ASYMPTOTIC FORMULAS FOR COEFFICIENTS OF KAC-WAKIMOTO CHARACTERS

KATHRIN BRINGMANN AND KARL MAHLBURG

Abstract. We study the coefficients of Kac and Wakimoto's character formulas for the affine Lie superalgebras $s\ell(n+1|1)^\wedge$. The coefficients of these characters are the weight multiplicities of irreducible modules of the Lie superalgebras, and their asymptotic study begins with Kac and Peterson's earlier use of modular forms to understand the coefficients of characters for affine Lie algebras. In the affine Lie superalgebra setting, the characters are products of weakly holomorphic modular forms and Appell-type sums, which have recently been studied using developments in the theory of mock modular forms and harmonic Maass forms. Using our previously developed extension of the Circle Method for products of mock modular forms along with the Saddle Point Method, we find asymptotic series expansions for the coefficients of the characters with polynomial error.

1. Introduction and statement of results

In a series of papers [20, 21], Kac and Wakimoto studied integrable irreducible highest weight modules over affine Lie superalgebras, which were previously introduced by Kac in [17]. Kac and Moody's original construction of infinite-dimensional (affine) Lie algebras is described in the book [18]. In this paper we prove asymptotic results for the coefficients of the traces of modules for certain affine Lie superalgebras; these coefficients are (essentially) the multiplicities of irreducible module weight space decompositions. Our approach is similar in spirit to Kac and Peterson's earlier work on the characters for affine Lie algebras [19], where they used the modularity properties of q-series and Tauberian theorems in order to prove asymptotic results for the coefficients of string functions.

In the setting of affine Lie superalgebras, the "modularity" properties do not come from the classical theory of theta functions and elliptic modular forms, but instead rely on more recent developments in the theory of mock modular forms, and the asymptotic formulas follow in part from our previous work extending the classical Hardy-Ramanujan Circle Method to this setting [5]. Our present approach also offers a significantly more precise expansion of the main asymptotic term as compared to [5], as we now incorporate an improved analysis using the Saddle Point Method (which further requires a modified approach to certain modular transformations). As such, the technical heart of this paper lies in the theory of modular and automorphic properties of q-series, and our main goal is to determine asymptotic expansions for the coefficients of the following series

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derived by Kac and Wakimoto in [21]:

$$tr_{L(\Lambda_{(s)})}q^{L_0} := 2q^{-\frac{s}{2}}\frac{\phi(q^2)^2}{\phi(q)^{n+3}}\sum_{k=(k_1,\ldots,k_n)\in\mathbb{Z}^n}\frac{q^{\frac{1}{2}\sum_{i=1}^n k_i(k_i+1)}}{1+q^{\sum_{i=1}^n k_i - s}}. \tag{1.1}$$

Here the parameters satisfy the restrictions $-\frac{n}{2}\le s\le\frac{n}{2}$, $s\in\mathbb{Z}$ (the restriction on the range of s is simply a convenience following from the Jacobi transformation laws stated in Proposition 2.4), $n\in\mathbb{N}$, and $\phi(q) := \prod_{m=1}^\infty(1-q^m)$. This formula gives a closed form for the trace of $L(\Lambda(s))$, which is the irreducible $s\ell(n+1,1)^\wedge$-module with highest weight $\Lambda_{(s)}$. The term q^{L_0} is the "energy operator", and for purposes of our discussion serves as a normalizing factor.

To motivate our study, we further recall some of the basic properties of Kac and Wakimoto's study of affine Lie superalgebras. If $\mathfrak{g}$ is a simple (or abelian) complex, finite-dimensional Lie algebra with a bilinear form, then the associated (infinite-dimensional) *affine Lie algebra* $\widetilde{\mathfrak{g}}$ is constructed by adding to the algebra an auxiliary Laurent variable, a *central* element, and a *derivation*, with the bilinear form also extended in a natural way. The structure of such an algebra can then be understood through the study of $\widetilde{\mathfrak{g}}$-modules; in particular, $L(\Lambda)$ is defined to be the $\widetilde{\mathfrak{g}}$-module with highest weight Λ.

An important feature of such a $\widetilde{\mathfrak{g}}$-module is that it has a *weight space decomposition*

$$L(\Lambda) = \bigoplus_\lambda L(\Lambda)_\lambda, \tag{1.2}$$

where the multiplicity of each weight space $L(\Lambda)_\lambda$ for $\lambda\in\widetilde{\mathfrak{h}}^\star$ is denoted by $\mathrm{mult}_\Lambda(\lambda)$. The *character* of the module is then defined as

$$\mathrm{ch}_{L(\Lambda)} := \sum_\lambda \mathrm{mult}_\Lambda(\lambda) q^\lambda, \tag{1.3}$$

where the sum runs over the dual root lattice.

In Section 4.7 of [19], Kac and Peterson addressed the asymptotic behavior of weight multiplicities for affine Lie algebras.

Theorem (Kac-Peterson [19])**.** *If $\widetilde{\mathfrak{g}}$ is an affine Lie algebra with $\ell+1$ simple roots, then as $m\to\infty$*

$$\mathrm{mult}_\Lambda(\lambda - m\delta) \sim 2^{-\frac{1}{2}} a^{\frac{\ell+1}{4}} b m^{-\frac{\ell+3}{4}} e^{4\pi\sqrt{am}}, \tag{1.4}$$

where a and b are certain constants that are determined by $\widetilde{\mathfrak{g}}$.

Remark. The constants a and b and the weight expression $\lambda - m\delta$ all depend on the Cartan subalgebra.

Remark. Their proof relied heavily on the modularity of the so-called "string functions" of the character, which arise by dissecting (1.3) using theta functions (see (2.18) in [19]). They first applied modular inversion formulas in order to determine the asymptotic behavior as $q\to 1^-$ (resulting in expressions analogous to (1.6)), and then applied Tauberian theorems in order to prove the corresponding asymptotics for the coefficients. It is important to note that the application of Tauberian theorems requires that the multiplicities be monotonically increasing.

For example, Kac and Peterson showed that in the case that $\widetilde{\mathfrak{g}}$ has type $A_1^{(1)}$, this theorem implies Hardy and Ramanujan's famous asymptotic result for Euler's partition function [14]

$$p(m) \sim \frac{1}{4m\sqrt{3}}\cdot e^{\pi\sqrt{\frac{2m}{3}}} \quad (m\to\infty).$$

Here $p(m)$ is the number of integer partitions of m, which have the generating function

$$\sum_{m\geq 0} p(m)q^m = \prod_{m\geq 1} \frac{1}{1-q^m} = \frac{q^{1/24}}{\eta(\tau)}.$$

Dedekind's eta-function is defined by $\eta(\tau) := q^{1/24}\prod_{m=1}^{\infty}(1-q^m)$, which is a weight 1/2 (cuspidal) modular form whose Fourier expansion is written using the uniformizer $q := e^{2\pi i\tau}$.

Hardy and Ramanujan actually proved a much stronger result that gave an asymptotic series for $p(m)$ with polynomial error instead of just the main term. Rademacher [23] then further refined their development of the Circle Method and obtained an exact formula for $p(m)$. In order to state their results, several definitions are necessary; first, if $k \geq 1$ and m are integers, then define the Kloosterman sum

$$A_k(m) := \sum_{\substack{0\leq h<k \\ (h,k)=1}} \omega_{h,k} e^{-\frac{2\pi i h m}{k}},$$

where $\omega_{h,k}$ is the multiplier of the partition generating function and is given explicitly by

$$\omega_{h,k} := \exp\left(\pi i s(h,k)\right),$$

where $s(h,k)$ is the usual Dedekind sum [13]. Finally, $I_s(x)$ denotes the usual modified Bessel function of order s.

Rademacher's main result was the series expansion

$$p(m) = 2\pi(24m-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(m)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24m-1}}{6k}\right). \tag{1.5}$$

Rademacher and Zuckerman later established similar formulas for the coefficients of any weakly holomorphic modular form of negative weight [24, 27].

However, in our present study of characters for affine Lie superalgebras, the general theory of Rademacher does not apply. In [7], the first author and Ono answered a question of Kac regarding the "modularity" of the traces in (1.1), and proved that they may be written as a modular form multiplied with an analog of one of Ramanujan's *mock theta functions* (this will be described more precisely in Section 2).

Ramanujan's mock theta functions have served as motivating examples for recent developments in the study of modularity properties of hypergeometric q-series. In Ramanujan's last letter to Hardy, he introduced 17 examples of hypergeometric q-series whose striking asymptotic and modular transformation properties resembled weakly holomorphic modular forms. Building on Ramanujan and Watson's studies of modular transformations for the mock theta function $f(q)$, Dragonette [11], and later Andrews [1], used the Circle Method in order to prove asymptotic series expansions for its coefficients $\alpha(n)$.

In [28], Zwegers finally explained the proper automorphic framework for $f(q)$ and the rest of Ramanujan's mock theta functions by showing that they are the holomorphic parts of *harmonic weak Maass forms*, which are certain automorphic forms introduced in [9]. The first author and Ono used this framework to prove a conjecture of Andrews and Dragonette that gave an exact series expansion for $\alpha(n)$ of a similar shape to that seen in (1.5). In order to prove this result, they used Maass-Poincaré series decompositions for harmonic weak Maass forms, and subsequently derived exact formulas for the holomorphic coefficients of all harmonic weak Maass forms of non-positive weight [8].

However, the present situation is rather more complicated, as there is no longer a Poincaré series decomposition for the product of mock modular forms. Nevertheless, the modular transformations of $tr_{L(\Lambda_{(s)})}q^{L_0}$ can be explicitly understood, and thus it is still possible to address asymptotic questions. One application of these modularity properties is found in work of the first author and Folsom [4], where they considered the behavior of these characters as $q \to 1^-$. Replacing τ by $it, t \in \mathbb{R}^+$, Kac and Wakimoto previously [21] established the following asymptotic behavior for $tr_{L(\Lambda_{(s)})}q^{L_0}$ as $t \to 0^+$:

$$tr_{L(\Lambda_{(s)})}q^{L_0} \sim \frac{\sqrt{t}}{2} e^{\frac{\pi(n+2)}{12t}}. \tag{1.6}$$

Kac and Wakimoto proved this result through series manipulations and the asymptotic behavior of the modular transformations of theta functions. This result was improved in [4] by employing the relation to harmonic weak Maass forms, which led to a full asymptotic expansion of the main exponential term, written as Taylor polynomials involving Euler numbers.

In contrast to Kac and Peterson's asymptotic results (cf. (1.4)), Tauberian theorems do not directly apply to (1.6) due to the potential sign changes in the trace formulas. Furthermore, the transformation theory of Appell sums had not yet been fully developed when Kac and Wakimoto's initial work was completed, and furthermore, the previously mentioned results regarding exact formulas for the coefficients of modular forms or mock modular forms do not apply to products of mock modular forms. The first results for such a situation are found in a previous paper of the authors [5], in which an asymptotic series expansion was given for partitions without sequences. In this work we developed a generalization of the Circle Method that addressed the "continuous" principal parts that arise in these sorts of examples.

