QUASIMODULAR FORMS AND $s\ell(m|m)^\wedge$ CHARACTERS

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Abstract. This note corrects the proof of Theorem 1.1 of [1], and extends the statement of the result to odd m.

1. Introduction and statement of results

Let for $m \in \mathbb{N}$

$$\varphi_m(z) = \varphi_m(z;\tau) := \left(\frac{\vartheta\left(z+\frac{1}{2}\right)}{\vartheta(z)}\right)^m,$$

where ($q := e^{2\pi i\tau}, \zeta := e^{2\pi iz}$ with $\tau \in \mathbb{H}, z \in \mathbb{C}$)

$$\vartheta(z) = \vartheta(z;\tau) := \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} e^{\pi i\nu\tau + 2\pi i\nu\left(z+\frac{1}{2}\right)}$$

is the *Jacobi theta function.* Note that in contrast to [1], we write φ_m in order to highlight the dependence on m. Denote the coefficients of the Fourier expansion (in z) by χ_r, so that

$$\varphi_m(z;\tau) =: \sum_{r\in\mathbb{Z}} \chi_r(\tau)\zeta^r. \tag{1.1}$$

Define the Nebentypus character ψ_m for matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(2)$ by

$$\psi_m(\gamma) := e^{\frac{\pi i m}{2}\left(\frac{c}{2}d+d-1\right)}. \tag{1.2}$$

Moreover, we require the well-known Eisenstein series $E_{2j}(\tau)$. For $j \geq 2$, they are holomorphic modular forms, while $E_2(\tau)$ is a *quasimodular* form. The Bernoulli numbers B_ℓ are defined for non-negative integers ℓ by the generating function

$$\frac{t}{e^t-1} = \sum_{\ell\geq 0} B_\ell \frac{t^\ell}{\ell!}.$$

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Theorem 1.1. *For $r \in \mathbb{Z}$ and $m \in \mathbb{N}$, we have*

$$\chi_r(\tau) = \frac{q^r}{1+(-1)^{m+1}q^r} \sum_{0 \leq \ell < \frac{m}{2}} r^{m-2\ell-1} \frac{D_{m-2\ell}(\tau)}{(m-2\ell-1)!} \qquad (\text{assuming that } r \neq 0 \text{ if } m \text{ is even}),$$

$$\chi_0(\tau) = D_0(\tau) + \sum_{1 \leq j \leq \frac{m}{2}} \frac{B_{2j}}{(2j)!} D_{2j}(\tau) E_{2j}(\tau) \qquad \text{for } m \text{ even},$$

where for each $0 \leq j \leq m$ such that $j \equiv m \pmod 2$, the function D_j is a modular form of weight $-j$ on $\Gamma_0(2)$ with Nebentypus character ψ_m, as defined in (1.2).

Remark. Theorem 1.1 was given for even m in [1]; above, we have extended the statement to hold for odd m. Moreover, the proof in [1] had a mistake: the second displayed formula in the proof of Proposition 3.3 was incorrect. We thank Sander Zwegers for pointing out the mistake and for fruitful discussion.

2. Proof of Theorem 1.1

Using that, for $\lambda, \mu \in \mathbb{Z}$, we have

$$\vartheta(z + \lambda\tau + \mu) = (-1)^{\lambda+\mu} q^{-\frac{\lambda^2}{2}} e^{-2\pi i \lambda z} \vartheta(z),$$
$$\vartheta\left(z + \frac{1}{2} + \lambda\tau + \mu\right) = (-1)^{\mu} q^{-\frac{\lambda^2}{2}} e^{-2\pi i \lambda z} \vartheta\left(z + \frac{1}{2}\right),$$

we obtain that

$$\varphi_m(z + \lambda\tau + \mu) = (-1)^{m\lambda} \varphi_m(z). \tag{2.1}$$

Let for $z_0 \in \mathbb{C},\ \tau \in \mathbb{H}$

$$P_{z_0} := \{z_0 + r\tau + s : 0 \leq r, s \leq 1\}.$$

Then, with z_0 such that no pole of φ_m lies at the boundary of P_{z_0}, we compute

