INFINITE FAMILIES OF CRANK FUNCTIONS, STANTON-TYPE CONJECTURES, AND UNIMODALITY

KATHRIN BRINGMANN, KEVIN GOMEZ, LARRY ROLEN, AND ZACK TRIPP

Abstract. Dyson's rank function and the Andrews–Garvan crank function famously give combinatorial witnesses for Ramanujan's partition function congruences modulo 5, 7, and 11. While these functions can be used to show that the corresponding sets of partitions split into 5, 7, or 11 equally sized sets, one may ask how to make the resulting bijections between partitions organized by rank or crank combinatorially explicit. Stanton recently made conjectures which aim to uncover a deeper combinatorial structure along these lines, where it turns out that minor modifications of the rank and crank are required. Here, we prove two of these conjectures. We also provide abstract criteria for quotients of polynomials by certain cyclotomic polynomials to have non-negative coefficients based on unimodality and symmetry. Furthermore, we extend Stanton's conjecture to an infinite family of cranks. This suggests further applications to other combinatorial objects. We also discuss numerical evidence for our conjectures, connections with other analytic conjectures such as the distribution of partition ranks.

1. Introduction and statement of results

1.1. **Partition congruences and invariants.** Let $p(n)$ be the integer partition function, which counts the number of non-increasing sequences of positive integers that sum to $n \in \mathbb{N}_0$. Ramanujan discovered the following three congruences for the partition function $p(n)$

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7, \quad p(11n+6) \equiv 0 \pmod{11},$$

and gave q-series proofs of the first two congruences [26]. Based on an unpublished manuscript of Ramanujan, Hardy found proofs of all three [27]. Additionally, Ramanujan conjectured that these were the only congruences of the form $p(\ell n+\beta) \equiv 0 \pmod{\ell}$ for a prime ℓ, which was later proved by Ahlgren and Boylan [2]. Since this time, there has been significant study of other congruences for the partition function [1, 7, 23, 25, 29].

In order to give a combinatorial explanation for Ramanujan's congruences, Dyson [15] defined the *rank* of a partition to be its largest part minus its number of parts and conjectured that partitions of $5n+4$ (resp. $7n+5$) can be split into 5 (resp. 7) sets of equal size by considering the rank modulo 5 (resp. 7). This equidistribution of the rank modulo 5 and 7 was later proved by Atkin and Swinnerton-Dyer [8]. Dyson also conjectured the existence of a partition statistic he called the crank of a partition that was equistributed modulo 11. Andrews and Garvan [6] found such a statistic which is equidistributed modulo 5, 7, and 11. For a partition λ, we let $\ell(\lambda)$ be the largest part of λ, $\omega(\lambda)$ be the number of 1's in λ, and $\mu(\lambda)$ be the number of parts of λ larger than $\omega(\lambda)$. The *crank* of λ is then defined as

$$\operatorname{crank}(\lambda) := \begin{cases} \ell(\lambda) & \text{if } \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0. \end{cases}$$

In addition to giving a combinatorial explanation for Ramanujan's congruences, the rank and the crank give interesting examples in the theory of modular and mock modular forms. The first author and Ono [11] showed that the rank generating function (2.1) is essentially a mock Jacobi form, while the crank generating function (2.2) is essentially a meromorphic Jacobi form. Specializing the rank (resp. crank) generating function in the elliptic variable z to a torsion point gives a mock modular (resp. modular) form.

1.2. **Cranks for colored partitions.** Since the discovery of Ramanujan's congruences, many papers have studied similar congruences for other partition related functions [3, 4, 24]. One such example is the *k-colored partition* function $p_k(n)$ defined as the number of partitions of n into k-colors or by the generating function

$$\sum_{n=0}^{\infty} p_k(n) q^n := \left(\sum_{n=0}^{\infty} p(n) q^n \right)^k = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^k}.$$

Boylan [9] and Dawsey and Wagner [13] have proved a number of congruences for $p_k(n)$ using the theory of CM forms and have given partial progress towards classifying all such congruences. Boylan classified all congruences of the form $p_k(\ell n+\beta) \equiv 0 \pmod{\ell}$ for a prime ℓ and $k \leq 47$ odd and found that all but three such congruences were explained by the following result or by other well-known families of congruences.

Theorem 1.1 ([9, 13])**.** *Let $k+h = \ell t$ for a prime ℓ and positive integers h and t, and let $\delta_{k,\ell} \in \mathbb{Z}$ be such that $24\delta_{k,\ell} \equiv k \pmod{\ell}$. Then we have the Ramanujan-type* *congruence*

$$p_k(\ell n + \delta_{k,\ell}) \equiv 0 \pmod{\ell}$$

if any of the following hold:

(1) *We have* $h \in \{4, 8, 14\}$ *and* $\ell \equiv 2 \pmod{3}$.
(2) *We have* $h \in \{6, 10\}$ *and* $\ell \equiv 3 \pmod{4}$.
(3) *We have* $h = 26$ *and* $\ell \equiv 11 \pmod{12}$.

Analogous to the rank and crank for $p(n)$, various statistics for k-colored partitions have been given. Hammond and Lewis [21] defined the birank for 2-colored partitions to explain congruences modulo 5, while Andrews [5] gave a combinatorial interpretation of a certain 2-color congruence using the ordinary crank. Garvan [18] was later able to provide extensions of both of these results in order to explain a certain infinite family of congruences for k-colored partitions. More recently, Wagner, the third, and the fourth author [28] have found two infinite families of cranks that together explain most known congruences for k-colored partitions. The generating function for these cranks are defined by certain products of the ordinary crank generating function. The shape of the crank (see (2.3)) is defined in such a way in order to utilize the theory of theta blocks set forth by Gritsenko, Skoruppa, and Zagier [20]. In [28], cranks of this form were multiplied by $1 = \frac{\theta_R}{\theta_R}$ for a given theta block θ_R depending on the congruence in order to apply a set of sum-to-product identities known as the Macdonald identities to the numerator and denominator separately. This allowed the authors to prove equidistribution in an infinite family of cases.

1.3. **Stanton's Conjectures.** While the rank and crank distribute partitions into congruence classes of equal size in order to explain Ramanujan's congruences, there is no known direct map between these equinumerous classes. The search for such a map led to a conjecture of Stanton. In order to state this conjecture, we need to modify the Laurent polynomials $\mathrm{rank}_n(\zeta)$ and $\mathrm{crank}_n(\zeta)$, which are defined in (2.1) and (2.2), respectively.

Definition. For $n \in \mathbb{N}_0$, the *modified rank* and *modified crank* are defined by

$$\mathrm{rank}^*_{\ell,n}(\zeta) := \mathrm{rank}_{\ell n+\beta}(\zeta) + \zeta^{\ell n+\beta-2} - \zeta^{\ell n+\beta-1} + \zeta^{2-\ell n-\beta} - \zeta^{1-\ell n-\beta}, \tag{1.1}$$

$$\mathrm{crank}^*_{\ell,n}(\zeta) := \mathrm{crank}_{\ell n+\beta}(\zeta) + \zeta^{\ell n+\beta-\ell} - \zeta^{\ell n+\beta} + \zeta^{\ell-\ell n-\beta} - \zeta^{-\ell n-\beta}, \tag{1.2}$$

where $\beta := \ell - \frac{\ell^2-1}{24}$.