For notational simplicity, in this paper we only consider the case that n is odd; the case of even n can be treated similarly. The coefficients of interest are defined by the normalized series

$$g_s(q) = tr_{L(\Lambda_{(s)})}q^{L_0}q^{-\frac{s}{2}} = \sum_{m=0}^{\infty} c_s(m)q^m. \tag{1.7}$$

To state our main theorem, we require some notation. We let $0 \leq h < k$ with $(h,k) = 1$, $\gamma := (n,k)$ and write $k = k'\gamma$ and $n = n'\gamma$. We define for $r, j \in \mathbb{Z}$ the sets

$$\begin{aligned} S_n &:= \left\{0 \leq j \leq n-1; \left|j - \frac{n-1}{2}\right| > \sqrt{\frac{n(n+2)}{6}}; 2j \equiv -1-2sh \pmod{\gamma}\right\}, \\ T_n &:= \left\{-\frac{n}{2} < j < \frac{n}{2}; |j| < \frac{n}{2} - \frac{1}{2}\sqrt{\frac{n(2n+1)}{2}}; j \equiv -sh \pmod{\gamma}\right\}, \end{aligned}$$

and the constants

$$\begin{aligned} \varepsilon_s(n) &:= \frac{n-1}{24} + \frac{s^2}{2n} - \frac{s}{2}, \\ \delta_j(n) &:= \frac{n}{24} - \frac{5}{12} + \frac{1}{2n}\Big(j^2 + (1-n)j\Big) + \frac{1}{8n}, \\ \widetilde{\delta_j}(n) &:= \frac{j^2}{2n} - \frac{|j|}{2} + \frac{n}{24} - \frac{1}{24}, \\ n_r &:= \frac{n+2}{12} - r. \end{aligned}$$

Moreover $A_j(r)$, $B_j(r)$, and $C_\ell(r)$ are the principal part coefficients of certain Fourier series that are given in (3.1), (3.2), and (4.4), respectively, and for real arguments satisfying $0 < a < \frac{b}{2}$ and

$|c| < \frac{1}{2}$, we define the principal value integral

$$P(a,b,c;M) := \int_{-1}^{1} \frac{\left(1-y^2\right)^{-\frac{3}{4}}}{\cosh\left(\pi(bMy+ci)\right)} I_{\frac{3}{2}}\left(aM\sqrt{1-y^2}\right)dy. \tag{1.8}$$

Finally $K_k(m,n)$, $\widetilde{K_k}(m,n)$, and $K^*_{k,\ell}(m,n)$ are certain sums of Kloosterman-type defined in (4.1), (4.3), and (4.7), respectively. Following our non-holomorphic generalization of the Circle Method found in [5] and strengthening the asymptotic analysis using the Saddle Point Method (note that the integrals in (1.8) are of a different shape than those in [5]), we find the following asymptotic series for the coefficients $c_s(m)$.

Theorem 1.1. *Assuming the notation above, we have as $m \to \infty$*

$$\begin{aligned}
c_s(m) = {} & \frac{2\pi}{\sqrt{\left(m-\varepsilon_s(n)\right)}} \sum_{\substack{1\le k\le\sqrt{m}\\ 2\nmid k}} \frac{1}{k}\frac{1}{\sqrt{n'}} \sum_{j\in S_n} \sum_{0\le r<2\delta_j(n)} A_j(r)\sqrt{\delta_j(n)-\frac{r}{2}} \\
& \times K_k(-m,r) I_1\left(\frac{4\pi}{k}\sqrt{\left(\delta_j(n)-\frac{r}{2}\right)\left(m-\varepsilon_s(n)\right)}\right) \\
& + \frac{2^{n+2}\pi}{\sqrt{\left(m-\varepsilon_s(n)\right)}} \sum_{\substack{1\le k\le\sqrt{m}\\ 2\mid k}} \frac{1}{k}\frac{1}{\sqrt{n'}} \sum_{j\in T_n} \sum_{0\le r<\widetilde{\delta}_j(n)} B_j(r)\sqrt{\widetilde{\delta}_j(n)-r} \\
& \times \widetilde{K_k}(-m,r) I_1\left(\frac{4\pi}{k}\sqrt{\left(\widetilde{\delta}_j(n)-r\right)\left(m-\varepsilon_s(n)\right)}\right) \\
& + \frac{2^{-\frac{1}{4}}\pi}{\left(m-\varepsilon_s(n)\right)^{\frac{1}{4}}\sqrt{n}} \sum_{2\nmid k}\sum_{n_r>0} C(r) n_r^{\frac{3}{4}} \sum_{-\frac{k-1}{2}\le\ell\le\frac{k-1}{2}} K^*_{k,\ell}(-m,r) \\
& P\left(\frac{2\pi}{k}\sqrt{2n_r}, \frac{\sqrt{2}}{k\sqrt{n}}, -\frac{1}{k}\left(\ell-\frac{s}{n}\right); \sqrt{m-\varepsilon_s(n)}\right) + O\left(m^{\frac{1}{4}}\right).
\end{aligned}$$

Remark. Although the Kac-Wakimoto characters are more exotic than classical modular functions, our asymptotic results for the coefficients are still "optimal" in the sense that one would not expect a more precise error even in the modular setting. Indeed, the characters essentially transform like (weight zero) modular functions, ignoring the non-holomorphic correction factors that are described more thoroughly in Section 2. Furthermore, just as one can explicitly calculate asymptotic expansions for the Bessel functions that arise in the classical Circle Method, we can in principle also use Taylor series and the method of steepest descent to calculate asymptotic expansions for the principal value integrals that arise in our formula.

We note that unlike in previous situations (including [5]), where the asymptotic main term arose from the q-series contribution, here the main term comes from the (non-holomorphic) obstruction to modularity. In particular, the dominant asymptotic term is the $k = 1$ term of the third sum, which has the equivalent asymptotic form given in the following result.

Corollary 1.2. *Assuming the notation above, as $m \to \infty$*

$$c_s(m) \sim \frac{\sqrt{n+2}}{8m\sqrt{3}} e^{2\pi\sqrt{\frac{(n+2)m}{6}}}.$$

Remark. This leading asymptotic would also follow from the application of Ingham's Tauberian theorem [15], although various technical requirements regarding the monotonicity of the coefficients must also be verified. Theorem 1.1 provides much more than the main asymptotic term, and we present it here solely in order to illustrate that it can be easily read off from the overall asymptotic series expansion.

The remainder of the paper is organized as follows. In Section 2 we review the transformation theory of Jacobi and mock Jacobi forms, and derive new explicit formulas for certain multipliers. In Section 3 we determine the principal parts of the characters at all cusps. Finally, in Section 4 we apply the Circle Method to determine asymptotic series expansions for the coefficients, thus proving our main results.

2. Transformation laws

Throughout we let $0 \leq h < k$ with $(h,k) = 1$ be given, and we let $[a]_b$ denotes the inverse of $a \pmod{b}$, where this notation extends to higher moduli in a consistent manner; we also assume (legal) divisibility properties as needed. In particular, we always choose $[-h]_{2k}$ instead of $[-h]_k$ if k is even, and assume that $8|[-h]_k$ if k is odd. Moreover, we restrict $z \in \mathbb{C}$ to the half-plane $\mathrm{Re}(z) > 0$. We further denote the local cuspidal parameters from the Circle Method as

$$\tau_{h,k} := \frac{1}{k}(h+iz), \quad \widetilde{\tau}_{h,k} := \frac{1}{k}\left([-h]_k + \frac{i}{z}\right), \quad q_{h,k} := e^{2\pi i \tau_{h,k}}, \quad \widetilde{q}_{h,k} := e^{2\pi i \widetilde{\tau}_{h,k}}.$$

We also adopt the standard notation $e(x) := e^{2\pi i x}$.

2.1. **Theta functions.**

In this section we show transformation laws for certain theta functions. Let χ be the multiplier of η, which is defined by

$$\eta\left(\tau_{h,k}\right) = \sqrt{\frac{i}{z}}\chi\left(h, [-h]_k, k\right)\eta\left(\widetilde{\tau}_{h,k}\right).$$

In the notation of the introduction we have

$$\chi\left(h, [-h]_k, k\right) = i^{-\frac{1}{2}}\omega_{h,k}^{-1} e^{-\frac{\pi i}{12k}\left([-h]_k - h\right)}.$$

Note that we must be careful when picking the representative of the inverse of $-h$ modulo k, as $\chi(h, [-h]_k, k)$ depends on the choice of $[-h]_k$ modulo $\mathrm{lcm}(24,k)$. Recalling that $q = e^{2\pi i\tau}$, we define shifted theta functions for $\ell \in \mathbb{Z}$ by

$$\Theta_{n,\ell}(u;\tau) := (-1)^\ell q^{\frac{\ell^2}{2n}} e^{2\pi i \ell u}\vartheta\left(nu + \ell\tau - \frac{n+1}{2}; n\tau\right),$$

where

$$\vartheta(u) = \vartheta(u;\tau) := \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} q^{\frac{\nu^2}{2}} e^{2\pi i\nu\left(u+\frac{1}{2}\right)}.$$

The following elementary identity which relates ϑ at $\frac{\tau}{n}$ and $n\tau$ is well known (see [26] or [28]).

Lemma 2.1. *For $n \in \mathbb{N}$, we have*

$$\vartheta\left(u;\frac{\tau}{n}\right) = \sum_{\ell \pmod{n}} q^{\frac{1}{2n}\left(\ell-\frac{n-1}{2}\right)^2} e^{2\pi i\left(\ell-\frac{n-1}{2}\right)\left(u+\frac{1}{2}\right)}\vartheta\left(nu + \left(\ell - \frac{n-1}{2}\right)\tau + \frac{n-1}{2}; n\tau\right).$$

The following transformation laws for ϑ and Θ follow from the classical theory of theta functions [22, 28].

Lemma 2.2. *Assume the notation above.*

(*i*) *The following transformations hold:*

$$\vartheta\Big(-iuz;\tau_{h,k}\Big)=\chi^3\Big(h,[-h]_k,k\Big)\sqrt{\frac{i}{z}}e^{\pi k u^2 z}\vartheta\Big(u;\widetilde{\tau}_{h,k}\Big),$$
$$\vartheta(u+1)=-\vartheta(u),$$
$$\vartheta(u;\tau+1)=e^{\frac{\pi i}{4}}\vartheta(u;\tau).$$

(*ii*) *The following (vector) transformations hold:*

$$\Theta_{n,\ell}(u;\tau+1)=e^{\frac{\pi i}{n}\left(\ell-\frac{n}{2}\right)^2}\Theta_{n,\ell}(u;\tau),$$
$$\Theta_{n,\ell}\left(\frac{u}{\tau};-\frac{1}{\tau}\right)=i^{-n}\sqrt{-\frac{i\tau}{n}}e^{\frac{\pi i n u^2}{\tau}}\sum_{j \pmod n}e^{-\frac{2\pi i j\ell}{n}}\Theta_{n,j}(u;\tau).$$

In order to apply the Circle Method, we require the explicit modular behavior of Θ on all of $SL_2(\mathbb{Z})$. The following result is largely contained in the literature, but for the readers convenience we give a proof.

