e., $H_1(M_3, \mathbb{Z}) = 0$ as explained further in Section 2.7 below). Due to the invariance of $\widehat{Z}_{\boldsymbol{\delta}}(q)$ under a Kirby move [15], we may assume that $b_j \geq 2$, $j \in \{1,2,5,6\}$ (graphs with $b_j = 1$, $j \in \{1,2,5,6\}$ reduce to 3-star graphs whose quantum modularity is well-understood [6, 9]). With these assumptions $\widehat{Z}_{\boldsymbol{\delta}}(q)$ (also denoted by $\widehat{Z}_0(q)$ in [15]) is the only homological block and therefore it conjecturally gives WRT invariants at roots of unity. An important feature of this family of graphs is that $\widehat{Z}_{\boldsymbol{\delta}}(q)$ can be expressed via rank two false theta functions ($\boldsymbol{\beta} \in \mathbb{Q}^2$, $a, b, c \in \mathbb{N}$)

$$\sum_{\mathbf{n} \in \mathbb{Z}^2} \mathrm{sgn}^*(n_1)\mathrm{sgn}^*(n_2) q^{a(n_1+\beta_1)^2 + b(n_2+\beta_2)^2 + c(n_1+\beta_1)(n_2+\beta_2)},$$

where $\mathrm{sgn}^*(x) := \mathrm{sgn}(x)$ for $x \in \mathbb{R} \setminus \{0\}$ and $\mathrm{sgn}^*(0) := 1$. Our first result is on quantum modularity of certain partial theta functions needed to study $\widehat{Z}_{\boldsymbol{\delta}}(q)$.

More generally, we prove quantum modularity of an infinite family of false theta functions which we now introduce. Define

$$F_{\mathcal{S},Q,\varepsilon}(\tau) := \sum_{\boldsymbol{\alpha} \in \mathcal{S}} \varepsilon(\boldsymbol{\alpha}) \sum_{\boldsymbol{n} \in \mathbb{N}_0^2} q^{KQ(\boldsymbol{n}+\boldsymbol{\alpha})},$$

where $\mathcal{S} \subset \mathbb{Q}^2 \cap (0,1)^2$ is a finite set with the property that $(1,1) - \boldsymbol{\alpha}$, $(1-\alpha_1, \alpha_2)$, $(\alpha_1, 1-\alpha_2) \in \mathcal{S}$ for $\boldsymbol{\alpha} \in \mathcal{S}$, $\varepsilon : \mathcal{S} \to \mathbb{C}$ satisfies $\varepsilon(\boldsymbol{\alpha}) = \varepsilon((1,1) - \boldsymbol{\alpha}) = \varepsilon((1-\alpha_1, \alpha_2))$, and $K \in \mathbb{N}$ is minimal such that $K\mathcal{S} \subset \mathbb{N}^2$. For convenience, we extend the domain of ε to $\mathcal{S} + \mathbb{Z}^2$ by letting $\varepsilon(\boldsymbol{\alpha}) = \varepsilon(\boldsymbol{\alpha} + \boldsymbol{n})$, $\boldsymbol{n} \in \mathbb{Z}^2$.

Theorem 1.1. *The function $F_{\mathcal{S},Q,\varepsilon}$ is a quantum modular form of depth two, weight one, and quantum set $\mathcal{Q}_{\mathcal{S},Q,\varepsilon}$, defined in (3.1).*

Theorem 1.1 is of independent interest and can be used to investigate other examples of quantum modular forms.

Next we move on to studying unimodular matrices arising from H-graphs. Since the graph has six vertices it is not surprising that there are only finitely many positive definite unimodular matrices. We prove the following result.

Theorem 1.2. *There are, up to graph isomorphism, precisely* 39 *equivalence classes of unimodular positive definite plumbing matrices (1.2) with* $b_j \geq 2$, $j \in \{1, 2, 5, 6\}$.

Then our main result is the following.

Theorem 1.3. *For any positive definite unimodular plumbing matrix as in Theorem 1.2 there exists some* $c_M \in \mathbb{Q}$ *such that* $q^{c_M}\widehat{Z}_0(q)$ *is a quantum modular form of depth two, weight one, and quantum set* $\mathbb{Q}$.

Based on the results here and in [6, 9, 14], we can slightly reformulate Gukov's conjecture mentioned in [6] on the quantum modularity of $\widehat{Z}_{\boldsymbol{a}}(q)$ and $\widehat{Z}(q)$.

Conjecture 1.4. *Let* T *be a plumbing graph (tree) with* r *nodes of degree at least three. Then* $\widehat{Z}_{\mathbf{a}}(q)$ *is a depth* r *quantum modular form. Moreover, for any unimodular plumbing matrix,* $\widehat{Z}(q)$ *is quantum of depth* r *with quantum set* $\mathbb{Q}$.

Although it should be possible to give a more quantitative statement relating the structure of the graph to the weight of $\widehat{Z}(q)$, we do not include such a claim here due to the lack of supporting evidence beyond n-star graphs. For n-star graphs we strongly believe this weight to be $\frac{1}{2}+(n-3)$ (see [6, 9, 14]).

Combined with the conjecture on $\widehat{Z}(q)$ mentioned above, Conjecture 1.4 would imply that (unified) WRT invariants of plumbed 3-manifolds are higher depth quantum modular forms. We expect that the higher depth property also holds true for higher rank $\mathrm{SU}(N)$ invariants (see [10]).

For the remainder of the paper we primarily work with a modified version of the invariants for plumbed 3-manifolds, which we denote by $Z(q)$ (see (2.9)). In the case that M is unimodular we see that $Z(q)$ is closely related to $\widehat{Z}_0(q)$, and thus our main result follows from determining the quantum modularity of $Z(q)$.

The paper is organized as follows. In Section 2, we discuss special functions, the Euler-Maclaurin summation formula, higher depth quantum modular forms, and double Eichler integrals. We also define $Z(q)$ and describe its relationship to $\widehat{Z}_0(q)$. In Section 3 we show the quantum modularity of $F_{\mathcal{S},Q_1,\varepsilon}$ (see Theorem 3.1). In Section 4, we prove our main result on the quantum modularity of $Z(q)$ for unimodular plumbing graphs (see Theorem 4.1). The proof of the classification of positive definite unimodular matrices (1.2) is given in Section 5. Finally, in the appendix we list data for all 39 equivalence classes of positive unimodular matrices needed to compute $Z(q)$.

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2. Preliminaries

2.1. **Special functions.** Following [1] (with slightly different notation), for each $\kappa \in \mathbb{R}$ we define a function $E_2 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ by

$$E_2(\kappa;\boldsymbol{x}) := \int_{\mathbb{R}^2} \operatorname{sgn}(w_1)\operatorname{sgn}(w_2+\kappa w_1)\, e^{-\pi\left((w_1-x_1)^2+(w_2-x_2)^2\right)}dw_1dw_2.$$

For $x_2, x_1-\kappa x_2\neq 0$, we set

$$M_2(\kappa;\boldsymbol{x}) := -\tfrac{1}{\pi^2}\int_{\mathbb{R}^2-i\boldsymbol{x}} \frac{e^{-\pi w_1^2-\pi w_2^2-2\pi i(x_1w_1+x_2w_2)}}{w_2(w_1-\kappa w_2)} dw_1dw_2.$$

The following formula relates M_2 and E_2 :

$$M_2(\kappa;x_1+\kappa x_2,x_2) = E_2(\kappa;x_1+\kappa x_2,x_2)+\operatorname{sgn}(x_1)\operatorname{sgn}(x_2) \\ -\operatorname{sgn}(x_2)E(x_1+\kappa x_2)-\operatorname{sgn}(x_1)E\left(\tfrac{\kappa}{\sqrt{1+\kappa^2}}x_1+\sqrt{1+\kappa^2}x_2\right), \quad (2.1)$$

where for $x\in\mathbb{R}$, we set $E(x):=2\int_0^x e^{-\pi w^2}dw$.

The proof of the next result follows from the proof of [4, Lemma 6.1]. Here $\tau=u+iv$.

Proposition 2.1. *For $\kappa, x_1, x_2\in\mathbb{R}$ we have*

$$M_2(\kappa;x_1,x_2) = -\frac{x_1}{2\sqrt{v}}\frac{x_2}{\sqrt{v}}q^{\frac{x_1^2}{4v}+\frac{x_2^2}{4v}}\int_{-\overline{\tau}}^{i\infty}\frac{e^{\frac{\pi i x_1^2 w_1}{2v}}}{\sqrt{-i(w_1+\tau)}}\int_{w_1}^{i\infty}\frac{e^{\frac{\pi i x_2^2 w_2}{2v}}}{\sqrt{-i(w_2+\tau)}}dw_2dw_1 \quad (2.2)$$

$$-\frac{x_2+\kappa x_1}{2\sqrt{(1+\kappa^2)v}}\frac{x_1-\kappa x_2}{\sqrt{(1+\kappa^2)v}}q^{\frac{(x_2+\kappa x_1)^2}{4(1+\kappa^2)v}+\frac{(x_1-\kappa x_2)^2}{4(1+\kappa^2)v}}\int_{-\overline{\tau}}^{i\infty}\frac{e^{\frac{\pi i(x_2+\kappa x_1)^2w_1}{2(1+\kappa^2)v}}}{\sqrt{-i(w_1+\tau)}}\int_{w_1}^{i\infty}\frac{e^{\frac{\pi i(x_1-\kappa x_2)^2w_2}{2(1+\kappa^2)v}}}{\sqrt{-i(w_2+\tau)}}dw_2dw_1.$$

2.2. **Euler-Maclaurin summation formula.** Let $B_m(x)$ be the m-th Bernoulli polynomial defined by $\frac{we^{xw}}{e^w-1}=:\sum_{m\geq 0}B_m(x)\frac{w^m}{m!}$. We require

$$B_m(1-x)=(-1)^mB_m(x). \quad (2.3)$$

The Euler-Maclaurin summation formula implies the following lemma.

Lemma 2.2. *For $\boldsymbol{\alpha}\in\mathbb{R}^2$, $F:\mathbb{R}^2\to\mathbb{R}$ a C^∞-function which has rapid decay, we have*

$$\sum_{\boldsymbol{n}\in\mathbb{N}_0^2}F((\boldsymbol{n}+\boldsymbol{\alpha})t)$$

$$\sim \frac{\mathcal{I}_F}{t^2}-\sum_{n_2\geq 0}\frac{B_{n_2+1}(\alpha_2)}{(n_2+1)!}\int_0^\infty F^{(0,n_2)}(x_1,0)dx_1t^{n_2-1}-\sum_{n_1\geq 0}\frac{B_{n_1+1}(\alpha_1)}{(n_1+1)!}\int_0^\infty F^{(n_1,0)}(0,x_2)dx_2t^{n_1-1}$$

$$+\sum_{n_1,n_2\geq 0}\frac{B_{n_1+1}(\alpha_1)}{(n_1+1)!}\frac{B_{n_2+1}(\alpha_2)}{(n_2+1)!}F^{(n_1,n_2)}(0,0)t^{n_1+n_2},$$

where $\mathcal{I}_F:=\int_0^\infty\int_0^\infty F(\boldsymbol{x})dx_1dx_2$. Here by $\sim$ we mean that the difference between the left- and the right-hand side is $O(t^N)$ for any $N\in\mathbb{N}$.