Remark. Note that these modifications only change the definition of rank and crank for the partitions n and $1 + \ldots + 1$. For instance, in the case of rank, this assigns the partition n the value $n-2$, although the classical rank assigns it the value $n-1$.

We see in Lemma 2.1 below that the explanation of Ramanujan's congruences for $p(n)$ by ranks and cranks is equivalent to the divisibility of rank and crank polynomials by cyclotomic polynomials. Stanton found that the quotients of these rank and crank polynomials by cyclotomic polynomials do not have positive coefficients. The modifications are designed to fix positivity, with the eventual goal of uncovering new combinatorial structure of what these positive coefficients count. Such an interpretation would hopefully yield a map between the congruence classes for the rank and crank. As we see in Lemma 3.1 below, this positivity is related to unimodality of coefficients. Stanton's modifications essentially fix this unimodality and maintain divisibility by cyclotomic polynomials.

We are now able to state Stanton's conjecture, which was given in his unpublished notes. Here and throughout the paper, $\Phi_\ell(\zeta) := 1 + \zeta + \ldots + \zeta^{\ell-1}$ denotes the *ℓ-th cyclotomic polynomial* and $\zeta_\ell := e^{\frac{2\pi i}{\ell}}$.

Conjecture 1.2 (Stanton)**.** *Let $n \in \mathbb{N}_0$.*

(1) *The following are Laurent polynomials with non-negative coefficients:*

$$\frac{\operatorname{rank}^*_{5,n}(\zeta)}{\Phi_5(\zeta)} \quad and \quad \frac{\operatorname{rank}^*_{7,n}(\zeta)}{\Phi_7(\zeta)}.$$

(2) *The following is a Laurent polynomial with positive coefficients:*

$$\frac{\operatorname{crank}_{5n+4}(\zeta)}{\Phi_5(\zeta^2)}.$$

(3) *The following are Laurent polynomials with non-negative coefficients:*

$$\frac{\operatorname{crank}^*_{5,n}(\zeta)}{\Phi_5(\zeta)}, \quad \frac{\operatorname{crank}^*_{7,n}(\zeta)}{\Phi_7(\zeta)}, \ and \quad \frac{\operatorname{crank}^*_{11,n}(\zeta)}{\Phi_{11}(\zeta)}.$$

Remark. In Conjecture 1.2 (2), Stanton used the term "positive" instead of "non-negative". However, given that the authors found that all examples of the given Laurent polynomial in this case have 0's between positive coefficients, we assume that "Laurent polynomial with positive coefficients" was meant to be the same as "Laurent polynomial with non-negative coefficients".

Stanton also stated a related conjecture of Garvan for the 5-core crank [19] that cannot be proven by the methods we give here due to the fact that the 5-core crank does not appear to be unimodal. A Laurent polynomial $f(\zeta) = \sum_{m=-M}^{N} a_m \zeta^m$ is said to be *unimodal* if there exists $k \in \mathbb{Z}$ such that $a_m \leq a_{m+1}$ for $m \leq k$ and $a_m \geq a_{m+1}$ for $m > k$. Our key to proving Conjecture 1.2 is to use the fact that the modified rank and crank given in Conjecture 1.2 are unimodal.

1.4. **Results.** It turns out that the modification Stanton gives for the crank along with known inequalities for the crank are sufficient to prove the following.

Theorem 1.3. *Parts* (2) *and* (3) *of Conjecture 1.2 are true.*

The analogous inequalities for rank are not known, and the authors are unaware of a reference in the literature for such a conjecture, which we give here.

Conjecture 1.4. *We have* $N(m,n) \geq N(m+1,n)$ *for* $0 \leq m \leq n-2$ *and* $n \geq 39$.

Unfortunately, Conjecture 1.4 appears to be out of reach with current methods, though there is strong computational evidence and partial progress towards it. The claim is known to be true for n sufficiently large, and for fixed m, this can be made explicit. Additionally, Dousse and Mertens [14] used methods of Dousse and the first author [10] to show that for $|m| \leq \frac{\sqrt{n}\log(n)}{\pi\sqrt{6}}$, we have as $n \to \infty$,

$$N(m,n) = \frac{\gamma}{4}\operatorname{sech}^2\left(\frac{\gamma m}{2}\right) p(n)\left(1 + O\left(\gamma^{\frac{1}{2}}|m|^{\frac{1}{3}}\right)\right),$$

where $\gamma := \frac{\pi}{\sqrt{6n}}$. Note that sech is decreasing, so this gives the claim asymptotically. If we assume Conjecture 1.4, we are able to prove another part of Conjecture 1.2.

Theorem 1.5. *Conjecture 1.4 implies part* (1) *of Conjecture 1.2.*

It is natural to ask whether Stanton's conjectures are part of a broader phenomenon. Searching for an extension of them may also shed light on their combinatorial interpretation. Recently, in [28], Wagner and two of the authors gave a procedure for generating infinite families of crank-type functions which "explain" most known congruences for the family of k-colored partitions. Thus, it is natural to ask whether a deeper phenomenon like Stanton's conjecture also holds in these cases. Numerically, the authors found that these functions do not typically satisfy Stanton-type conjectures. Moreover, the authors were unable to find simple modifications like Stanton found for rank and crank which "fixed" positivity of the quotients by cyclotomic polynomials. However, the method for producing such functions in [28] is flexible, and the crank-type functions produced are not unique.

This paper suggests new families of crank-type functions $\mathcal{A}_k(z;\tau)$ and $\mathcal{B}_k(z;\tau)$ (see (2.4) for the definition), produced using the same machinery. These still explain most congruences of the colored partitions. The proof is sketched in Section 3.4. These invariants appear to satisfy Stanton-type conjectures without any modifications, which suggests that they may be more natural to consider than the original crank-type functions in [28], and that Stanton's conjecture appears to be a very general phenomenon deserving an explanation.

Along these lines, extensive numerical evidence suggests the following.

Conjecture 1.6. *For all $n \geq 15$, (resp. 24) and all $k \geq 7$, $[q^n]\mathcal{A}_k(z;\tau)$ (resp. $[q^n]\mathcal{B}_k(z;\tau)$) are unimodal Laurent polynomials.*

Similar to Conjecture 1.4, proving this seems to require delicate analytic techniques that are currently out of reach. Assuming this conjecture, we have the following.