Proposition 2.3. *Assume the notation above. We have*

$$\Theta_{n,0}\Big(-iuz;\tau_{h,k}\Big)=\sqrt{\frac{i}{n'z}}e^{\pi n k u^2 z}(-1)^{\frac{n'-1}{2}}\chi^3\Big(hn',[-h]_k[n']_{8k'},k'\Big)$$
$$\times\sum_{\ell \pmod{n'}}e^{\frac{\pi i\ell^2}{n'k'}\left([n']_{8k'}n'-1\right)[-h]_k}\Theta_{n,\ell\gamma}\Big(u;\widetilde{\tau}_{h,k}\Big).$$

Proof. Using Lemma 2.2 (*ii*), we easily see that

$$\Theta_{n,0}\left(-iuz;\tau_{h,k}\right)=\sqrt{\frac{i}{z}}e^{\pi n k u^2 z}\sum_{\ell \pmod n}\chi_\ell\Big(h,[-h]_k,k\Big)\Theta_{n,\ell}\Big(u;\widetilde{\tau}_{h,k}\Big)$$

for certain multipliers χ_ℓ. We note that χ_ℓ is independent of u, so we may determine χ_ℓ by picking $u=\frac{1}{2}$. Using Lemma 2.2 (*i*), we conclude that

$$\Theta_{n,0}\left(-\frac{iz}{2};\tau_{h,k}\right)=(-1)^{\frac{n+1}{2}}\chi^3\Big(hn',\left[-hn'\right]_{k'},k'\Big)\sqrt{\frac{i}{zn'}}e^{\frac{\pi k n z}{4}}\vartheta\left(\frac{\gamma}{2};\frac{1}{k'}\left(\left[-hn'\right]_{k'}+\frac{i}{n'z}\right)\right).$$

Using Lemma 2.1 and changing $\ell\mapsto\ell+\frac{n'-1}{2}$, we dissect the theta function as

$$\vartheta\left(\frac{\gamma}{2};\frac{1}{k'}\left(\left[-hn'\right]_{k'}+\frac{i}{n'z}\right)\right)=\sum_{\ell \pmod{n'}}e^{\frac{\pi i\ell^2}{n'k'}\left([-hn']_{k'}n'+\frac{i}{z}\right)}e^{2\pi i\ell\left(\frac{\gamma+1}{2}\right)}$$
$$\times\vartheta\left(\frac{n}{2}+\frac{\ell}{k'}\left(\left[-hn'\right]_{k'}n'+\frac{i}{z}\right)+\frac{n'-1}{2};\frac{n'}{k'}\left([-h]_{k'}\left[n'\right]_{k'}n'+\frac{i}{z}\right)\right).$$

Using Lemma 2.2 and the definition of $\Theta_{n,\ell}$, we obtain

$$\vartheta\left(\frac{n}{2}+\frac{\ell}{k'}\left([-h]_{k'}[n']_{k'}n'+\frac{i}{z}\right)+\frac{n'-1}{2};\frac{n'}{k'}\left([-h]_{k'}[n']_{k'}n'+\frac{i}{z}\right)\right)=(-1)^{\frac{n'+n}{2}}\widetilde{q}_{h,k}^{\,-\frac{\ell^2\gamma^2}{2n}}\Theta_{n,\ell\gamma}\left(\frac{1}{2};\widetilde{\tau}_{h,k}\right).$$

Here we have chosen the inverse of n' modulo $8k'$ instead of k', and the inverse of $-h$ modulo k instead of k'. Simplifying we then easily conclude that

$$\Theta_{n,0}\left(-\frac{iz}{2};\tau_{h,k}\right)=(-1)^{\frac{n'-1}{2}}\chi^3\Big(hn',[-h]_k[n']_{8k'},k'\Big)\sqrt{\frac{i}{n'z}}e^{\frac{\pi knz}{4}}$$
$$\times\sum_{\ell\pmod{n'}}e^{\frac{\pi i\ell^2}{n'k'}\left([n']_{8k'}n'-1\right)[-h]_k}\Theta_{n,\ell\gamma}\left(\frac{1}{2};\widetilde{\tau}_{h,k}\right).$$

This then gives Proposition 2.3. □

2.2. **Transformation laws for for Lerch-type sums $\boldsymbol{\mu_n}$.** We next recall certain Appell sums introduced by Zwegers [28]. We denote for $n\in\mathbb{N}$

$$\mu_n(u,v)=\mu_n(u,v;\tau):=\frac{e^{\pi iu}}{\prod_{j=1}^n\vartheta(v_j;\tau)}\sum_{k\in\mathbb{Z}^n}\frac{(-1)^{|k|}q^{\frac{1}{2}||k||^2+\frac{1}{2}|k|}e^{2\pi ik\cdot v}}{1-e^{2\pi iu}q^{|k|}},$$

where $|k|:=\sum_{i=1}^n k_i$, $||k||^2:=\sum_{i=1}^n k_i^2$, and $u\in\mathbb{C}, v\in\mathbb{C}^n,\tau\in\mathbb{H}$. If $a\in\mathbb{C}$, we will throughout write $\boldsymbol{a}$ to denote the scalar vector $(a,\ldots,a)$. To state the transformation laws of μ_n, we require shifted Appell sums, which are given by

$$\mu_{n,\ell}(u,v)=\mu_{n,\ell}(u,v;\tau):=(-1)^\ell q^{-\frac{\ell^2}{2n}}e^{-\frac{2\pi i\ell}{n}(u-|v|)}\mu_n(u+\ell\tau,v;\tau).$$

Finally, we require the "completed function"

$$\widehat{\mu}_n(u,v)=\widehat{\mu}_n(u,v;\tau):=\mu_n(u,v;\tau)-\frac{i}{2}R\left(u-|v|-\frac{n+1}{2};n\tau\right). \tag{2.1}$$

Here the real-analytic function R is defined by

$$R(u)=R(u;\tau):=\sum_{\nu\in\frac{1}{2}+\mathbb{Z}}\left\{\operatorname{sgn}(\nu)-E\left((\nu+a)\sqrt{2y}\right)\right\}(-1)^{\nu-\frac{1}{2}}e^{-\pi i\nu^2\tau-2\pi i\nu u},$$

where $y:=\operatorname{Im}(\tau)$, $a:=\operatorname{Im}(u)/\operatorname{Im}(\tau)$, and the function E is defined by

$$E(u):=2\int_0^u e^{-\pi u^2}du=\operatorname{sgn}(u)\Big(1-\beta(u^2)\Big),$$

where for positive real x we let $\beta(x):=\int_x^\infty u^{-\frac{1}{2}}e^{-\pi u}du$. The shifted completions $\widehat{\mu}_{n,\ell}$ are then also defined analogously. In [28] the following transformation laws are shown.

Proposition 2.4. *Let $u\in\mathbb{C}$, $v\in\mathbb{C}^n,\tau\in\mathbb{H}$, $\lambda_1,\nu_1\in\mathbb{Z}$ and $\lambda_2,\nu_2\in\mathbb{Z}^n$ such that $\lambda_1-|\lambda_2|\in n\mathbb{Z}$. Then the following are true:*

(i) $\widehat{\mu}_n(u,v)=(-1)^{\lambda_1+|\lambda_2|+\nu_1+|\nu_2|}e^{-\frac{2\pi i}{n}\left(\lambda_1-|\lambda_2|\right)\left(u-|v|\right)}q^{-\frac{1}{2n}(\lambda_1-|\lambda_2|)^2}$

$$\times\widehat{\mu}_n\left(u+\lambda_1\tau+\nu_1,v+\lambda_2\tau+\nu_2\right),$$

(ii) $\widehat{\mu}_{n,\ell}(u,v;\tau+1)=e^{-\frac{\pi i}{n}\left(\ell-\frac{n}{2}\right)^2}\widehat{\mu}_{n,\ell}(u,v;\tau),$

(iii) $\widehat{\mu}_{n,\ell}\left(\frac{u}{\tau},\frac{v}{\tau};-\frac{1}{\tau}\right)=i^n\sqrt{\frac{i\tau}{n}}e^{-\frac{\pi i}{n\tau}\left(u-|v|\right)^2}\sum_{j\pmod n}e^{\frac{2\pi i\ell j}{n}}\widehat{\mu}_{n,\ell}(u,v;\tau),$

(iv) $R(u+1)=-R(u),$

(v) $R(u;\tau+1)=e^{-\frac{\pi i}{4}}R(u;\tau)$,

(vi) $R(u;\tau)=H(u;\tau)-\frac{1}{\sqrt{-i\tau}}e^{\frac{\pi i u^2}{\tau}}R\left(\frac{u}{\tau};-\frac{1}{\tau}\right)$,

where the Mordell integral is defined by

$$H(u)=H(u;\tau):=\int_{\mathbb{R}}\frac{e^{\pi i\tau x^2-2\pi ux}}{\cosh(\pi x)}\,dx.$$

Moreover, similarly as for ϑ, we require an identity which dissects R (see Proposition 2.3 of [4]).

Proposition 2.5. *For $n\in\mathbb{N}, u\in\mathbb{C}$ and $\tau\in\mathbb{H}$, we have*

$$R\left(u;\frac{\tau}{n}\right)=\sum_{\ell=0}^{n-1}q^{-\frac{1}{2n}\left(\ell-\frac{n-1}{2}\right)^2}e^{-2\pi i\left(\ell-\frac{n-1}{2}\right)\left(u+\frac12\right)}R\left(nu+\left(\ell-\frac{n-1}{2}\right)\tau+\frac{n-1}{2};n\tau\right).$$

We next turn to the transformation law for $\widehat{\mu}_n$ under all of $SL_2(\mathbb{Z})$. From Lemma 2.2, the proof of Proposition 2.3, and Proposition 2.7 we may conclude the following.

Corollary 2.6. *Assuming the notation above, we have*

$$\begin{aligned}\widehat{\mu}_n\Big(-iuz,-ivz;\tau_{h,k}\Big)&=\sqrt{\frac{i}{n'z}}e^{-\frac{\pi kz}{n}\left(u-|v|\right)^2}(-1)^{\frac{n'-1}{2}}\chi^{-3}\Big(hn',[-h]_k[n']_{8k'},k'\Big)\\ &\times\sum_{\ell\pmod{n'}}e^{-\frac{\pi i\ell^2}{n'k'}\left([n']_{8k'}n'-1\right)[-h]_k}\widehat{\mu}_{n,\ell\gamma}\Big(u,v;\widetilde{\tau}_{h,k}\Big).\end{aligned}\tag{2.2}$$

From this we now derive the transformation law for the "mock part"of the character (cf. (2.10)), $\mu_n\left(\frac12-s\tau,\mathbf{\frac12};\tau\right)$. Before stating the transformations, we define several multipliers. If k is odd, then set

$$\begin{aligned}\psi_j'\Big(h,[-h]_k,k\Big)&:=\chi^{-3}\Big(hn',[-h]_k[n']_{8k'},k'\Big)(-1)^{\frac{n'-1}{2}+sh}\\ &\times e\Bigg[-\frac{[n']_{8k'}[-h]_kh^2s^2}{2k\gamma}-\frac{s}{2n}-\frac{s}{2nk}(k-1)(n-1)\Big(1+[n']_{8k'}n'[-h]_kh\Big)\\ &\qquad+\frac{[-h]_k}{2kn}\left(\frac{n-1}{2}\right)^2\left(1-[n']_{8k'}n'\right)+\frac{[n']_{8k'}[-h]_k}{4\gamma}+\frac{[-h]_k}{2nk}\Big(1-[n']_{8k'}n'\Big)j^2\\ &\quad+j\left(-\frac{s}{nk}\Big(1+[n']_{8k'}n'[-h]_kh\Big)-\frac{(n-1)[-h]_k}{2nk}\Big(1-[n']_{8k'}n'\Big)+\frac{[n']_{8k'}[-h]_k}{2\gamma}\right)\Bigg],\\ \chi_\ell'\Big(h,k\Big)&:=-i(-1)^{sh}e^{\frac{\pi i(k-1)}{2}-\frac{\pi ihnk}{4}}(-1)^{\ell(h+1)}e^{-\frac{\pi inh\ell^2}{k}+\frac{2\pi i\ell sh}{k}}.\end{aligned}$$