2.3. **Gauss sums.** We define for $a,b,c\in\mathbb{Z}$ with $c>0$ the *quadratic Gauss sums*

$$G_c(a,b):=\sum_{n \pmod{c}}e^{\frac{2\pi i}{c}(an^2+bn)};$$

see [2, Section 1.5] for some basic properties. We use the following elementary result on the vanishing of $G_c(a,b)$.

Proposition 2.3. *If* $\gcd(a,c)\nmid b$*, then* $G_c(a,b)=0$*.*

2.4. **Shimura theta functions.** We require certain theta functions studied, for example, by Shimura [22]. For $\nu\in\{0,1\}$, $h\in\mathbb{Z}$, $N,A\in\mathbb{N}$, with $A|N$, $N|hA$, define

$$\vartheta_\nu(A,h,N;\tau):=\sum_{\substack{m\in\mathbb{Z}\\ m\equiv h \pmod N}} m^\nu q^{\frac{Am^2}{2N^2}}.$$

Define the *slash operator* of weight $k\in\frac{1}{2}\mathbb{Z}$ ($\left(\frac{\cdot}{\cdot}\right)$ the Jacobi symbol)

$$f\big|_k\gamma(\tau):=\left(\tfrac{c}{d}\right)^{2k}\varepsilon_d^{2k}(c\tau+d)^{-k}f(\gamma\tau),\qquad \gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z}).$$

Note that if $k\in\mathbb{Z}+\frac{1}{2}$, we require that $\gamma\in\Gamma_0(4)$. Recall that Shimura's modular transformation formula [22, Proposition 2.1] states that for $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(2N)$, with $2|b$, we have

$$\vartheta_\nu(A,h,N;\tau)\mid_{\frac{3}{2}}\gamma=e\left(\frac{abAh^2}{2N^2}\right)\left(\frac{-2A}{d}\right)\vartheta_\nu(A,ah,N;\tau). \tag{2.4}$$

Here $e(x):=e^{2\pi ix}$ and for odd d, we define $\varepsilon_d=1$ or i, depending on whether $d\equiv 1 \pmod 4$ or $d\equiv 3 \pmod 4$.

2.5. **Integral evaluations.** We require, for $m\in\mathbb{Z}$,

$$\int_{|w|=1}\left(w-w^{-1}\right)w^m\frac{dw}{w}=\int_{|w|=1}w^m dw-\int_{|w|=1}w^{m-2}dw=2\pi i\left(\delta_{m,-1}-\delta_{m,1}\right), \tag{2.5}$$

where $\delta_{m,a}=0$ unless $m=a$ in which case it equals 1 and

$$\frac{1}{2\pi i}\mathrm{PV}\int_{|w|=1}\frac{w^m}{w-w^{-1}}\frac{dw}{w}=\frac{1}{2}\mathrm{sgn}_o(m), \tag{2.6}$$

where $\mathrm{sgn}_o(m):=\frac{1}{2}\mathrm{sgn}(m)(1-(-1)^m)$.

2.6. **Higher depth quantum modular forms.** We now give the formal definition of quantum modular forms, following [23].

Definition 2.4. *A function* $f:\mathcal{Q}\to\mathbb{C}$ *(*$\mathcal{Q}\subseteq\mathbb{Q}$*) is called a quantum modular form of weight* $k\in\frac{1}{2}\mathbb{Z}$ *for a subgroup* Γ *of* $\mathrm{SL}_2(\mathbb{Z})$ *(of* $\Gamma_0(4)$ *if* $k\in\mathbb{Z}+\frac{1}{2}$*) and quantum set* $\mathcal{Q}$ *if for* $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$*, the function*

$$f(\tau)-f\big|_k\gamma(\tau)$$

can be extended to an open subset of $\mathbb{R}$ *on which it is real-analytic. We denote the vector space of such forms by* $\mathcal{Q}_k(\Gamma)$*.*

We next turn to the definition of higher-depth quantum modular forms (see Definition 3 of [4]).

Definition 2.5. *A function $f : \mathcal{Q} \to \mathbb{C}$ ($\mathcal{Q} \subset \mathbb{Q}$) is called a quantum modular form of depth $N \in \mathbb{N}$, weight $k \in \frac{1}{2}\mathbb{Z}$, and quantum set $\mathcal{Q}$ for Γ if for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$*

$$f - f\big|_k \gamma \in \bigoplus_j \mathcal{Q}_{\kappa_j}^{N_j}(\Gamma)\mathcal{O}(R),$$

where j runs through a finite set, $\kappa_j \in \frac{1}{2}\mathbb{Z}$, $N_j \in \mathbb{N}$ with $\max_j(N_j) = N - 1$, $\mathcal{Q}_k^1(\Gamma) := \mathcal{Q}_k(\Gamma)$, $\mathcal{Q}_k^0(\Gamma) := 1$, and $\mathcal{Q}_k^N(\Gamma)$ is the space of quantum modular forms of weight k and depth N for Γ.

For $f_j \in S_{k_j}(\Gamma)$, the space of cusp forms of weight k_j for Γ with $k_j > \frac{1}{2}$, define the *(non-holomorphic) Eichler integrals*

$$I_f(\tau) := \int_{-\overline{\tau}}^{i\infty} \frac{f(w)}{(-i(w+\tau))^{2-k}} dw,$$
$$I_{f_1,f_2}(\tau) := \int_{-\overline{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{f_1(w_1)f_2(w_2)}{(-i(w_1+\tau))^{2-k_1}(-i(w_2+\tau))^{2-k_2}} dw_2 dw_1,$$

and the *errors of modularity*, for $\varrho \in \mathbb{Q}$

$$r_{f,\varrho}(\tau) := \int_{\varrho}^{i\infty} \frac{f(w)}{(-i(w+\tau))^{2-k}} dw,$$
$$r_{f_1,f_2,\varrho}(\tau) := \int_{\varrho}^{i\infty} \int_{w_1}^{\varrho} \frac{f_1(w_1)f_2(w_2)}{(-i(w_1+\tau))^{2-k_1}(-i(w_2+\tau))^{2-k_2}} dw_2 dw_1.$$

The next result is [4, Theorem 5.1].

Theorem 2.6. *We have, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma^* := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \Gamma \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$,*

$$I_{f_1,f_2}(\tau) - I_{f_1,f_2} \,|_{k_1+k_2-4}\, \gamma(\tau) = r_{f_1,f_2,\tau,\frac{d}{c}}(\tau) + I_{f_1}(\tau) r_{f_2,\frac{d}{c}}(\tau).$$

Moreover $r_{f_1,f_2,\frac{d}{c}} \in \mathcal{O}(\mathbb{R})$.

2.7. **Definitions and notation.** In this section we recall the construction of $Z(q)$ following [6], which is another invariant that is closely related to $\widehat{Z}_{\boldsymbol{a}}(q)$ from (1.1). Consider a tree G with N vertices labeled by integers m_{jj}, $1 \le j \le N$, which is called a *plumbing graph.* To this data we associate an $N \times N$ matrix $M = (m_{jk})_{1 \le j,k \le N}$, called its *linking* (or *plumbing*) *matrix*, such that $m_{jk} = -1$ if vertex j is connected to vertex k and zero otherwise. We say that two plumbing matrices M and M' are *equivalent* if their underlying graphs are isomorphic, and there is a graph isomorphism that maps M to M'. The first homology group of $M_3(G)$ (the plumbed 3-manifold constructed from G and M) is

$$H_1(M_3(G), \mathbb{Z}) \cong \operatorname{coker}(M) = \mathbb{Z}^N / M\mathbb{Z}^N.$$

If M is invertible, then this group is finite and if $M \in \mathrm{SL}_N(\mathbb{Z})$, then $H_1(M_3, \mathbb{Z}) = 0$; this is the case for the main results of this paper, as M is positive definite and unimodular.

To each edge $j-k$ in G we associate a rational function

$$f(w_j,w_k):=\frac{1}{\left(w_j-w_j^{-1}\right)\left(w_k-w_k^{-1}\right)} \tag{2.7}$$

and to each vertex w_j a Laurent polynomial

$$g(w_j):=\left(w_j-w_j^{-1}\right)^2. \tag{2.8}$$

For a fixed tree G and positive definite M, set

$$Z(q):=\frac{q^{\frac{-3N+\sum_{\nu=1}^N a_\nu}{2}}}{(2\pi i)^N}\mathrm{PV}\int_{|w_j|=1}\prod_{j=1}^N g(w_j)\prod_{(k,\ell)\in E} f(w_k,w_\ell)\Theta_M(q;\boldsymbol{w})\frac{dw_j}{w_j}, \tag{2.9}$$

where we let $a_j:=m_{jj}$ for the vertex labels, $w_j:=e^{2\pi i z_j}$, and

$$\Theta_M(q;\boldsymbol{w}):=\sum_{\boldsymbol{n}\in\mathbb{Z}^N} q^{\frac12\boldsymbol{n}^TM\boldsymbol{n}}e^{2\pi i\boldsymbol{n}^TM\boldsymbol{z}}.$$

Note that we may write

$$\Theta_M(q;\boldsymbol{w})=\sum_{\boldsymbol{m}\in M\mathbb{Z}^N} q^{\frac12\boldsymbol{m}^TM^{-1}\boldsymbol{m}}e^{2\pi i\boldsymbol{m}^T\boldsymbol{z}}. \tag{2.10}$$

The following result is given in Proposition 3.4 of [6].

