CONVOLUTION BOOTSTRAP PERCOLATION MODELS, MARKOV-TYPE STOCHASTIC PROCESSES, AND MOCK THETA FUNCTIONS

KATHRIN BRINGMANN, KARL MAHLBURG, AND ANTON MELLIT

Abstract. We introduce a new family of directed, multi-state bootstrap percolation models that naturally occur as the "convolution" of classical bootstrap percolation models as well as generalized k-cross models studied by Bringmann, Gravner, Holroyd, Liggett, and Mahlburg. We prove bounds for the probability of indefinite growth by relating the percolation process to sequences of random variables that characterize the percolation growth combinatorics. The corresponding stochastic processes are of independent interest, and we prove a general bound for the limiting density of their probability distributions. We prove these bounds using new results for the convexity and monotonicity of linear operators; these results are of independent interest and the techniques also apply to other stochastic processes with "forbidden patterns".

In the simplest case of the new multi-state percolation models, we prove a stronger result that gives the precise asymptotic behavior for the limiting probability density of the corresponding stochastic process. This follows from the surprising appearance of Ramanujan's mock theta functions, whose cuspidal asymptotics are closely connected to the limiting probabilities.

1. Introduction and statement of results

In this paper we build on work of Andrews, Gravner, Holroyd, Liggett, and the first two authors in [3, 11, 16, 22], and study the relation between probability bounds for bootstrap percolation models, bounds for the probabilities of stochastic processes, the combinatorics of overpartitions, and the cuspidal behavior of mock theta functions and modular forms. We introduce an infinite family of percolation processes on the square lattice that can be understood as a nontrivial convolution of two (directed) copies of the k-percolation models from [22] and [11]. In particular, the case $k = 2$ is a "mixed" superposition of two copies of the classical directed bootstrap percolation model [24].

The process begins with an initial configuration that is generated by the Bernoulli random process in which each site $x \in \mathbb{N}^2$ is independently set to one of four possible states, which are represented by the following names/symbols:

$$\text{Empty: } \square \text{, Right: } \rightarrow \text{, Up: } \uparrow \text{, or Active: } + .$$

The state of x is denoted by $\boldsymbol{\mathcal{S}}(x)$.

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For a fixed parameter $0 \leq q \leq 1$, the corresponding probabilities of each state are given by

$$\begin{aligned} \mathbf{P}\left(\boldsymbol{\mathcal{S}}(x) = \square\right) &= q^2, \\ \mathbf{P}\left(\boldsymbol{\mathcal{S}}(x) = \rightarrow\right) = \mathbf{P}\left(\boldsymbol{\mathcal{S}}(x) = \uparrow\right) &= q(1-q), \\ \mathbf{P}\left(\boldsymbol{\mathcal{S}}(x) = \boldsymbol{+}\right) &= (1-q)^2. \end{aligned} \tag{1.1}$$

Note that if we interpret the state $\boldsymbol{+}$ as the union of $\rightarrow$ and $\uparrow$, then these probabilities are equivalent to the Cartesian product of the disjoint two-state configurations with probabilities

$$\begin{aligned} &\mathbf{P}\left(\square\right) = q, \ \mathbf{P}\left(\rightarrow\right) = (1-q), \ \text{and} \\ &\mathbf{P}\left(\square\right) = q, \ \mathbf{P}\left(\uparrow\right) = (1-q). \end{aligned} \tag{1.2}$$

Given an initial configuration $\mathcal{C}$, the system then evolves according to deterministic neighborhood growth rules that are defined below. The viewpoint in (1.2) is essential for understanding a vital feature of the system: it is "increasing" relative to each of the independent processes in the Right and Up directions. This means a site that is Empty relative to either Right or Up states may eventually become occupied, but an occupied Right or Up state will never become Empty. In other words, an Empty site may evolve to a Right or Up state, which may then evolve to an Active state (which then remains stationary throughout the remainder of the process).

An alternative characterization uses the following simple partial order on states:

$$\square \leq \rightarrow, \ \square \leq \uparrow; \qquad \rightarrow \leq \boldsymbol{+}, \ \uparrow \leq \boldsymbol{+}.$$

We also use the notation $A + B$ to represent the ordered lattice *join* operation on two such states; for example, $\rightarrow + \uparrow = \boldsymbol{+}$. This lattice is represented graphically in Figure 1.

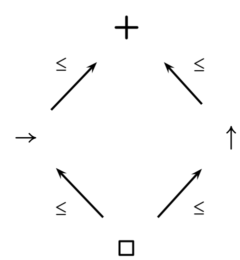

FIGURE 1. Lattice ordering of percolation states

Then the percolation process has the property that the set of configurations in which some site x eventually becomes active forms a lattice ideal. This can also be viewed at the level of individual growth rules, which must also then be monotonic with respect to the partial order described above.

The specific growth of a lattice site x is determined by the states in the surrounding $(k-1)$-*cross*, which is the neighborhood defined by

$$N(x) = N_k(x) := \{x + w \ : \ w = (v, 0) \text{ or } (0, v), -(k-1) \leq v \leq (k-1), \ v \neq 0\}. \tag{1.3}$$

Definition 1.1. The *k-convolution bootstrap percolation* process on a given initial configuration $\mathcal{C}$ evolves according to the following rules.

(i) **Disjoint** directed growth (see Figure 2):

(R) Suppose $\boldsymbol{\mathcal{S}}(x) = \boldsymbol{\sigma} \ngeq \rightarrow$.

$$\begin{aligned} &\text{If } \boldsymbol{\mathcal{S}}(x-(v,0)) \geq \rightarrow \text{ for all } 1 \leq v \leq k-1, \\ &\text{and } \boldsymbol{\mathcal{S}}(x \pm (0,v)) \geq \rightarrow \text{ for at least one } 1 \leq v \leq k-1, \\ &\text{then update } \boldsymbol{\mathcal{S}}(x) \coloneqq \boldsymbol{\sigma} + \rightarrow . \end{aligned}$$

(U) Suppose $\boldsymbol{\mathcal{S}}(x) = \boldsymbol{\sigma} \ngeq \uparrow$.

$$\begin{aligned} &\text{If } \boldsymbol{\mathcal{S}}(x-(0,v)) \geq \uparrow \text{ for all } 1 \leq v \leq k-1, \\ &\text{and } \boldsymbol{\mathcal{S}}(x \pm (v,0)) \geq \uparrow \text{ for at least one } 1 \leq v \leq k-1, \\ &\text{then update } \boldsymbol{\mathcal{S}}(x) \coloneqq \boldsymbol{\sigma} + \uparrow . \end{aligned}$$

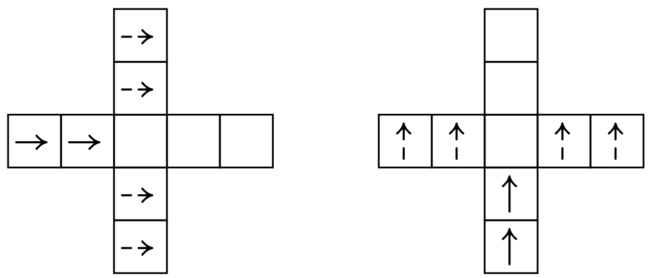

FIGURE 2. The required configurations for Rule (i) parts (R) and (U), listed respectively, when $k = 3$. The solidly marked states must all occur, and the dashed arrows indicate that at least one of the specified states occurs.

(ii) **Skew** directed growth (see Figure 3):

(R) Suppose $\boldsymbol{\mathcal{S}}(x) = \boldsymbol{\sigma} \neq \boldsymbol{+}$.

$$\begin{aligned} &\text{If } \boldsymbol{\mathcal{S}}(x-(v,0)) = \boldsymbol{+} \text{ for all } 1 \leq v \leq k-1, \\ &\text{and } \boldsymbol{\mathcal{S}}(x \pm (0,v)) \geq \uparrow \text{ for at least one } 1 \leq v \leq k-1, \\ &\text{then update } \boldsymbol{\mathcal{S}}(x) \coloneqq \boldsymbol{+} . \end{aligned}$$

(U) Suppose $\boldsymbol{\mathcal{S}}(x) = \boldsymbol{\sigma} \neq \boldsymbol{+}$.

$$\begin{aligned} &\text{If } \boldsymbol{\mathcal{S}}(x-(0,v)) = \boldsymbol{+} \text{ for all } 1 \leq v \leq k-1, \\ &\text{and } \boldsymbol{\mathcal{S}}(x \pm (v,0)) \geq \rightarrow \text{ for at least one } 1 \leq v \leq k-1, \\ &\text{then update } \boldsymbol{\mathcal{S}}(x) \coloneqq \boldsymbol{+} . \end{aligned}$$

(iii) **Gap** jumping (see Figure 4):

(R) Suppose $\boldsymbol{\mathcal{S}}(x) = \boldsymbol{\sigma} \neq \boldsymbol{+}$.

$$\begin{aligned} &\text{If } \boldsymbol{\mathcal{S}}(x-(v,0)) = \boldsymbol{+} \text{ for all } 1 \leq v \leq k-1, \text{ and } j \text{ is such that} \\ &\quad \boldsymbol{\mathcal{S}}(x+(v,0)) = \rightarrow \text{for all } 1 \leq v \leq j-1 \leq k-2,\ \boldsymbol{\mathcal{S}}(x+(j,0)) = \boldsymbol{+} , \\ &\text{and } \boldsymbol{\mathcal{S}}(x \pm (0,v)) \geq \rightarrow \text{ for at least one } 1 \leq v \leq k-1, \\ &\text{then update } \boldsymbol{\mathcal{S}}(x) \coloneqq \boldsymbol{+} . \end{aligned}$$

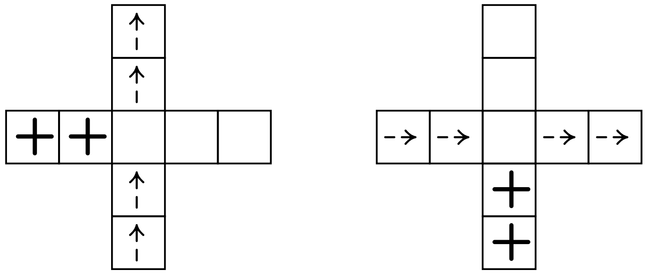

FIGURE 3. The required configurations for Rule (ii) parts (R) and (U) when $k = 3$.

(U) Suppose $\boldsymbol{\mathcal{S}}(x) = \boldsymbol{\sigma} \neq +$.

If $\boldsymbol{\mathcal{S}}(x - (0,v)) = +$ for all $1 \leq v \leq k-1$, and j is such that

$\boldsymbol{\mathcal{S}}(x + (0,v)) = \rightarrow$ for all $1 \leq v \leq j-1 \leq k-2$, $\boldsymbol{\mathcal{S}}(x + (0,j)) = +$,

and $\boldsymbol{\mathcal{S}}(x \pm (v,0)) \geq \rightarrow$ for at least one $1 \leq v \leq k-1$,

then update $\boldsymbol{\mathcal{S}}(x) \coloneqq +$.

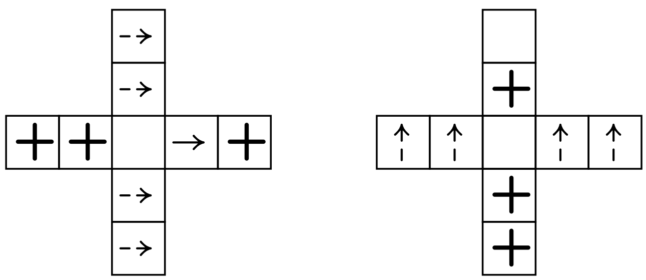

FIGURE 4. Two examples of the required configurations for Rule (iii) parts (R) and (U) when $k = 3$. In the example shown for part (R), the gap is followed by a Right state until another Active state occurs; in part (U), the gap is immediately followed by an Active state.

Remark 1.2. We refer to these models as the "convolution" of individual directed bootstrap percolation models due to the fact that if the growth rules are only considered relative to Right states, then the growth rules allow for only adjacent growth to the right, but upward gap-jumping across at most $k-1$ empty sites. Similarly, relative to Up states, the process proceeds upward only one row at a time, but can jump to the right across a gap of length $k-1$.

The main question that we address concerns long-term behavior of the k-convolution percolation process. We say that a configuration has *indefinite growth* if all of $\mathbb{N}^2$ eventually becomes active. In order to describe the probability of indefinite growth, we define the function $\overline{f}_k(u)$ to be the unique solution to the functional equation

$$\overline{f}_k^k = u\overline{f}_k^{k-1} + \sum_{j=2}^{k} u^{j+1}(1-u)^{j-1}\overline{f}_k^{k-j} \tag{1.4}$$

that satisfies $\overline{f}_k(u) > u(1-u)$ on $(0,1)$. See Section 2 for a rigorous discussion of the existence of $\overline{f}_k$, as well as its many useful properties. Using this function we define the constants

$$\overline{\lambda}_k := -\int_0^1 \log\left(\overline{f}_k\left(1-e^{-z}\right)\right)\frac{dz}{z}. \tag{1.5}$$

We will also see in Section 4 that the function $\overline{f}_k$ is the "limiting density" of the probability for certain sequences of random variables with "forbidden patterns" that arise in the characterization of growing configurations in the k-convolution process.