Similarly, if k is even, then set

$$\widetilde{\psi}_j'\Big(h,[-h]_k,k\Big) := \chi^{-3}\Big(hn',[-h]_{2k}[n']_{8k'},k'\Big)(-1)^{\frac{n'-1}{2}}$$
$$\times e\Bigg(-\frac{[n']_{8k'}[-h]_{2k}h^2s^2}{2k\gamma}-\frac{[n']_{8k'}s}{2\gamma}+\frac{j^2[-h]_{2k}}{2nk}\Big(1-[n']_{8k'}n'\Big)$$
$$+j\left(-\frac{s}{nk}\Big(1+h[-h]_{2k}[n']_{8k'}n'\Big)+\frac{1}{2\gamma}[n']_{8k'}[-h]_{2k}+\frac{1}{2n}\right)\Bigg),$$
$$\widetilde{\chi}_\ell'\Big(h,k\Big) := -i(-1)^s e^{\frac{\pi ihn}{2}\left(\frac{k}{2}-1\right)}e^{-\frac{\pi ihn\ell^2}{k}+\frac{2\pi i\ell sh}{k}}.$$

Proposition 2.7. *Assuming the notation above, we have the following transformation laws:*

(*i*) *If k is odd, then we have*

$$\mu_n\left(\frac{1}{2}-s\tau_{h,k},\frac{\mathbf{1}}{\mathbf{2}};\tau_{h,k}\right)=\sqrt{\frac{i}{n'z}}e^{-\frac{\pi s^2z}{nk}}\sum_{\substack{0\le j\le n-1\\ 2j\equiv -1-2sh \pmod{\gamma}}}\psi_j'\Big(h,[-h]_k,k\Big)\widetilde{q}_{h,k}^{\,-\frac{(n-1)^2}{8n}}$$
$$\times\mu_{n,j}\left(\frac{\widetilde{\tau}_{h,k}}{2},\frac{\widetilde{\boldsymbol{\tau}}_{\boldsymbol{h,k}}}{\mathbf{2}};\widetilde{\tau}_{h,k}\right)+\frac{1}{2}e^{-\frac{\pi s^2z}{nk}}\sum_{-\frac{k-1}{2}\le\ell\le\frac{k-1}{2}}\chi_\ell'\Big(h,k\Big)H\Big(iz(n\ell-s);inkz\Big)e^{\frac{\pi z(n\ell-s)^2}{nk}}.$$

(*ii*) *If k is even, then we have*

$$\mu_n\left(\frac{1}{2}-s\tau_{h,k},\frac{\mathbf{1}}{\mathbf{2}};\tau_{h,k}\right)=\sqrt{\frac{i}{n'z}}e^{-\frac{\pi s^2z}{nk}}\sum_{\substack{-\frac{n}{2}<j<\frac{n}{2}\\ j\equiv -sh \pmod{\gamma}}}\widetilde{\psi}_j'\Big(h,[-h]_k,k\Big)\mu_{n,j}\left(\frac{1}{2},\frac{\mathbf{1}}{\mathbf{2}};\widetilde{\tau}_{h,k}\right)$$
$$+\frac{1}{2}e^{-\frac{\pi s^2z}{nk}}\sum_{\substack{-\frac{k-1}{2}\le\ell\le\frac{k-1}{2}\\ \ell\in\frac{1}{2}+\mathbb{Z}}}\widetilde{\chi}_\ell'\Big(h,k\Big)H\Big(iz(n\ell-s);inkz\Big)e^{\frac{\pi z(n\ell-s)^2}{nk}}.$$

Proof. For the proof we use the definition of $\widehat{\mu}_n$ (2.1) to write μ_n in tems of $\widehat{\mu}_n$ and R and prove transformations for these components individually,

$$\mu_n\left(\frac{1}{2}-s\tau_{h,k},\frac{\mathbf{1}}{\mathbf{2}};\tau_{h,k}\right)=\widehat{\mu}_n\left(\frac{1}{2}-s\tau_{h,k},\frac{\mathbf{1}}{\mathbf{2}};\tau_{h,k}\right)-\frac{i}{2}R\Big(-s\tau_{h,k};n\tau_{h,k}\Big). \tag{2.3}$$

We first dissect the R-function. Using Proposition 2.5 gives that

$$R\Big(-s\tau_{h,k};n\tau_{h,k}\Big)=\sum_{\ell=0}^{k-1}q_{h,k}^{-\frac{n}{2}\left(\ell-\frac{k-1}{2}\right)^2}e^{-2\pi i\left(\ell-\frac{k-1}{2}\right)\left(-\frac{s}{k}(h+iz)+\frac{1}{2}\right)}$$
$$\times R\left(-s(h+iz)+\left(\ell-\frac{k-1}{2}\right)n(h+iz)+\frac{k-1}{2};nk(h+iz)\right). \tag{2.4}$$

We use Proposition 2.4 (iv) and (v) to pull out of the R-function a factor of

$$(-1)^{sh+\ell nh-\frac{(k-1)(nh-1)}{2}}e^{-\frac{\pi ihnk}{4}}.$$

We then change $\ell\mapsto\ell+\frac{k-1}{2}$. By Proposition 2.4 (vi) we find that

$$R\Big(iz(n\ell-s);inkz\Big)=-\frac{1}{\sqrt{nkz}}e^{-\frac{\pi z}{nk}(n\ell-s)^2}R\left(\frac{s-\ell n}{nk};\frac{i}{nkz}\right)+H\Big(iz(n\ell-s);inkz\Big).$$

Thus we have shown that (2.4) equals

$$-(-1)^{sh} e^{\frac{\pi i(k-1)}{2}-\frac{\pi i nhk}{4}} e^{-\frac{\pi s^2 z}{nk}} \sum_{\substack{-\frac{k-1}{2}\leq \ell \leq \frac{k-1}{2}\\ \ell\in\frac{k-1}{2}+\mathbb{Z}}} e^{-\frac{\pi i nh\ell^2}{k}+2\pi i\ell\left(\frac{sh}{k}+\frac{nh-1}{2}\right)} \tag{2.5}$$
$$\times\left(\frac{1}{\sqrt{nkz}}R\left(\frac{1}{k}\left(\frac{s}{n}-\ell\right);\frac{i}{nkz}\right)-e^{\frac{\pi z(n\ell-s)^2}{nk}}H\Big(iz(n\ell-s);inkz\Big)\right).$$

We next turn to the function $\widehat{\mu}_n$ in (2.3). By (2.2) we obtain that

(2.6)
$$\widehat{\mu}_n\left(\frac{1}{2}-s\tau_{h,k},\frac{\mathbf{1}}{\mathbf{2}};\tau_{h,k}\right)=\sqrt{\frac{i}{n'z}}e^{-\frac{\pi kz}{n}\left(\frac{i}{z}\left(\frac{1-n}{2}-\frac{sh}{k}\right)+\frac{s}{k}\right)^2}(-1)^{\frac{n'-1}{2}}\chi^{-3}\Big(hn',[-h]_k[n']_{8k'},k'\Big)$$
$$\times\sum_{\ell\pmod{n'}} e^{-\frac{\pi i\ell^2}{n'k'}\Big([n']_{8k'}n'-1\Big)[-h]_k}\widehat{\mu}_{n,\ell\gamma}\left(\frac{i}{z}\left(\frac{1}{2}-\frac{sh}{k}\right)+\frac{s}{k},\frac{\boldsymbol{i}}{\mathbf{2z}};\widetilde{\tau}_{h,k}\right).$$

We now fix representatives ℓ satisfying

$$-\frac{n'}{2}+\frac{(n-1)k'}{2}+\frac{sh}{\gamma}<\ell<\frac{n'}{2}+\frac{(n-1)k'}{2}+\frac{sh}{\gamma}$$

and denote the associated sum on ℓ by $\sum_\ell'$. We use the definition of $\mu_{n,\ell\gamma}$ and then decompose $\widehat{\mu}_n$ into μ_n and R and simplify the exponential terms. This gives that (2.6) may be written as

$$\sqrt{\frac{i}{n'z}}e^{-\frac{2\pi is}{n}\left(\frac{1-n}{2}-\frac{sh}{k}\right)}(-1)^{\frac{n'-1}{2}}\chi^{-3}\Big(hn',[-h]_k[n']_{8k'},k'\Big)e^{-\frac{\pi s^2 z}{nk}}\sum_\ell{}' e^{-\frac{\pi i\ell^2[n']_{8k'}[-h]_k}{k'}}(-1)^\ell$$
$$\times e^{-\frac{2\pi i\ell s}{n'k}+\frac{\pi k}{nz}\left(\frac{1-n}{2}-\frac{sh}{k}+\frac{\ell}{k'}\right)^2}\left(\mu_n\left(\frac{i}{z}\left(\frac{1}{2}-\frac{sh}{k}\right)+\frac{s}{k}+\ell\gamma\widetilde{\tau}_{h,k},\frac{\boldsymbol{i}}{\mathbf{2z}};\widetilde{\tau}_{h,k}\right)\right. \tag{2.7}$$
$$\left.-\frac{i}{2}R\left(\frac{i}{z}\left(\frac{1-n}{2}-\frac{sh}{k}+\frac{\gamma\ell}{k}\right)+\frac{s}{k}+\frac{\ell[-h]_k}{k'}-\frac{n+1}{2};n\widetilde{\tau}_{h,k}\right)\right).$$

We next show that the R-functions in (2.7) and (2.5) cancel. For this we multiply all terms by $e^{\frac{\pi s^2 z}{nk}}$. Set $\tau=\frac{i}{kz}$, $y=\mathrm{Im}(\tau)$. It is not hard to see that we are finished if we can show that each of the occuring R-terms has a Fourier expansion of the form ($q=e^{2\pi i\tau}$)

$$\sum_{n\in\mathbb{Q}\setminus\{0\}} a(n)\Gamma\left(\frac{1}{2};4\pi|n|y\right)q^{-n},$$

where $\Gamma(a;x):=\int_x^\infty e^{-t}t^{a-1}dt$ is the incomplete gamma function. This may easily be concluded from the identity,

$$E\left(\left(\nu+\frac{\mathrm{Im}(u)}{y}\right)\sqrt{2y}\right)=\mathrm{sgn}\left(\nu+\frac{\mathrm{Im}(u)}{y}\right)\left(1+\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2};2\pi\left(\nu+\frac{\mathrm{Im}(u)}{y}\right)^2y\right)\right)$$

and the specific restrictions on ℓ.