Theorem 1.7. *Assume Conjecture 1.6 is true.*

(1) *If $\ell n + \delta_{k,\ell} \geq 15$ and $p_k(\ell n + \delta_{k,\ell}) \equiv 0 \pmod{\ell}$ is a Ramanujan-type congruence coming from Theorem 1.1 with $h \notin \{14, 26\}$ if k is odd and $h \neq 26$ if k is even, then*
$$\frac{\left[q^{\ell n+\delta_{k,\ell}}\right] \mathcal{A}_k(z;\tau)}{\Phi_\ell(\zeta)}$$
is a Laurent polynomial with non-negative coefficients.

(2) *If $\ell n + \delta_{k,\ell} \geq 24$ and $p_k(\ell n + \delta_{k,\ell}) \equiv 0 \pmod{\ell}$ is a Ramanujan-type congruence coming from Theorem 1.1 with $h \notin \{4, 8, 10, 26\}$ and $k \geq 7$ is odd, then*
$$\frac{\left[q^{\ell n+\delta_{k,\ell}}\right] \mathcal{B}_k(z;\tau)}{\Phi_\ell(\zeta)}$$
is a Laurent polynomial with non-negative coefficients.

Remark. Throughout this paper, we are studying the n-th Fourier coefficient (in τ) for a fixed n of objects with modularity properties. Traditionally, it has been more common to study the n-th Fourier coefficient (in z) for a fixed m instead; see e.g. [12].

The paper is organized as follows. In Section 2, we provide a lemma illustrating the connection between divisibility and equidistribution, along with known results about crank equalities and inequalities. We then provide proofs in Section 3 of a lemma, our main theorem, and its corollaries. Finally, Section 4 provides computational evidence related to Conjecture 1.6 and directions for future research.

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2. Preliminaries

We begin by formally defining terms used in the introduction. We let $N(r,t;n)$ (resp. $M(r,t;n)$) be the number of partitions of n with rank (resp. crank) congruent to $r \pmod t$. The equidistribution of the rank modulo 5 and 7 is equivalent to

$$N(0,5;5n+4) = N(1,5;5n+4) = \cdots = N(4,5;5n+4),$$
$$N(0,7;7n+5) = N(1,7;7n+5) = \cdots = N(6,7;7n+5).$$

Equidistribution of the crank modulo 5, 7, and 11 may be written similarly in terms of $M(r,t;n)$. Additionally, we let $N(m,n)$ (resp. $M(m,n)$) be the number of partitions of n with rank m (resp. crank m). Letting $\zeta := e^{2\pi i z}$ and $q := e^{2\pi i \tau}$, [8] showed that we have the two-parameter generating function

$$\mathcal{R}(z;\tau) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} N(m,n)\zeta^m q^n =: \sum_{n=0}^{\infty} \mathrm{rank}_n(\zeta) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{k=1}^{n}(1-\zeta q^k)(1-\zeta^{-1}q^k)}. \tag{2.1}$$

Andrews and Garvan [6] proved that aside from the anomalous case of $M(m,n)$ if $n=1$ (where the correct values are $M(0,1) := 1$ and $M(m,1) := 0$ for $m \neq 0$) the crank generating function is given by

$$\mathcal{C}(z;\tau) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} M(m,n)\zeta^m q^n =: \sum_{n=0}^{\infty} \mathrm{crank}_n(\zeta) q^n = \prod_{n=1}^{\infty} \frac{1-q^n}{(1-\zeta q^n)(1-\zeta^{-1}q^n)}. \tag{2.2}$$

The cranks that the authors in [28] used are of the form

$$\mathcal{C}_k\left(a_1, a_2, \ldots, a_{\frac{k+\delta_{2\nmid k}}{2}}; z; \tau\right) := \mathcal{C}(0;\tau)^{\frac{k-\delta_{2\nmid k}}{2}} \prod_{j=1}^{\frac{k+\delta_{2\nmid k}}{2}} \mathcal{C}(a_j z;\tau), \tag{2.3}$$

where $a_j \in \mathbb{Z}$ for $j = 1, \ldots, \frac{k+\delta_{2\nmid k}}{2}$ and where $\delta_S := 1$ if a statement S is true and 0 otherwise. Note that the notation differs slightly from that of [28] and we choose it since it is more convenient for our purposes. Recalling (2.3), we define (in the case of $\mathcal{B}_k$, for $k \geq 7$),

$$\begin{aligned} \mathcal{A}_k(z;\tau) &:= \mathcal{C}_k\left(\frac{k+\delta_{2\nmid k}}{2}+1, \frac{k+\delta_{2\nmid k}}{2}, \ldots, 3, 2; z; \tau\right), \\ \mathcal{B}_k(z;\tau) &:= \mathcal{C}_k\left(\frac{k+\delta_{2\nmid k}}{2}+2, \frac{k+\delta_{2\nmid k}}{2}+1, \ldots, 6, 5, 3, 2; z; \tau\right). \end{aligned} \tag{2.4}$$

We begin with a lemma illustrating how Conjecture 1.2 is a statement related to the equidistribution of the rank and crank. This is implicit in the existing literature, but we provide a proof here for the convenience of the reader. By $\Phi_\ell(\zeta)|f(\zeta)$ for $f(\zeta) \in \mathbb{Q}[\zeta^{-1},\zeta]$, we mean that $f(\zeta) = g(\zeta)\Phi_\ell(\zeta)$ for $g(\zeta) \in \mathbb{Q}[\zeta^{-1},\zeta]$, i.e., that the quotient $\frac{f(\zeta)}{\Phi_\ell(\zeta)}$ is a Laurent polynomial as well. Define

$$\widehat{f}_{r,\ell} := \sum_{j\equiv r \pmod{\ell}} \left[\zeta^j\right] f(\zeta).$$

Lemma 2.1. *Let $f(\zeta)$ be a Laurent polynomial in $\mathbb{Q}[\zeta^{-1},\zeta]$ and ℓ a prime. Then $\Phi_\ell(\zeta) \mid f(\zeta)$ in $\mathbb{Q}[\zeta^{-1},\zeta]$ if and only if for $r \in \{0,\ldots,\ell-2\}$*

$$\widehat{f}_{r,\ell} = \widehat{f}_{\ell-1,\ell}.$$

Remark. Letting $f(\zeta)$ be $\mathrm{rank}_{\ell n+\beta}(\zeta)$ or $\mathrm{crank}_{\ell n+\beta}(\zeta)$, we see that divisibility by $\Phi_\ell(\zeta)$ is equivalent to equidistribution modulo ℓ. We use this result frequently.