The following theorem describes the probability of indefinite growth in the k-convolution process, and is analogous to the growth bounds for local k-percolation models stated as Theorem 1.1 of [11] and Theorem 1 of [16] in that it provides a double-sided bound with explicit second-order terms.

Theorem 1.3. *If we write $q = e^{-s}$ with $s \geq 0$, then for each $k \geq 1$, there exists a sufficiently small n_0, and constants c_1, c_2 such that for $s < n_0$ we have*

$$\exp\left(-2\overline{\lambda}_k s^{-1} + c_1 s^{-\frac{1}{2}}\right) \leq \mathbf{P}(\textit{indefinite growth}) \leq \exp\left(-2\overline{\lambda}_k s^{-1} + c_2 s^{-\frac{2k-1}{2k}}\left(\log s^{-1}\right)^2\right).$$

Remark 1.4. This result is weaker than the theorems in [11, 16] because the second-order terms are different s-powers, whereas in the earlier papers the upper bound also had $s^{-1/2}$.

The proof of this theorem proceeds by first characterizing configurations with indefinite growth by certain sequence conditions, and then by proving a general bound for the limiting densities of those sequences. In particular, consider a sequence of independent random variables $\{E_j\}_{j=1}^n$ such that

$$\mathbf{P}(E_j = A) = u_j^2, \quad \mathbf{P}(E_j = B) = \mathbf{P}(E_j = C) = u_j(1-u_j), \quad \mathbf{P}(E_j = D) = (1-u_j)^2,$$

where $0 \leq u_j \leq 1$ for each $1 \leq j \leq n$. Next, we consider events defined by certain "word" restrictions on the sequence of random variables $\{E_j\}_{j=1}^n$. In particular, we define an event that satisfies the k-*sequence condition* (cf. Section 3) and denote its probability by

$$\overline{\rho}_k(n) := \mathbf{P}\Big(\{E_j\}_{j=1}^n \text{ has no } D, CB, \text{ or } C^k\Big), \tag{1.6}$$

where C^k denotes a sequence of k consecutive Cs. Note that this event includes sequences that end with at most $k-1$ Cs.

The following result provides a very tight double-sided bound for the probabilities $\overline{\rho}_k(n)$.

Theorem 1.5. *If $0 \leq u_1 \leq \cdots \leq u_n \leq 1$, then*

$$\prod_{j=1}^{n} \overline{f}_k(u_j) \leq \overline{\rho}_k(n) \leq \prod_{j=k}^{n} \overline{f}_k(u_j).$$

Remark. The strength of these bounds is seen in the fact that the relative error is constant as $n \to \infty$.

Remark. If $u_1 = 0$, then the first two expressions in the statement of Theorem 1.5 are zero, so we need only consider the cases where $0 < u_1$.

The first two authors proved bounds for the probability of sequences without k-gaps in terms of the analogous function f_k in [11] (this work generalized and refined results in [16, 22]). The proofs in that paper used rather intricate and specialized arguments involving monotonic functions on simplices. In the current work we develop a new method for proving such bounds that instead follows from "spectral" estimates for the eigenvalues of associated (Markov-type) stochastic processes. This approach is much more adaptable and indeed shows that there are natural families of pattern-avoiding sequences that satisfy similar bounds.

In the application to k-convolution bootstrap percolation models, Theorem 1.5 is specialized by setting

$$u_j := 1 - q^n = 1 - e^{-ns}$$

and letting $n \to \infty$ (cf. Section 3). Define $\overline{A}_k$ as the event that the corresponding probability sequence $\{E_j\}_{j=1}^{\infty}$ has no D, CB, or C^k.

Corollary 1.6. *For $k \geq 2$, we have the following asymptotic as $s \to 0$:*

$$\exp\left(-\overline{\lambda}_k s^{-1}\right) \leq \mathbf{P}\left(\overline{A}_k\right) \leq s^{-\frac{(2k-1)}{2}(1+o(1))} \exp\left(-\overline{\lambda}_k s^{-1}\right).$$

The case $k = 2$ is also of number theoretical interest due to the striking appearance of one of Ramanujan's famous mock theta functions. Indeed, we will show in Section 6 that $\mathbf{P}(\overline{A}_2)$ is equal (up to a q-power) to the product of quotients of elliptic theta functions and one of Ramanujan's third-order mock theta functions, namely (as in [26])

$$\phi(q) := \sum_{n \geq 0} \frac{q^{n^2}}{\prod_{j=1}^{n}(1+q^{2j})}. \tag{1.7}$$

Ramanujan introduced this and 16 other mock theta functions, each given by simple hypergeometric q-series, in his final letter to Watson, and observed that they behave "nearly" like modular forms. Although these functions were a frequent subject of study in the ensuing eighty years, it was not until the doctoral thesis of Zwegers [28] that mock theta functions were put in a proper theoretical framework. It is now understood that the mock theta functions are the holomorphic components of *harmonic weak Maass forms* [12], the theory of which grew from Borcherds' work on automorphic representations [8].

The precise definitions of the above concepts are unnecessary for the present work, as the key feature is simply that the complex analytic symmetries (so-called "modular transformations") of modular and mock modular forms provide extremely precise asymptotic expansions for such functions near their zeros and poles. Our main result for the $k = 2$ case is therefore a significant improvement on the general statement of Theorem 1.5, as instead of unequal lower and upper bounds, in this case we describe the exact asymptotic behavior.

Theorem 1.7. *As $s \to 0^+$,*

$$\mathbf{P}\left(\overline{A}_2\right) \sim 2\sqrt{\pi} s^{-\frac{1}{2}} e^{-\frac{\pi^2}{8s}}.$$

This case also combinatorially corresponds to overpartitions without sequences, which are the subject of further study in the first two authors' work in progress with Holroyd [9]. This forthcoming work will provide an alternative proof that $\overline{\lambda}_2 = \frac{\pi^2}{8}$ (along with explicit formulas for other constants related to overpartition asymptotics) by relating certain logarithmic integrals of algebraic functions to Dilogarithm evaluations; see [4, 22] for the calculation of related constants. In this paper we will instead follow the approach of Andrews [3], who used properties of another of Ramanujan's mock theta functions, $\chi(q)$, in the study of partitions without sequences and bootstrap percolation. We will similarly prove Theorem 1.7 by relating the probability sequences to cuspidal asymptotics for modular eta-quotients and $\phi(q)$.

Remark 1.8. Our present use of $\phi(q)$ and Andrews' work on $\chi(q)$ illustrate the striking fact that the mock theta functions naturally arise in the setting of stochastic processes.

Remark 1.9. The appearance of theta functions and mock theta functions has further applications. In [10] the first two authors extended the Hardy-Ramanujan Circle Method in order to find very precise coefficient formulas for partitions without sequences; this method is also applicable to overpartitions without sequences.

The remainder of the paper is structured as follows. Section 2 contains the proper definition of $\overline{f}_k$ and proofs of many useful properties, including the convexity of an important auxiliary function $\overline{g}_k$. In Section 3 we explain how growing configurations in the k-convolution process can be characterized by simple combinatorial sequence conditions on columns and rows. Section 4 is a self-contained development of a new method for proving bounds for the limiting probability densities of certain stochastic processes, which is then applied to the specific case of the probabilities from Theorem 1.5 and Corollary 1.6. In Section 5 we prove the probability bounds for the k-convolution process stated in Theorem 1.3. The main body of the paper concludes in Section 6 with an explanation of the use of q-series and mock theta functions in the case $k = 2$, as well as the proof of Theorem 1.7.

2. Properties of $\overline{f}_k$

In this section we rigorously study monotonicity properties of $\overline{f}_k$ and the auxiliary function

$$\overline{g}_k(z) := -\log \overline{f}_k \left(1 - e^{-z}\right). \tag{2.1}$$

These properties will be needed in order to apply variational results in later sections. The key insight of this section is that the functional equation (1.4) can be re-scaled so that the resulting algebraic equation is linear in the variable u; we therefore find that it is easiest to understand $\overline{f}_k$ by studying its inverse.

Proposition 2.1. *Define $F_k : [0,1) \to [0,\infty)$ by $F_k(x) := \frac{x(1-x^k)}{(1-x)^2}$. The following statements then hold.*

(*i*) *The function F_k is smooth and increasing, with boundary values $F_k(0) = 0$ and $\lim_{x\to 1^-} F_k(x) = +\infty$.*

(*ii*) *The functional equation (1.4) has a unique solution that satisfies $\overline{f}_k(u) > u(1-u)$ on $u \in (0,1)$. This solution is given by*

$$\overline{f}_k(u) := \frac{u(1-u)}{F_k^{-1}\left(\frac{1}{u} - 1\right)},$$

where $F_k^{-1} : [0,\infty) \to [0,1)$ denotes the inverse function of F_k.

Proof. (*i*) This follows easily from the equivalent formula

$$F_k(x) = \left(x + x^2 + \cdots + x^k\right) \frac{1}{1-x}.$$

(*ii*) Consider the defining functional equation (1.4). Since the only solutions we are interested in are non-zero and $u \neq 0$ or 1, we may multiply both sides by $\frac{1-u}{u\overline{f}_k^k}$:

$$\frac{1-u}{u} = \frac{1-u}{\overline{f}_k} + \sum_{i=2}^{k} \left(\frac{u(1-u)}{\overline{f}_k}\right)^i.$$

Subtracting $\frac{(1-u)^2}{\overline{f}_k}$ from both sides produces

$$\frac{1-u}{u}\left(1 - \frac{u(1-u)}{\overline{f}_k}\right) = \sum_{i=1}^{k} \left(\frac{u(1-u)}{\overline{f}_k}\right)^i.$$

Since the right hand side is positive for all positive values of $\overline{f}_k$, $1-\frac{u(1-u)}{\overline{f}_k}$ cannot be zero, so the equation is equivalent to

$$\frac{1-u}{u}=F_k\left(\frac{u(1-u)}{\overline{f}_k}\right).$$

The restriction $\overline{f}_k>u(1-u)$ is equivalent to $\frac{u(1-u)}{\overline{f}_k}\in(0,1)$. Since $\frac{1-u}{u}\in(0,\infty)$ and the map F_k bijectively maps $(0,1)$ to $(0,\infty)$ we conclude that $\overline{f}_k$ is unique and $\overline{f}_k(u)=\frac{u(1-u)}{F_k^{-1}(\frac{1}{u}-1)}$, as required.

□

We will describe the behavior of $\overline{f}_k$ and $\overline{g}_k$ by carefully examining their Taylor series expansions, and we thus present several simple technical results before proceeding further.

Lemma 2.2. *Let $\{a_i\}_0^\infty,\{b_i\}_0^\infty$ be sequences of non-negative real numbers, and denote the partial sums by $a'_n:=\sum_{i=0}^n a_i$, $b'_n:=\sum_{i=0}^n b_i$ for $n\in\mathbb{Z}_{\geq 0}$. Define the power series $a(x)=\sum_{n=0}^\infty a_nx^n$ and $b(x)=\sum_{n=0}^\infty b_nx^n$ (where any undefined coefficients are set to 0), and assume that their radii of convergence are at least 1.*

- (i) *If all b_n are positive and the sequence $\frac{a_n}{b_n}$ is non-decreasing, then the sequence $\frac{a'_n}{b'_n}$ is also non-decreasing.*
- (ii) *If all b_n are positive and the sequence $\frac{a_n}{b_n}$ is non-decreasing and non-constant, then the ratio $\frac{a(x)}{b(x)}$ is an increasing function on $(0,1)$.*
- (iii) *If all b'_n are positive and the sequence $\frac{a'_n}{b'_n}$ is non-decreasing and non-constant, then the ratio $\frac{a(x)}{b(x)}$ is an increasing function on $(0,1)$.*

Proof. (i) For $n\geq 0$, we calculate

$$\frac{a'_{n+1}}{b'_{n+1}}-\frac{a'_n}{b'_n}=\frac{a'_n+a_{n+1}}{b'_n+b_{n+1}}-\frac{a'_n}{b'_n}=\frac{a_{n+1}b'_n-a'_nb_{n+1}}{b'_n(b'_n+b_{n+1})}.$$

The denominator is positive and the numerator can be written as

$$\sum_{i=0}^n(a_{n+1}b_i-a_ib_{n+1}),$$

where each summand is non-negative.

(ii) Note that the conditions guarantee that $b(x)>0$ for $x\in(0,1)$. We differentiate the ratio to obtain

$$\left(\frac{a(x)}{b(x)}\right)'=\frac{a'(x)b(x)-a(x)b'(x)}{b(x)^2}.$$

It is enough to check that the numerator is positive; we rewrite it as

$$a'(x)b(x)-a(x)b'(x)=\sum_{0\leq i<j}(j-i)(a_jb_i-a_ib_j)x^{i+j-1}.$$

It is clear that the right hand side has only non-negative terms with at least one being positive. The claim then follows.