Proposition 2.7. *If M is unimodular, then $Z(q)=\widehat{Z}_{\boldsymbol{\delta}}(q^2)$, where $\widehat{Z}_{\boldsymbol{\delta}}(q)$ is defined in (1.1).*

3. A general construction

In this section we construct an infinite family of quantum modular forms of depth two closely following the arguments in [4]. Define

$$\mathcal{Q}_{\mathcal{S},Q,\varepsilon}:=\left\{\frac hk\in\mathbb{Q}\,:\,\gcd(h,k)=1,\,k\in\mathbb{N},\,\sum_{\boldsymbol{\alpha}\in\mathcal{S}}\varepsilon(\boldsymbol{\alpha})\sum_{\boldsymbol{\ell}\pmod k}e^{2\pi i\frac hk KQ(\boldsymbol{\ell}+\boldsymbol{\alpha})}=0\right\}. \tag{3.1}$$

We write $Q(\boldsymbol{n})=:\sigma_1n_1^2+2\sigma_2n_1n_2+\sigma_3n_2^2$, and denote its discriminant by $D:=\sigma_1\sigma_3-\sigma_2^2$. We also regularly use the relationship between the quadratic form and the associated bilinear form, namely

$$Q(\boldsymbol{x}+\boldsymbol{y})-Q(\boldsymbol{x})-Q(\boldsymbol{y})=B(\boldsymbol{x},\boldsymbol{y}). \tag{3.2}$$

Theorem 3.1. *The functions $F_{\mathcal{S},Q,\varepsilon}$ are quantum modular forms of depth two, weight one, on some congruence subgroup containing $\Gamma(8\cdot\mathrm{lcm}(\sigma_1,\sigma_3)KD)$, and quantum set $\mathcal{Q}_{\mathcal{S},Q,\varepsilon}$.*

Before proving Theorem 3.1, we require some auxiliary lemmas. Set

$$\mathbb{E}_{\mathcal{S},Q,\varepsilon}(\tau):=\sum_{\boldsymbol{\alpha}\in\mathcal{S}}\varepsilon(\boldsymbol{\alpha})\mathbb{F}_{Q,\boldsymbol{\alpha}}(\tau),$$

where

$$\mathbb{F}_{Q,\boldsymbol{\alpha}}(\tau) := \frac{1}{2}\sum_{\boldsymbol{n}\in\mathbb{Z}^2+\boldsymbol{\alpha}} M_2\left(\kappa; (a_1n_1+a_2n_2, b_2n_2)\sqrt{Kv}\right) q^{-KQ(\boldsymbol{n})}$$

with

$$\kappa := \frac{\sigma_2}{\sqrt{D}}, \qquad a_1 := 2\sqrt{\sigma_1}, \qquad a_2 := \frac{2\sigma_2}{\sqrt{\sigma_1}}, \qquad b_2 := 2\sqrt{\frac{D}{\sigma_1}}.$$

We begin by determining the asymptotic expansions of these functions.

Lemma 3.2. *If $\frac{h}{k}\in\mathcal{Q}_{\mathcal{S},\varepsilon}$, then we have the asymptotic expansions (as $t\to 0^+$)*

$$F_{\mathcal{S},Q,\varepsilon}\left(\frac{h}{k}+\frac{it}{2\pi}\right) =: \sum_{m\geq 0} a_{h,k}(m)t^m, \qquad \mathbb{E}_{\mathcal{S},Q,\varepsilon}\left(\frac{h}{k}+\frac{it}{2\pi}\right) = \sum_{m\geq 0} a_{-h,k}(m)(-t)^m. \tag{3.3}$$

Proof. For the proof we abbreviate

$$F := F_{\mathcal{S},Q,\varepsilon}, \qquad \mathbb{E} := \mathbb{E}_{\mathcal{S},Q,\varepsilon}, \qquad \mathcal{Q} := \mathcal{Q}_{\mathcal{S},Q,\varepsilon}.$$

We first determine the asymptotic expansion of F using the Euler-Maclaurin summation formula. We let $\boldsymbol{n}\mapsto\boldsymbol{\ell}+k\boldsymbol{n}$ with $0\leq\boldsymbol{\ell}\leq k-1$ (i.e., $0\leq\ell_j\leq k-1$, $j\in\{1,2\}$), $\boldsymbol{n}\in\mathbb{N}_0^2$. The assumption that $K\mathcal{S}\subset\mathbb{N}^2$ implies that $\frac{h}{k}KQ(\boldsymbol{\ell}+\boldsymbol{\alpha}+k\boldsymbol{n})\equiv\frac{h}{k}KQ(\boldsymbol{\ell}+\boldsymbol{\alpha}) \pmod 1$, thus

$$F\left(\frac{h}{k}+\frac{it}{2\pi}\right) = \sum_{\boldsymbol{\alpha}\in\mathcal{S}}\varepsilon(\boldsymbol{\alpha})\sum_{0\leq\boldsymbol{\ell}\leq k-1} e^{2\pi i\frac{h}{k}KQ(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{\boldsymbol{n}\in\mathbb{N}_0^2+\frac{1}{k}(\boldsymbol{\ell}+\boldsymbol{\alpha})} g\left(k\sqrt{t}\boldsymbol{n}\right),$$

where $g(\boldsymbol{x}) := e^{-KQ(\boldsymbol{x})}$. The main term in Lemma 2.2 is

$$\frac{\mathcal{I}_g}{k^2t}\sum_{\boldsymbol{\alpha}\in\mathcal{S}}\varepsilon(\boldsymbol{\alpha})\sum_{0\leq\boldsymbol{\ell}\leq k-1} e^{2\pi i\frac{h}{k}KQ(\boldsymbol{\ell}+\boldsymbol{\alpha})}.$$

Using that $K\mathcal{S}\subset\mathbb{N}^2$ we may let $\boldsymbol{\ell}$ run $\pmod k$. Since $\frac{h}{k}\in\mathcal{Q}$ the sum vanishes.

The second term in Lemma 2.2 yields

$$-\sum_{\boldsymbol{\alpha}\in\mathcal{S}}\varepsilon(\boldsymbol{\alpha})\sum_{0\leq\boldsymbol{\ell}\leq k-1} e^{2\pi i\frac{h}{k}KQ(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_2\geq 0}\frac{B_{n_2+1}\left(\frac{1}{k}(\ell_2+\alpha_2)\right)}{(n_2+1)!}\int_0^\infty g^{(0,n_2)}(x_1,0)dx_1\left(k\sqrt{t}\right)^{n_2-1}. \tag{3.4}$$

Making the change of variables $\boldsymbol{\ell}\mapsto(k-1)(1,1)-\boldsymbol{\ell}$ and using that $(1,1)-\boldsymbol{\alpha}\in\mathcal{S}$ if $\boldsymbol{\alpha}\in\mathcal{S}$, (2.3) yields that only the odd values of n_2 survive, and (3.4) becomes

$$-\sum_{\boldsymbol{\alpha}\in\mathcal{S}}\varepsilon(\boldsymbol{\alpha})\sum_{0\leq\boldsymbol{\ell}\leq k-1} e^{2\pi i\frac{h}{k}KQ(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_2\geq 0}\frac{B_{2n_2+2}\left(\frac{1}{k}(\ell_2+\alpha_2)\right)}{(2n_2+2)!}\int_0^\infty g^{(0,2n_2+1)}(x_1,0)dx_1k^{2n_2}t^{n_2}.$$

In exactly the same way we obtain that the third term in Lemma 2.2 equals

$$-\sum_{\boldsymbol{\alpha}\in\mathcal{S}}\varepsilon(\boldsymbol{\alpha})\sum_{0\leq\boldsymbol{\ell}\leq k-1} e^{2\pi i\frac{h}{k}KQ(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_1\geq 0}\frac{B_{2n_1+2}\left(\frac{1}{k}(\ell_1+\alpha_1)\right)}{(2n_1+2)!}\int_0^\infty g^{(2n_1+1,0)}(0,x_2)dx_2k^{2n_1}t^{n_1}.$$

For the final term in Lemma 2.2 we obtain, pairing in exactly the same way

$$\sum_{\boldsymbol{\alpha}\in\mathcal{S}}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le k-1}e^{2\pi i\frac{h}{k}KQ(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{\substack{n_1,n_2\ge 0\\ n_1\equiv n_2 \pmod 2}}\frac{B_{n_1+1}\left(\frac{1}{k}(\ell_1+\alpha_1)\right)}{(n_1+1)!}\frac{B_{n_2+1}\left(\frac{1}{k}(\ell_2+\alpha_2)\right)}{(n_2+1)!}$$
$$\times g^{(n_1,n_2)}(0,0)\left(k\sqrt{t}\right)^{n_1+n_2}.$$

In particular we obtain that the asymptotic expansion of F has the shape as claimed in (3.3).

We now turn to the asymptotic behavior of $\mathbb{E}$. We use (2.1) and let M_2^* denote the function such that the sgn in (2.1) is replaced by sgn^*, where $\mathrm{sgn}^*(x) := \mathrm{sgn}(x)$ if $x \in \mathbb{R}\setminus\{0\}$ and $\mathrm{sgn}^*(0) := 1$. We obtain

$$M_2^*(\kappa; a_1n_1+a_2n_2, b_2n_2) = E_2(\kappa; a_1n_1+a_2n_2, b_2n_2) + \mathrm{sgn}^*(n_1)\mathrm{sgn}^*(n_2)$$
$$-\,\mathrm{sgn}^*(n_2)E\left(\tfrac{2}{\sqrt{\sigma_1}}(\sigma_1n_1+\sigma_2n_2)\right) - \mathrm{sgn}^*(n_1)E\left(\tfrac{2}{\sqrt{\sigma_3}}(\sigma_2n_1+\sigma_3n_2)\right).$$

Proceeding as above

$$\mathbb{E}\left(\tfrac{h}{k}+\tfrac{it}{2\pi}\right) = \sum_{\boldsymbol{\alpha}\in\mathcal{S}}\varepsilon(\boldsymbol{\alpha})\left(\sum_{0\le\boldsymbol{\ell}\le k-1}e^{-2\pi i\frac{h}{k}KQ(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{\boldsymbol{n}\in\mathbb{N}_0^2+\boldsymbol{\alpha}}G\left(k\sqrt{t}\boldsymbol{n}\right)\right.$$
$$\left.+\sum_{0\le\boldsymbol{\ell}\le k-1}e^{-2\pi i\frac{h}{k}K\widetilde{Q}(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{\boldsymbol{n}\in\mathbb{N}_0^2+\boldsymbol{\alpha}}\widetilde{G}\left(k\sqrt{t}\boldsymbol{n}\right)\right),$$

where

$$\widetilde{Q}(x_1,x_2) := Q(-x_1,x_2)$$

$$G(\boldsymbol{x}) := \tfrac{1}{2}M_2^*\left(\kappa; \sqrt{\tfrac{K}{2\pi}}(a_1x_1+a_2x_2, b_2x_2)\right)e^{KQ(\boldsymbol{x})}, \qquad \widetilde{G}(x_1,x_2) := G(-x_1,x_2).$$

We again use the Euler-Maclaurin summation formula. The main term in Lemma 2.2 is

$$\tfrac{4\mathcal{I}_G}{k^2t}\sum_{\boldsymbol{\alpha}\in\mathcal{S}}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le k-1}e^{-2\pi i\frac{h}{k}KQ(\boldsymbol{\ell}+\boldsymbol{\alpha})} + \tfrac{4\mathcal{I}_{\widetilde{G}}}{k^2t}\sum_{\boldsymbol{\alpha}\in\mathcal{S}}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le k-1}e^{-2\pi i\frac{h}{k}K\widetilde{Q}(\boldsymbol{\ell}+\boldsymbol{\alpha})} = 0$$

by conjugating the condition in $\mathcal{Q}$.