Thus we have shown that

$$\mu_n\left(\frac{1}{2}-s\tau_{h,k},\boldsymbol{\frac{1}{2}};\tau_{h,k}\right)=-\frac{i}{2}(-1)^{sh}e^{\frac{\pi i(k-1)}{2}-\frac{\pi ihnk}{4}}e^{-\frac{\pi s^2z}{nk}}$$

$$\times\sum_{\substack{-\frac{k-1}{2}\le\ell\le\frac{k-1}{2}\\ \ell\in\frac{k-1}{2}+\mathbb{Z}}}e^{-\frac{\pi inh\ell^2}{k}+2\pi i\ell\left(\frac{sh}{k}+\frac{nh-1}{2}\right)}e^{\frac{\pi z(n\ell-s)^2}{nk}}H\Big(iz(n\ell-s);inkz\Big)$$

(2.8)

$$+\sqrt{\frac{i}{n'z}}e^{-\frac{2\pi is}{n}\left(\frac{1-n}{2}-\frac{sh}{k}\right)}(-1)^{\frac{n'-1}{2}}\chi^{-3}\Big(hn',[-h]_k[n']_{8k'},k'\Big)e^{-\frac{\pi s^2z}{nk}}$$

$$\sum_{\ell}{}' e^{-\frac{\pi i\ell^2[n']_{8k'}[-h]_k}{k'}}(-1)^\ell e^{-\frac{2\pi i\ell s}{n'k}+\frac{\pi k}{nz}\left(\frac{1-n}{2}-\frac{sh}{k}+\frac{\ell}{k'}\right)^2}\mu_n\left(\frac{i}{z}\left(\frac{1}{2}-\frac{sh}{k}\right)+\frac{s}{k}+\ell\gamma\widetilde{\tau}_{h,k},\boldsymbol{\frac{i}{2z}};\widetilde{\tau}_{h,k}\right).$$

We now further simplify the μ_n-terms. For this we require the easily verified identity ($r\in\mathbb{Z}$)

$$\mu_n\Big(u+nr\tau,v+\boldsymbol{r\tau};\tau\Big)=\mu_n(u,v;\tau). \tag{2.9}$$

Distinguishing the cases k even and k odd and simplifying easily yields the terms involving H in Proposition 2.7. To consider the μ_n-terms first consider the case k is odd. Using that $4|[-h]_k$ and $\frac{s}{k}(1+[-h]_kh)\equiv s \pmod 2$ and (2.9) then yields

$$\mu_n\left(\frac{i}{z}\left(\frac{1}{2}-\frac{sh}{k}\right)+\frac{s}{k}+\ell\gamma\widetilde{\tau}_{h,k},\boldsymbol{\frac{i}{2z}};\widetilde{\tau}_{h,k}\right)=(-1)^s\mu_n\left(\left(k\left(\frac{1}{2}-\frac{sh}{k}\right)+\ell\gamma\right)\widetilde{\tau}_{h,k},\boldsymbol{\frac{k\widetilde{\tau}_{h,k}}{2}};\widetilde{\tau}_{h,k}\right)$$
$$=(-1)^s\mu_n\left(\left(k\left(\frac{1}{2}-\frac{sh}{k}\right)+\ell\gamma-\frac{n(k-1)}{2}\right)\widetilde{\tau}_{h,k},\boldsymbol{\frac{\widetilde{\tau}_{h,k}}{2}};\widetilde{\tau}_{h,k}\right).$$

Set $j:=\frac{(k-1)(1-n)}{2}-sh+\ell\gamma$. Then $0\le j\le n-1$ and $2j\equiv-1-2sh \pmod\gamma$. After a lengthy but straightforward calculation, using the definition of $\mu_{n,j}$, Proposition 2.4 and simplifying the exponential terms, the proof of part (i) of the Proposition is complete.

Next we consider the case k even. Proposition 2.7, equation (2.9) and additional simplifications imply

$$\mu_n\left(\frac{i}{z}\left(\frac{1}{2}-\frac{sh}{k}\right)+\frac{s}{k}+\ell\gamma\widetilde{\tau}_{h,k},\boldsymbol{\frac{i}{2z}};\widetilde{\tau}_{h,k}\right)=\mu_n\left(\left(k\left(\frac{1}{2}-\frac{sh}{k}\right)+\ell\gamma-\frac{nk}{2}\right)\widetilde{\tau}_{h,k}+\frac{1}{2};\boldsymbol{\frac{1}{2}};\widetilde{\tau}_{h,k}\right)$$

Set $j:=\frac{k(1-n)}{2}-sh+\ell\gamma$. Then $-\frac{n}{2}<j<\frac{n}{2}$ and $j\equiv-sh \pmod\gamma$. Another lengthy calculation now gives part (ii) of the Proposition. □

2.3. **The final transformation law.**

From (1.1) it is not hard to see that

$$tr_{L(\Lambda_{(s)})}q^{L_0}=i\cdot2^{n+1}q^{\frac{n-1}{24}}\frac{\eta^{2n+2}(2\tau)}{\eta^{2n+3}(\tau)}\mu_n\left(\frac{1}{2}-s\tau,\boldsymbol{\frac{1}{2}};\tau\right). \tag{2.10}$$

To state the transformation law we require some multipliers. We define for odd k

$$\psi_j^*\Big(h,[-h]_k,k\Big):=ie^{\frac{\pi i(n-1)h}{12k}}\frac{\chi\big(2h,[-h]_k[2]_{3k},k\big)^{2n+2}}{\chi\big(h,[-h]_k,k\big)^{2n+3}}\psi_j'\Big(h,[-h]_k,k\Big),$$

$$\chi_\ell^*\Big(h,[-h]_k,k\Big):=i^{\frac{1}{2}}e^{\frac{\pi i(n-1)h}{12k}}\frac{\chi\big(2h,[-h]_k[2]_{3k},k\big)^{2n+2}}{\chi\big(h,[-h]_k,k\big)^{2n+3}}\chi_\ell'\Big(h,k\Big).$$

Moreover, for k even we let

$$\widetilde{\psi}_j^*\Big(h,[-h]_k,k\Big) := ie^{\frac{\pi i(n-1)h}{12k}} \frac{\chi\big(h,[-h]_{2k},k\big)^{2n+2}}{\chi\big(h,[-h]_k,k\big)^{2n+3}} \widetilde{\psi}_j'\Big(h,[-h]_k,k\Big),$$

$$\widetilde{\chi}_\ell^*\Big(h,[-h]_k,k\Big) := i^{\frac{1}{2}} e^{\frac{\pi i(n-1)h}{12k}} \frac{\chi\big(h,[-h]_{2k},k\big)^{2n+2}}{\chi\big(h,[-h]_k,k\big)^{2n+3}} \widetilde{\chi}_\ell'\Big(h,k\Big).$$

Combined with Proposition 2.7, this now gives transformation formulas for the characters.

Theorem 2.8. *(i) If $2\nmid k$, the trace is*

$$tr_{L(\Lambda_{(s)})} q_{h,k}^{L_0}$$
$$= e^{-\frac{\pi z}{k}\left(\frac{n-1}{12}+\frac{s^2}{n}\right)} \widetilde{q}_{h,k}^{-\frac{(n-1)^2}{8n}} \frac{\eta^{2n+2}\left(\frac{\widetilde{\tau}_{h,k}}{2}\right)}{\eta^{2n+3}(\widetilde{\tau}_{h,k})} \frac{1}{\sqrt{n'}} \sum_{\substack{0\le j\le n-1\\ 2j\equiv -2sh-1 \pmod{\gamma}}} \psi_j^*\Big(h,[-h]_k,k\Big)\mu_{n,j}\left(\frac{\widetilde{\tau}_{h,k}}{2},\frac{\widetilde{\boldsymbol{\tau}}_{\boldsymbol{h,k}}}{\mathbf{2}};\widetilde{\tau}_{h,k}\right)$$
$$+ e^{-\frac{\pi z}{k}\left(\frac{n-1}{12}+\frac{s^2}{n}\right)} \frac{\sqrt{z}}{2} \frac{\eta^{2n+2}\left(\frac{\widetilde{\tau}_{h,k}}{2}\right)}{\eta^{2n+3}(\widetilde{\tau}_{h,k})} \sum_{-\frac{k-1}{2}\le \ell\le \frac{k-1}{2}} \chi_\ell^*\Big(h,[-h]_k,k\Big) e^{\frac{\pi z(n\ell-s)^2}{nk}} H\Big(iz(n\ell-s);inkz\Big).$$

(ii) If $2|k$, the trace is

$$tr_{L(\Lambda_{(s)})} q_{h,k}^{L_0}$$
$$= e^{-\frac{\pi z}{k}\left(\frac{n-1}{12}+\frac{s^2}{n}\right)} \frac{\eta^{2n+2}(2\widetilde{\tau}_{h,k})}{\eta^{2n+3}(\widetilde{\tau}_{h,k})} \frac{2^{n+1}}{\sqrt{n'}} \sum_{\substack{-\frac{n}{2}<j<\frac{n}{2}\\ j\equiv -sh \pmod{\gamma}}} \widetilde{\psi_j}^*\Big(h,[-h]_k,k\Big)\mu_{n,j}\left(\frac{1}{2},\frac{\mathbf{1}}{\mathbf{2}};\widetilde{\tau}_{h,k}\right)$$
$$+ 2^n\sqrt{z} e^{-\frac{\pi z}{k}\left(\frac{n-1}{12}+\frac{s^2}{n}\right)} \frac{\eta^{2n+2}(2\widetilde{\tau}_{h,k})}{\eta^{2n+3}(\widetilde{\tau}_{h,k})} \sum_{\substack{-\frac{k-1}{2}\le\ell\le\frac{k-1}{2}\\ \ell\in\frac{1}{2}+\mathbb{Z}}} \widetilde{\chi_\ell}^*\Big(h,[-h]_k,k\Big) e^{\frac{\pi z(n\ell-s)^2}{nk}} H\Big(iz(n\ell-s);inkz\Big).$$

3. Principal Parts

In this section we determine the principal parts of the holomorphic q-series that arise in the transformation formulas from Theorem 2.8, and in particular, we determine the values of j for which these principal parts exist. We postpone the non-holomorphic terms from the character transformation formulas until the next section, since their "continuous" principal parts exist in all cases. All of the asymptotic estimates in the section are relative to the limit $z\to 0$ in the complex right half-plane.

3.1. The holomorphic parts for k odd.

We aim to determine the principal parts of the functions ($0\le j\le n-1$ with $2j\equiv -1-2sh \pmod{\gamma}$)

$$f_{n,j}(\tau) := q^{-\frac{(n-1)^2}{8n}} \frac{\eta^{2n+2}\left(\frac{\tau}{2}\right)}{\eta^{2n+3}(\tau)} \mu_{n,j}\left(\frac{\tau}{2},\frac{\boldsymbol{\tau}}{\mathbf{2}};\tau\right).$$

It is easy to see that

$$\frac{\eta^{2n+2}\left(\frac{\tau}{2}\right)}{\eta^{2n+3}(\tau)} = q^{-\frac{1}{24}(n+2)}\Big(1+O\Big(q^{\frac{1}{2}}\Big)\Big).$$

Moreover, we have (see the proof of Proposition 3.1 of [4])

$$\mu_{n,j}\left(\frac{\tau}{2},\frac{\boldsymbol{\tau}}{\mathbf{2}};\tau\right)=q^{-\frac{1}{2n}\left(j^2+(1-n)j\right)+\frac{1}{4}+\frac{n}{8}}\left(c+O\left(q^{\frac{1}{2}}\right)\right)$$

for some constant $c\neq 0$. Using the notation from the introduction, we may therefore write

$$f_{n,j}(\tau)=q^{-\delta_j(n)}\widetilde{f}_{n,j}(\tau)$$

with

$$\widetilde{f}_{n,j}(\tau):=\sum_{r\geq 0}A_j(r)q^{\frac{r}{2}}, \tag{3.1}$$

and $A_j(0)\neq 0$. We easily see that only the $j\in S_n$ lead to a nonzero principal part.