Proof of Lemma 2.1. Multiplying $f(\zeta)$ by a sufficiently large power of ζ and using the fact that $\gcd(\zeta,\Phi_\ell(\zeta)) = 1$, we may assume that $f(\zeta) \in \mathbb{Q}[\zeta]$. Since $\Phi_\ell(\zeta)$ is irreducible over $\mathbb{Q}[\zeta]$, it is a standard fact from algebra that $\Phi_\ell(\zeta) \mid f(\zeta)$ is equivalent to $f(\zeta_\ell) = 0$. Writing $f(\zeta) = \sum_{j=0}^n a_j\zeta^j$, we see that

$$f(\zeta_\ell) = \sum_{j=0}^{n} a_j\zeta_\ell^j = \sum_{r=0}^{\ell-1} \sum_{\substack{0\leq j\leq n\\ j\equiv r \pmod{\ell}}} a_j\zeta_\ell^r = \sum_{r=0}^{\ell-1} \widehat{f}_{r,\ell}\zeta_\ell^r = \sum_{r=0}^{\ell-2} \left(\widehat{f}_{r,\ell} - \widehat{f}_{\ell-1,\ell}\right)\zeta_\ell^r,$$

where for the last equality we use the fact that $\Phi_\ell(\zeta_\ell) = 0$. Since $1, \zeta_\ell, \ldots, \zeta_\ell^{\ell-2}$ is a basis for $\mathbb{Q}[\zeta]$ over $\mathbb{Q}$, the claim follows. □

Lemma 2.1 can be generalized to equidistribution modulo prime powers by requiring divisibility by multiple cyclotomic polynomials, but we omit the details since we are only interested in equidistribution modulo primes in this paper. However, we utilize a modified version of the above lemma when we require divisibility by $\Phi_5(\zeta^2)$ for the proof of part (2) of Conjecture 1.2. The following is used in conjunction with Theorem 2.3, which in particular satisfies the conditions of this lemma.

Lemma 2.2. *Let $f(\zeta)$ be a Laurent polynomial and ℓ an odd prime. Then $\Phi_\ell(-\zeta) \mid f(\zeta)$ in $\mathbb{Q}[\zeta^{-1}, \zeta]$ if and only if for $r \in \{0, 1, \ldots, \ell-2\}$,*

$$(-1)^r\left(\widehat{f}_{r,2\ell} - \widehat{f}_{r+\ell,2\ell}\right) = \widehat{f}_{\ell-1,2\ell} - \widehat{f}_{2\ell-1,2\ell}.$$

Proof. As in the proof of Lemma 2.1, we may assume that $f(\zeta) \in \mathbb{Q}[\zeta]$. As $\Phi_\ell(-\zeta)$ is also an irreducible polynomial in $\mathbb{Q}[\zeta]$, $\Phi_\ell(-\zeta) \mid f(\zeta)$ is equivalent to $f(-\zeta_\ell) = 0$ since $-\zeta_\ell$ is a root of $\Phi_\ell(-\zeta)$. Writing $f(\zeta) = \sum_{j=0}^n a_j\zeta^j$, we see that

$$\begin{aligned} f(-\zeta_\ell) = \sum_{j=0}^{n} a_j(-\zeta_\ell)^j &= \sum_{r=0}^{\ell-1}\left(\sum_{\substack{0\leq j\leq n\\ j\equiv r \pmod{2\ell}}} (-1)^j a_j + \sum_{\substack{0\leq j\leq n\\ j\equiv \ell+r \pmod{2\ell}}} (-1)^j a_j\right)\zeta_\ell^r \\ &= \sum_{r=0}^{\ell-1}(-1)^r\left(\widehat{f}_{r,2\ell} - \widehat{f}_{r+\ell,2\ell}\right)\zeta_\ell^r \\ &= \sum_{r=0}^{\ell-2}\left((-1)^r\left(\widehat{f}_{r,2\ell} - \widehat{f}_{r+\ell,2\ell}\right) - \left(\widehat{f}_{\ell-1,2\ell} - \widehat{f}_{2\ell-1,2\ell}\right)\right)\zeta_\ell^r. \end{aligned}$$

Since $1, \zeta_\ell, \ldots, \zeta_\ell^{\ell-2}$ is a basis for $\mathbb{Q}[\zeta_\ell]$ over $\mathbb{Q}$, we conclude the claim. □

We now review known results concerning the crank that allow us to prove Theorem 1.3. First, in order to prove (2) of Conjecture 1.2, we need additional relationships among cranks modulo 10 given by Garvan [17].

Theorem 2.3 ((1.17) and (1.18) of [17])**.** *For $n \in \mathbb{N}_0$, $0 \leq k \leq 4$, $j \in \{0, 1\}$*

$$M(2k+j, 10; 5n+4) = \frac{1}{5} M(j, 2; 5n+4).$$

Additionally, in order to utilize Theorem 3.1, we need the following result of Ji and Zang [22] related to crank unimodality.

Theorem 2.4 (Theorem 1.7 of [22])**.** *For $n \geq 44$ and $1 \leq m \leq n-1$,*

$$M(m-1, n) \geq M(m, n).$$

Unfortunately, $\mathrm{crank}_n(\zeta)$ is not actually unimodal for $n \geq 44$ because $M(n-1,n) = 0$ and $M(n,n) = 1$ for $n \geq 2$. In order to see this, we have the following result, which is needed below in the paper.

Lemma 2.5. *For fixed $k \in \mathbb{N}$, the sequence $M(n-k, n)$ is constant for $n \geq 2k$.*

Remark. As alluded to above, Lemma 2.5 proves that $\mathrm{crank}_n(\zeta)$ is not unimodal. We see in the proof of Theorem 1.3 below that $\mathrm{crank}^*_{\ell,n}(\zeta)$ is unimodal, thus illustrating the need for the modified crank function that Stanton provides.

Proof of Lemma 2.5. We utilize the following summation formula for $m \in \mathbb{N}$ [16, Theorem 7.19]

$$\sum_{n=0}^{\infty} M(m,n) q^n = \frac{1}{\prod_{k=0}^{\infty}(1-q^k)} \sum_{n=1}^{\infty} (-1)^{n-1} q^{\frac{n(n-1)}{2}+mn}. \tag{2.5}$$

By replacing m by $m+1$ and dividing both sides by q, we find that

$$\sum_{n=-1}^{\infty} M(m+1, n+1) q^n = \frac{1}{\prod_{k=0}^{\infty}(1-q^k)} \sum_{n=1}^{\infty} (-1)^{n-1} q^{\frac{n(n-1)}{2}+(m+1)n-1}.$$

However, since $m+1 > 0$, we may conclude that $M(m+1, 0) = 0$, so

$$\sum_{n=0}^{\infty} M(m+1, n+1) q^n = \frac{1}{\prod_{k=0}^{\infty}(1-q^k)} \sum_{n=1}^{\infty} (-1)^{n-1} q^{\frac{n(n-1)}{2}+(m+1)n-1}. \tag{2.6}$$

Subtracting (2.5) from (2.6) yields

$$\sum_{n=0}^{\infty}(M(m+1,n+1)-M(m,n))q^n$$
$$=\frac{1}{\prod_{k=0}^{\infty}(1-q^k)}\sum_{n=1}^{\infty}(-1)^{n-1}\left(q^{\frac{n(n-1)}{2}+(m+1)n-1}-q^{\frac{n(n-1)}{2}+mn}\right). \quad (2.7)$$

We now claim that the j-th Fourier coefficient vanishes for $j\leq 2m$. Note that the term $n=1$ vanishes in (2.7). On the other hand for $n\geq 2$,

$$\frac{n(n-1)}{2}+(m+1)n-1\geq\frac{n(n-1)}{2}+mn\geq 2m+1,$$

so the smallest power of q in (2.7) is at least $2m+1$. Comparing coefficients on both sides of the equality in (2.7), this tells us that $M(m+1,n+1)=M(m,n)$ for $n\leq 2m$. Replacing m by $n-k$ yields the result. □

3. Proof of the main results

3.1. **A general result.** We first need the following lemma, where a Laurent polynomial $f(\zeta)$ is called *symmetric* if $f(\zeta^{-1})=f(\zeta)$.