The second term in Lemma 2.2 is, pairing terms as before,

$$-\sum_{\boldsymbol{\alpha}\in\mathcal{S}}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le k-1}e^{-2\pi i\frac{h}{k}KQ(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{n_2\ge 0}\frac{B_{2n_2+2}\left(\frac{1}{k}(\ell_2+\alpha_2)\right)}{(2n_2+2)!}$$
$$\times\int_0^\infty\left(G^{(0,2n_2+1)}(x_1,0)+\widetilde{G}^{(0,2n_2+1)}(x_1,0)\right)dx_1\left(k^2t\right)^{n_2}.$$

It is now straightforward to verify, as in [4], that

$$\int_0^\infty\left(G^{(0,2n_2+1)}(x_1,0)+\widetilde{G}^{(0,2n_2+1)}(x_1,0)\right)dx_1 = (-1)^{n_2}\int_0^\infty g^{(0,2n_2+1)}(x_1,0)dx_1.$$

Via symmetry the third term in Lemma 2.2 is treated in exactly the same way.

The fourth term in Lemma 2.2 is, pairing as before,

$$\sum_{\boldsymbol{\alpha}\in\mathcal{S}}\varepsilon(\boldsymbol{\alpha})\sum_{0\le\boldsymbol{\ell}\le k-1}e^{-2\pi i\frac{h}{k}KQ(\boldsymbol{\ell}+\boldsymbol{\alpha})}\sum_{\substack{n_1,n_2\ge0\\ n_1\equiv n_2 \pmod 2}}\frac{B_{n_1+1}\left(\frac{1}{k}(\ell_1+\alpha_1)\right)}{(n_1+1)!}\frac{B_{n_2+1}\left(\frac{1}{k}(\ell_2+\alpha_2)\right)}{(n_2+1)!}$$
$$\times\left(G^{(n_1,n_2)}(0,0)+(-1)^{n_1+1}\widetilde{G}^{(n_1,n_2)}(0,0)\right)\left(k\sqrt{t}\right)^{n_1+n_2}.$$

It can now be shown that

$$G^{(n_1,n_2)}(0,0)+(-1)^{n_1+1}\widetilde{G}^{(n_1,n_2)}(0,0)=i^{n_1+n_2}g^{(n_1,n_2)}(0,0).$$

Comparing terms gives the claim. □

Write $\mathcal{A}:=K\mathcal{S}$, and define

$$\mathcal{B}:=\{0\le\boldsymbol{B}<\sigma_1K : B_1=\sigma_1A_1+\sigma_2A_2+\varrho\sigma_2K,\ B_2=A_2+\varrho K,\ \text{for some }\boldsymbol{A}\in\mathcal{A},\varrho \pmod{\sigma_1}\},$$
$$\mathcal{C}:=\{0\le\boldsymbol{C}<\sigma_3K : C_1=\sigma_2A_1+\sigma_3A_2+\varrho\sigma_2K,\ C_2=A_2+\varrho K,\ \text{for some }\boldsymbol{A}\in\mathcal{A},\varrho \pmod{\sigma_3}\}.$$

The following lemma rewrites $\mathbb{E}$ as a two-dimensional theta integral, which is essential in order to calculate modular transformations.

Lemma 3.3. *We have*

$$\mathbb{E}_{\mathcal{S},Q,\varepsilon}(\tau)=-\frac{\sqrt{D}}{2\sigma_1K}\sum_{\boldsymbol{B}\in\mathcal{B}}\varepsilon\left(\tfrac{B_1-\sigma_2B_2}{\sigma_1K},\tfrac{B_2}{K}\right)I_{T_1,T_2}(\tau)-\frac{\sqrt{D}}{2\sigma_3K}\sum_{\boldsymbol{C}\in\mathcal{C}}\varepsilon\left(\tfrac{C_2}{K},\tfrac{C_2-\sigma_2C_1}{\sigma_3K}\right)I_{U_1,U_2}(\tau),$$

where

$$T_1(w):=\vartheta_1(\sigma_1K,B_1,\sigma_1K;2w),\qquad T_2(w):=\vartheta_1(\sigma_1K,B_2,\sigma_1K;2Dw),$$
$$U_1(w):=\vartheta_1(\sigma_3K,C_1,\sigma_3K;2w),\qquad U_2(w):=\vartheta_1(\sigma_3K,C_2,\sigma_3K;2Dw).$$

Proof. Using (2.2) we obtain

$$M_2\left(\kappa;(a_1n_1+a_2n_2,bn_2)\sqrt{Kv}\right)q^{-KQ(\boldsymbol{n})}$$
$$=-\frac{2\sqrt{D}}{\sigma_1}(\sigma_1n_1+\sigma_2n_2)n_2\int_{-K\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{e^{\frac{2\pi i}{\sigma_1}(\sigma_1n_1+\sigma_2n_2)^2w_1+\frac{2\pi iDn_2^2}{\sigma_1}w_2}}{\sqrt{-i(w_1+K\tau)}\sqrt{-i(w_2+K\tau)}}dw_2dw_1$$
$$-\frac{2\sqrt{D}}{\sigma_3}(\sigma_2n_1+\sigma_3n_2)n_1\int_{-K\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{e^{\frac{2\pi i}{\sigma_3}(\sigma_2n_1+\sigma_3n_2)^2w_1+\frac{2\pi iDn_1^2}{\sigma_3}w_2}}{\sqrt{-i(w_1+K\tau)}\sqrt{-i(w_2+K\tau)}}dw_2dw_1.$$

This yields

$$\mathbb{E}_{\mathcal{S},Q,\varepsilon}(\tau)=-\frac{K\sqrt{D}}{\sigma_1}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{\theta_1(\boldsymbol{w})}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1$$

$$- \frac{K\sqrt{D}}{\sigma_3}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{\theta_2(\boldsymbol{w})}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1,$$

where

$$\theta_1(\boldsymbol{w}) := \sum_{\boldsymbol{\alpha}\in\mathcal{S}}\varepsilon(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\mathbb{Z}^2+\boldsymbol{\alpha}}(\sigma_1n_1+\sigma_2n_2)n_2e^{\frac{2\pi iK}{\sigma_1}(\sigma_1n_1+\sigma_2n_2)^2w_1+\frac{2\pi iDK}{\sigma_1}n_2^2w_2},$$
$$\theta_2(\boldsymbol{w}) := \sum_{\boldsymbol{\alpha}\in\mathcal{S}}\varepsilon(\boldsymbol{\alpha})\sum_{\boldsymbol{n}\in\mathbb{Z}^2+\boldsymbol{\alpha}}(\sigma_2n_1+\sigma_3n_2)n_1e^{\frac{2\pi iK}{\sigma_3}(\sigma_2n_1+\sigma_3n_2)^2w_1+\frac{2\pi iDK}{\sigma_3}n_1^2w_2}.$$

We now rewrite the θ_j in terms of the Shimura theta functions. Letting $\boldsymbol{n}\mapsto\frac{\boldsymbol{n}}{K}$, we obtain

$$\theta_1(\boldsymbol{w}) = \tfrac{1}{K^2}\sum_{\boldsymbol{A}\in\mathcal{A}}\varepsilon\left(\tfrac{\boldsymbol{A}}{K}\right)\sum_{\boldsymbol{n}\equiv\boldsymbol{A}\pmod{K}}(\sigma_1n_1+\sigma_2n_2)n_2e^{\frac{2\pi i}{\sigma_1K}(\sigma_1n_1+\sigma_2n_2)^2w_1+\frac{2\pi iD}{\sigma_1K}n_2^2w_2}.$$

Set $\nu_1 := \sigma_1n_1+\sigma_2n_2$ and $\nu_2 := n_2$, so that $n_1 = \frac{\nu_1-\sigma_2\nu_2}{\sigma_1}$. Plugging in the restrictions on $\boldsymbol{n}$ yields

$$\nu_2 \equiv A_2+\varrho K \pmod{\sigma_1K} \qquad \text{for } 0\le\varrho\le\sigma_1-1,$$
$$\nu_1 = \sigma_1n_1+\sigma_2n_2 \equiv \sigma_1A_1+\sigma_2A_2+\varrho\sigma_2K \pmod{\sigma_1K}.$$

This shows that $\boldsymbol{\nu}\in\mathcal{B}$. Furthermore, if $\boldsymbol{\alpha}\in\mathcal{A}$, there exists a corresponding $\boldsymbol{B}\in\mathcal{B}$ such that

$$\boldsymbol{\alpha} = \left(\tfrac{A_1}{K},\tfrac{A_2}{K}\right) \equiv \left(\tfrac{B_1-\sigma_2B_2}{\sigma_1K},\tfrac{B_2}{K}\right) \pmod{1}.$$

Overall, we therefore have

$$\theta_1(\boldsymbol{w}) = \frac{1}{K^2}\sum_{\boldsymbol{B}\in\mathcal{B}}\varepsilon\Big(\tfrac{B_1-\sigma_2B_2}{\sigma_1K},\tfrac{B_2}{K}\Big)\sum_{\nu_1\equiv B_1\pmod{\sigma_1K}}\nu_1e^{\frac{2\pi i\nu_1^2w_1}{\sigma_1K}}\sum_{\nu_2\equiv B_2\pmod{\sigma_1K}}\nu_2e^{\frac{2\pi iD\nu_2^2w_2}{\sigma_1K}}$$
$$= \frac{1}{K^2}\sum_{\boldsymbol{B}\in\mathcal{B}}\varepsilon\left(\tfrac{B_1-\sigma_2B_2}{\sigma_1K},\tfrac{B_2}{K}\right)\vartheta_1\left(\sigma_1K,B_1,\sigma_1K;2w_1\right)\vartheta_1\left(\sigma_1K,B_2,\sigma_1K;2Dw_2\right).$$

In the same way, by setting $\nu_1 := \sigma_2n_1+\sigma_3n_2$ and $\nu_2 := n_1$, we can show that

$$\theta_2(\boldsymbol{w}) = \frac{1}{K^2}\sum_{\boldsymbol{C}\in\mathcal{C}}\varepsilon\left(\tfrac{C_2}{K},\tfrac{C_2-\sigma_2C_1}{\sigma_3K}\right)\vartheta_1(\sigma_3K,C_1,\sigma_3K;2w_1)\vartheta_1(\sigma_3K,C_2,\sigma_3K;2Dw_2). \qquad \square$$

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Suppose that f is one of the theta functions from Lemma 3.3 and $\gamma\in\Gamma(8\cdot\mathrm{lcm}(\sigma_1,\sigma_3)KD)$. Then the transformation (2.4) implies (after a short calculation) that $f|_{\frac{3}{2}}\gamma = f$. The theorem statement now follows from Lemmas 3.2 and 3.3, and Theorem 2.6. $\square$