(iii) This is an easy consequence of part (ii) using the observation that $\sum_{n=0}^\infty a'_nx^n=\frac{a(x)}{1-x}$ and $\sum_{n=0}^\infty b'_nx^n=\frac{b(x)}{1-x}$.

□

We are now prepared to state and prove the main result of this section.

Theorem 2.3. *Assume the notation and definitions from above.*

(*i*) *The function* $\overline{f}_k$ *is positive and increasing on* $(0,1)$*, with boundary limits* $\lim_{u\to 0^+}\overline{f}_k(u)=0$*, and* $\lim_{u\to 1^-}\overline{f}_k(u)=1$.

(*ii*) *The function* $\overline{g}_k$ *is convex, positive and decreasing on* $(0,\infty)$*, with boundary limits* $\lim_{z\to 0^+}\overline{g}_k(z)=\infty$*, and* $\lim_{z\to\infty}\overline{g}_k(u)=0$.

Proof. First we note that F_k extends to an analytic function of x in a neighborhood of 0 and $F_k(x)=x+O(x^2)$. This implies that F_k^{-1} extends to an analytic function in a neighborhood of 0 and $F_k^{-1}(x)=x+O(x^2)$. Thus $\overline{f}_k$ extends to an analytic function in a neighborhood of 1 and

$$\overline{f}_k(u)=\frac{u(1-u)}{1-u+O((1-u)^2)}=1+O(1-u),$$

so that $\lim_{u\to 1^-}\overline{f}_k(u)=1$.

When u tends to 0, the expression $\frac{1}{u}-1$ tends to ∞, so $\lim_{u\to 0^+}F_k^{-1}\left(\frac{1}{u}-1\right)=1$, which implies $\lim_{u\to 0^+}\overline{f}_k(u)=0$. The boundary values for $\overline{g}_k$ follow immediately from those for $\overline{f}_k$.

We now introduce an auxiliary variable $y:=F_k^{-1}\left(\frac{1}{u}-1\right)$, so that $F_k(y)=\frac{1}{u}-1$. The corresponding differential is then $F_k'(y)dy=-\frac{du}{u^2}$, which means that the logarithmic differential of f can be computed as

$$d\log\overline{f}_k(u)=\frac{du}{u}-\frac{du}{1-u}+\frac{du}{u^2F_k'(y)y}.$$

Using the fact that $u=1-e^{-z}$ and the corresponding differential $du=e^{-z}dz=(1-u)dz$, we further compute

$$-\frac{d\overline{g}_k(z)}{dz}=\left(\frac{1}{u}-\frac{1}{1-u}+\frac{1}{u^2F_k'(y)y}\right)(1-u). \tag{2.2}$$

To show convexity of $\overline{g}_k$ (which will imply the monotonicity of $\overline{f}_k$), it is now enough to check that the right hand side of (2.2) is a positive decreasing function of z. Due to the monotonicity of the changes of variables, this is equivalent to showing that it is a positive decreasing function of u, which is further equivalent to showing that it is a positive increasing function of y. We thus rewrite the expression as a function of y (recalling that $u=\frac{1}{1+F_k(y)}$), obtaining

$$\left(F_k(y)+1-\frac{F_k(y)+1}{F_k(y)}+\frac{(F_k(y)+1)^2}{F_k'(y)y}\right)\frac{F_k(y)}{1+F_k(y)}=\frac{yF_k(y)F_k'(y)+(1+F_k(y))F_k(y)}{yF_k'(y)}-1. \tag{2.3}$$

Using the Taylor expansion $F_k(y)=y+O(y^2)$, we find that the limiting value of the right hand side of (2.3) at $y=0$ is exactly 0, so we need only prove that

$$\varphi(y):=\frac{yF_k(y)F_k'(y)+(1+F_k(y))F_k(y)}{yF_k'(y)}$$

is an increasing function on $(0,1)$.

Consider the two series defined by

$$a(y)=\sum_{n=0}^{\infty}a_ny^n:=\left(F_k(y)F_k'(y)+(1+F_k(y))\frac{F_k(y)}{y}\right)(1-y)^2,$$

$$b(y)=\sum_{n=0}^{\infty}b_ny^n:=F_k'(y)(1-y)^2,$$

so that $\varphi(y) = \frac{a(y)}{b(y)}$. We will show that this rational decomposition of $\varphi(y)$ satisfies the requirements of Lemma 2.2 part (iii), thus proving the required monotonicity.

By definition,

$$F_k(y)(1-y) = \sum_{i=1}^{k} y^i, \tag{2.4}$$

and thus the second term of $a(y)$ can be expanded as

$$(1 + F_k(y))\frac{F_k(y)}{y}(1-y)^2 = \left(1 + \sum_{i=2}^{k} y^i\right)\sum_{i=0}^{k-1} y^i. \tag{2.5}$$

Differentiating (2.4) we obtain

$$F_k'(y)(1-y) - F_k(y) = \sum_{i=0}^{k-1}(i+1)y^i, \tag{2.6}$$

which implies the following expansion for the first term of $a(y)$:

$$F_k(y)F_k'(y)(1-y)^2 = \left(\sum_{i=1}^{k} y^i\right)\left(\sum_{i=0}^{k-1}(i+1)y^i + F_k(y)\right). \tag{2.7}$$

Note that excluding the constant term (which is zero), all of the Taylor coefficients of $F_k(y)F_k'(y)(1-y)^2$ are strictly positive. Furthermore, all of the Taylor coefficients of $(1 + F_k(y))\frac{F_k(y)}{y}(1-y)^2$ are non-negative, with constant term 1. Adding (2.5) and (2.7) then shows that $a_n > 0$ for all n.

We next explicitly determine a_n for $0 \le n \le k-1$, using the fact that $F_k(y) = \frac{y}{(1-y)^2} + O(y^{k+1})$. Therefore $F_k'(y) = \frac{1}{(1-y)^2} + \frac{2y}{(1-y)^3} + O(y^k)$, and hence

$$\begin{aligned}\left(F_k(y)F_k'(y) + (1 + F_k(y))\frac{F_k(y)}{y}\right)(1-y)^2 &= \frac{y}{(1-y)^2} + \frac{2y^2}{(1-y)^3} + 1 + \frac{y}{(1-y)^2} + O(y^k)\\ &= 1 + \frac{2y}{(1-y)^3} + O(y^k) = 1 + \sum_{i=1}^{k-1} i(i+1)y^i + O(y^k).\end{aligned}$$

We have thus computed that $a_0 = 1$ and $a_n = n(n+1)$ for $1 \le n \le k-1$.

To compute b_n we use (2.6) to write $b(y)$ as

$$F_k'(y)(1-y)^2 = F_k(y)(1-y) + (1-y)\sum_{i=0}^{k-1}(i+1)y^i = \sum_{i=1}^{k} y^i + \sum_{i=0}^{k-1} y^i - ky^k = 1 + 2\sum_{i=1}^{k-1} y^i - (k-1)y^k.$$

We see immediately that all b_n' are positive. Moreover $1 = \frac{a_0}{b_0} = \frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_{k-1}}{b_{k-1}}$, and Lemma 2.2 part (i) then implies that $\frac{a_0'}{b_0'} \le \frac{a_1'}{b_1'} \le \cdots \le \frac{a_{k-1}'}{b_{k-1}'}$. Since $a_n > 0$ for all n, we have $a_{k-1}' < a_k' < \dots$, but $b_n = 0$ for $n > k$, so $b_{k-1}' > b_k' = b_{k+1}' = \dots$. Thus $\frac{a_{k-1}'}{b_{k-1}'} < \frac{a_k'}{b_k'} < \dots$ and all conditions of Lemma 2.2 part (iii) are fulfilled. □

3. Column/row sequence characterization of growth

In this section we define column and row events that can be used to describe simple necessary and sufficient conditions for a lattice to become Active.

Definition 3.1. Suppose that $\mathcal{M}$ is a finite (partial) row or column of length n in the square directed lattice, so that for some $r \in \mathbb{Z}$,

$$\mathcal{M} = \{(x, v) \,:\, r \le v \le r + n - 1\} \qquad \text{or} \qquad \mathcal{M} = \{(v, y) \,:\, r \le v \le r + n - 1\}.$$

All possible configurations for $\mathcal{M}$ are then split into the following four disjoint events:

A : $\mathcal{M}$ contains two (possibly equal) sites x and y such that $\boldsymbol{\mathcal{S}}(x) \ge \rightarrow$ and $\boldsymbol{\mathcal{S}}(y) \ge \uparrow$.
B : $\mathcal{M}$ contains x such that $\boldsymbol{\mathcal{S}}(x) = \uparrow$, and no state of $\mathcal{M}$ satisfies $\boldsymbol{\mathcal{S}}(y) \ge \rightarrow$.
C : $\mathcal{M}$ contains x such that $\boldsymbol{\mathcal{S}}(x) = \rightarrow$, and no state of $\mathcal{M}$ satisfies $\boldsymbol{\mathcal{S}}(y) \ge \uparrow$.
D : Every site in $\mathcal{M}$ satisfies $\boldsymbol{\mathcal{S}}(x) = \square$.

These four possible states for $\mathcal{M}$ are denoted by $\boldsymbol{\mathcal{S}}(\mathcal{M})$.

Remark 3.2. Recalling (1.1) and (1.2), the corresponding probabilities for the four column/row events of length n are

$$\mathbf{P}(A) = (1 - q^n)^2, \quad \mathbf{P}(B) = \mathbf{P}(C) = q^n(1 - q^n), \quad \mathbf{P}(D) = q^{2n}.$$

We now consider sequences of (typically adjacent) columns or rows, and the following definition helps characterize those which have "good" growth properties relative to the k-convolution process.

Definition 3.3. Suppose that $\{\mathcal{M}_j\}_{j=1}^m$ is a sequence of finite columns or rows.

- The *row k-sequence condition* is satisfied if $\{\mathcal{M}_j\}$ contains no consecutive subsequence with state sequences $D, BC,$ or B^k.
- The *column k-sequence condition* is satisfied if $\{\mathcal{M}_j\}$ contains no consecutive subsequence with state sequences $D, CB,$ or C^k.

Remark 3.4. An alternative characterization of the row k-sequence condition is that the states contain *only* subsequences of the form $A, C,$ or $B^j A$ for $1 \le j \le k - 1$ (the column k-sequence condition is analogous, with B and C events interchanged).

The next result shows that the row and column conditions described above are a necessary condition that any subconfiguration on a rectangular sublattice must satisfy in order for the process to conclude with Active sites.

Proposition 3.5. *Suppose that a rectangular lattice region of dimensions (m, n) becomes active in the k-convolution percolation process. Then the rows $\{\mathcal{R}_j\}_{j=1}^m$ (indexed bottom-to-top) and columns $\{\mathcal{C}_j\}_{j=1}^n$ (indexed left-to-right) satisfy the row and column k-sequence conditions, respectively.*

Proof. We present the proof for the column conditions only; the row conditions are entirely analogous, with the role of B and C events interchanged. The argument is by contradiction, so suppose that $\{\mathcal{C}_j\}_{j=1}^n$ does not satisfy the column k-sequence condition. Suppose that the first disallowed pattern occurs at position m, so that for some j, $\mathcal{C}_m, \ldots, \mathcal{C}_{m+j}$ has the state pattern $D, CB,$ or C^k. We may now use the monotonicity of the percolation process and assume without loss of generality that $\mathcal{C}_{m-k+1}, \ldots, \mathcal{C}_{m-1}$ are completely filled with Active sites, adding additional columns with nonpositive indices if $m < k$.

We first consider the case that $\boldsymbol{\mathcal{S}}(\mathcal{C}_m) = D$, so that $\boldsymbol{\mathcal{S}}(x) = \square$ for each site x in the m-th column. It is clear that none of the growth rules from Definition 1.1 can lead to an increase in the state of x. See the first image in Figure 5.

Next, consider the case that $\boldsymbol{\mathcal{S}}(\mathcal{C}_m) = C$ and $\boldsymbol{\mathcal{S}}(\mathcal{C}_{m+1}) = B$. Then Definition 1.1 (i) allows us to assign each site in $\mathcal{C}_m$ to state $\rightarrow$. However, note that in order to increase the states in $\mathcal{C}_{m+1}$, parts (ii) and (iii) of Definition 1.1 require additional $\rightarrow$ states in $\mathcal{C}_{m+1}$, which are precluded by the assumption that $\boldsymbol{\mathcal{S}}(\mathcal{C}_{m+1}) = B$. This case is depicted in the second image of Figure 5.

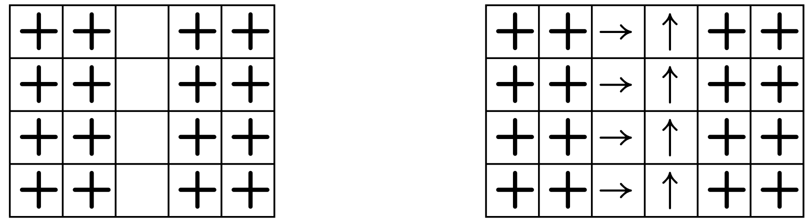

FIGURE 5. Maximal configurations with the column subsequences D and CB.