Lemma 3.1. *Let $f(\zeta)$ be a symmetric unimodal Laurent polynomial that is divisible by $\Phi_\ell(\zeta)$ for an odd prime ℓ. Then the coefficients of the Laurent polynomial $\frac{f(\zeta)}{\Phi_\ell(\zeta)}$ are non-negative.*

Remark. Note that if $f(\zeta)$ is strictly unimodal, then the coefficients of $\frac{f(\zeta)}{\Phi_\ell(\zeta)}$ are positive.

Proof of Lemma 3.1. Write

$$(1-\zeta)f(\zeta)=:\sum_m a_m\zeta^m.$$

By the symmetry of $f(\zeta)$, we have

$$\sum_m a_m\zeta^{-m}=\left(1-\zeta^{-1}\right)f\left(\zeta^{-1}\right)=-\zeta^{-1}(1-\zeta)f(\zeta)=-\sum_m a_m\zeta^{m-1}.$$

Comparing coefficients, we conclude that

$$a_m=-a_{-m+1}. \quad (3.1)$$

We next show that

$$a_m\geq 0 \text{ for } m\leq 0 \text{ and } a_m\leq 0 \text{ for } m\geq 1. \quad (3.2)$$

By (3.1), we only need to prove $a_m \leq 0$ for $m \geq 1$. If $f(\zeta) = \sum_m c_m\zeta^m$, then comparing coefficients for $(1-\zeta)f(\zeta)$ yields $a_m = c_m - c_{m-1}$. By the unimodality of $f(\zeta)$, we conclude that $c_m \leq c_{m-1}$ for $m \geq 1$ yielding $a_m \leq 0$ as desired.

Now, we consider $\frac{f(\zeta)}{\Phi_\ell(\zeta)}$. Note that $(1-\zeta)\Phi_\ell(\zeta) = 1-\zeta^\ell$. Thus for $|\zeta| < 1$,

$$\sum_m b_m\zeta^m := \frac{f(\zeta)}{\Phi_\ell(\zeta)} = \frac{(1-\zeta)f(\zeta)}{1-\zeta^\ell} = \left(1+\zeta^\ell+\zeta^{2\ell}+\dots\right)\sum_m a_m\zeta^m.$$

Comparing coefficients, we find that $b_m = \sum_{j=0}^\infty a_{m-\ell j}$. By (3.2), the non-negativity of b_m is then clear for $m \leq 0$. For a fixed $m \geq 1$, let k be sufficiently large so that $b_{m+\ell k} = 0$. Such a k exists because $\Phi_\ell(\zeta) \mid f(\zeta)$ implies that $b_j = 0$ for sufficiently large j. Then

$$b_m = b_m - b_{m+\ell k} = \sum_{j=0}^\infty a_{m-\ell j} - \sum_{j=0}^\infty a_{m+\ell k-\ell j} = -\sum_{j=0}^{k-1} a_{m+\ell k-\ell j} \geq 0$$

by (3.2). Thus, we conclude $b_m \geq 0$ for all m. □

In order to prove Conjecture 1.2 (2), we require a slight modification of Lemma 3.1. The proof is similar to the proof of Lemma 3.1.

Lemma 3.2. *Let $f(\zeta)$ be a symmetric Laurent polynomial that is divisible by $\Phi_\ell(\zeta^2)$ for a prime ℓ and such that $[\zeta^{m-1}]f(\zeta) \geq [\zeta^{m+1}]f(\zeta)$ for $m \in \mathbb{N}$. Then $\frac{\zeta^{\ell-1}f(\zeta)}{\Phi_\ell(\zeta^2)}$ is a symmetric Laurent polynomial with non-negative coefficients.*

Proof. We write

$$(\zeta^{-1}-\zeta)f(\zeta) =: \sum_{m=-N}^{N} d_m\zeta^m \quad \text{and} \quad \frac{\zeta^{\ell-1}f(\zeta)}{\Phi_\ell(\zeta^2)} =: \sum_{k=-N+\ell}^{N-\ell} e_k\zeta^k.$$

By the symmetry of $f(\zeta)$, we have

$$\sum_m d_m\zeta^{-m} = \left(\zeta-\zeta^{-1}\right)f\left(\zeta^{-1}\right) = -\left(\zeta^{-1}-\zeta\right)f(\zeta) = -\sum_m d_m\zeta^m.$$

Comparing coefficients, we conclude that

$$d_m = -d_{-m}. \tag{3.3}$$

Note that in particular $d_0 = -d_0$, so $d_0 = 0$. We next show that

$$d_m \geq 0 \text{ for } m \leq 0 \text{ and } d_m \leq 0 \text{ for } m \geq 1. \tag{3.4}$$

By (3.3), we only need to prove that $d_m \leq 0$ for $m \geq 1$. Writing $f(\zeta) = \sum_m c_m \zeta^m$, then comparing coefficients for $(\zeta^{-1} - \zeta) f(\zeta)$ yields the equality

$$d_m = c_{m+1} - c_{m-1}.$$

Additionally, by assumption of the lemma we have that

$$c_{m-1} = \left[\zeta^{m-1}\right] f(\zeta) \geq \left[\zeta^{m+1}\right] f(\zeta) = c_{m+1},$$

from which $d_m \leq 0$ follows.

We next note that $\frac{\zeta^{\ell-1} f(\zeta)}{\Phi_\ell(\zeta^2)}$ is symmetric since $f(\zeta)$ and $\zeta^{\frac{1-\ell}{2}}\Phi_\ell(\zeta)$ are.