4. A family with quantum set $\mathbb{Q}$ and unimodular matrices

In this section we construct a family of depth two quantum modular forms with quantum set $\mathbb{Q}$. Let $N_1, N_2 \in 2\mathbb{N}$ and write $L := \gcd(N_1, N_2)$, $N_1 := LR_1$, $N_2 := LR_2$, so that $\gcd(R_1, R_2) = 1$. Set $Q(\boldsymbol{n}) = \sigma_1 n_1^2 + 2\sigma_2 n_1 n_2 + \sigma_3 n_2^2$. We assume the factorizations $\sigma_1 = R_1\mu_1$, with $\gcd(R_1, \mu_1) = 1$, and $\sigma_3 = R_2\mu_3$, with $\gcd(\mu_3, R_2) = 1$. Moreover we assume that $2\sigma_2 = LR_1R_2 = \operatorname{lcm}(N_1, N_2)$ and that $\gcd(\mu_1, \mu_3)$ consists of at most one odd prime factor, and always satisfies $\gcd(L, \gcd(\mu_1, \mu_3)) = 1$. If $4 \nmid L$, then we also require that exactly one of R_1, R_2, μ_3 is even. Set, with $r_1, r_2, s_1, s_2 \in \mathbb{N}$ satisfying $\gcd(r_j, N_j) = \gcd(s_j, N_j) = 1$, $r_j^2 \equiv s_j^2 \pmod{2N_j}$,

$$\begin{aligned}\mathcal{S}_1 := \Big\{ &\left(\tfrac{r_1}{N_1}, \tfrac{r_2}{N_2}\right), \left(1-\tfrac{r_1}{N_1}, \tfrac{r_2}{N_2}\right), \left(\tfrac{r_1}{N_1}, 1-\tfrac{r_2}{N_2}\right), \left(1-\tfrac{r_1}{N_1}, 1-\tfrac{r_2}{N_2}\right), \\ &\left(\tfrac{s_1}{N_1}, \tfrac{s_2}{N_2}\right), \left(1-\tfrac{s_1}{N_1}, \tfrac{s_2}{N_2}\right), \left(\tfrac{s_1}{N_1}, 1-\tfrac{s_2}{N_2}\right), \left(1-\tfrac{s_1}{N_1}, 1-\tfrac{s_2}{N_2}\right)\Big\}, \\ \mathcal{S}_2 := \Big\{ &\left(\tfrac{r_1}{N_1}, \tfrac{s_2}{N_2}\right), \left(1-\tfrac{r_1}{N_1}, \tfrac{s_2}{N_2}\right), \left(\tfrac{r_1}{N_1}, 1-\tfrac{s_2}{N_2}\right), \left(1-\tfrac{r_1}{N_1}, 1-\tfrac{s_2}{N_2}\right), \\ &\left(\tfrac{s_1}{N_1}, \tfrac{r_2}{N_2}\right), \left(1-\tfrac{s_1}{N_1}, \tfrac{r_2}{N_2}\right), \left(\tfrac{s_1}{N_1}, 1-\tfrac{r_2}{N_2}\right), \left(1-\tfrac{s_1}{N_1}, 1-\tfrac{r_2}{N_2}\right)\Big\}.\end{aligned} \tag{4.1}$$

We define

$$\mathcal{Z}_{Q,\boldsymbol{r},\boldsymbol{s}}(q) := \sum_{j\in\{1,2\}} (-1)^{j+1} \sum_{\boldsymbol{\alpha}\in\mathcal{S}_j} \sum_{\boldsymbol{n}\in\mathbb{N}_0^2} q^{LQ(\boldsymbol{n}+\boldsymbol{\alpha})} = F_{\mathcal{S},Q,\varepsilon}\left(\frac{\tau}{R_1R_2}\right),$$

where $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2$ and $\varepsilon(\boldsymbol{\alpha}) := (-1)^{j+1}$ if $\boldsymbol{\alpha} \in \mathcal{S}_j$. We see in the proof of Theorem 4.1 that the assumptions imply that the asymptotic expansion of $\mathcal{Z}_{Q,\boldsymbol{r},\boldsymbol{s}}(q)$ consists of several leading terms with identical Gauss sums that always cancel, and thus the series converges for all $\mathbb{Q}$.

Theorem 4.1. *Under the assumption above, the function $\mathcal{Z}_{Q,\boldsymbol{r},\boldsymbol{s}}(q)$ is a quantum modular form of depth two, weight one, group $\Gamma(8 \cdot \operatorname{lcm}(\sigma_1, \sigma_2)LR_1R_2)$, and quantum set $\mathbb{Q}$.*

Proof. Note that the conditions of Theorem 3.1 are satisfied. We are left to show that we have quantum set $\mathbb{Q}$, which follows if we show that

$$\sum_{j\in\{1,2\}} (-1)^{j+1} \sum_{\boldsymbol{\alpha}\in\mathcal{S}_j} \sum_{\boldsymbol{\ell} \pmod{k}} e^{2\pi i \frac{h}{k} LQ(\boldsymbol{\ell}+\boldsymbol{\alpha})} = 0. \tag{4.2}$$

Write $L = 2^\Lambda L_1$, $k = gk_1$, where L_1, k_1 are odd and where $g := \gcd(k, L)$. We claim that the sum on $\boldsymbol{\ell}$ vanishes unless $\gcd(LR_1R_2, k_1) = 1$ and $g \in \{1, 2\}$. For this we first consider the (one-dimensional) Gauss sum in ℓ_1, which is ($a_j := N_j\alpha_j$)

$$\sum_{\ell_1 \pmod{k}} e^{2\pi i \frac{h}{k}\left(LR_1\mu_1\ell_1^2 + \left(2\mu_1a_1 + L^2R_1R_2\ell_2 + LR_1a_2\right)\ell_1\right)}. \tag{4.3}$$

The linear term reduces to $2\mu_1a_1 \pmod{R_1}$, and μ_1a_1 is coprime to R_1 by assumption. Thus by Proposition 2.3 the expression in (4.3) is zero if $\gcd(R_1, k_1) > 1$. Similarly, the linear term reduces to $2\mu_1a_1 \pmod{L}$. The Gauss sum (4.3) vanishes if $g > 1$ and $g \nmid 2\mu_1$. Now write an alternative Gauss sum by grouping the ℓ_2 terms in (4.2), obtaining an analogous version of (4.3).

As before, this immediately shows that (4.2) is zero if $\gcd(R_2, k_1) > 1$, and also vanishes if $g > 1$ and $g \nmid 2\mu_3$. If $g > 1$, then the only way the sum fails to vanish is if $g \mid \gcd(2\mu_1, 2\mu_3)$, which implies that $g = 2$ by assumption. This shows that (4.2) vanishes if $4 \mid L$.

Next, assuming $g = 2$, $4 \nmid L$, and $4 \mid k$, we also show that (4.2) vanishes in this case. Recalling the corresponding assumptions on the R_j and μ_j, one possibility is that $2|R_1$ and $2 \nmid R_2\mu_1\mu_2$ (or the analogous condition with ℓ_1 and ℓ_2 swapped if necessary). Then 4 divides the factor in front of ℓ_1^2 in (4.3), and the linear term is congruent to 2 modulo 4 since a_1 is odd. The sum therefore vanishes by Proposition 2.3. Otherwise the condition on R_j and μ_j is that $2 \nmid R_1R_2\mu_1, 2 \mid \mu_3$, and we again consider the analog of (4.3) for the sum in ℓ_2. Now 4 divides the coefficient in front of ℓ_2^2, and the linear term is congruent to $2 \pmod 4$ so Proposition 2.3 again applies.

We next assume that $\gcd(LR_1R_2, k_1) = 1$, and $g \in \{1, 2\}$ and prove that the sum on $\boldsymbol{\ell}$ in (4.2) is the same for all choices of $\boldsymbol{\alpha}$. We note that the multiplicative inverses $\overline{N_j} \pmod{k_1}$ exist. Using (3.2), we write

$$\frac{h}{k}L\left(Q\left(\boldsymbol{\ell}+\boldsymbol{\alpha}\right)-Q\left(\boldsymbol{\ell}+\left(\overline{N_1}a_1,\overline{N_2}a_2\right)\right)\right)$$
$$=\frac{hL}{k}\left(Q(\boldsymbol{\alpha})-Q\left(\overline{N_1}a_1,\overline{N_2}a_2\right)+B\left(\boldsymbol{\ell},\boldsymbol{\alpha}\right)-B\left(\boldsymbol{\ell},\left(\overline{N_1}a_1,\overline{N_2}a_2\right)\right)\right). \quad (4.4)$$

Since $B(\boldsymbol{\ell}, \boldsymbol{\alpha}) - B(\boldsymbol{\ell}, (\overline{N_1}a_1, \overline{N_2}a_2)) \equiv 0 \pmod{k_1}$ by construction, (4.4) implies that

$$\frac{hL}{k_1}Q\left(\boldsymbol{\ell}+\boldsymbol{\alpha}\right)\equiv\frac{hL}{k_1}\bigg(Q\left(\boldsymbol{\ell}+\left(\overline{N_1}a_1,\overline{N_2}a_2\right)\right)+Q(\boldsymbol{\alpha})-Q\left(\overline{N_1}a_1,\overline{N_2}a_2\right)\bigg)\pmod 1.$$

We now calculate

$$\frac{hL}{k_1}\left(Q(\boldsymbol{\alpha})-Q\left(\overline{N_1}a_1,\overline{N_2}a_2\right)\right)=\frac{h}{kLR_1R_2}X,$$

where $X := R_2\mu_1a_1^2 + LR_1R_2a_1a_2 + R_1\mu_3a_2^2 - N_1N_2Q(\overline{N_1}a_1, \overline{N_2}a_2)$.

If p is an odd prime such that p^λ exactly divides LR_1R_2, then the assumptions on the parameters easily imply that

$$X \equiv R_2\mu_1a_1^2 + R_1\mu_3a_2^2 \pmod{p^\lambda}$$

is independent from $\boldsymbol{\alpha}$.

Finally, suppose that 2^λ exactly divides LR_1R_2. Then the final congruence is

$$X \equiv R_2\mu_1a_1^2 + R_1R_2 + R_1\mu_3a_2^2 \pmod{2^\lambda g},$$

which is independent from $\boldsymbol{\alpha}$ due to the assumption that $r_j^2 \equiv s_j^2 \pmod{2^{\lambda+1}}$.