$$\begin{aligned}\int_{\partial P_{z_0}} \varphi_m(w) e^{-2\pi i r w} dw &= \left(\int_{z_0}^{z_0+1} + \int_{z_0+1}^{z_0+1+\tau} + \int_{z_0+1+\tau}^{z_0+\tau} + \int_{z_0+\tau}^{z_0}\right) \varphi_m(w) e^{-2\pi i r w} dw \\ &= \int_0^1 \varphi_m(z_0+t) e^{-2\pi i r(z_0+t)} dt + \tau \int_0^1 \varphi_m(z_0+1+t\tau) e^{-2\pi i r(z_0+t\tau)} dt \\ &\quad - \int_0^1 \varphi_m(z_0+\tau+t) e^{-2\pi i r(z_0+\tau+t)} dt - \tau \int_0^1 \varphi_m(z_0+t\tau) e^{-2\pi i r(z_0+t\tau)} dt.\end{aligned} \tag{2.2}$$

Using (2.1) gives

$$\varphi_m(z_0+1+t\tau) = \varphi_m(z_0+t\tau), \quad \varphi_m(z_0+t+\tau) = (-1)^m \varphi_m(z_0+t).$$

Thus (2.2) becomes

$$e^{-2\pi i r z_0}\left(1 + (-1)^{m+1} e^{-2\pi i r \tau}\right) \int_0^1 \varphi_m(z_0+t) e^{-2\pi i r t} dt.$$

Inserting the Fourier expansion of φ_m yields

$$\int_0^1 \varphi_m\left(z_0+t\right)e^{-2\pi irt}dt=\sum_{\ell\in\mathbb{Z}}\chi_\ell(\tau)e^{2\pi i\ell z_0}\int_0^1 e^{2\pi i(\ell-r)t}dt=\chi_r(\tau)e^{2\pi irz_0}.$$

So (assuming $r\neq 0$ if m is even)

$$\chi_r(\tau)=\frac{(-1)^{m+1}q^r}{1+(-1)^{m+1}q^r}\int_{\partial P_{z_0}}\varphi_m(w)e^{-2\pi irw}dw. \tag{2.3}$$

We now compute (2.2) in another way, picking $z_0=-\frac{1}{2}-\frac{\tau}{2}$. Then the only pole of φ_m in P_{z_0} is at $z=0$. So, using the Residue Theorem, (2.2) equals

$$2\pi i\operatorname{Res}_{z=0}\left(\varphi_m(z)e^{-2\pi irz}\right). \tag{2.4}$$

Write (noting that φ_m is even or odd, depending on the parity of m)

$$\varphi_m(z)=\sum_{m-2\ell>0}\frac{D_{m-2\ell}(\tau)}{(2\pi iz)^{m-2\ell}}+O(1). \tag{2.5}$$

Inserting the series expansion of $e^{-2\pi irz}$, (2.4) becomes

$$(-1)^{m+1}\sum_{0\leq\ell<\frac{m}{2}}r^{m-2\ell-1}\frac{D_{m-2\ell}(\tau)}{(m-2\ell-1)!}.$$

Thus, for $r\in\mathbb{Z}$ (with the restriction that $r\neq 0$ if m is even) we obtain by comparing with (2.3),

$$\chi_r(\tau)=\frac{q^r}{1+(-1)^{m+1}q^r}\sum_{0\leq\ell<\frac{m}{2}}r^{m-2\ell-1}\frac{D_{m-2\ell}(\tau)}{(m-2\ell-1)!}.$$

This gives the first equation in Theorem 1.1.