Finally, if $\boldsymbol{\mathcal{S}}(\mathcal{C}_{m+j}) = C$ for $0 \leq j \leq k-1$, then Definition 1.1 (i) implies that each of these columns can be increased to the state $\rightarrow$, as seen in Figure 6. However, there is no further growth possible, as the gap is then too large for Definition 1.1 (iii). This completes the proof, as the rectangle does not become Active in the case of any of the restricted sequences.

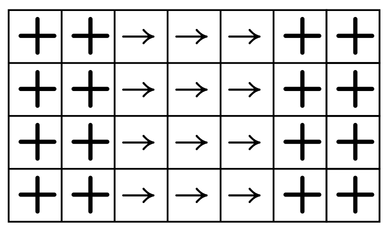

FIGURE 6. Maximal configuration with the column subsequence C^k.

□

Next, we show that the column/row conditions also define a sufficient growth condition for certain diagonal configurations that originate from the origin in the lower-left corner.

Definition 3.6. Consider the "stair-step" columns and rows defined such that for $i \geq k+1$, the column $\mathcal{C}_i$ at a distance i to the right of the origin has height $i-k$, and the row $\mathcal{R}_i$ at height i also has width $i-k$. If $b > a \geq k$, then the *diagonal growth event* $\overline{\mathcal{D}}_k(a,b)$ is the event that the columns $\{\mathcal{C}_{a+1}, \ldots, \mathcal{C}_b\}$ and rows $\{\mathcal{R}_{a+1}, \ldots, \mathcal{R}_b\}$ satisfies the k-sequence conditions for the k-convolution percolation process.

Proposition 3.7. *If the lower-left square of side-length a is filled with Active sites and event $\overline{\mathcal{D}}_k(a,b)$ is satisfied, then some rectangle with dimensions $(b-s, b-t)$ for $0 \leq s,t, \leq k-1$ is eventually filled with Active sites. Furthermore, the final s columns will have state $\rightarrow$ in each cell (x,y) with $a-s+1 \leq x \leq a$ and $1 \leq y \leq a-t$. Similarly, cells (x,y) with $1 \leq x \leq a-s$ and $a-t+1 \leq y \leq a$ will have state $\uparrow$.*

Proof. Following Remark 3.4, we need consider only row/column sequences of the prescribed form. Observe that if columns $a+1, \ldots, a+j$ (of height a) and rows $a+1, \ldots, a+\ell$ (of width a) eventually become Active, then by Definition 1.1 (ii), the lower-left square of length $a + \min\{j, \ell\}$ will become Active. Thus it is sufficient to prove that the permissible column and row events independently lead to extended column and row growth.

Without loss of generality, consider the column states and the left-to-right growth of the percolation process. If $\boldsymbol{\mathcal{S}}(\mathcal{C}_{a+1}) = A$ or B, then Definition 1.1 (ii) directly implies that the column becomes filled with Active sites.

The other possibility is that the initial column states satisfy $\boldsymbol{\mathcal{S}}(\mathcal{C}_{a+j}) = C$ for $1 \leq j \leq m \leq k-1$, with $\boldsymbol{\mathcal{S}}(\mathcal{C}_{a+m+1}) = A$. In this case, the evolution first proceeds according to Definition 1.1 (i), which

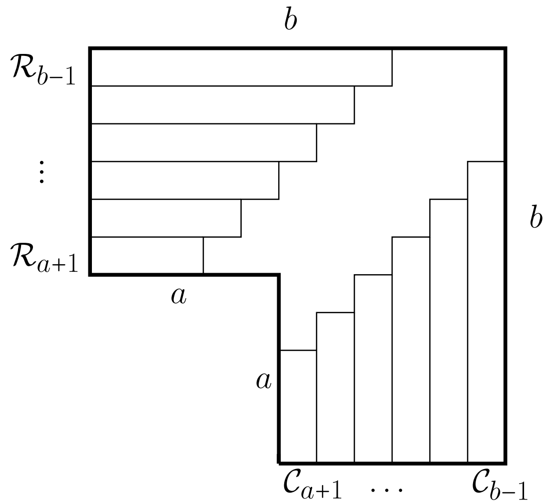

FIGURE 7. The "stair-step" columns and rows defined for the event $\overline{\mathcal{D}}_k(a,b)$ for $k = 3$.

implies that the columns $\mathcal{C}_{a+j}$ for $1 \leq j \leq m$ are completely filled with the state $\rightarrow$. The same rule also then applies to the sites of column $\mathcal{C}_{a+m+1}$, which are therefore set to states $\boldsymbol{\mathcal{S}}(x) \geq \rightarrow$, with strict inequality for at least one $x \in \mathcal{C}_{a+m+1}$; strictly speaking, the columns here are extended to height a. Next, Definition 1.1 (iii) then implies that for any state $x \in \mathcal{C}_{a+m+1}$ with $\boldsymbol{\mathcal{S}}(x) = \boldsymbol{+}$, the sites directly to the left of x will also be set to the state $\boldsymbol{+}$. Finally, Definition 1.1 (ii) then fills in the rest of the columns $\mathcal{C}_{a+j}$ with Active sites, until the final column with states A or B is reached. Beyond this point, there remain at most $k-1$ columns with state C; Definition 1.1 (i) then fills their cells entirely with $\rightarrow$ states. $\square$

Remark 3.8. The above arguments show that a near-converse to Proposition 3.5 also holds: if $\boldsymbol{\mathcal{S}}(\mathcal{C}_1) = \dots \boldsymbol{\mathcal{S}}(\mathcal{C}_{k-1}) = A$ and $\{\mathcal{C}_k, \dots, \mathcal{C}_n\}$ satisfy the k-column sequence conditions, then the process will evolve such that every column $\mathcal{C}_j$ is completely Active for $k \leq j \leq n-s$, where $0 \leq s \leq k-1$, and $\mathcal{C}_j$ is completely filled with $\rightarrow$ states for $n-s+1 \leq j \leq n$. The analogous statement holds for row sequences.

Thus the sequences of column/row events in a configuration give a natural description of both necessary and sufficient conditions that the configuration eventually become Active. The next section addresses such probabilistic sequence events in more generality, and we return to the specific setting of k-convolution percolation in Section 5.

4. BOUNDS FOR THE LIMITING PROBABILITIES OF COLUMN/ROW SEQUENCES

In this section we present a new method for proving double-sided bounds for the probabilities of stochastic sequences, which uses monotonicity properties of the eigenvalues and eigenvectors for the associated Markov-type processes. In particular, we apply this method to probability sequences satisfying the column k-sequence conditions from Definition 3.3 (the row k-sequence conditions are equivalent following a simple renaming), and prove Theorem 1.5. The probability bounds in this section are then a key part of the proof of Theorem 1.3. We note that some of our technical results also follow from the theory of stochastic matrices [15] (for example, the uniqueness of the largest, real eigenvalue is implied by the Perron-Frobenius theorem). However, we prefer the present approach, as our situation is also rather special due to the nature of the characteristic polynomial, which is in turn related to the defining functional equations for our $\overline{f}_k$. As previously mentioned in Section 2, this algebraic equation can be re-scaled so that it is linear in the inverse function, and we use this property extensively as we prove the bounds.

4.1. **Definitions and statement of probability bound.** Consider a sequence of independent random variables $\{E_j\}_{j=1}^{\infty}$ such that

$$\mathbf{P}(E_j = A) = u_j^2, \quad \mathbf{P}(E_j = B) = \mathbf{P}(E_j = C) = u_j(1-u_j), \quad \mathbf{P}(E_j = D) = (1-u_j)^2.$$

Next, define notation for the probability of the events that satisfy k-sequence conditions, namely

$$\overline{\rho}_k(m,n) := \mathbf{P}\Big(\{E_j\}_{j=m}^{n} \text{ has no } D, CB, C^k\Big),$$

for $1 \leq m \leq n$. Note that a sequence that ends with at most $k-1$ Cs is allowed.

The following recurrence relation is easily verified by conditioning on the number of C events that occur at the beginning of the sequence. For $n \geq m+k-1$ we have

$$\overline{\rho}_k(m,n) = \underbrace{u_m}_{E_m = A \text{ or } B} \overline{\rho}_k(m+1,n) + \sum_{j=2}^{k} \underbrace{u_m(1-u_m)\cdots u_{m+j-2}(1-u_{m+j-2})}_{E_m=\cdots=E_{m+j-2}=C} \underbrace{u_{m+j-1}^2}_{E_{m+j-1}=A} \overline{\rho}_k(m+j,n). \tag{4.1}$$

The initial conditions for the recurrence are defined by recalling the "extendable" property of the sequences. The values are most easily described by first setting $\overline{\rho}_k(n+1,n) := 1$, and then using a modified, truncated recurrence, which states that for $n \leq m+k-2$,

$$\begin{aligned}\overline{\rho}_k(m,n) = u_m\overline{\rho}_k(m+1,n) &+ \sum_{j=2}^{n-m+1} u_m(1-u_m)\cdots u_{m+j-2}(1-u_{m+j-2})u_{m+j-1}^2\overline{\rho}_k(m+j,n) \\ &+ \underbrace{u_m(1-u_m)\cdots u_n(1-u_n)}_{E_m=\cdots=E_n=C}.\end{aligned} \tag{4.2}$$

The first term and summation record similar events as in (4.1), and the final term is due to the fact that the sequence is allowed to end with a truncated string of at most $k-1$ Cs.

The *limiting density* for these probabilities is found by setting all $u_j := u$ in (4.1), and replacing each $\overline{\rho}_k(m,n)$ by $\overline{f}_k^{\,m-n+1}$. This immediately gives the functional equation (1.4), and also leads to a simple inductive proof of Theorem 1.5 in the case that all u_j are equal. Indeed, this is nothing more than the standard approach for finding the limiting distribution of a Markov chain with fixed probabilities [19].

4.2. **A "Markov-type" stochastic process with variable transition probabilities.** In order to address the general case of (monotonic) unequal u_js, we modify the simple Markov process that naturally corresponds to the case where all $u_j = u$. The states of this Markov process encode the number of consecutive C events at the end of the sequence of random variables, and the transition probabilities are polynomial functions of u. Adapting these ideas to the present case, we refine the probabilities $\overline{\rho}_k(m,n)$ by also encoding the number of C events at the end of the sequence. In particular, for $0 \leq a \leq k-1$, define

$$\overline{\rho}_k(a;m,n) := \mathbf{P}\Big(\{E_j\}_{j=m}^{n} \text{ has no } D, CB, C^k \text{ and ends with exactly } a \ C\text{s}\Big).$$

This last condition means that $E_{n-a} \neq C$ and $E_{n-a+1} = \cdots = E_n = C$.

Heuristically, we are now considering a Markov-type process in which the transition probabilities after j steps are a function of u_j. This means that we can easily describe recurrence relations between the $\overline{\rho}_k(a;m,n)$, namely

$$\overline{\rho}_k(a;m,n+1) = \begin{cases} u_{n+1}^2 \sum_{j=0}^{k-1} \overline{\rho}_k(j;m,n) + u_{n+1}(1-u_{n+1})\overline{\rho}_k(0;m,n) & \text{if } a = 0, \\ u_{n+1}(1-u_{n+1})\overline{\rho}_k(a-1;m,n) & \text{if } 1 \leq a \leq k-1. \end{cases} \tag{4.3}$$

In the case $a = 0$, the sequence ends with exactly zero Cs, which means that the final event is A or B. If $E_{n+1} = A$, it may be preceded by any number of Cs up to length $k-1$, which accounts for the first term in the top formula above, whereas if $E_{n+1} = B$, it must be preceded by an A or B, which is the second term. The cases $1 \leq a \leq k-1$ are simpler, because if $\{E_j\}_m^{n+1}$ ends with exactly a Cs, then $E_{n+1} = C$ and $\{E_j\}_m^n$ ends with exactly $a-1$ Cs.