Therefore the sum on $\boldsymbol{\ell}$ in (4.2) equals

$$e^{2\pi i\frac{hX}{kLR_1R_2}}\sum_{\boldsymbol{\ell}\pmod k}e^{2\pi i\frac{h}{k}LQ\left(\boldsymbol{\ell}+\left(\overline{N_1}a_1,\overline{N_2}a_2\right)\right)}=e^{2\pi i\frac{hX}{LR_1R_2}}\sum_{\boldsymbol{\ell}\pmod k}e^{2\pi i\frac{h}{k}LQ(\boldsymbol{\ell})}$$

by shifting $\boldsymbol{\ell}$; this overall expression is now clearly independent from choice of $\boldsymbol{\alpha}$. □

5. Classification of positive unimodular H-matrices and the proofs of Theorem 1.2 and Theorem 1.3

5.1. **Proof of Theorem 1.2.** Let

$$M = M(b_1, b_2, b_3, b_4, b_5, b_6) := \begin{pmatrix} b_1 & 0 & -1 & 0 & 0 & 0 \\ 0 & b_2 & -1 & 0 & 0 & 0 \\ -1 & -1 & b_3 & -1 & 0 & 0 \\ 0 & 0 & -1 & b_4 & -1 & -1 \\ 0 & 0 & 0 & -1 & b_5 & 0 \\ 0 & 0 & 0 & -1 & 0 & b_6 \end{pmatrix}. \tag{5.1}$$

In this section, we classify all positive, unimodular (PU) matrices M with the additional property that $b_j \geq 2$ ($j \in \{1, 2, 5, 6\}$). The determinant of M can be written as follows:

$$\begin{aligned} D = D(b_1, b_2, b_3, b_4, b_5, b_6) &:= \det(M) \\ &= b_1b_2b_3b_4b_5b_6 - b_1b_2b_3b_5 - b_1b_2b_3b_6 - b_1b_2b_5b_6 - b_1b_4b_5b_6 - b_2b_4b_5b_6 + (b_1 + b_2)(b_5 + b_6) \\ &= b_1b_2b_5b_6 \left(\left(b_3 - \tfrac{1}{b_1} - \tfrac{1}{b_2} \right) \left(b_4 - \tfrac{1}{b_5} - \tfrac{1}{b_6} \right) - 1 \right). \end{aligned}$$

The goal of this section is to show the following.

Proposition 5.1. *If $M(b_1, b_2, b_3, b_4, b_5, b_6)$ is a* PU *matrix with $b_j \geq 2$ ($j \in \{1, 2, 5, 6\}$), then (up to equivalence)*

$$b_1 \leq 23, \ b_2 \leq 133, \ 2 \leq b_3 \leq 7, \ b_4 = 1, \ b_5 \leq 13, \ b_6 \leq 97.$$

In particular, there are finitely many PU *matrices M.*

This then enables us to prove Theorem 1.2.

Proof of Theorem 1.2. Proposition 5.1 together with a computer search quickly shows there are 312 PU matrices. Since the group of automorphisms of an H-graph is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, we have 39 equivalence classes of such matrices; these are listed in the appendix. This gives the claim. □

We now prove the main statement of this section, namely Proposition 5.1.

Proof of Proposition 5.1. It is clear that $\gcd(b_1, b_2) \mid D$, thus b_1 and b_2 must be coprime and without loss of generality we may assume $b_1 < b_2$ and $b_5 < b_6$. This further implies that $b_1b_2, b_5b_6 \geq 6$, and $\frac{1}{b_1} + \frac{1}{b_2} \leq \frac{5}{6}$. If $b_3, b_4 \geq 2$ we therefore have

$$D \geq b_1b_2b_5b_6 \left(\tfrac{7}{6} \cdot \tfrac{7}{6} - 1 \right) = \tfrac{13}{36} b_1b_2b_5b_6 > 1,$$

and thus M is not unimodular. Furthermore, the fact that $1 - \frac{1}{b_1} - \frac{1}{b_2} < 1$ immediately shows that if $b_3 = b_4 = 1$, then $\det(M) < 0$. Thus without loss of generality we assume that $b_4 = 1$ and $b_3 \neq 1$. If $b_3 \geq 8$, then

$$D > b_1b_2b_5b_6 \left((b_3 - 1)\tfrac{1}{6} - 1 \right) \geq b_1b_2b_5b_6 \frac{1}{6} > 1.$$

Thus we must have $b_3 \leq 7$.

Now suppose that $b_5 \geq 14$. Then, since $b_6 > b_5$,

$$D \geq b_1b_2b_5b_6 \left(\tfrac{7}{6} \left(1 - \tfrac{1}{14} - \tfrac{1}{15} \right) - 1 \right) = 2 \cdot 3 \cdot 14 \cdot 15 \tfrac{1}{180} > 1.$$

Thus we must have $b_5 \leq 13$.

The remaining bounds require a case by case analysis based on the values of b_5. If $b_5 = 2$, then for $b_2 = 2$, $D \leq 0$, thus we must have $b_3 \geq 3$. If $b_6 \geq 28$, then

$$D \geq 6 \cdot 2 \cdot 27 \left(\left(3 - \tfrac{1}{2} - \tfrac{1}{3}\right) \left(1 - \tfrac{1}{2} - \tfrac{1}{28}\right) - 1 \right) \geq 2.$$

We therefore conclude that $b_6 \leq 27$. However, in order to have D positive we also need

$$3 \left(\frac{1}{2} - \frac{1}{b_6} \right) > 1,$$

which implies that $b_6 \geq 7$.

We next determine the possible values of b_1. In order to have $D = 1$, it must be true that $D > 0$, thus

$$b_3 - \frac{1}{b_1} - \frac{1}{b_2} > \left(\frac{1}{2} - \frac{1}{b_6} \right)^{-1}. \tag{5.2}$$

Now suppose that $3 \leq b_3 \leq 7$ and $14 \leq b_6 \leq 27$ are fixed. Now suppose that $b_1 \geq 11$. Then

$$D \geq 11 \cdot 12 \cdot 2 \cdot 7 \left(\left(3 - \tfrac{1}{11} - \tfrac{1}{12}\right) \left(\tfrac{1}{2} - \tfrac{1}{7}\right) - 1 \right) = 17,$$

so we must have $b_1 \leq 10$.

In this case a Maple calculation shows that the right-side is at most 5 (which occurs for $b_3 = 3$ and $b_6 = 7$), and thus all $b_1 > 10$ are not possible; in other words, we must have $b_1 \leq 10$. To complete this case, we now consider fixed $2 \leq b_1 \leq 10, 3 \leq b_3 \leq 7$, and $3 \leq b_6 \leq 27$. If there is a solution, then following (5.2), it must be for the minimal value of b_2 such that

$$b_2 > -\left(\left(\frac{1}{2} - \frac{1}{b_6} \right)^{-1} - b_3 + \frac{1}{b_1} \right)^{-1}. \tag{5.3}$$

A Maple search shows that the maximum value of the right-side is 30 (which occurs with $b_1 = 6, b_3 = 3$ and $b_6 = 7$), $b_2 \leq 31$.

Next, let $b_5 = 3$. If $b_3 \geq 4$, then

$$D \geq 6 \cdot 3 \cdot 4 \left(\left(4 - \tfrac{1}{2} - \tfrac{1}{3}\right) \left(1 - \tfrac{1}{3} - \tfrac{1}{4}\right) - 1 \right) = 23.$$

Thus $b_3 \leq 3$, and we begin with $b_3 = 3$. Very similar calculations show, in turn, that $b_6 \leq 5$, and $b_1 \leq 3$. As in (5.3), checking

$$b_2 > -\left(\left(\frac{2}{3} - \frac{1}{b_6} \right)^{-1} - b_3 + \frac{1}{b_1} \right)^{-1}.$$

in these ranges now gives a maximum right-side value of 10 (with $b_1 = 2, b_3 = 3$, and $b_6 = 4$), then $b_2 \leq 11$.

For the case $b_5 = 3$ and $b_3 = 2$, if $b_6 \leq 6$, then

$$D = b_1 b_2 b_5 b_6 \left(\left(2 - \frac{1}{b_1} - \frac{1}{b_2}\right) \left(1 - \frac{1}{3} - \frac{1}{b_6}\right) - 1 \right) < b_1 b_2 b_5 b_6 \left(2 \cdot \tfrac{1}{2} - 1\right) = 0,$$

and thus we must have $b_6 \geq 7$. However, in order for $D > 0$, it also must be true that

$$2 - \frac{1}{b_1} - \frac{1}{b_2} > \left(\frac{2}{3} - \frac{1}{b_6} \right)^{-1} > \frac{3}{2}. \tag{5.4}$$

The largest values of b_6 occurs when the left side is as close to $\frac{3}{2}$ as possible (while being larger, so $b_1 \geq 3$), which occurs for $b_1 = 3$ and $b_2 = 7$ (and then $2 - \frac{1}{3} - \frac{1}{7} = \frac{32}{21}$). Plugging in to (5.4), this implies that the first inequality holds for $b_6 > 96$, and again by monotonicity, this gives the bound $b_6 \leq 97$.

Furthermore, if $b_1 \geq 24$, then

$$D \geq 24 \cdot 25 \cdot 3 \cdot 7 \left(\left(2 - \tfrac{1}{24} - \tfrac{1}{25}\right)\left(1 - \tfrac{1}{3} - \tfrac{1}{7}\right) - 1\right) = 61,$$

thus we must have $b_1 \leq 23$. Finally, checking

$$b_2 > -\left(\left(\frac{2}{3} - \frac{1}{b_6}\right)^{-1} - 2 + \frac{1}{b_1}\right)^{-1}$$

over the ranges $3 \leq b_1 \leq 23$, and $7 \leq b_6 \leq 97$ shows that the right-side is at most 132 (which occurs at $b_1 = 12$ and $b_6 = 7$), so $b_2 \leq 133$.

For the remaining values $4 \leq b_5 \leq 13$, we proceed similarly. First, if $b_3 \geq 3$, then

$$D \geq 2 \cdot 3 \cdot 4 \cdot 5 \left(\tfrac{13}{6}\left(1 - \tfrac{1}{4} - \tfrac{1}{5}\right) - 1\right) = 23,$$

thus we must have $b_3 = 2$. Furthermore, if $b_1 \geq 11$, then

$$D \geq 11 \cdot 12 \cdot 4 \cdot 5 \left(\left(2 - \tfrac{1}{11} - \tfrac{1}{12}\right) \cdot \tfrac{11}{20} - 1\right) = 11,$$

thus $b_1 \leq 10$.