Next we find a formula for e_k in terms of d_m. For $|\zeta| < 1$, we have

$$\sum_k e_k \zeta^k = \frac{\zeta^\ell \left(\zeta_{-1} - \zeta\right) f(\zeta)}{1 - \zeta^{2\ell}} = \zeta^\ell \left(1 + \zeta^{2\ell} + \zeta^{4\ell} + \dots\right) \sum_m d_m \zeta^m.$$

Comparing coefficients, we obtain that

$$e_k = \sum_{j=0}^{\infty} d_{k-\ell(2j+1)}.$$

By (3.4), the non-negativity of e_k is then clear for $k \leq \ell$ since $k - \ell(2j+1) \leq 0$ for such values. For a fixed $k \geq \ell + 1$, let r be sufficiently large so that $e_{k+2\ell r} = 0$. Such an r exists because $\Phi_\ell(\zeta^2) \mid f(\zeta)$ implies that $e_n = 0$ for sufficiently large n. Then

$$e_k = e_k - e_{k+2\ell r} = \sum_{j=0}^{\infty} d_{k-\ell(2j+1)} - \sum_{j=0}^{\infty} d_{k+2\ell r-\ell(2j+1)} = -\sum_{j=0}^{k-1} d_{k+2\ell(r-j)-\ell} \geq 0$$

by (3.4) since all of the indices in the final sum are negative. Thus, we conclude $e_k \geq 0$ for all m. □

3.2. **Proof of Theorem 1.3.**

Proof. We begin with the proof of Conjecture 1.2 (3). The polynomials under consideration are of the form (1.2) for $\ell \in \{5, 7, 11\}$. We check the conditions of Lemma 3.1 for these polynomials. The symmetry of (1.2) follows from the fact that $M(m, n) = M(-m, n)$ for all $m, n \in \mathbb{Z}$, which can be seen from the symmetry of $\mathcal{C}(z; \tau)$ under $z \mapsto -z$. Now, we check that (1.2) is divisible by $\Phi_\ell(\zeta)$. By Lemma 2.1, the divisibility of $\mathrm{crank}_{\ell n+\beta}(\zeta)$ by $\Phi_\ell(\zeta)$ is equivalent to

$$M(0, \ell; \ell n + \beta) = M(1, \ell; \ell n + \beta) = \dots = M(\ell - 1, \ell; \ell n + \beta).$$

For $\ell \in \{5,7,11\}$, this is simply the statement of the well-known equidistribution of the crank (see [6, Vector-crank Theorem]). On the other hand,

$$\zeta^{\ell n+\beta-\ell} - \zeta^{\ell n+\beta} + \zeta^{\ell-\ell n-\beta} - \zeta^{-\ell n-\beta} = \left(\zeta^{\ell n+\beta-\ell} - \zeta^{-\ell n-\beta}\right)\left(1-\zeta^{\ell}\right)$$
$$= \left(\zeta^{\ell n+\beta-\ell} - \zeta^{-\ell n-\beta}\right)(1-\zeta)\Phi_\ell(\zeta)$$

proves the divisibility of the remaining part of (1.2). Now, we show the unimodality of $\mathrm{crank}^*_{\ell n+\beta}(\zeta)$ for $\ell n+\beta \geq 44$. By symmetry, it suffices to show that the coefficients are decreasing for non-negative powers of ζ. First, note that for $m \geq \ell n+\beta+1$,

$$[\zeta^m]\mathrm{crank}^*_{\ell,n}(\zeta) = M(m, \ell n+\beta) = 0.$$

Additionally, if $m = \ell n+\beta$, then

$$\left[\zeta^{\ell n+\beta}\right]\mathrm{crank}^*_{\ell,n}(\zeta) = M(\ell n+\beta, \ell n+\beta) - 1 = 0.$$

Now, for $0 \leq m \leq \ell n+\beta-1$, we note that $[\zeta^m]\mathrm{crank}^*_{\ell,n}(\zeta) = M(m, \ell n+\beta)$ except if $m = \ell n+\beta-\ell$. Thus, by Theorem 2.4,

$$[\zeta^m]\,\mathrm{crank}^*_{\ell,n}(\zeta) - \left[\zeta^{m+1}\right]\mathrm{crank}^*_{\ell,n}(\zeta) \geq M(m, \ell n+\beta) - M(m+1, \ell n+\beta) \geq 0$$

for $0 \leq m \leq \ell n+\beta-2$ and $m \neq \ell n+\beta-\ell-1$. In order to prove the inequality for $m = \ell n+\beta-\ell-1$, we note for a fixed value k, the sequence $\{M(n-k,n)\}_{n=1}^{\infty}$ is constant for $n \geq 2k$ by Lemma 2.5, so it suffices to check that

$$\left[\zeta^{\ell n+\beta-\ell-1}\right]\mathrm{crank}^*_{\ell,n}(\zeta) - \left[\zeta^{\ell n+\beta-\ell}\right]\mathrm{crank}^*_{\ell,n}(\zeta)$$
$$= M(\ell n+\beta-\ell-1, \ell n+\beta) - M(\ell n+\beta-\ell, \ell n+\beta) - 1 \geq 0$$

for $\ell \in \{5,7,11\}$ and $n = 22$, which the authors have checked by computer. As a result, we have unimodality of $\mathrm{crank}^*_{\ell,n}(\zeta)$ for $\ell n+\beta \geq 44$. For $\ell n+\beta < 44$, the result can be checked manually. Additionally, if $m = \ell n+\beta-1$, then

$$\left[\zeta^{\ell n+\beta-1}\right]\mathrm{crank}^*_{\ell,n}(\zeta) - \left[\zeta^{\ell n+\beta}\right]\mathrm{crank}^*_{\ell,n}(\zeta) = M(\ell n+\beta-1, \ell n+\beta) - M(\ell n+\beta, \ell n+\beta) + 1 = 0,$$

completing the proof of unimodality.

We now move to the proof of Conjecture 1.2 (2). In order to check the inequality condition of Lemma 3.2, note that $M(m, 5n+4) \geq M(m+2, 5n+4)$ for $0 \leq m \leq 5n+1$ and $5n+4 \geq 44$ by Theorem 2.4 and can be checked manually for $5n+4 < 44$. As for $m = 5n+2$, it is easy to check that $M(5n+2, 5n+4) = M(5n+4, 5n+4) = 1$ for $5n+4 \geq 2$, and for $m \geq 5n+3$, the inequality follows from the fact that $M(5n+3, 5n+4) = 0$ and $M(m, 5n+4) = 0$ for $m > 5n+4$. This proves that $[\zeta^m]\mathrm{crank}_{5n+4}(\zeta) \geq [\zeta^{m+2}]\mathrm{crank}_{5n+4}(\zeta)$ for $5n+4 \geq 2$. Additionally, the divisibility by $\Phi_5(\zeta^2)$ follows directly from Lemma 2.2 and Theorem 2.3, so the result follows from Lemma 3.2. □

3.3. **Proof of Theorem 1.5.**

Proof. We again use Lemma 3.1. The polynomials under consideration are given in (1.1) for $\ell \in \{5,7\}$. The symmetry $N(m,n) = N(-m,n)$ can be seen by the invariance of $\mathcal{R}(z;\tau)$ under $z \mapsto -z$. The symmetry for the remaining terms in (1.1) is clear. As for unimodality, the assumption of Conjecture 1.4 means that it suffices to show that for $m \geq \ell n + \beta - 3$,

$$[\zeta^m]\,\mathrm{rank}^*_{\ell,n}(\zeta) \geq \left[\zeta^{m+1}\right]\mathrm{rank}^*_{\ell,n}(\zeta)\,.$$