We now use these relations in order to rewrite our probabilities $\overline{\rho}_k(m,n)$ in terms of linear transformations. For the remainder of the section, we will frequently suppress the subscript k for notational convenience. Define the vectors

$$\overline{P}_k(m,n) = \overline{P}(m,n) := \begin{pmatrix} \overline{\rho}(0;m,n) & \overline{\rho}(1;m,n) & \cdots & \overline{\rho}(k-1;m,n) \end{pmatrix}^T.$$

Observe that the probabilities can then be recovered as

$$\overline{\rho}_k(m,n) = \left\| \overline{P}(m,n) \right\|_1 = \mathbf{s}^T \overline{P}(m,n),$$

where $\mathbf{s} = \mathbf{s}_k := \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}^T$ is a k-dimensional vector. Furthermore, the relations (4.3) are easily rewritten as

$$\overline{P}(m,n) = \overline{M}(u_n)\overline{P}(m,n-1),$$

where the transition matrices are given by

$$\overline{M}(u) = \overline{M}_k(u) := \begin{pmatrix} u & u^2 & \cdots & \cdots & u^2 \\ u(1-u) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u(1-u) & 0 \end{pmatrix}.$$

Finally, the initial conditions described in (4.2) are then seen to be equivalent to setting $\overline{P}(n+1,n) = \mathbf{e}$, where

$$\mathbf{e} = \mathbf{e}_k := \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T.$$

This then means that

$$\begin{aligned} \overline{\rho}(m,n) &= \mathbf{s}^T \overline{P}(m,n) = \mathbf{s}^T \overline{M}(u_n)\overline{P}(m,n-1) = \cdots = \mathbf{s}^T \overline{M}(u_n)\cdots\overline{M}(u_m)\overline{P}(m,m-1) \\ &= \mathbf{s}^T\, \overline{M}(u_n)\cdots\overline{M}(u_m)\,\mathbf{e} = \mathbf{e}^T\, \overline{M}(u_m)^T\cdots\overline{M}(u_n)^T\,\mathbf{s}; \end{aligned} \tag{4.4}$$

the final equality follows because the expression is a scalar overall, and is thus its own transpose.

It is well known from the theory of vector spaces over a field $\mathbb{F}$ that a matrix of the form

$$M = \begin{pmatrix} \alpha_{k-1} & 1 & 0 & \cdots & 0 \\ \alpha_{k-2} & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ \alpha_0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

corresponds to the linear action given by multiplication by x on the vector space

$$V = \left\langle x^{k-1}, \dots, x, 1 \right\rangle \cong \mathbb{F}[x]/p(x),$$

where $p(x) := x^k - \alpha_{k-1}x^{k-1} - \cdots - \alpha_0$. Furthermore, the characteristic polynomial of this transformation is also equal to $p(x)$, and the eigenvector corresponding to an eigenvalue λ is simply the vector whose coordinates are the coefficients of $p(x)/(x-\lambda)$. Throughout this section we also index vectors in this linear space with respect to the corresponding polynomial degrees, writing $\mathbf{v} = \begin{pmatrix} v_{k-1} & \cdots & v_0 \end{pmatrix}^T$.

Returning to the situation at hand, consider the scaled transformation

$$\frac{1}{u(1-u)}\overline{M}(u)^T = \begin{pmatrix} \frac{1}{1-u} & 1 & 0 & \cdots & 0 \\ \frac{u}{1-u} & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ \frac{u}{1-u} & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

which then has the characteristic polynomial

$$\overline{Q}(u;x) = \overline{Q}_k(u;x) := x^k - \frac{1}{1-u}x^{k-1} - \frac{u}{1-u}\left(x^{k-2} + \cdots + 1\right). \tag{4.5}$$

Now let $\overline{\lambda}(u) = \overline{\lambda}_k(u)$ denote the positive real root of $\overline{Q}(u;x)$; a unique such root exists by Descartes' Rule of Signs and the fact that the constant term of $\overline{Q}(u;x)$ is negative (as $u > 0$). Note that in fact

$$\overline{\lambda}_k(u) = \frac{\overline{f}_k(u)}{u(1-u)}, \tag{4.6}$$

as the characteristic polynomial of $\overline{M}_k(u)^T$ is easily seen to be

$$u^k(1-u)^k\overline{Q}_k\left(u; \frac{x}{u(1-u)}\right) = x^k - ux^{k-1} - u^3(1-u)x^{k-2} - \cdots - u^{k+1}(1-u)^{k-1},$$

which can be compared with (1.4). We continue working with the normalized matrix as it will simplify various computations.

Proposition 4.1. *The eigenvalue $\overline{\lambda}(u)$ is a strictly increasing function of u, with boundary values $\overline{\lambda}(0) = 1$ and $\overline{\lambda}(1) = \infty$.*

Proof. By definition, the eigenvalue satisfies

$$\lambda(u)^k - \frac{1}{1-u}\lambda(u)^{k-1} - \frac{u}{1-u}\left(\lambda(u)^{k-2} + \cdots + 1\right) = 0. \tag{4.7}$$

Differentiating this equation with respect to u gives

$$\frac{\partial}{\partial x}\overline{Q}(u;x)\Big|_{x=\lambda} \cdot \frac{d\lambda}{du} - \frac{1}{(1-u)^2}\left(\lambda^{k-1} + \cdots + 1\right) = 0.$$

From the definition of $\lambda(u)$ as the unique (and therefore largest) positive root of $\overline{Q}(u;x)$, the derivative $\frac{\partial}{\partial x}\overline{Q}(u;x)$ is positive at $x = \lambda(u)$, and thus we find that $\frac{d\lambda}{du}$ is always positive.

For the boundary values, it is clear that the positive real root of $\overline{Q}(0;x) = x^k - x^{k-1}$ is $\overline{\lambda}(0) = 1$. It is also clear that as $u \to 1^-$,

$$\lambda(u)^k \geq \frac{u}{1-u} \to \infty.$$

$\square$

4.3. **Proof of lower bound for Theorem 1.5.** In order to prove the lower probability bound, we work with the scaled transpose matrices $\frac{1}{u(1-u)}\overline{M}_k(u)^T$. We begin by calculating the eigenvectors associated to $\overline{\lambda}(u)$ and will next prove monotonicity properties about their entries. Using the polynomial vector space basis, the eigenvectors are

$$\overline{\Lambda}(u) = \overline{\Lambda}_k(u) = x^{k-1} + \sum_{j=0}^{k-2} \overline{a}_j(u)x^j := \frac{\overline{Q}_k(u)}{x - \overline{\lambda}_k(u)}. \tag{4.8}$$

Comparing with (4.5), we find that

$$x^k - \frac{1}{1-u}x^{k-1} - \frac{u}{1-u}\left(x^{k-2} + \cdots + 1\right) = x^k - \overline{\lambda}(u)x^{k-1} + \sum_{j=0}^{k-2} \overline{a}_j(u)x^j\Big(x - \overline{\lambda}(u)\Big).$$

Comparing like powers of x gives the recurrences $\overline{a}_j = \overline{\lambda}^{-1}\left(\overline{a}_{j-1} + \frac{u}{1-u}\right)$ for $1 \le j \le k-2$, which leads to the formulas

$$\overline{a}_j = \frac{u}{1-u}\left(\overline{\lambda}^{-1} + \cdots + \overline{\lambda}^{-(j+1)}\right) \tag{4.9}$$

for $0 \le j \le k-2$. We also have $\overline{a}_{k-1}(u) \equiv 1$.

Proposition 4.2. *For $0 \le j \le k-2$, the coefficients $\overline{a}_j(u)$ are increasing functions of u. Furthermore, $\overline{a}_j(0) = 0$ and $\overline{a}_j(1) = 1$.*

Proof. By Proposition 4.1, the first statement is equivalent to the claim that $\overline{a}_j(u)$ is an increasing function of $\lambda(u)$. Recall from (4.7) that $\overline{\lambda}(u)$ is defined by an algebraic expression linear in u; solving gives

$$u = \frac{\overline{\lambda}^k - \overline{\lambda}^{k-1}}{\overline{\lambda}^k + \overline{\lambda}^{k-2} + \overline{\lambda}^{k-3} + \cdots + 1}.$$

Combined with (4.9) this implies that

$$\overline{a}_j = \frac{\overline{\lambda}^{k-j-2}\left(\overline{\lambda}^{j+1} - 1\right)\left(\overline{\lambda} - 1\right)}{\overline{\lambda}^k - 1}. \tag{4.10}$$

Write $t := \overline{\lambda}^{-1}$, so that

$$\overline{a}_j = \frac{\left(1 - t^{j+1}\right)(1-t)}{1-t^k}. \tag{4.11}$$

By Proposition 4.1 we have the range $0 \le t \le 1$, so $\overline{a}_j$ is clearly nonnegative. Our task is now to show that the expression in (4.11) is a decreasing function of t on $[0,1]$.

In fact, we observe that $1-t$ is a decreasing function of t, and then also show that $F_{a,b}(t) := \frac{1-t^a}{1-t^b}$ is decreasing in t for any $0 < a < b$. The derivative with respect to t of this rational function is

$$F'_{a,b}(t) = \frac{at^{a-1}(t^b-1) - bt^{b-1}(t^a-1)}{(t^b-1)^2},$$

and thus we need to show that the numerator is negative. This is equivalent to showing that

$$t^b \le \frac{b-a}{b}t^{a+b} + \frac{a}{b}t^a,$$

which follows directly from Jensen's inequality applied to the convex function $z \mapsto t^z$ (with both $t, z > 0$).

For the boundary values, simply evaluate (4.11) at $t = 0$ and $t = 1$ (which correspond to $u = 1$ and $u = 0$, respectively). $\square$

We can now easily complete the proof of the lower bound of Theorem 1.5. We adopt a partial order on vectors given by $\mathbf{v} \le \mathbf{w}$ if and only if $v_j \le w_j$ for each individual component (recall that we denote the vector components by $\mathbf{v} = \begin{pmatrix} v_{k-1} & \cdots & v_0 \end{pmatrix}^T$). Note that this ordering is compatible with (positive) matrix transformations: if all of the coefficients of M are nonnegative, then $\mathbf{v} \le \mathbf{w}$ implies $M\mathbf{v} \le M\mathbf{w}$.

Recalling (4.4), (4.6) and the eigenvectors from (4.8), we have

$$\begin{aligned}\overline{\rho}(1,n) &= \mathbf{e}^T\, \overline{M}(u_1)^T \cdots \overline{M}(u_n)^T\, \mathbf{s} \\ &\ge \mathbf{e}^T\, \overline{M}(u_1)^T \cdots \overline{M}(u_n)^T\, \overline{\Lambda}(u_n) = \overline{f}_k(u_n)\cdot \mathbf{e}^T\, \overline{M}(u_1)^T \cdots \overline{M}(u_{n-1})^T\, \overline{\Lambda}(u_n),\end{aligned} \tag{4.12}$$

where the inequality follows from Proposition 4.2, which gives $\overline{a}_j(u) \le 1$. We now use the assumption that the u_j are increasing. The monotonicity property found in Proposition 4.2 then also implies that $\overline{\Lambda}(u_{j+1}) \ge \overline{\Lambda}(u_j)$ for all j, and we apply this iteratively to (4.12), finding

$$\begin{aligned}\overline{\rho}(1,n) &\ge \overline{f}_k(u_n)\cdot \mathbf{e}^T\, \overline{M}(u_1)^T \cdots \overline{M}(u_{n-1})^T\, \overline{\Lambda}(u_{n-1}) \\ &= \overline{f}_k(u_{n-1})\overline{f}_k(u_n)\cdot \mathbf{e}^T\, \overline{M}(u_1)^T \cdots \overline{M}(u_{n-2})^T\, \overline{\Lambda}(u_{n-1}) \\ &\ge \cdots \ge \overline{f}_k(u_1)\cdots\overline{f}_k(u_n)\cdot \mathbf{e}^T\, \overline{\Lambda}(u_1) = \overline{f}_k(u_1)\cdots\overline{f}_k(u_n).\end{aligned}$$

For the final equality we used the fact that $\mathbf{e}^T\, \overline{\Lambda}(u_1) = \overline{a}_{k-1}(u_1) \equiv 1$.

4.4. **Proof of upper bound for Theorem 1.5.** In this section we will at times vary k, and thus it is much more important that we take care with these subscripts in formulas. In order to study the upper bound, we now work directly with the matrices $\overline{M}_k(u)$. The calculation of the eigenvalues is unchanged from Section 4.3, and we again denote the unique positive eigenvalue of $\frac{1}{u(1-u)}\overline{M}_k(u)$ by $\overline{\lambda}_k(u) = \frac{\overline{f}_k(u)}{u(1-u)}$. However, without the transpose matrix we now no longer have a polynomial ring representation of the vector space. Fortunately, the eigenvectors of $\overline{M}_k(u)$ have a very simple explicit form; in particular, the normalized eigenvector corresponding to the eigenvalue $\overline{f}_k(u)$ is

$$\widetilde{\Lambda}_k(u) := \frac{\begin{pmatrix} \overline{\lambda}_k(u)^{k-1} & \overline{\lambda}_k(u)^{k-2} & \dots & 1 \end{pmatrix}^T}{\overline{\lambda}_k(u)^{k-1} + \overline{\lambda}_k(u)^{k-2} + \cdots + 1}.$$

For the proof of the upper bound, we use a different partial ordering on vectors; this order is simply a continuous version of the natural "dominance order" for partitions from the theory of representations of finite symmetric groups [23].

Definition 4.3. For vectors in $\mathbb{R}^k$,

$$\mathbf{v} \preceq \mathbf{w} \quad \text{if and only if} \quad \mathbf{s}_k(i)^T\mathbf{v} \le \mathbf{s}_k(i)^T\mathbf{w} \text{ for all } 1 \le i \le k,$$

where $\mathbf{s}_k(i) := \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}^T$ denotes the length k vector with exactly i 1s.