Now we bound b_6 as in the previous case. For example, if $b_5 = 4$, then $D > 0$ requires that

$$2 - \frac{1}{b_1} - \frac{1}{b_2} > \left(\frac{3}{4} - \frac{1}{b_6}\right)^{-1} > \frac{4}{3}.$$

This is only possible if $\frac{1}{b_1} + \frac{1}{b_2} < \frac{2}{3}$, and the largest value of b_6 occurs when the sum is as close as possible to $\frac{2}{3}$. This occurs with $b_1 = 2, b_2 = 7$, which implies that $b_6 \leq 77$. Repeating the argument for $b_5 \geq 5$ never gives a larger range for b_6 (and $b_5 \geq 8$ can be treated as a single case, since then the maximal case is always $\frac{1}{2} + \frac{1}{3} < \frac{b_5-2}{b_5-1}$). Finally, plugging in $b_1 \leq 10, 4 \leq b_5 \leq 13$, and $b_6 \leq 77$ to

$$b_2 > -\left(\left(1 - \frac{1}{b_5} - \frac{1}{b_6}\right)^{-1} - 2 + \frac{1}{b_1}\right)^{-1}$$

gives the bound $b_2 \leq 71$. □

5.2. **Calculation of $Z(q)$ and the proof of Theorem 1.3.** Let M be as in (5.1), with inverse matrix $M^{-1} = (\ell_{jk})_{1 \leq j,k \leq 6}$. We need the central 2×2 sub-matrix of M^{-1}, which we write as

$$A := \begin{pmatrix} \ell_{33} & \ell_{34} \\ \ell_{43} & \ell_{44} \end{pmatrix} = \begin{pmatrix} b_1b_2(b_4b_5b_6 - b_5 - b_6) & b_1b_2b_5b_6 \\ b_1b_2b_5b_6 & \frac{b_5b_6(b_1b_2b_5b_6+1)}{b_4b_5b_6-b_5-b_6} \end{pmatrix}.$$

In order to write $Z(q)$ as a double series of the type found in Section 4, we use a linear algebra identity, which can be verified by a Maple computation.

Lemma 5.2. *If* $\boldsymbol{r} = (\varepsilon_1, \varepsilon_2, 2n_1+1, 2n_2+1, \varepsilon_5, \varepsilon_6)^T$ *with* $n_1, n_2 \in \mathbb{Z}$ *and* $\varepsilon_j \in \{\pm 1\}$*, then*

$$\frac{1}{2}\boldsymbol{r}^T M^{-1}\boldsymbol{r} = \frac{1}{2}\left(2n_1+2\alpha_1,\, 2n_2+2\alpha_2\right) A \begin{pmatrix} 2n_1+2\alpha_1 \\ 2n_2+2\alpha_2 \end{pmatrix} + c,$$

where

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}(\varepsilon) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} := \frac{1}{2}\begin{pmatrix} 1 + \frac{\varepsilon_1}{b_1} + \frac{\varepsilon_2}{b_2} \\ 1 + \frac{\varepsilon_5}{b_5} + \frac{\varepsilon_6}{b_6} \end{pmatrix}, \qquad c := \tfrac{1}{2}\left(\tfrac{1}{b_1} + \tfrac{1}{b_2} + \tfrac{1}{b_5} + \tfrac{1}{b_6}\right).$$

Remark. *Importantly, note that* c *is independent of the* ε_j*'s.*

We can now evaluate $Z(q)$ for any positive unimodular M.

Proposition 5.3. *With* $\mathcal{S} := \{\boldsymbol{\alpha}(\varepsilon)\}$*, we have*

$$Z(q) = \frac{q^{-9+\frac{\mathrm{tr}(M)}{2}+c}}{4} \sum_{\boldsymbol{\alpha}\in\mathcal{S}} (-1)^{j+1} \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \mathrm{sgn}^*(n_1)\mathrm{sgn}^*(n_2) q^{Q_1(\boldsymbol{n}+\boldsymbol{\alpha})}, \tag{5.5}$$

where $Q_1(\boldsymbol{n}) := \frac{1}{2}\boldsymbol{m}^T M^{-1}\boldsymbol{m}$*, with* $\boldsymbol{m} := (0,0,2n_1,2n_2,0,0)^T$*.*

Proof. An application of formula (2.9) for the H-graph gives

$$Z(q) := \frac{q^{-9+\frac{\mathrm{tr}(M)}{2}}}{(2\pi i)^6} \mathrm{PV}\int_{|w_j|=1} \frac{\left(w_1 - w_1^{-1}\right)\left(w_2 - w_2^{-1}\right)\left(w_5 - w_5^{-1}\right)\left(w_6 - w_6^{-1}\right)}{\left(w_3 - w_3^{-1}\right)\left(w_4 - w_4^{-1}\right)} \Theta_M(q;\boldsymbol{w}) \prod_{j=1}^{6} \frac{dw_j}{w_j},$$

where by (2.10) (because M is unimodular) we have

$$\Theta_M(q;\boldsymbol{w}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^6} q^{\frac{1}{2}\boldsymbol{m}^T M^{-1}\boldsymbol{m}} e^{2\pi i \boldsymbol{m}^T \boldsymbol{z}}.$$

Applying (2.5) and (2.6) we find that

$$Z(q) = \frac{q^{-9+\frac{\mathrm{tr}(M)}{2}}}{4} \sum_{\substack{\boldsymbol{r}=(\varepsilon_1,\varepsilon_2,2n_1,2n_2,\varepsilon_5,\varepsilon_6)^T \\ \varepsilon_j\in\{\pm1\},(n_1,n_2)\in\mathbb{Z}^2}} (\varepsilon_1\varepsilon_2\varepsilon_5\varepsilon_6)\mathrm{sgn}^*(n_1)\mathrm{sgn}^*(n_2) q^{\frac{1}{2}\boldsymbol{r}^T M^{-1}\boldsymbol{r}}.$$

Applying Lemma 5.2 completes the proof. ∎

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. By splitting the summation over $\mathbb{Z}^2$ in (5.5) into summations over $\mathbb{N}_0\times\mathbb{N}_0$, $(-\mathbb{N})\times(-\mathbb{N})$, $\mathbb{N}_0\times(-\mathbb{N})$, and $(-\mathbb{N})\times\mathbb{N}_0$, a case-by-case computation for each unimodular matrix (5.1) gives

$$\sum_{\boldsymbol{\alpha}\in\mathcal{S}} (-1)^{j+1} \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \mathrm{sgn}^*(m_1)\mathrm{sgn}^*(m_2) q^{Q(\boldsymbol{m}+\boldsymbol{\alpha})} = \mathcal{Z}_1(q) - \mathcal{Z}_2(q)$$

where

$$\mathcal{Z}_1(q) := \mathcal{Z}_{Q,\boldsymbol{r},\boldsymbol{s}}(q) = \sum_{j\in\{1,2\}} (-1)^{j+1} \sum_{\boldsymbol{\alpha}\in\mathcal{S}_j} \sum_{\boldsymbol{n}\in\mathbb{N}_0^2} q^{LQ(\boldsymbol{n}+\boldsymbol{\alpha})}$$

$$\mathcal{Z}_2(q) := \mathcal{Z}_{Q^*,\boldsymbol{r},\boldsymbol{s}}(q) = \sum_{j\in\{1,2\}} (-1)^{j+1} \sum_{\boldsymbol{\alpha}\in\mathcal{S}_j} \sum_{\boldsymbol{n}\in\mathbb{N}_0^2} q^{LQ^*(\boldsymbol{n}+\boldsymbol{\alpha})}$$

and $Q^*(\boldsymbol{n}) := Q(-n_1, n_2)$.

The quadratic form Q and constants $N_1, N_2, r_1, r_2, s_1, s_2$ (recall, $L = \gcd(N_1, N_2)$) are given in the appendix. In Section 4, Theorem 4.1 establishes that $\mathcal{Z}_1(q)$ is a quantum modular form of weight one and depth two on $\mathbb{Q}$. The same result also applies to $\mathcal{Z}_2(q)$. Finally, we let $c_M := 9 - \frac{1}{2}\mathrm{tr}(M) - c$, where c is also listed in the appendix. □

Appendix: Data for positive unimodular matrices

Here we list all positive unimodular matrices of the form (5.1), and the corresponding parameters that appear in $Z(q)$ (see (4.1) and Proposition 5.3). In each case one can directly check that the assumptions in Section 4 are satisfied.

The value of c and the quadratic form Q are given below, and the data for $\mathcal{S}_j$ are presented in condensed form.

1. $M(2,3,7,1,2,3)$
$Q(\boldsymbol{n}) = n_1^2 + 12n_1n_2 + 37n_2^2$, $c = \frac{5}{6}$, $N_1 = N_2 = 12$, $r_1 = r_2 = 1$, $s_1 = s_2 = 5$.

2. $M(2,7,4,1,5,2)$
$Q(\boldsymbol{n}) = 21n_1^2 + 140n_1n_2 + 235n_2^2$, $c = \frac{47}{70}$, $N_1 = 28$, $N_2 = 20$, $r_1 = 5$, $s_1 = 9$, $r_2 = 3$, $s_2 = 7$.

3. $M(6,31,3,1,2,7)$
$Q(\boldsymbol{n}) = 465n_1^2 + 2604n_1n_2 + 3647n_2^2$, $c = \frac{274}{651}$, $N_1 = 372$, $N_2 = 28$, $r_1 = 149$, $s_1 = 161$, $r_2 = 5$, $s_2 = 23$.

4. $M(7,18,3,1,2,7)$
$Q(\boldsymbol{n}) = 45n_1^2 + 252n_1n_2 + 353n_2^2$, $c = \frac{53}{126}$, $N_1 = 252$, $N_2 = 28$, $r_1 = 101$, $s_1 = 115$, $r_2 = 5$, $s_2 = 9$.

5. $M(3,11,3,1,2,9)$
$Q(\boldsymbol{n}) = 77n_1^2 + 396n_1n_2 + 510n_2^2$, $c = \frac{205}{396}$, $N_1 = 66$, $N_2 = 36$, $r_1 = 19$, $s_1 = 25$, $r_2 = 7$, $s_2 = 11$.

6. $M(2,19,3,1,2,11)$
$Q(\boldsymbol{n}) = 171n_1^2 + 836n_1n_2 + 1023n_2^2$, $c = \frac{239}{418}$, $N_1 = 76$, $N_2 = 44$, $r_1 = 17$, $s_1 = 21$, $r_2 = 9$, $s_2 = 13$.

7. $M(2,3,3,1,2,27)$
$Q(\boldsymbol{n}) = 25n_1^2 + 108n_1n_2 + 117n_2^2$, $c = \frac{37}{54}$, $N_1 = 12$, $N_2 = 108$, $r_1 = 1$, $s_1 = 5$, $r_2 = 25$, $s_2 = 29$.

8. $M(2,3,3,1,3,5)$
$Q(\boldsymbol{n}) = 14n_1^2 + 60n_1n_2 + 65n_2^2$, $c = \frac{41}{60}$, $N_1 = 12$, $N_2 = 30$, $r_1 = 1$, $s_1 = 5$, $r_2 = 7$, $s_2 = 13$.