3.2. **The holomorphic part for k even.** We now determine the principal part of the functions

$$g_{n,j}(\tau):=\frac{\eta^{2n+2}(2\tau)}{\eta^{2n+3}(\tau)}\mu_{n,j}\left(\frac{1}{2},\frac{\mathbf{1}}{\mathbf{2}};\tau\right),$$

where $-\frac{n}{2}<j<\frac{n}{2}$ and $j\equiv -sh \pmod{\gamma}$. Since

$$g_{n,-j}(\tau)=e^{\frac{2\pi ij}{n}}g_{n,j}(\tau),$$

we may restrict ourselves to the case $0\leq j<\frac{n}{2}$. We note that

$$\frac{\eta^{2n+2}(2\tau)}{\eta^{2n+3}(\tau)}=q^{\frac{n}{12}+\frac{1}{24}}\Big(1+O(q)\Big).$$

We also recall that by definition,

$$\mu_{n,j}\left(\frac{1}{2},\frac{\mathbf{1}}{\mathbf{2}};\tau\right)=\frac{(-1)^jq^{-\frac{j^2}{2n}}e^{\frac{\pi ij(1-n)}{n}}e^{\pi i\left(\frac{1}{2}+j\tau\right)}}{\vartheta^n\left(\frac{1}{2};\tau\right)}\sum_{r\in\mathbb{Z}^n}\frac{q^{\frac{1}{2}||r||^2+\frac{1}{2}|r|}}{1+q^{|r|+j}}.$$

To determine the main term of this series, first observe that

$$\vartheta\left(\frac{1}{2};\tau\right)=q^{\frac{1}{8}}\Big(-2+O(q)\Big).$$

To analyze the sum, we split it into 3 pieces and use geometric summation

$$\frac{1}{2}\sum_{\substack{r\in\mathbb{Z}^n\\|r|=-j}}q^{\frac{1}{2}||r||^2-\frac{j}{2}}+\sum_{\substack{r\in\mathbb{Z}^n\\|r|>-j\\m\geq 0}}(-1)^mq^{\frac{1}{2}||r||^2+\frac{1}{2}|r|+m(|r|+j)}+\sum_{\substack{r\in\mathbb{Z}^n\\|r|<-j\\m\geq 1}}(-1)^{m+1}q^{\frac{1}{2}||r||^2+\frac{1}{2}|r|-m(|r|+j)},$$

and determine the main term of each of the sums separately. The first sum has the smallest exponent if j of the n components are (-1) and the remaining ones are 0, giving a total exponent of 0. The exponent of the second sum is growing in m. For $m=0$, we complete the square to write the exponent as $\frac{1}{2}\left|\left|r+\frac{1}{2}\right|\right|^2-\frac{n}{8}\geq 0$. Note that there cannot be cancellation with the first summand since we are considering summands with all positive coefficients. The exponent of the third sum is again growing in m. For $m=1$, completing the square gives the exponent $\frac{1}{2}\left|\left|r-\frac{1}{2}\right|\right|^2-j-\frac{n}{8}$. Since the summation condition requires that $|r|<-j$ this is minimized if $(j+1)$ of the components of r are -1 and the rest are 0. This gives us the total exponent 1. Combining the above considerations we see that we may write

$$g_{n,j}(\tau)=q^{-\widetilde{\delta}_j(n)}\widetilde{g}_{n,j}(\tau),$$

where $\widetilde{\delta}_j(n)$ was defined in the introduction and

$$\widetilde{g}_{n,j}(\tau) := \sum_{r\geq 0} B_j(r)q^r, \tag{3.2}$$

with $B_j(0) \neq 0$. We easily see that only $j \in T_n$ contribute to the main asymptotic term.

4. Proofs of main results

In this section we apply the Circle Method to obtain the coefficient asymptotics of Theorem 1.1. We then use asymptotics for principal value integrals to identify the leading term and prove Corollary 1.2.

Proof of Theorem 1.1. By Cauchy's Theorem, the coefficients of $g_s(q)$ may be recovered from a residue calculation, as for $m > 0$ we obtain

$$c_s(m) = \int_0^1 g_s\left(e^{-\frac{2\pi}{m}+2\pi it}\right) e^{2\pi-2\pi imt}dt.$$

Define $\vartheta'_{h,k} := \frac{1}{k(k_1+k)}$ and $\vartheta''_{h,k} := \frac{1}{k(k_2+k)}$, where $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ are adjacent fractions in the Farey sequence of order $N := \lfloor\sqrt{m}\rfloor$. We now decompose the path of integration into the Farey arcs $-\vartheta_{h,k} \leq \phi \leq \vartheta''_{h,k}$, with $0 \leq h < k \leq N$ and $(h,k) = 1$, and where $\phi := t - \frac{h}{k}$. Adopting the additional notation $z = \frac{k}{m} - ik\phi$ and applying Theorem 2.8 along each arc then gives

$$c_s(m) = \sum_{h,k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} g_s\left(e^{\frac{2\pi i}{k}(h+iz)}\right) q_{h,k}^{-m} d\phi = \sum_{h,k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} tr_{L(\Lambda_{(s)})} q_{h,k}^{L_0} q_{h,k}^{-\frac{s}{2}-m} d\phi = \sum_1 + \sum_2 + \sum_3 + \sum_4.$$

Here we have separated the q-series and Mordell integrals, and also split the Farey fractions based on the parity of k, as

$$\sum_1 := \sum_{\substack{h,k\\ 2\nmid k}} \frac{1}{\sqrt{n'}} \sum_{\substack{0\le j\le n-1\\ 2j\equiv -1-2sh \pmod{\gamma}}} \psi_j^*\Big(h,[-h]_k,k\Big)$$
$$\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \frac{\eta^{2n+2}\left(\frac{\widetilde{\tau}_{h,k}}{2}\right)}{\eta^{2n+3}\left(\widetilde{\tau}_{h,k}\right)}\mu_{n,j}\left(\frac{\widetilde{\tau}_{h,k}}{2},\frac{\boldsymbol{\widetilde{\tau}_{h,k}}}{\mathbf{2}};\widetilde{\tau}_{h,k}\right) q_{h,k}^{-m-\frac{s}{2}}\widetilde{q}_{h,k}^{-\frac{(n-1)^2}{8n}} e^{-\frac{\pi z}{k}\left(\frac{n-1}{12}+\frac{s^2}{n}\right)}d\phi,$$

$$\sum_2 := 2^{n+1}\sum_{\substack{h,k\\ 2\mid k}} \frac{1}{\sqrt{n'}} \sum_{\substack{-\frac{n}{2}<j<\frac{n}{2}\\ j\equiv -sh \pmod{\gamma}}} \widetilde{\psi}_j^*\Big(h,[-h]_k,k\Big)$$
$$\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \frac{\eta^{2n+2}\left(2\widetilde{\tau}_{h,k}\right)}{\eta^{2n+3}\left(\widetilde{\tau}_{h,k}\right)}\mu_{n,j}\left(\frac{1}{2},\frac{\mathbf{1}}{\mathbf{2}};\widetilde{\tau}_{h,k}\right) q_{h,k}^{-m-\frac{s}{2}} e^{-\frac{\pi z}{k}\left(\frac{n-1}{12}+\frac{s^2}{n}\right)}d\phi,,$$

$$\sum_3 := \frac{1}{2}\sum_{\substack{h,k\\ 2\nmid k}} \sum_{-\frac{k-1}{2}\le \ell\le \frac{k-1}{2}} \chi_\ell^*\Big(h,[-h]_k,k\Big)\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \frac{\eta^{2n+2}\left(\frac{\widetilde{\tau}_{h,k}}{2}\right)}{\eta^{2n+3}(\widetilde{\tau}_{h,k})}\sqrt{z}$$
$$\times H\Big(iz(n\ell-s);iknz\Big) q_{h,k}^{-m-\frac{s}{2}} e^{-\frac{\pi z}{k}\left(\frac{n-1}{12}-n\ell^2+2\ell s\right)}d\phi,$$

$$\sum_4 := 2^n\sum_{\substack{h,k\\ 2\mid k}} \sum_{\substack{-\frac{k-1}{2}\le \ell\le \frac{k-1}{2}\\ \ell\in\frac12+\mathbb{Z}}} \widetilde{\chi}_\ell^*\Big(h,[-h]_k,k\Big)\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \frac{\eta^{2n+2}\left(2\widetilde{\tau}_{h,k}\right)}{\eta^{2n+3}\left(\widetilde{\tau}_{h,k}\right)}\sqrt{z}$$
$$\times H\Big(iz(n\ell-s);iknz\Big) q_{h,k}^{-m-\frac{s}{2}} e^{-\frac{\pi z}{k}\left(\frac{n-1}{12}-n\ell^2+2\ell s\right)}d\phi.$$

We first consider $\sum_1$ and $\sum_2$. Following Hardy, Ramanujan, and Rademacher's original development of the Circle Method [24], we may show that

$$\sum_1 = \sum_{\substack{h,k\\ 2\nmid k}} e^{-\frac{2\pi i h m}{k}} \frac{1}{\sqrt{n'}} \sum_{j\in S_n} \psi_j\Big(h,[-h]_k,k\Big)$$
$$\times \sum_{0\le r<2\delta_j(n)} A_j(r) e^{\frac{\pi i [-h]_k r}{k}} \int_{-\frac{1}{kN}}^{\frac{1}{kN}} e^{\frac{2\pi}{k}\left(\left(m-\varepsilon_s(n)\right)z+\left(\delta_j(n)-\frac{r}{2}\right)\frac{1}{z}\right)} d\phi + O(1),$$

where

$$\psi_j\Big(h,[-h]_k,k\Big) := e^{-\frac{2\pi i \delta_j(n)[-h]_k}{k}-\frac{\pi i s h}{k}}\psi_j^*\Big(h,[-h]_k,k\Big).$$

We define the Kloosterman sum

$$K_k(\alpha,\beta) := \sum_{\substack{0\le h<k\\ (h,k)=1}} \psi_j\Big(h,[-h]_k,k\Big) e^{\frac{\pi i}{k}\left(2\alpha h+\beta[-h]_k\right)}, \tag{4.1}$$

and use the following well-known integral evaluation (see [24]) for $b>0$:

$$\int_{-\frac{1}{kN}}^{\frac{1}{kN}} e^{\frac{2\pi}{k}\left(az+\frac{b}{z}\right)}d\phi = \frac{2\pi}{k}\sqrt{\frac{b}{a}}I_1\left(\frac{4\pi}{k}\sqrt{ab}\right)+O\left(\frac{1}{Nk}\right). \tag{4.2}$$