We state several simple properties of the matrices $\overline{M}_k(u)$ that describe the compatibility of the partial ordering with vector and sequence inequalities (recall once more that we index vectors as $\mathbf{v} = \begin{pmatrix} v_{k-1} & \cdots & v_0 \end{pmatrix}^T$).

Proposition 4.4. *Suppose that $\mathbf{v}, \mathbf{w} \in \mathbb{R}^k_{\ge 0}$ and $u \in [0,1]$.*

(*i*) *The vector norm satisfies $\left\|\overline{M}_k(u)\mathbf{v}\right\|_1 \le \left\|\mathbf{v}\right\|_1$.*
(*ii*) *If $\mathbf{v} \preceq \mathbf{w}$, then $\overline{M}_k(u)\mathbf{v} \preceq \overline{M}_k(u)\mathbf{w}$.*
(*iii*) *If $v_{k-1} \ge \cdots \ge v_0$, then $v'_{k-1} \ge \cdots \ge v'_0$ as well, where $\mathbf{v}' := \overline{M}_k(u)\mathbf{v}$.*
(*iv*) *If $v_{k-1} \ge \cdots \ge v_0$ and $u \le u'$, then $\overline{M}_k(u)\mathbf{v} \preceq \overline{M}_k(u')\mathbf{v}$.*

(v) *If $u \le u'$, then $\widetilde{\Lambda}_k(u) \le \widetilde{\Lambda}_k(u')$.*

Proof. (i) Direct computation shows

$$\left\|\overline{M}_k(u)\mathbf{v}\right\|_1 = \left(2u-u^2\right)v_{k-1} + u\left(v_{k-2}+\cdots+v_1\right) + u^2 v_0 \le v_{k-1}+\cdots+v_0 = \left\|\mathbf{v}\right\|_1.$$

(ii) For any $1 \le i \le k$, a simple computation shows that

$$\mathbf{s}_k(i)^T\overline{M}_k(u) = u(1-u)\,\mathbf{s}_k(1)^T + u\,\mathbf{s}_k(i)^T,$$

and thus Definition 4.3 implies the conclusion.

(iii) This follows from the computation

$$\overline{M}_k(u)\mathbf{v} = \begin{pmatrix} uv_{k-1} + u^2\left(v_{k-2}+\cdots+v_0\right) \\ u(1-u)v_{k-1} \\ \vdots \\ u(1-u)v_1 \end{pmatrix}, \tag{4.13}$$

combined with the observation that $u \ge u(1-u)$.

(iv) Using (4.13), we calculate

$$\mathbf{s}_k(i)^T\overline{M}_k(u)\mathbf{v} = \begin{cases} uv_{k-1} + u^2\left(v_{k-2}+\cdots+v_0\right) & \text{if } i=1; \\ \left(2u-u^2\right)v_{k-1} + u\left(v_{k-2}+\cdots+v_{k-i+1}\right) + u^2\left(v_{k-i}+\cdots+v_0\right) & \text{if } 2 \le i \le k. \end{cases}$$

The proof is completed by noting that all terms are increasing functions of u.

(v) The statement is equivalent to showing that $\mathbf{s}_k(i)^T\widetilde{\Lambda}_k(u)$ is an increasing function of u for each i. Explicitly, this expression is

$$\mathbf{s}_k(i)^T\widetilde{\Lambda}_k(u) = \frac{\overline{\lambda}_k(u)^{k-1}+\cdots+\overline{\lambda}_k(u)^{k-i}}{\overline{\lambda}_k(u)^{k-1}+\cdots+1} = \frac{\overline{\lambda}_k(u)^{k-i}\left(\overline{\lambda}_k(u)^i-1\right)}{\overline{\lambda}_k(u)^k-1}.$$

Recalling (4.10) and the proof of Proposition 4.2, as well as Proposition 4.1, we find that we have already proven that this expression is increasing in u.

□

We will also need an additional monotonicity property of the eigenvalues $\overline{\lambda}_k(u)$, as we will shortly compare their values for different k in an inductive argument.

Proposition 4.5. *For fixed u, $\overline{\lambda}_{k-1}(u) \le \overline{\lambda}_k(u)$.*

Proof. By (4.5),

$$\overline{Q}_k(u;x) = x\overline{Q}_{k-1}(u;x) - \frac{u}{1-u}.$$

Therefore, using the definition of $\lambda_{k-1}(u)$ as the positive root of $\overline{Q}_{k-1}(u;x)$, we find that

$$\overline{Q}_k(u;\lambda_{k-1}(u)) = -\frac{u}{1-u}.$$

But $\overline{Q}_k(u;x) > 0$ for any $x > \lambda_k(u)$, and thus the contrapositive proves the claimed statement. □

We require one more preliminary result before proving the upper bound.

Proposition 4.6. *For all $k \ge 1$,*

$$\overline{M}_k(u)^{k-1}\mathbf{e}_k \le \widetilde{\Lambda}_k(u).$$

Proof. The proof proceeds by induction on k, and the base case $k = 1$ is immediate. For general k, it is clear due to the shape of $\overline{M}_k$ that

$$\overline{M}_k(u)^{k-1}\mathbf{e}_k = \overline{M}_k(u) \cdot \overline{M}_k(u)^{k-2}\mathbf{e}_k = \overline{M}_k(u)\left(\left(\overline{M}_{k-1}(u)^{k-2}\mathbf{e}_{k-1}\right)^T \quad 0\right)^T. \tag{4.14}$$

By induction, this final vector satisfies

$$\left(\left(\overline{M}_{k-1}(u)^{k-2}\mathbf{e}_{k-1}\right)^T \quad 0\right)^T \le \frac{\left(\overline{\lambda}_{k-1}(u)^{k-2} \quad \dots \quad 1 \quad 0\right)^T}{\overline{\lambda}_{k-1}(u)^{k-2} + \dots + 1}$$

Plugging this in to (4.14) and expanding gives

$$\begin{aligned}\overline{M}_k(u)^{k-1}\mathbf{e}_k &\le \overline{M}_k(u)\frac{\left(\overline{\lambda}_{k-1}(u)^{k-2} \quad \dots \quad 1 \quad 0\right)^T}{\overline{\lambda}_{k-1}(u)^{k-2} + \dots + 1} \\ &= \frac{u(1-u)\cdot\left(\overline{\lambda}_{k-1}(u)^{k-1} \quad \overline{\lambda}_{k-1}(u)^{k-2} \quad \cdots \quad 1\right)}{\overline{\lambda}_{k-1}(u)^{k-2} + \dots + 1};\end{aligned}$$

the equality follows by recalling the definition of $\lambda_{k-1}(u)$.

We now claim that

$$\frac{u(1-u)}{\overline{\lambda}_{k-1}(u)^{k-2} + \dots + 1} \le \frac{1}{\overline{\lambda}_{k-1}(u)^{k-1} + \dots + 1}. \tag{4.15}$$

We first use the bound

$$\overline{\lambda}_{k-1}(u)^{k-1} \le \frac{1}{1-u}\left(\overline{\lambda}_{k-1}(u)^{k-2} + \dots + 1\right),$$

which is immediately implied by the fact that $\overline{\lambda}_k(u)$ is a root of (4.5). This then reduces (4.15) to the simple inequality

$$u \le \frac{1}{2-u},$$

which holds for $u \in [0,1]$. The proof is now finished by applying Proposition 4.5. □

With all of these technical results, the upper bound of Theorem 1.5 is now a simple calculation. Recall (4.4) and observe that if $n \le k$, then the statement is trivial, so we henceforth assume that $n > k$. We implicitly use Proposition 4.4 (i) in all of the following calculations. First, by Proposition 4.4 parts (iii) and (iv),

$$\begin{aligned}\overline{\rho}_k(1,n) &= \mathbf{s}_k^T\, \overline{M}_k(u_n)\cdots\overline{M}_k(u_1)\,\mathbf{e}_k \\ &\le \mathbf{s}_k^T\, \overline{M}_k(u_n)\cdots\overline{M}_k(u_k)\overline{M}_k(u_{k-1})^{k-1}\,\mathbf{e}_k.\end{aligned}$$

Proposition 4.6 then implies

$$\overline{\rho}_k(1,n) \le \mathbf{s}_k^T\, \overline{M}_k(u_n)\cdots\overline{M}_k(u_k)\,\widetilde{\Lambda}_k(u_{k-1}).$$

We conclude the proof by successively using the definition of the $\widetilde{\Lambda}_k$ as the eigenvectors of the $\overline{M}_k$, as well as Proposition 4.4 (v) to obtain the bounds

$$\begin{aligned}\overline{\rho}_k(1,n) &\le \mathbf{s}_k^T\, \overline{M}_k(u_n)\cdots\overline{M}_k(u_k)\,\widetilde{\Lambda}_k(u_k) \\ &= \overline{f}_k(u_k)\cdot\mathbf{s}_k^T\, \overline{M}_k(u_n)\cdots\overline{M}_k(u_{k+1})\,\widetilde{\Lambda}_k(u_{k+1}) \\ &\le \dots \le \overline{f}_k(u_k)\cdots\overline{f}_k(u_n)\cdot\mathbf{s}_k^T\,\widetilde{\Lambda}_k(u_n) \\ &= \overline{f}_k(u_k)\cdots\overline{f}_k(u_n).\end{aligned}$$

The final equality follows from the fact that $\widetilde{\Lambda}_k(u)$ is a normalized eigenvector.

4.5. **Proof of Corollary 1.6.** This proof is analogous to the proofs in Section 3 of [22] and in Section 6 of [11], and we therefore provide only a brief sketch. Since the events from Theorem 1.5 form a decreasing, nested sequence, we may specialize to $u_j = 1 - e^{-js}$ and take the limit as $n \to \infty$, which gives

$$\exp\left(-\sum_{j=1}^{\infty} \overline{g}_k(js)\right) \leq \mathbf{P}\left(\overline{A}_k\right) \leq \exp\left(-\sum_{j=k}^{\infty} \overline{g}_k(js)\right).$$

Note that (2.1) implies that we can rewrite the constants from (1.5) as

$$\overline{\lambda}_k = \int_0^1 \overline{g}_k(z)\, \frac{dz}{z}. \tag{4.16}$$

We next use the Integral Comparison Theorem and the fact that $\overline{g}_k(z)$ is a decreasing, convex function (cf. Theorem 2.3) to obtain

$$\exp\left(-\int\limits_0^{\infty} \log \overline{g}_k(zs)dz\right) \leq \mathbf{P}\left(\overline{A}_k\right) \leq \exp\left(-\frac{\overline{g}_k(ks)}{2} - \int\limits_0^{\infty} \overline{g}_k(zs)dz + \int\limits_0^{k} \overline{g}_k(zs)dz\right).$$

The main exponential term follows from the definition of $\overline{\lambda}_k$ in (4.16), and the error terms in the upper bound follow similarly as in [11], where the only key difference is the asymptotic

$$\overline{g}_k(z) \sim \log z^{-1}$$

as $z \to 0^+$; the corresponding statement in [11] was scaled by a factor of $1/k$.

5. Growth bounds for k-convolution percolation

We now follow the approach of Gravner and Holroyd in [17, 16] (which the first two authors generalized and optimized in [11]) in order to prove Theorem 1.3. We will need to bound the probability that an initial configuration with parameter q on the square of side length L eventually becomes Active under the k-convolution percolation process. Much of the technical machinery is similar to the preceding works, and thus we primarily focus on the new adjustments that are necessary in the current k-convolution models. The differences mainly arise in the combinatorics of growth events, as the multiple states of the present models lead to significantly different properties than those seen in previously studied examples of bootstrap percolation. Most importantly, the concept of "internally spanned" configurations from [2] is no longer valid for the upper bounds in Section 5.2.

5.1. **Lower bound.** In this section we prove the lower bound in Theorem 1.3 by generalizing the combinatorial construction used in [17] and [11]. The general idea is to consider configurations that are *sufficient* for growth and that occur with large enough probability to give a good lower bound.

For a rectangle $R = \{a, \ldots, c\} \times \{b, \ldots, d\}$ in $\mathbb{N}^2$, denote its *dimensions* by

$$\dim(R) := (c - a + 1, d - b + 1).$$

We also let $R(a, b)$ denote a rectangle with dimensions (a, b) whose position may or may not be specified. Moreover, we visualize $(0, 0)$ as the lower-left corner of the quarter-lattice $\mathbb{N}^2$.

Recall the event $\overline{\mathcal{D}}_k(a, b)$ from Definition 3.6, which we now use to characterize "diagonal" growth; this signifies the case where $R(a, a)$ grows to $R(a + 1, a + 1)$, then to $R(a + 2, a + 2)$, and so on until $R(b, b)$ is active (with deviations from the diagonal of at most distance k). Following [11], we also introduce an additional (horizontal) "skew" event, where growth occurs first in the horizontal direction only, and then continues in the vertical direction only.