From the definition of the rank, it is easy to check that the partition $\ell n+\beta$ of $\ell n+\beta$ is the only partition with rank $\ell n + \beta - 1$. Similarly, we may check that there are no partitions of rank $\ell n+\beta-2$ or of rank $m \geq \ell n+\beta$, while the partition $(\ell n+\beta-1, 1)$ is the only partition of rank $\ell n+\beta-3$ for $n \geq 1$. Therefore, from the definition (1.1), we see that the unimodality of $\mathrm{rank}^*_{\ell,n}(\zeta)$ holds. Finally, the divisibility of $\mathrm{rank}_{\ell n+\beta}(\zeta)$ by $\Phi_\ell(\zeta)$ for $\ell \in \{5,7\}$ follows from the well-known equidistribution of the rank modulo 5 and 7 and from Lemma 2.1. Additionally,

$$\zeta_\ell^{\ell n+\beta-2} - \zeta_\ell^{\ell n+\beta-1} + \zeta_\ell^{2-\ell n-\beta} - \zeta_\ell^{1-\ell n-\beta} = \zeta_\ell^{\beta-2} - \zeta_\ell^{\beta-1} + \zeta_\ell^{2-\beta} - \zeta_\ell^{1-\beta},$$

and by plugging in 5 and 7 for ℓ, we can see that the latter term of (1.1) vanishes under ζ_ℓ and hence is divisible by $\Phi_\ell(\zeta)$. Thus, Lemma 3.1 completes the proof of Theorem 1.5. □

3.4. **Sketch of proof of Theorem 1.7.** We provide only a sketch of the proof in order to avoid reintroducing the entire framework of [28] for a simple modification of those results. We leave it to the interested reader to make the necessary changes to the proofs in [28]. For such a reader, we point out that there are results analogous to Lemmas 4.1 and 4.2 that apply to $\mathcal{A}_k$ and $\mathcal{B}_k$. The first lemma allows one to analyze $\mathcal{A}_k$, which explains almost all of the congruences coming from Theorem 1.1. Below, we use the notation $g(\zeta;q) \equiv h(\zeta;q) \pmod{\Phi_\ell(\zeta)}$ to mean that $\Phi_\ell(\zeta)$ divides $g(\zeta;q) - h(\zeta;q)$ in the ring $\mathbb{Q}[[q]][\zeta,\zeta^{-1}]$.

Lemma 3.3. *Suppose that $\{\ell n + \delta_{k,\ell}\}_{n\in\mathbb{N}_0}$ is an arithmetic progression coming from Theorem 1.1 for $p_k(n)$ with $k + h = \ell t$ and $h \in \{4,6,8,10\}$ if k is odd and $k \in \{4,6,8,10,14\}$ if k is even. Then if $\phi_R(\boldsymbol{z};\tau)$ is the theta block associated to h in [28, Table 1], then there is a choice of $a,b \in \mathbb{Z}$ such that if $\boldsymbol{z} = (az,bz)$, then*

$$\phi_R(\boldsymbol{z};\tau)\left(\zeta^{\pm 2}q\right)_\infty \cdot \ldots \cdot \left(\zeta^{\pm\frac{k+\delta_{2\nmid k}}{2}}q\right)_\infty \left(\zeta^{\pm\left(\frac{k+\delta_{2\nmid k}}{2}+1\right)}q\right)_\infty \equiv q^{\frac{h}{24}} f(\zeta)(q)_\infty^{\delta_{2\nmid k}}\left(q^\ell;q^\ell\right)_\infty^t \pmod{\Phi_\ell(\zeta)}$$

for some $f(\zeta)\in\mathbb{Q}[\zeta^{\frac{1}{2}},\zeta^{-\frac{1}{2}}]$.

Unfortunately, Lemma 3.3 does not appear to hold if $k\equiv -14 \pmod{\ell}$ and k is odd. However, we have the following analogous result that provides a combinatorial description in this case using $\mathcal{B}_k$.

Lemma 3.4. *Suppose that* $\{\ell n+\delta_{k,\ell}\}_{n\in\mathbb{N}_0}$ *is an arithmetic progression coming from Theorem 1.1 for* $p_k(n)$ *with* $k+h=\ell t$ *and* $h\in\{6,14\}$ *and* k *odd. Then if* $\phi_R(\boldsymbol{z};\tau)$ *is the theta block associated to* h *in [28, Table 1], then there is a choice of* $a,b\in\mathbb{Z}$ *such that*

$$\phi_R(\boldsymbol{z};\tau)\left(\zeta^{\pm 2}q\right)_\infty\left(\zeta^{\pm 3}q\right)_\infty\left(\zeta^{\pm 5}q\right)_\infty \cdot \ldots \cdot \left(\zeta^{\pm\frac{k+3}{2}}q\right)_\infty \left(\zeta^{\pm\frac{k+5}{2}}q\right)_\infty \equiv q^{\frac{h}{24}}(q)_\infty f(\zeta)\left(q^\ell;q^\ell\right)_\infty^t \pmod{\Phi_\ell(\zeta)}$$

for some $f(\zeta)\in\mathbb{Q}[\zeta^{\frac{1}{2}},\zeta^{-\frac{1}{2}}]$.

One difference between the above lemmas and those in [28] is that the missing residues in our case may depend on the value of k. However, they are still easy to determine and can be filled in by an appropriate choice of a and b as was done in [28]. Following the proof of [28, Theorem 1.3], we have the following result.

Corollary 3.5. *If* $\{\ell n+\delta_{k,\ell}\}_{n\in\mathbb{N}_0}$ *is an arithmetic progression coming from Theorem 1.1 with* $k+h=\ell t$ *for* $h\in\{4,6,8,10,14\}$ *if* k *is even and* $h\in\{4,6,8,10\}$ *if* k *is odd, then*

$$\Phi_\ell(\zeta)\mid\left[q^{\ell n+\delta_{k,\ell}}\right]\mathcal{A}_k(z;\tau).$$

Similarly, if $\{\ell n+\delta_{k,\ell}\}_{n\in\mathbb{N}_0}$ *is an arithmetic progression coming from Theorem 1.1 with* $k+h=\ell t$, $h\in\{6,14\}$, *and* $k\geq 7$ *odd, then*

$$\Phi_\ell(\zeta)\mid\left[q^{\ell n+\delta_{k,\ell}}\right]\mathcal{B}_k(z;\tau).$$

Using this result, we are able to prove Theorem 1.7.