Then we obtain

$$\sum_1 = \frac{2\pi}{\sqrt{(m-\varepsilon_s(n))}} \sum_{\substack{1\le k\le N\\ 2\nmid k}} \frac{1}{k}\frac{1}{\sqrt{n'}} \sum_{j\in S_n} \sum_{0\le r<2\delta_j(n)} A_j(r)\sqrt{\delta_j(n)-\frac{r}{2}}$$

$$\times K_k(-m,r) I_1\left(\frac{4\pi}{k}\sqrt{\left(\delta_j(n)-\frac{r}{2}\right)(m-\varepsilon_s(n))}\right) + O(1).$$

We next treat $\sum_2$ in a similar manner. Define

$$\widetilde{K_k}(\alpha,\beta) := \sum_{\substack{0\le h<k\\ (h,k)=1}} \widetilde{\psi}_j\Big(h,[-h]_k,k\Big), e^{\frac{2\pi i}{k}\left(\alpha h+\beta[-h]_{2k}\right)} \tag{4.3}$$

with

$$\widetilde{\psi}_j\Big(h,[-h]_k,k\Big) := e^{-\frac{2\pi i\delta_j(n)[-h]_{2k}}{k}-\frac{\pi ish}{k}} \widetilde{\psi}^*_j\Big(h,[-h]_k,k\Big).$$

Then we have, using again (4.2)

$$\sum_2 = \frac{2^{n+2}\pi}{\sqrt{(m-\varepsilon_s(n))}} \sum_{\substack{1\le k\le N\\ 2\mid k}} \frac{1}{k}\frac{1}{\sqrt{n'}} \sum_{j\in T_n} \sum_{0\le r<\widetilde{\delta}_j(n)} B_j(r)\sqrt{\widetilde{\delta}_j(n)-r}$$

$$\times \widetilde{K_k}(-m,r) I_1\left(\frac{4\pi}{k}\sqrt{\Big(\widetilde{\delta}_j(n)-r\Big)(m-\varepsilon_s(n))}\right) + O(1).$$

We now turn to the non-holomorphic terms, beginning with $\sum_3$. Define the coefficients of the principal part expansion by

$$\frac{\eta^{2n+2}\left(\frac{\tau}{2}\right)}{\eta^{2n+3}(\tau)} =: \sum_{n_r>0} C(r) q^{-\frac{n_r}{2}} + O\left(q^{\frac{1}{24}}\right); \tag{4.4}$$

this holds because n is odd and n_r is thus never an integer. We will also use a change of variables to rewrite the Mordell integrals as

$$H\Big(iz(n\ell-s); iknz\Big) = e^{-\frac{\pi(n\ell-s)^2 z}{kn}} \int_{\mathbb{R}} \frac{e^{-\pi knzx^2}}{\cosh\left(\pi\left(x-\frac{i}{k}\left(\ell-\frac{s}{n}\right)\right)\right)} dx. \tag{4.5}$$

Therefore the overall sum is

$$\sum_3 = \frac{1}{2}\sum_{\substack{h,k\\ 2\nmid k}} \sum_{n_r>0} C(r) e^{\frac{\pi i[-h]_k r}{k}-\frac{\pi i[-h]_k(n+2)}{12k}} \sum_{-\frac{k-1}{2}\le\ell\le\frac{k-1}{2}} \chi^*_\ell\Big(h,[-h]_k,k\Big) e^{-\frac{2\pi ih}{k}\left(m+\frac{s}{2}\right)} \tag{4.6}$$

$$\int_{\mathbb{R}} \frac{1}{\cosh\left(\pi\left(x-\frac{i}{k}\left(\ell-\frac{s}{n}\right)\right)\right)} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \sqrt{z} e^{\frac{2\pi}{k}\left(z\left(m-\varepsilon_s(n)-\frac{x^2k^2n}{2}\right)+\frac{n_r}{2z}\right)} d\phi dx + O\left(m^{\frac{1}{4}}\right).$$

Similarly, since the holomorphic terms in the final sums have no principal part, we find the bound

$$\sum_4 = O\left(m^{\frac{1}{4}}\right).$$

Returning to $\sum_3$, we can change the integral in ϕ to the range $-\frac{1}{kN}$ to $\frac{1}{kN}$ without affecting the overall error bound. We then write the sum on h in terms of the Kloosterman sums

$$K^*_{k,\ell}(\alpha,\beta) := \sum_{\substack{0\le h<k\\(h,k)=1}} \chi_\ell\Big(h,[-h]_k,k\Big)e^{\frac{2\pi i}{k}\left(\alpha h+\frac{\beta}{2}[-h]_k\right)}, \tag{4.7}$$

where

$$\chi_\ell\Big(h,[-h]_k,k\Big) := e^{-\frac{\pi i s h}{k}}e^{-\pi i[-h]_k(n+2)12k}\chi^*_\ell\Big(h,[-h]_k,k\Big).$$

The next task is then to understand the integrals

$$\int_{-\frac{1}{kN}}^{\frac{1}{kN}} \sqrt{z}e^{\frac{2\pi}{k}\left(uz+\frac{n_r}{2z}\right)}d\phi = \left(\frac{\pi n_r}{k}\right)^{\frac32}\frac{2\pi}{k}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} t^{-\frac52}e^{t+\frac{\alpha}{t}}dt + O\left(m^{-\frac14}\right),$$

with $\alpha := \frac{2\pi u n_r}{k^2}$. A modification of the Bessel function representation (4.2) now implies that the integral (including the factor $\frac{1}{2\pi i}$) equals

$$|\alpha|^{-\frac34}\begin{cases} I_{\frac32}\Big(2\sqrt{\alpha}\Big) & \text{if } \alpha>0,\\ J_{\frac32}\Big(2\sqrt{-\alpha}\Big) & \text{if } \alpha<0.\end{cases}$$

Here $J_n(x)$ denotes the (index n) Bessel function of the first kind.

It is a standard fact that as $x\to\infty$, the J-Bessel functions decay proportionally to $x^{-1/2}$; this implies the asymptotic simplification for (4.6)

$$\sum_3 = \pi 2^{-\frac34}\sum_{2\nmid k}\frac1k\sum_r C(r)n_r^{\frac34}\sum_\ell K^*_{k,\ell}(-m,r)\int\limits_{\frac{x^2k^2n}{2}<m-\varepsilon_s(n)}\frac{\left(m-\varepsilon_s(n)-\frac{x^2k^2n}{2}\right)^{-\frac34}}{\cosh\left(\pi\left(x-\frac ik\left(\ell-\frac sn\right)\right)\right)} \tag{4.8}$$
$$I_{\frac32}\left(\frac{2\pi}{k}\sqrt{2n_r\left(m-\varepsilon_s(n)-\frac{x^2k^2n}{2}\right)}\right)dx + O\left(m^{-\frac14}\right).$$

Making the change of variables $y = kx\sqrt{\frac{n}{2(m-\varepsilon_s(n))}}$, we rewrite the integrals in (4.8) as

$$\frac{k\big(m-\varepsilon_s(n)\big)^{-\frac14}\sqrt2}{n^{\frac12}}\int\limits_{|y|\le1}\frac{\left(1-y^2\right)^{-\frac34}}{\cosh\left(\pi\left(\frac{y\sqrt{2(m-\varepsilon_s(n))}}{k\sqrt n}-\frac ik\left(\ell-\frac sn\right)\right)\right)}$$
$$I_{\frac32}\left(\frac{2\pi}{k}\sqrt{2n_r\big(m-\varepsilon_s(n)\big)\big(1-y^2\big)}\right)dy.$$

Recalling the definition of the principal value integrals in (1.8), we find that the $\sum_3$ contributes the following non-error terms:

$$\frac{2^{-\frac14}\pi}{\big(m-\varepsilon_s(n)\big)^{\frac14}\sqrt n}\sum_{2\nmid k}\sum_{n_r>0}C(r)n_r^{\frac34}\sum_\ell K^*_{k,\ell}(-m,r)P\left(\frac{2\pi}{k}\sqrt{2n_r},\frac{\sqrt2}{k\sqrt n},-\frac1k\left(\ell-\frac sn\right);\sqrt{m-\varepsilon_s(n)}\right).$$

This completes the proof of Theorem 1.1. □

The remainder of the paper is devoted to isolating the main term from the series expansion for the coefficients $c_s(m)$. Note that our principal value integrals $P(a,b,c;M)$ in this paper are different from those in our earlier paper [5] (the current integrals correspond to "non-inverted" mock theta function transformations). The present form allows us to use the method of stationary phase in order to obtain better error bounds.

Lemma 4.1. *Recall the definition of the principal value integrals from* (1.8)*. If* $a, b,$ *and* c *are fixed reals with* $0 < a < \frac{b}{2}$ *and* $|c| < \frac{1}{2}$*, then as* $M \to \infty$*,*

$$P(a, b, c; M) \sim \frac{e^{aM}}{\sqrt{2\pi aM} \cdot bM}.$$

Remark. It will be clear from the proof of Lemma 4.1 that the asymptotic expansion of $P(a, b, c; M)$ can also be determined to an arbitrary degree of precision (i.e., a partial expansion with relative error $O(M^{-\alpha})$ can be written for any positive half-integer α). Indeed, the asymptotic expansion of the modified Bessel function (cf. (4.11)) has been extensively studied, and the other terms in the integrand have convergent Taylor expansions around zero. Note that the main asymptotic term depends only on a and b; however, the subsequent terms in the asymptotic expansion will also depend on c.

Proof. The integral in question is

$$P(a, b, c; M) = \int_{-1}^{1} \frac{\left(1 - y^2\right)^{-\frac{3}{4}}}{\cosh\left(\pi(bMy + ci)\right)} I_{\frac{3}{2}}\left(aM\sqrt{1 - y^2}\right) dy. \tag{4.9}$$

In order to identify the dominant asymptotic behavior, we split the integral into two ranges. First, consider the range around $y = 0$,

$$\int_{|y| \le \frac{1}{2}} \frac{\left(1 - y^2\right)^{-\frac{3}{4}}}{\cosh\left(\pi(bMy + ci)\right)} I_{\frac{3}{2}}\left(aM\sqrt{1 - y^2}\right) dy. \tag{4.10}$$

It is a standard fact that the general asymptotic behavior for modified Bessel functions is given by

$$I_\ell(x) = \frac{e^x}{\sqrt{2\pi x}} + O\left(\frac{e^x}{x\sqrt{x}}\right) \tag{4.11}$$

as $x \to \infty$ (regardless of ℓ). This means that when $|y| < \frac{1}{2}$ we may apply the asymptotic estimate

$$I_{\frac{3}{2}}\left(aM\sqrt{1 - y^2}\right) \sim \frac{e^{aM\sqrt{1-y^2}}}{\sqrt{2\pi aM}(1 - y^2)^{\frac{1}{4}}} \left(1 + O\left(\frac{1}{M}\right)\right),$$

and the integral (4.10) is therefore asymptotically equivalent to

$$\left(1 + O\left(\frac{1}{M}\right)\right) \int_{|y| \le \frac{1}{2}} \frac{e^{aM\sqrt{1-y^2}}}{\sqrt{2\pi aM}(1 - y^2)\cosh\left(\pi bMy + c\pi i\right)} dy. \tag{4.12}$$

In order to determine the behavior of (4.12) as $M \to \infty$, we use the Saddle Point Method. In particular, we take the convergent Taylor expansion (around $y = 0$) of the integrand from (4.12), excluding the hyperbolic cosine, and also apply the change of variables $x = \sqrt{M}y$, writing

$$\frac{e^{aM\sqrt{1-y^2}}}{1 - y^2} = e^{aM} \cdot e^{-\frac{ax^2}{2}} \left(1 + c_1\frac{x^2}{M} + c_2\frac{x^4}{M^2} + \dots\right) \tag{4.13}$$

for certain constants c_i. The hyperbolic cosine factor was not included because as $M \to \infty$, it does not have a convergent Taylor series in any neighborhood of $y = 0$. This factor will instead be simplified through the use of a complex contour shift after the other terms have been further approximated.