Definition 5.1. Suppose that $b-a \geq 3$, let $\mathcal{C}_{a+i}$ have height $a+i-k$ for $1 \leq i \leq k$, and let all other $\mathcal{C}_i$ have height $a+1$ $(a+k+1 \leq i \leq b)$. For the rows, let $\mathcal{R}_{a+1}$ have width $a-k+1$, let $\mathcal{R}_{a+2}$ have width $b-1$, and let $\mathcal{R}_i$ have width b for $a+3 \leq i \leq b$. The *(horizontally) skew event* $\overline{\mathcal{J}}_k(a,b)$ is the event that the following occur:

- $\boldsymbol{\mathcal{S}}(\mathcal{R}_{a+1}) = \boldsymbol{\mathcal{S}}(\mathcal{C}_{a+1}) = \boldsymbol{\mathcal{S}}(\mathcal{R}_b) = \boldsymbol{\mathcal{S}}(\mathcal{C}_b) = A$,
- $\boldsymbol{\mathcal{S}}(\mathcal{R}_{a+2}) = D$,
- the cell $x = (b, a+k+1)$ has $\boldsymbol{\mathcal{S}}(x) \geq \rightarrow$,
- $\{\mathcal{C}_{a+2}, \dots, \mathcal{C}_{b-1}\}$ satisfy the k-convolution column conditions, and $\{\mathcal{R}_{a+k+2}, \dots, \mathcal{R}_{b-1}\}$ satisfy the k-convolution row conditions.

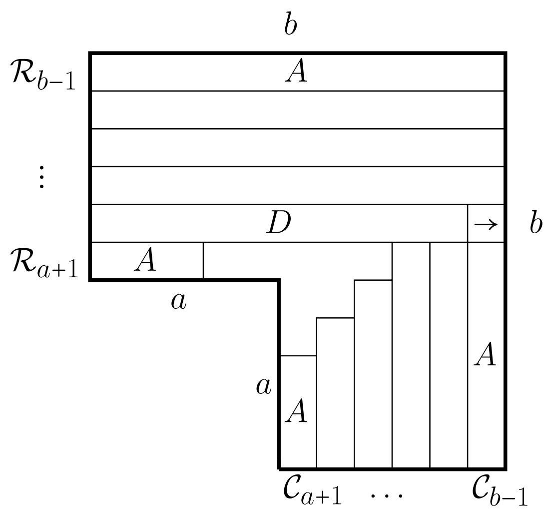

FIGURE 8. The columns and rows defined in the event $\overline{\mathcal{J}}_k(a,b)$ for $k=3$.

Note that since a single empty row is the minimal obstruction to growth in this model, regardless of k, the dimensions of this definition are slightly modified from [11].

The growth rules (1.1) immediately imply that both types of events will lead to further growth; the case of $\overline{\mathcal{D}}_k$ events was already addressed in Proposition 3.7.

Proposition 5.2. *Given a configuration $\mathcal{C}$, consider those rectangles whose lower-left corner is at 0. If $R(a-s, a-t)$ eventually becomes Active for some $0 \leq s,t \leq k-1$, with $\boldsymbol{\mathcal{S}}(\mathcal{C}_j) = C$ for $a-s+1 \leq j \leq a$ and $\boldsymbol{\mathcal{S}}(\mathcal{R}_j) = B$ for $a-t+1 \leq j \leq a$, and $\overline{\mathcal{J}}_k(a,b)$ also occurs, then $R(b,b)$ also becomes Active.*

Proof. Using the monotonicity of the growth process, consider the worst case in the given conditions, which occurs when $s=t=k-1$. The first column of the $\overline{\mathcal{J}}_k$ event has height $a-k+1$ and has $\boldsymbol{\mathcal{S}}(\mathcal{C}_{a+1}) = A$. Since columns $\mathcal{C}_j$ are entirely filled with $+$ sites for $j \leq a-k$, Definition 1.1 (iii) then successively fills each column $\mathcal{C}_{a-k+1}, \dots, \mathcal{C}_a$ with $+$ sites up to height $a-k$. Analogous arguments show that the rows $\mathcal{R}_{a-k+1}, \dots, \mathcal{R}_a$ are similarly filled with Active sites. Definition 1.1 (ii) then also fills $\mathcal{R}_{a+1}$ and $\mathcal{C}_{a+1}$ with Active sites.

Next, use the result described in Remark 3.8 and the fact that $\boldsymbol{\mathcal{S}}(\mathcal{C}_b) = A$ to conclude that all columns become filled with Active sites up to height $a+1$. By minimal assumption, $\boldsymbol{\mathcal{S}}((b, a+2)) \geq \rightarrow$, and thus Definition 1.1 (ii) then fills all of $\mathcal{R}_{a+2}$ with Active sites. Finally, Remark 3.8 now fills in all remaining rows with Active sites. $\square$

Definition 5.3. For $2 \leq a_1 \leq b_1 \leq \dots a_m \leq b_m \leq L$ with $b_i - a_i \geq 3$ for all i, define the *growth event* corresponding to these parameters as

$$\begin{aligned}\overline{E}_k(a_1, b_1, \dots, a_m, b_m) :=& \overline{\mathcal{D}}_k(k, a_1) \cap \bigcap_{i=1}^{m} \overline{\mathcal{J}}_k(a_i, b_i) \cap \bigcap_{i=1}^{m-1} \overline{\mathcal{D}}_k(b_i, a_{i+1}) \cap \overline{\mathcal{D}}_k(b_m, L-1) \\ & \bigcap \{(k \times k) \text{ lower left rectangle and cells } (1, L-1), (L-1, 1) \text{ are Active}\}.\end{aligned}$$

Again following previous work, define a collection of disjoint events which each lead to a square filled completely with Active sites. The proof of the following result is analogous to Lemma 4.5 from [11], relying on Propositions 3.7 and 5.2 from above in order to show that the growth occurs as claimed.

Lemma 5.4. *Suppose that $\{a_i, b_i\}$ satisfy the conditions in Definition 5.3.*

(*i*) *The various events appearing in the definition of single occurrence of $\overline{E}_k(a_1, \dots, b_m)$ are independent.*

(*ii*) *If $\overline{E}_k(a_1, \dots, b_m)$ occurs, then $R(L, L)$ is eventually Active.*

(*iii*) *For different choices of $a_1, \dots, b_m$ the events $\overline{E}_k(a_1, \dots, b_m)$ are disjoint.*

The next result bounds the probabilities of the events $\overline{\mathcal{D}}_k$ and $\overline{\mathcal{J}}_k$ in terms of the function $\overline{g}_k$. The proof is analogous to that of Lemma 3.7 in [11], and is based in part on the fact that the combinatorics of the events $\overline{\mathcal{D}}_k$ and $\overline{\mathcal{J}}_k$ are similar in shape to the events $\mathcal{D}_k$ and $\mathcal{J}_k$ from the previous paper. In particular, in the stated range of dimensions a, b, the probability of the required states in the events $\mathcal{J}_k$ are simply polynomials in q that can be uniformly bounded. The proof also requires the fact that $\overline{g}_k$ is both decreasing and convex (cf. Theorem 2.3).

Lemma 5.5. *The probability of the growth events satisfies the following lower bounds.*

(*i*) *If $b > a \geq k$, then*

$$\mathbf{P}\left(\overline{\mathcal{D}}_k(a, b)\right) \geq \exp\left(-2 \sum_{i=a-(k-1)}^{b-k} \overline{g}_k(is)\right).$$

(*ii*) *Let $c_- < c_+$ be positive constants, $s \in \left(0, \frac{1}{2}\right)$, and $b \geq a + k + 2$, with $a, b \in \left[c_- s^{-1}, c_+ s^{-1}\right]$. Then*

$$\mathbf{P}\left(\overline{\mathcal{J}}_k(a, b)\right) \underset{\text{unif}}{\gg} s \exp\left(\overline{g}_k'(c_-) s (b-a)^2 - 2 \sum_{i=a-(k-1)}^{b-k} \overline{g}_k(is)\right),$$

where the asymptotic inequality is uniform over all a, b in the given range.

The remainder of the proof of the lower bound in Theorem 1.3 follows the arguments from Section 3 of [16], which were also modified and used in Section 4 from [11], which is where similar bounds for bootstrap percolation and bootstrap k-percolation were proven in both papers. The argument proceeds by using a simple combinatorial estimate for the number of distinct events $\overline{E}_k$, appealing to Lemma 5.4, and then writing all bounds in terms of $\overline{g}_k$ using Lemma 5.5. As there are no serious additional technicalities that arise, we do not provide any further details here.

5.2. **Upper bound.** We now turn to the upper bound in Theorem 1.3, whose proof roughly follows the arguments outlined in Section 5 of [11]. We modify the definition of "rectangle growth sequences" from [11] in order to encode the row and column sequence conditions that can occur in growing configurations in the k-convolution models. One key difference for the k-convolution percolation process is that the notion of "internally spanned" sets is no longer valid; such sets were a vital tool in the study of both global and local bootstrap percolation models in [2, 20, 17, 16].

In order to illustrate this important difference, consider the 3-by-5 configuration $\mathcal{C}$ in which the first column has state $+$ in the bottom cell, the middle three columns each have state $\uparrow$ in the middle cell, and the fifth column has state $+$ in both the bottom and the top cell (see Figure 9). This is a growing configuration for the case $k = 2$, but it has the property that there is no proper subrectangle and corresponding subconfiguration that becomes completely Active.

				$+$
	$\uparrow$	$\uparrow$	$\uparrow$	
$+$				$+$

FIGURE 9. An internally spanned configuration with no internally spanned subrectangles for $k = 2$.

In [11] the first two authors showed that the use of internally spanned rectangles in [16] was not essential. Instead, growing configurations were characterized by row/column sequence conditions for k-percolation. We use this same approach to precisely characterize growing configurations in k-convolution percolation.

Following the general approach of [11], one can find the optimal parameter choices for various constants. For simplicity, we simply state these values and fix them as definitions. In particular, set

$$\begin{aligned} (\textit{Lower dimension}) \quad & A := s^{-\frac{2k-1}{2k}} \log s^{-1}, \\ (\textit{Upper dimension}) \quad & B := s^{-1} \log s^{-1}, \\ (\textit{Growth ratio}) \quad & D := s^{\frac{2k-1}{2k}}. \end{aligned}$$

Next, we define the new combinatorial structure that we will use to encode and approximate the spread of active sites in the k-convolution percolation model.

Definition 5.6. A *rectangle growth sequence* for an initial configuration $\mathcal{C}$ on $\mathbb{N}^2$ is denoted by $\Omega(\mathcal{C})$, and is defined to be a sequence of rectangles

$$\Omega(\mathcal{C}) := \left\{ 0 = R_1' \subsetneq \cdots \subsetneq R_m' \subsetneq \dots \right\}$$

such that

(*i*) $S(\mathcal{C})$ is empty if the origin is not Active in $\mathcal{C}$, and otherwise $R_1' = \{0\}$.
(*ii*) Each R_i' satisfies the k-convolution row and column sequence conditions.
(*iii*) $R_{i+1}' \smallsetminus R_i'$ is contained in $\operatorname{Shell}(R_i')$, where $\operatorname{Shell}(R)$ is defined to be the width 1 boundary in the upward and right directions around any rectangle R.

As in [11], we are only interested in the *maximal* growth sequences, which are those sequences that continue as long as the row and column conditions are still satisfied.

Definition 5.7. A *good configuration* is an initial configuration $\mathcal{C}$ on $\mathbb{N}^2$ such that any maximal rectangle growth sequence is infinite.

Lemma 5.8. *If $\mathcal{C}$ has indefinite (Active) growth, then $\mathcal{C}$ is a good configuration.*

Proof. The proof proceeds by contradiction. If $\mathcal{C}$ is not a good configuration, then both the row and column sequences must end in disallowed states. Although there are several possible combinations of such states, corresponding to all pairings of disallowed sequences in Definition 3.3, the proofs for all cases proceed similarly. We therefore only present the details for one specific pair of disallowed states.

Consider the case that there is a rectangle R of dimensions (s,t) in the growth sequence such that $\boldsymbol{\mathcal{S}}(\mathcal{C}_s) = C$ with $\boldsymbol{\mathcal{S}}(\mathcal{C}_{s+1}) = B$, and $\boldsymbol{\mathcal{S}}(\mathcal{R}_t) = B$ with $\boldsymbol{\mathcal{S}}(\mathcal{R}_{t+1}) = C$. Note that the upper-right corner therefore has $\boldsymbol{\mathcal{S}}((s,t)) = \square$. By the monotonicity of the growth process, we may assume that all other sites in $\mathcal{C}_s$ have state $\rightarrow$, all other sites in $\mathcal{R}_t$ have state $\uparrow$, and all other sites in R have state $\boldsymbol{+}$. This maximal situation is illustrated in Figure 10. However, reviewing Definition 1.1 shows that there are no growth rules that can lead to any further activity in $\mathcal{C}_s, \mathcal{C}_{s+1}, \mathcal{R}_t$, or $\mathcal{R}_{t+1}$, which contradicts the assumption that the sequence leads to an entirely Active configuration.