9. $M(2,11,3,1,3,4)$
$Q(\boldsymbol{n}) = 55n_1^2 + 264n_1n_2 + 318n_2^2$, $c = \frac{155}{264}$, $N_1 = 44$, $N_2 = 24$, $r_1 = 9$, $s_1 = 13$, $r_2 = 5$, $s_2 = 11$.

10. $M(3,4,3,1,3,4)$
$Q(\boldsymbol{n}) = 5n_1^2 + 24n_1n_2 + 29n_2^2$, $c = \frac{155}{264}$, $N_1 = N_2 = 24$, $r_1 = 5$, $s_1 = 11$, $r_2 = 5$, $s_2 = 11$.

11. $M(3,7,2,1,3,97)$
$Q(\boldsymbol{n}) = 1337n_1^2 + 4074n_1n_2 + 3104n_2^2$, $c = \frac{835}{2037}$, $N_1 = 42$, $N_2 = 582$, $r_1 = 11$, $s_1 = 17$, $r_2 = 191$, $s_2 = 197$.

12. $M(3,8,2,1,3,56)$
$Q(\boldsymbol{n}) = 109n_1^2 + 336n_1n_2 + 259n_2^2$, $c = \frac{17}{42}$, $N_1 = 48$, $N_2 = 336$, $r_1 = 13$, $s_1 = 19$, $r_2 = 109$, $s_2 = 115$.

13. $M(3,47,2,1,3,17)$
$Q(\boldsymbol{n}) = 1457n_1^2 + 4794n_1n_2 + 3944n_2^2$, $c = \frac{895}{2397}$, $N_1 = 282$, $N_2 = 102$, $r_1 = 91$, $s_1 = 97$, $r_2 = 31$, $s_2 = 37$.

14. $M(3,88,2,1,3,16)$
$Q(\boldsymbol{n}) = 319n_1^2 + 1056n_1n_2 + 874n_2^2$, $c = \frac{391}{1056}$, $N_1 = 528$, $N_2 = 96$, $r_1 = 173$, $s_1 = 179$, $r_2 = 29$, $s_2 = 35$.

15.$M(4,5,2,1,3,47)$
$Q(\boldsymbol{n}) = 1820n_1^2 + 5640n_1n_2 + 4371n_2^2$, $c = \frac{2263}{5640}$, $N_1 = 40$, $N_2 = 282$, $r_1 = 11$, $s_1 = 19$, $r_2 = 91$, $s_2 = 97$.

16. $M(4,77,2,1,3,11)$
$Q(\boldsymbol{n}) = 532n_1^2 + 1848n_1n_2 + 1605n_2^2, c = \frac{635}{1848}$, $N_1 = 616$, $N_2 = 66$, $r_1 = 227$, $s_1 = 235$, $r_2 = 19$, $s_2 = 25$.

17. $M(5,16,2,1,3,11)$
$Q(\boldsymbol{n}) = 1520n_1^2 + 5280n_1n_2 + 4587n_2^2, c = \frac{1813}{5280}$, $N_1 = 160$, $N_2 = 66$, $r_1 = 59$, $s_1 = 69$, $r_2 = 19$, $s_2 = 25$.

18. $M(7,92,2,1,3,8)$
$Q(\boldsymbol{n}) = 2093n_1^2 + 7728n_1n_2 + 7134n_2^2, c = \frac{2365}{7728}$, $N_1 = 1288$, $N_2 = 48$, $r_1 = 545$, $s_1 = 559$, $r_2 = 13$, $s_2 = 19$.

19. $M(8,35,2,1,3,8)$
$Q(\boldsymbol{n}) = 455n_1^2 + 1680n_1n_2 + 1551n_2^2), c = \frac{257}{840}$, $N_1 = 560$, $N_2 = 48$, $r_1 = 237$, $s_1 = 253$, $r_2 = 13$, $s_2 = 19$.

20. $M(11,16,2,1,3,8)$
$Q(\boldsymbol{n}) = 286n_1^2 + 1056n_1n_2 + 975n_2^2, c = \frac{323}{1056}$, $N_1 = 352$, $N_2 = 48$, $r_1 = 149$, $s_1 = 171$, $r_2 = 13$, $s_2 = 19$.

21. $M(12,133,2,1,3,7)$
$Q(\boldsymbol{n}) = 836n_1^2 + 3192n_1n_2 + 3047n_2^2, c = \frac{905}{3192}$, $N_1 = 3192$, $N_2 = 42$, $r_1 = 1451$, $s_1 = 1475$, $r_2 = 11$, $s_2 = 17$.

22. $M(13,72,2,1,3,7)$
$Q(\boldsymbol{n}) = 3432n_1^2 + 13104n_1n_2 + 12509n_2^2, c = \frac{3715}{13104}, N_1 = 1872, N_2 = 42, r_1 = 851, s_1 = 877, r_2 = 11, s_2 = 17$.

23. $M(3,4,2,1,4,23)$
$Q(\boldsymbol{n}) = 195n_1^2 + 552n_1n_2 + 391n_2^2$, $c = \frac{121}{276}$, $N_1 = 24$, $N_2 = 184$, $r_1 = 5$, $s_1 = 11$, $r_2 = 65$, $s_2 = 73$.

24. $M(3,10,2,1,4,9)$
$Q(\boldsymbol{n}) = 115n_1^2 + 360n_1n_2 + 282n_2^2, c = \frac{143}{360}$, $N_1 = 60$, $N_2 = 72$, $r_1 = 17$, $s_1 = 23$, $r_2 = 23$, $s_2 = 31$.

25. $M(3,52,2,1,4,7)$
$Q(\boldsymbol{n}) = 663n_1^2 + 2184n_1n_2 + 1799n_2^2, c = \frac{407}{1092}$, $N_1 = 312$, $N_2 = 56$, $r_1 = 101$, $s_1 = 107$, $r_2 = 17$, $s_2 = 25$.

26. $M(6,67,2,1,4,5)$

$Q_1(\boldsymbol{n}) = 2211n_1^2 + 8040n_1n_2 + 7310n_2^2$, $c = \frac{2539}{8040}$, $N_1 = 804$, $N_2 = 40$, $r_1 = 329$, $s_1 = 341$, $r_2 = 11$, $s_2 = 19$.

27. $M(2, 7, 2, 1, 4, 77)$
$Q(\boldsymbol{n}) = 227n_1^2 + 616n_1n_2 + 418n_2^2)$, $c = \frac{279}{616}$, $N_1 = 28$, $N_2 = 616$, $r_1 = 5$, $s_1 = 9$, $r_2 = 227$, $s_2 = 235$.

28. $M(7, 26, 2, 1, 4, 5)$

$Q(\boldsymbol{n}) = 1001n_1^2 + 3640n_1n_2 + 3310n_2^2$, $c = \frac{1149}{3640}$, $N_1 = 364$, $N_2 = 40$, $r_1 = 149$, $s_1 = 163$, $r_2 = 11$, $s_2 = 19$.

29. $M(2, 11, 2, 1, 4, 25)$
$Q(\boldsymbol{n}) = 781n_1^2 + 2200n_1n_2 + 1550n_2^2$, $c = \frac{969}{2200}$, $N_1 = 44$, $N_2 = 200$, $r_1 = 9$, $s_1 = 13$, $r_2 = 71$, $s_2 = 79$.

30. $M(2, 19, 2, 1, 4, 17)$
$Q(\boldsymbol{n}) = 893n_1^2 + 2584n_1n_2 + 1870n_2^2$, $c = \frac{1113}{2584}$, $N_1 = 76$, $N_2 = 136$, $r_1 = 17$, $s_1 = 21$, $r_2 = 47$, $s_2 = 55$.

31. $M(2, 71, 2, 1, 4, 13)$

$Q(\boldsymbol{n}) = 2485n_1^2 + 7384n_1n_2 + 5486n_2^2$, $c = \frac{3105}{7384}$, $N_1 = 284$, $N_2 = 104$, $r_1 = 69$, $s_1 = 73$, $r_2 = 35$, $s_2 = 43$.

32. $M(3, 7, 2, 1, 5, 7)$
$Q(\boldsymbol{n}) = 69n_1^2 + 210n_1n_2 + 160n_2^2$, $c = \frac{43}{105}$, $N_1 = 42$, $N_2 = 70$, $r_1 = 11$, $s_1 = 17$, $r_2 = 23$, $s_2 = 33$.

33. $M(2, 5, 2, 1, 5, 33)$
$Q(\boldsymbol{n}) = 254n_1^2 + 660n_1n_2 + 429n_2^2$, $c = \frac{307}{660}$, $N_1 = 20$, $N_2 = 330$, $r_1 = 3$, $s_1 = 7$, $r_2 = 127$, $s_2 = 137$.

34. $M(2, 7, 2, 1, 5, 16)$
$Q(\boldsymbol{n}) = 413n_1^2 + 1120n_1n_2 + 760n_2^2$, $c = \frac{507}{1120}$, $N_1 = 28$, $N_2 = 160$, $r_1 = 5$, $s_1 = 9$, $r_2 = 59$, $s_2 = 69$.

35.$M(2, 21, 2, 1, 5, 9)$
$Q(\boldsymbol{n}) = 434n_1^2 + 1260n_1n_2 + 915n_2^2$, $c = \frac{541}{1260}$, $N_1 = 84$, $N_2 = 90$, $r_1 = 19$, $s_1 = 23$, $r_2 = 31$, $s_2 = 41$.

36. $M(2, 55, 2, 1, 5, 8)$
$Q(\boldsymbol{n}) = 297n_1^2 + 880n_1n_2 + 652n_2^2$, $c = \frac{371}{880}$, $N_1 = 220$, $N_2 = 80$, $r_1 = 53$, $s_1 = 57$, $r_2 = 27$, $s_2 = 37$.

37. $M(2, 3, 2, 1, 8, 57)$
$Q(\boldsymbol{n}) = 391n_1^2 + 912n_1n_2 + 532n_2^2$, $c = \frac{445}{912}$, $N_1 = 12$, $N_2 = 912$, $r_1 = 1$, $s_1 = 5$, $r_2 = 391$, $s_2 = 407$.

38. $M(2, 3, 2, 1, 9, 32)$
$Q(\boldsymbol{n}) = 247n_1^2 + 576n_1n_2 + 336n_2^2$, $c = \frac{281}{576}$, $N_1 = 12$, $N_2 = 576$, $r_1 = 1$, $s_1 = 5$, $r_2 = 247$, $s_2 = 265$.

39. $M(2, 3, 2, 1, 12, 17)$
$Q(\boldsymbol{n}) = 175n_1^2 + 408n_1n_2 + 238n_2^2$, $c = \frac{199}{408}$, $N_1 = 12$, $N_2 = 408$, $r_1 = 1$, $s_1 = 5$, $r_2 = 175$, $s_2 = 199$.

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