The Taylor expansion (4.13) implies that the main asymptotic term from (4.12) is

$$\frac{e^{aM}}{\sqrt{2\pi a} \cdot M} \int\limits_{\mathbb{R}} \frac{e^{-\frac{ax^2}{2}}}{\cosh\left(\pi b\sqrt{M}x + c\pi i\right)} dx. \tag{4.14}$$

Note that after the change of variables, the integration range became $|x| \leq \frac{\sqrt{M}}{2}$. We extend this range to all of $\mathbb{R}$ in (4.14) since the "tail" (the neighborhoods with $|x| \geq \frac{\sqrt{M}}{2}$) is of exponentially lower magnitude; indeed, the magnitude of this part of the integral can be trivially bounded by $C \cdot \exp\left(-\frac{aM}{8} - \frac{bM}{2}\right)$.

The integral (4.14) may be further transformed to

$$\begin{aligned} \frac{e^{aM}}{\sqrt{2\pi aM} \cdot bM} \int\limits_{\mathbb{R}} \frac{e^{-\frac{ax^2}{2b^2M}}}{\cosh\left(\pi x + c\pi i\right)} dx &= \frac{e^{aM}}{\sqrt{2\pi aM} \cdot bM} \int\limits_{\mathbb{R}+ic} \frac{e^{-\frac{a}{2b^2M}(x-ci)^2}}{\cosh(\pi x)} dx \\ &= \frac{e^{aM}}{\sqrt{2\pi aM} \cdot bM} \int\limits_{\mathbb{R}} \frac{e^{-\frac{a}{2b^2M}(x-ci)^2}}{\cosh(\pi x)} dx, \end{aligned} \tag{4.15}$$

where the final equality follows from a contour shift and the fact that the poles of the integrand occur at $\mathrm{Im}(x) \in \frac{1}{2} + \mathbb{Z}$. It is straightforward to check that as $M \to \infty$, the above integral goes to 1. This gives the main asymptotic term for $P(a, b, c; M)$ as claimed.

Recalling (4.13), the subsequent terms in the asymptotic expansion of (4.12) all have the form

$$C \cdot \frac{1}{M^{j+1}} \cdot \int\limits_{\mathbb{R}} \frac{x^{2k} e^{-\frac{ax^2}{2}}}{\cosh\left(\pi b\sqrt{M}x + c\pi i\right)} dx, \tag{4.16}$$

where $j \geq k \geq 1$. Following the transformation and contour shift from (4.15), and using the fact that $\int_{\mathbb{R}} \frac{x^{2j}}{\cosh(x)} dx$ is bounded (as a function of j), we find that (4.16) is $O\left(\frac{1}{M^{j+3/2}}\right)$. We therefore conclude that (4.12) has the asymptotic form

$$\left(1 + O\left(\frac{1}{M}\right)\right) \frac{e^{aM}}{\sqrt{2\pi aM} \cdot bM}.$$

In order to complete the proof, we must show that the remaining integration range in (4.9) is of asymptotically lower order; this portion is

$$\int\limits_{\frac{1}{2} \leq |y| \leq 1} \frac{\left(1 - y^2\right)^{-\frac{3}{4}}}{\cosh\left(\pi(bMy + ci)\right)} I_{\frac{3}{2}}\left(aM\sqrt{1 - y^2}\right) dy. \tag{4.17}$$

Here we use the additional standard fact that the modified Bessel functions are monotonically increasing on $[0, \infty)$, which implies the uniform bound

$$I_{\frac{3}{2}}\left(aM\sqrt{1 - y^2}\right) \leq I_{\frac{3}{2}}\left(\frac{\sqrt{3}aM}{2}\right)$$

on $\frac{1}{2} \leq |y| \leq 1$. The Bessel function asymptotic (4.11) now implies that (4.17) is of exponentially lower order than (4.10) in the limit as $M \to \infty$. □

Proof of Corollary 1.2. We claim that the main contribution is found in the third sum, specifically, the term corresponding to $k = 1, r = 0$, and $\ell = 0$. We begin by calculating the asymptotic behavior of this term. It can be immediately checked that that $K^*_{1,0}(-m, 0) = 1$ and $C(0) = 1$, so Lemma 4.1 implies that as $m \to \infty$, this term is asymptotically

$$\frac{\pi n_0^{\frac{3}{4}}}{2^{\frac{1}{4}} \left(m - \varepsilon_s(n)\right)^{\frac{1}{4}} \sqrt{n}} P\left(2\pi\sqrt{2n_0}, \sqrt{\frac{2}{n}}, -\left(\ell - \frac{2}{n}\right); \sqrt{m - \varepsilon_s(n)}\right) \sim \frac{\sqrt{n+2}}{8\sqrt{3}m} e^{2\pi\sqrt{\frac{(n+2)m}{6}}}. \tag{4.18}$$

Finally, we show that (4.18), which has exponential argument $2\pi\sqrt{\frac{(n+2)m}{6}}$, is in fact the dominant term in Theorem 1.1. In the first summand of the theorem the dominant term clearly occurs when $k = 1$ and $r = 0$, which has an exponential argument

$$\begin{aligned} 4\pi\sqrt{\delta_j(n)m} &= 4\pi\sqrt{\frac{n}{24} - \frac{5}{12} + \frac{1}{2n}\left(j^2 + (1-n)j\right) + \frac{1}{8n}}\sqrt{m} \\ &\leq 4\pi\sqrt{\frac{n}{24} - \frac{5}{12} + \frac{1}{8n}}\sqrt{m} < 2\pi\sqrt{\frac{n+2}{6}}\sqrt{m}. \end{aligned}$$

The main term in the second summand occurs when $k = 2$ and $r = 0$, which has an exponential argument

$$2\pi\sqrt{\widetilde{\delta}_j(n)m} = 2\pi\sqrt{\frac{n}{24} - \frac{1}{24} + \frac{j^2}{2n} - \frac{j}{2}}\sqrt{m} \leq 2\pi\sqrt{\frac{n}{24} - \frac{1}{24}}\sqrt{m} < 2\pi\sqrt{\frac{n+2}{6}}\sqrt{m}.$$

Thus the dominant term is indeed (4.18). □

References

[1] G. Andrews, *On the theorems of Watson and Dragonette for Ramanujan's mock theta functions*, Amer. J. Math. **88** (1966), 454-490.
[2] G. Andrews, *Partitions with short sequences and mock theta functions*, Proc. Natl. Acad. Sci. (USA) **102** (2005), 4666-4671.
[3] R. Borcherds, *Monstrous Moonshine and Monstrous Lie Superalgebras*, Invent. Math. **109** (1992), 405-444.
[4] K. Bringmann and A. Folsom, *On the asymptotic behavior of Kac-Wakimoto characters*, to appear in Proc. of the Amer. Math. Soc.
[5] K. Bringmann and K. Mahlburg, *An extension of the Hardy-Ramanujan Circle Method and applications to partitions without sequences*, Amer. Journal of Math. **133** (2011), 1151-1178.
[6] K. Bringmann and K. Ono, *The $f(q)$ mock theta function conjecture and partition ranks*, Invent. Math. **165** (2006), 243–266.
[7] K. Bringmann and K. Ono, *Some characters of Kac and Wakimoto and nonholomorphic modular functions*, Math. Annalen **345** (2009), 547-558.
[8] K. Bringmann and K. Ono, *Coefficients of harmonic Maass forms*, Proceedings of the 2008 University of Florida Conference on Partitions, q-series, and modular forms, accepted for publication.
[9] J. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. Journal **125** (2004), 45–90.
[10] J. Conway and S. Norton, *Monstrous Moonshine*, Bull. London Math. Soc. **11** (1979), 308-339.
[11] L. Dragonette, *Some asymptotic formulae for the mock theta series of Ramanujan*, Trans. Amer. Math. Soc. **72** (1952), 474–500.
[12] F. Dyson, *Some guesses in the theory of partitions*, Eureka (Cambridge) **8** (1944), pages 10-15.
[13] E. Grosswald and H. Rademacher, *Dedekind Sums*, Carus Math. Monographs, Math. Assoc. of America, 1972.
[14] G. Hardy and S. Ramanujan, *Asymptotic Formulae in Combinatory Analysis*, Proc. London Math. Soc. **17** (1918), 75–115.

[15] A. Ingham, *Some Tauberian theorems connected with the prime number theorem,* J . London Math. Soc. **20** (1945), 171–180.
[16] V. Kac, *Infinite-dimensional Lie algebras and Dedekind's eta function*, Funct. Anal. Appl. **8** (1974), 68-70.
[17] V. Kac, *Lie superalgebras*, Adv. Math. **26** (1977), 8–96.
[18] V. Kac, *Infinite dimensional Lie algebras, 3rd edition,* Cambridge University Press, 1990.
[19] V. Kac and D. Peterson, *Infinite-dimensional Lie algebras, theta functions and modular forms,* Adv. in Math. **53** (1984), 125–264.
[20] V. Kac and M. Wakimoto, *Integrable highest weight modules over affine superalgebras and number theory,* Progress in Math. 123, Birkh auser, Boston, 1994, 415–456.
[21] V. Kac and M. Wakimoto, *Integrable highest weight modules over affine superalgebras and Appell's function*, Comm. Math. Phys. **215** (2001), 631-682.
[22] D. Mumford, *Tata Lectures on Theta I*, Progress in Mathematics, no. 28, Birkhäuser, 1983.
[23] H. Rademacher, *Topics in analytic number theory*, Die Grundlehren der mathematischen Wissenschaften, Band **169**, Springer Verlag New York- Heidelberg, 1973.
[24] H. Rademacher and H. Zuckerman, *On the Fourier coefficients of certain modular forms of positive dimension*, Ann. of Math. **39** (1938), 433–462.
[25] S. Ramanujan, *The lost notebook and other unpublished papers*, Narosa Publishing House, New Delhi, 1987.
[26] G. Shimura, *On modular forms of half integral weight*, Annals of Math. **97** (1973), 440–481.
[27] H. Zuckerman, *Certain functions with singularities on the unit circle*, Duke Math. J. **10** (1943), 381–395.
[28] S. Zwegers, *Mock theta functions*, Ph.D. Thesis, Universiteit Utrecht, (2002).

Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
E-mail address: kbringma@math.uni-koeln.de

Department of Mathematics, Princeton University, NJ 08544, U.S.A.
E-mail address: mahlburg@math.princeton.edu

Department of Mathematics, Louisiana State University, LA 70803, U.S.A.
E-mail address: mahlburg@math.lsu.edu