Proof of Theorem 1.7. To illustrate the symmetry of our polynomials, recall that $\mathrm{crank}_n(\zeta)$ is symmetric. Thus, the coefficients of $\mathcal{A}_k(z;\tau)$ and $\mathcal{B}_k(z;\tau)$ are products and sums of symmetric polynomials, so they must be symmetric as well. Additionally, the unimodality of the coefficients is given as an assumption, while the divisibility of the coefficients is given by Corollary 3.5. As a result, the proof is finished by Lemma 3.1. □

4. Numerical evidence of conjectures and ideas for further work

4.1. **Computations.** The computational evidence supporting Conjecture 1.6 was found initially through an exhaustive search of crank generating functions to find likely eventually unimodal examples. The search space is given by

$$\mathcal{S} := \bigcup_{3\leq k\leq 11} \left\{ \mathcal{C}_k \left(a_1, a_2, \ldots, a_{\frac{k+\delta_{2\nmid k}}{2}}; z; \tau \right) : k \geq a_1 > a_2 > \cdots > a_{\frac{k+\delta_{2\nmid k}}{2}} > 0 \right\}.$$

Note that the space is unrestricted by any consideration of the cranks' ability to explain k-colored partition congruences.

For each crank generating function $\mathcal{D} \in \mathcal{S}$, the minimum value of m such that $[q^n]\mathcal{D}$ is unimodal for all $m < n < 75$ was found, with $\mathcal{D}$ being considered a likely eventually unimodal candidate if such an m exists. The entire search was completed in approximately 56 hours on a single-threaded Intel i7-8750H CPU. The search space for larger k increases exponentially since the number of possible cranks is given by the $\lfloor \frac{k}{2} \rfloor$-th central binomial coefficient, with the total computation runtime then being in $O(2^k)$.

The set of all likely eventually unimodal examples was then searched manually for general trends and potential infinite families, with $\mathcal{A}_k$ and $\mathcal{B}_k$ emerging as the families of choice due to their determined unimodality and ability to explain almost all known k-colored partition congruences. Conjecture 1.6 was then formulated and more extensively verified for all $3 \leq k \leq 20$ and the respective $n \leq 99$; this verification took approximately 4 hours on the same hardware.

Evaluating the set of likely eventually unimodal examples also led to the following general conjecture. See Table 1 for a partial summary of our computations.

Crank	Unimodal?
$\mathcal{C}_3(2,1;z;\tau)$	$\forall n > 7$
$\mathcal{C}_3(3,1;z;\tau)$	no
$\mathcal{C}_3(3,2;z;\tau)$	$\forall n > 6$

(A) $k = 3$

Crank	Unimodal?
$\mathcal{C}_4(2,1;z;\tau)$	$\forall n > 1$
$\mathcal{C}_4(3,1;z;\tau)$	no
$\mathcal{C}_4(4,1;z;\tau)$	no
$\mathcal{C}_4(3,2;z;\tau)$	$\forall n > 1$
$\mathcal{C}_4(4,2;z;\tau)$	no
$\mathcal{C}_4(4,3;z;\tau)$	$\forall n > 23$

(B) $k = 4$

Crank	Unimodal?
$\mathcal{C}_5(3,2,1;z;\tau)$	$\forall n > 9$
$\mathcal{C}_5(4,2,1;z;\tau)$	no
$\mathcal{C}_5(5,2,1;z;\tau)$	no
$\mathcal{C}_5(4,3,1;z;\tau)$	$\forall n > 11$
$\mathcal{C}_5(5,3,1;z;\tau)$	no
$\mathcal{C}_5(5,4,1;z;\tau)$	$\forall n > 9$
$\mathcal{C}_5(4,3,2;z;\tau)$	$\forall n > 10$
$\mathcal{C}_5(5,3,2;z;\tau)$	no
$\mathcal{C}_5(5,4,2;z;\tau)$	$\forall n > 13$
$\mathcal{C}_5(5,4,3;z;\tau)$	$\forall n > 13$

(C) $k = 5$

Crank	Unimodal?
$\mathcal{C}_6(3,2,1;z;\tau)$	$\forall n > 1$
$\mathcal{C}_6(4,2,1;z;\tau)$	no
$\mathcal{C}_6(5,2,1;z;\tau)$	no
$\mathcal{C}_6(6,2,1;z;\tau)$	no
$\mathcal{C}_6(4,3,1;z;\tau)$	$\forall n > 5$
$\mathcal{C}_6(5,3,1;z;\tau)$	no
$\mathcal{C}_6(6,3,1;z;\tau)$	no
$\mathcal{C}_6(5,4,1;z;\tau)$	$\forall n > 11$
$\mathcal{C}_6(6,4,1;z;\tau)$	no
$\mathcal{C}_6(6,5,1;z;\tau)$	$\forall n > 21$
$\mathcal{C}_6(4,3,2;z;\tau)$	$\forall n > 14$
$\mathcal{C}_6(5,3,2;z;\tau)$	no
$\mathcal{C}_6(6,3,2;z;\tau)$	no
$\mathcal{C}_6(5,4,2;z;\tau)$	$\forall n > 19$
$\mathcal{C}_6(6,4,2;z;\tau)$	no
$\mathcal{C}_6(6,5,2;z;\tau)$	$\forall n > 20$
$\mathcal{C}_6(5,4,3;z;\tau)$	$\forall n > 7$
$\mathcal{C}_6(6,4,3;z;\tau)$	no
$\mathcal{C}_6(6,5,3;z;\tau)$	$\forall n > 32$
$\mathcal{C}_6(6,5,4;z;\tau)$	$\forall n > 19$

(D) $k = 6$

TABLE 1. Cranks for the given value of k

Conjecture 4.1. *Let* $\mathcal{D}(z;\tau) := \mathcal{C}_k(a_1, a_2, \ldots, a_{\frac{k+\delta_{2\nmid k}}{2}}; z; \tau)$ *for some* $a_1 > a_2 > \cdots > a_{\frac{k+\delta_{2\nmid k}}{2}} > 0$ *and* $k \geq 3$*. Then* $\mathcal{D}(z;\tau)$ *is eventually unimodal if and only if* $a_1 - a_2 = 1$.

4.2. **Questions and ideas for further research.** We conclude with open questions for further study.

(1) Recalling the motivation for Conjecture 1.2 given in Section 1.3, are there combinatorial interpretations of the coefficients in the conjectures?
(2) Similarly, is there a combinatorial explanation of the non-negativity of the coefficients given in Theorem 1.7?
(3) Use Lemma 2.1 and Lemma 3.1 (or modifications of them such as ones we have given) to prove non-negativity of coefficients for polynomials related to other families of partition functions, to congruences of the partition function modulo higher powers of primes, or to other combinatorial objects.
(4) Prove or give partial results towards any of the unimodality conjectures given such as Conjecture 1.4 and Conjecture 1.6.

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University of Cologne, Department of Mathematics and Computer Science, Weyertal 86-90, 50931 Cologne, Germany
Email address: kbringma@math.uni-koeln.de

Department of Mathematics, Vanderbilt University, Nashville, TN 37240
Email address: kevin.j.gomez@vanderbilt.edu
Email address: larry.rolen@vanderbilt.edu
Email address, Corresponding author: zachary.d.tripp@vanderbilt.edu