To determine χ_0 (for m even), we plug in to (1.1), which implies

$$\varphi_m(z)=\sum_{1\leq\ell\leq\frac{m}{2}}\frac{D_{2\ell}(\tau)}{(2\ell-1)!}\sum_{r\in\mathbb{Z}\setminus\{0\}}\frac{r^{2\ell-1}q^r\zeta^r}{1-q^r}+\chi_0(\tau). \tag{2.6}$$

We now insert the Laurent expansions around $z=0$ on both sides. We write the sum on r as

$$\sum_{r\geq 1}\frac{r^{2\ell-1}q^r\zeta^r}{1-q^r}+\sum_{r\geq 1}\frac{r^{2\ell-1}\zeta^{-r}}{1-q^r}. \tag{2.7}$$

It is not hard to see that both sums converge absolutely for $-v<y<0$, where $v:=\operatorname{Im}(\tau), y:=\operatorname{Im}(z)$. We write the second summand in (2.7) as

$$\sum_{r\geq 1}\frac{r^{2\ell-1}\zeta^{-r}}{1-q^r}=\sum_{r\geq 1}r^{2\ell-1}\zeta^{-r}+\sum_{r\geq 1}\frac{r^{2\ell-1}\zeta^{-r}q^r}{1-q^r}. \tag{2.8}$$

The first summand equals

$$\left(-\frac{1}{2\pi i}\frac{\partial}{\partial z}\right)^{2\ell-1}\sum_{r\geq 1}\zeta^{-r} = \left(-\frac{1}{2\pi i}\frac{\partial}{\partial z}\right)^{2\ell-1}\frac{1}{\zeta-1}$$
$$= \left(-\frac{1}{2\pi i}\frac{\partial}{\partial z}\right)^{2\ell-1}\left(\frac{B_0}{2\pi i z}+\frac{B_{2\ell}(2\pi i z)^{2\ell-1}}{(2\ell)!}\right)+O\left(z^2\right)$$
$$= \frac{(2\ell-1)!}{(2\pi i z)^{2\ell}}-\frac{B_{2\ell}}{2\ell}+O\left(z^2\right).$$

The second summand combines with the first summand in (2.7) as using that φ_m is an even function of z,

$$2\sum_{r\geq 1}\frac{r^{2\ell-1}q^r}{1-q^r}+O\left(z^2\right).$$

Thus the right hand side in (2.6) becomes

$$\sum_{1\leq \ell\leq \frac{m}{2}}\frac{D_{2\ell}(\tau)}{(2\ell-1)!}\left(2\sum_{r\geq 1}\frac{r^{2\ell-1}q^r}{1-q^r}-\frac{B_{2\ell}}{2\ell}+\frac{(2\ell-1)!}{(2\pi i z)^{2\ell}}\right)+\chi_0(\tau)+O\left(z^2\right)$$
$$= -\sum_{1\leq \ell\leq \frac{m}{2}}\frac{D_{2\ell}(\tau)}{(2\ell)!}B_{2\ell}E_{2\ell}(\tau)+\sum_{1\leq \ell\leq \frac{m}{2}}\frac{D_{2\ell}(\tau)}{(2\pi i z)^{2\ell}}+\chi_0(\tau)+O\left(z^2\right).$$

Picking off the constant term on both sides of (2.5) then gives

$$\chi_0(\tau) = D_0(\tau)+\sum_{1\leq \ell\leq \frac{m}{2}}\frac{D_{2\ell}(\tau)}{(2\ell)!}B_{2\ell}E_{2\ell}(\tau),$$

as claimed.

The proof of the modularity follows from the fact that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(2)$, we have that

$$\varphi_m\left(\frac{z}{c\tau+d};\gamma\tau\right) = \psi_m(\gamma)\varphi_m(z;\tau).$$

References

[1] K. Bringmann, A. Folsom, K. Mahlburg, *Quasimodular forms and $s\ell(m|m)^\wedge$ characters*, Ramanujan Journal **36** (2015), 103-116.

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