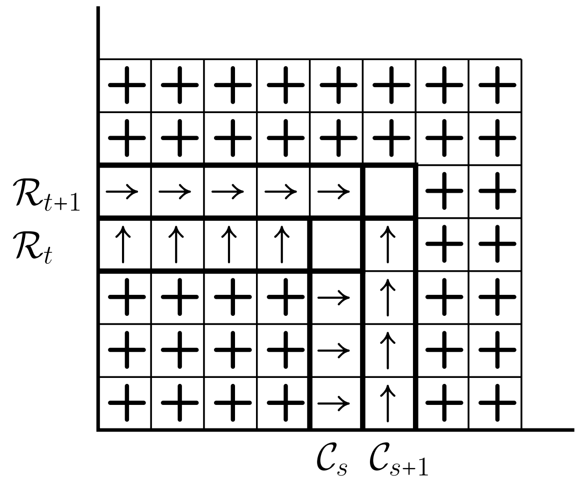

FIGURE 10. A maximal configuration with (diagonal) column sequence CB and row sequence BC.

$\square$

We can thus use good configurations as an upper bound for indefinite growth, and we follow [11, 16] in further classifying good configurations into two types of behavior.

Definition 5.9. A rectangle growth sequence *escapes* if there is an R_i' with dimensions (a', b') such that $a' \in [B, B+1]$ and $b' \leq A$, or such that $a' \leq A$ and $b' \in [B, B+1]$.

Definition 5.10. A *good sequence* is a sequence of rectangles $0 \in R_1 \subsetneq \ldots \subsetneq R_{n+1}$ that satisfies the following conditions on the dimensions $\dim R_i = (a_i, b_i)$:

(i) $\min\{a_1, b_1\} \in [A, A+1]$
(ii) $a_n + b_n \leq B$
(iii) $a_{n+1} + b_{n+1} > B$
(iv) For $i = 1, \ldots, n$ we have $s_i \geq a_i D$ or $t_i \geq b_i D$, where $s_i := a_{i+1} - a_i$ and $t_i := b_{i+1} - b_i$ are the successive dimension differences.
(v) For $i = 1, \ldots, n$ we have $s_i < a_i D + 2$ and $t_i < b_i D + 2$.

For a rectangle R define the event

$$\overline{G}(R) := \{R \text{ satisfies the column and row } k\text{-sequence conditions}\}.$$

Furthermore for two rectangles $R \subseteq R'$ whose lower-left corners coincide, we define the subrectangles S_1, S_2, S_3 (some of which may be empty).

Definition 5.11. Let $\overline{D}(R, R')$ denote the event that the rectangle $S_1 \cup S_2$ satisfies the column k-sequence condition, and that $S_2 \cup S_3$ satisfies the row k-sequence condition.

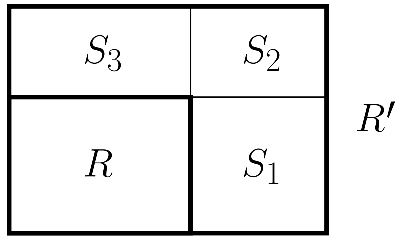

FIGURE 11. The subrectangles S_1, S_2 and S_3.

As in [11], the dichotomy in the next result follows by considering dimensions in rectangle growth sequence.

Lemma 5.12. *A good configuration $\mathcal{C}$ either has a good sequence $R_1 \subsetneq \ldots \subsetneq R_{n+1}$ such that $\overline{G}(R_1)$ and $\bigcap_{i=1}^{n} \overline{D}(R_i, R_{i+1})$ occur, or $\Omega(\mathcal{C})$ escapes.*

We have now characterized the upper bound for growth in the k-convolution process solely in terms of combinatorial sequence conditions on rows and columns, and the proof framework from Section 5.2 of [11] can now be translated directly (with one exception noted below), where occurrences of f_k and g_k are replaced by $\overline{f}_k$ and $\overline{g}_k$, and some of the precise constants and exponents are modified due to the technical estimates

$$\begin{aligned} \overline{g}_k(z) &\sim \log z^{-1} && \text{for } z \sim 0^+, \\ \overline{g}_k'(z) &\sim z^{-1} && \text{for } z \sim 0^+, \\ \overline{g}_k(z) &\sim c \cdot e^{-2z} && \text{for } z \to \infty. \end{aligned} \tag{5.1}$$

The proofs of these asymptotics are analogous to those for Lemma 3.1 of [11].

The precise probability estimates are as follows.

Lemma 5.13. *If s is sufficiently small, then there exists a constant $c > 0$ such that*

$$\mathbf{P}\Big(\Omega(\mathcal{C}) \text{ escapes}\Big) \leq \exp\left(-cB \log s^{-1}\right).$$

Lemma 5.14. *Let $R_1, \ldots, R_{n+1}$ be a good sequence of rectangles and let $a_0 = b_0 = A$, $s_0 = a_1 - a_0$, $t_0 = b_1 - b_0$. Then we have for some constant $c > 0$*

$$\mathbf{P}\left(\overline{G}(R_1)\right) \leq \exp\Big(-s_0 \overline{g}_k(b_0 s) - t_0 \overline{g}_k(a_0 s) + cA^{-1}B\Big).$$

Lemma 5.15. *If $R \subseteq R'$ are two rectangles with dimensions (a, b) and $(a + \ell, b + m)$, respectively, then*

$$\mathbf{P}\left(\overline{D}(R, R')\right) \leq \exp\Big(-(m - 2(k-1))\, \overline{g}_k(as) - (\ell - 2(k-1))\, \overline{g}_k(bs) + \ell m s \exp\left(k\left(\overline{g}_k(as) + \overline{g}_k(bs)\right)\right)\Big).$$

The following analog to Corollary 5.14 of [11] relies on a variational principle from [20], which again requires the fact that $\overline{g}_k$ is convex (cf. Theorem 2.3).

Lemma 5.16. *If s is sufficiently small, then there exists a constant $c > 0$ such that*

$$\sum_{i=1}^{n} \Big(s_i \overline{g}_k(b_i s) + t_i \overline{g}_k(a_i s)\Big) \geq 2\lambda_k s^{-1} - c\Big(A \log s^{-1} - s^{-1} \exp(-2Bs) - B \exp(-Bs)\Big).$$

The primary reason that the upper bound in Theorem 1.3 does not share the same s-powers as the lower bound is due to the fact that the asymptotics in (5.1) are uniform in k, which was not the case for the asymptotics of g_k in [11]. This difference also necessitates a modified version of Lemma 5.15 (*ii*) from [11], which is replaced by the following result, whose proof is also analogous.

Lemma 5.17. *Let n and $a_i, b_i (i = 1, \dots, n+1)$ be positive integers and denote the successive differences by $s_i := a_{i+1} - a_i \geq 0$ and $t_i := b_{i+1} - b_i \geq 0$ for $i = 1, \dots, n$. Further assume that the dimensions satisfy all of the properties of a good sequence. Then as $s \to 0$,*

$$\sum_{i=1}^{n} \frac{s_i t_i}{a_i^k b_i^k} \ll \frac{D}{A^{2k-2}}.$$

We continue with two final technical estimates following [11].

Lemma 5.18. *The number of good sequences of rectangles is at most*

$$\exp\left(cD^{-1}\left(\log s^{-1}\right)^2\right),$$

where $c > 0$ is some constant.

Lemma 5.19. *In the range $0 \leq a \leq B$, there is a uniform asymptotic bound*

$$e^{\overline{g}_k(as)} \underset{unif}{\ll} \frac{B}{a} \qquad \text{as } s \to 0.$$

Proof of Theorem 1.3 Upper Bound. Continuing to follow Section 5.3 of [11], we have now reached the overall upper bound

$$\mathbf{P}\left(\text{indefinite growth}\right) \leq \mathbf{P}\left(\Omega(\mathcal{C}) \text{ escapes}\right) + \sum_{\substack{\text{good sequences} \\ \{R_1,\dots,R_{n+1}\}}} \mathbf{P}\left(G(R_1)\right) \prod_{i=1}^{n} \mathbf{P}\left(D\left(R_i, R_{i+1}\right)\right). \tag{5.2}$$

Lemmas 5.13 – 5.19 lead to the upper bound

$$\exp\Big(-2\lambda_k s^{-1} + c\Big(A\log s^{-1} + A^{-1}B + D^{-1}\left(\log s^{-1}\right)^2 + sA^{-(2k-2)}B^{2k}D + s^{-1}e^{-2Bs} + Be^{-Bs}\Big)\Big),$$

which upon recalling the parameter values, implies the stated upper bound. ☐

6. The case $k = 2$ and mock theta functions

In this section we explain the connection between the case $k = 2$ and Ramanujan's mock theta function $\phi(q)$ from (1.7), which allows us to prove the precise asymptotic result of Theorem 1.7. Following the approach in [22], we write

$$\mathbf{P}\left(\overline{A}_2\right) = \sum_{\substack{\text{Sequences } \{X_j\}_1^\infty \\ X_j \in \{A,B,C,D\} \\ \text{No } D, CC \text{ or } CB}} \prod_{j\colon X_j = A} \left(1 - q^j\right)^2 \prod_{j\colon X_j = B \text{ or } C} q^j \left(1 - q^j\right) \prod_{j\colon X_j = D} q^{2j}.$$

This can be factored and rewritten as

$$\mathbf{P}\left(\overline{A}_2\right) = \prod_{n\geq 1} \left(1 - q^n\right)^2 \sum_{\substack{\text{Sequences } \{X_j\}_1^\infty \\ X_j \in \{A,B,C\} \\ \text{No } CC \text{ or } CB}} \prod_{j\colon X_j = A} 1 \prod_{j\colon X_j = B} \frac{q^j}{1 - q^j} \prod_{j\colon X_j = C} \frac{q^j}{1 - q^j}. \tag{6.1}$$

We introduce additional notation for the above sum, writing

$$\phi^*(q) := \frac{\mathbf{P}\left(\overline{A}_2\right)}{\prod_{n\geq 1}(1-q^n)^2}. \tag{6.2}$$

This is equal to a hypergeometric q-series of combinatorial interest.

Proposition 6.1. *For* $0 < q < 1$,

$$\phi^*(q) := 1 + 2\sum_{n\geq 1}\frac{q^n}{1-q^n}\prod_{j=1}^{n-1}\frac{1+q^{2j}}{1-q^j}$$

Proof. We present a brief proof using the combinatorics of overpartitions with sequence restrictions, which are studied in greater detail in [9]. The equality of (6.2) and the hypergeometric series in the proposition statement follows from the further observation that

$$\phi^*(q) = \sum_{n\geq 0}\overline{p}_2(n)q^n,$$

where $\overline{p}_2(n)$ denotes the number of *overpartitions of n without sequences* (following [13] and [22]). Precisely, these overpartitions must satisfy either the *upper* or *lower* sequence property, which states that if an overlined part $\overline{m}$ occurs, then there are no regular or overlined parts of size $m+1$ or $m-1$, respectively (a simple bijection shows that the two characterizations lead to the same overpartition function $\overline{p}_2(n)$, cf. [9]).

It is then easy to see that (6.2) generates overpartitions with the upper sequence property, as the event A_j corresponds to no parts of size j, B_j corresponds to only regular parts of size j, and C_j corresponds to an overlined part of size j. The hypergeometric series generates overpartitions with the property that if $\overline{m}$ occurs and m is not the largest part size, then m must also occur at least once. Under conjugation, this is equivalent to overpartitions with the lower sequence property. □

The combinatorial q-series $\phi^*(q)$ is also closely related to one of Ramanujan's famous third-order mock theta functions. In particular, Ramanujan defined the third-order mock theta function $\phi(q)$ as in (1.7), and in equation (26.32) on page 57 of [14], Fine showed the alternative representation

$$\phi(q) = \prod_{n\geq 1}\frac{1-q^n}{1+q^{2n}}\cdot\phi^*(q). \tag{6.3}$$

Using (6.2) and (6.3), we can therefore write the sequence probability in the case $k=2$ as the following product of a weight 1/2 weakly holomorphic modular form (up to a factor of $q^{1/8}$) and a mock theta function:

$$\mathbf{P}\left(\overline{A}_2\right) = \prod_{n\geq 1}\frac{(1-q^n)\left(1-q^{4n}\right)}{1-q^{2n}}\cdot\phi(q). \tag{6.4}$$

Proof of Theorem 1.7. By writing (6.4), we have reduced the evaluation of the limiting behavior to a simple cuspidal calculation using the modular inversion map. A standard result from the theory of modular forms (found, for example, as a special case of Theorem 3.4 of [5]) states that as $s\to 0^+$,

$$\prod_{n\geq 1}(1-q^n) \sim \sqrt{\frac{2\pi}{s}}\,e^{-\frac{\pi^2}{6s}}.$$

Furthermore, although there is also a similar theory of modular transformations for mock theta functions [26], the required asymptotics can also be evaluated directly from (1.7), giving

$$\phi(q) \sim \sum_{n\geq 0} \frac{1}{2^n} = 2, \qquad \text{as } s \to 0^+.$$

Applying these asymptotics to (6.4) proves the theorem statement. □

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