KAC-WAKIMOTO CHARACTERS AND NON-HOLOMORPHIC JACOBI FORMS

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Abstract. In this paper, we investigate the automorphic properties of certain characters introduced by Kac and Wakimoto pertaining to $s\ell(m,n)^\wedge$ highest weight modules. Extending previous work of the first author and Ono, the first author and Folsom, and the second author, we investigate the general case, not specializing the Jacobi variables. We prove that the Kac-Wakimoto characters are essentially holomorphic parts of multivariable mixed H-harmonic Maass-Jacobi forms, which are certain non-holomorphic generalizations of classical holomorphic Jacobi forms. This also gives extra structure to the previous considered cases.

1. Introduction and Statement of Results

It is now well-known that there is a connection between classical modular forms and the representation theory of infinite dimensional Lie algebras. Probably the most famous example is given by "Monstrous moonshine" (observed by McKay, generalized by Conway and Norton [11], and proved by Frenkel-Lepowsky-Meurman [15] and the Fields medalist Borcherds [2]). It relates the coefficients of the modular invariant $J(\tau)$ to dimensions of irreducible representations of the Monster group, the largest sporadic finite simple group. Note that there is also an interesting connection to conformal field theory and string theory. Prior to this, important work of Kac [17] established the so-called Kac-Weyl character formula and denominator identity. Among numerous consequences one obtains many beautiful identities involving modular forms. For example, the classical Rogers-Ramanujan identity

$$\sum_{r\geq 0}\frac{q^{r^2}}{(q;q)_r}=\prod_{r\geq 1}\left(1-q^{5r-1}\right)^{-1}\left(1-q^{5r-4}\right)^{-1} \tag{1.1}$$

may be viewed as a specialized character formula for the standard modules for $A_1^{(1)}$. Here, for $r\in\mathbb{N}_0\cup\{\infty\}$, let $(a;q)_r:=\prod_{j=0}^{r-1}\left(1-aq^j\right)$. Note that the right-hand side of (1.1) may easily be seen as a weakly holomorphic modular form, i.e., a meromorphic modular form whose poles can only lie at the cusps.

Recent works [4, 6, 14] showed that harmonic Maass forms and their generalizations also play an important role in understanding modularity properties of certain characters. *Harmonic weak Maass forms*, as originally defined by Bruinier and Funke [10], are non-holomorphic modular forms which naturally generalize classical modular forms. They satisfy the same modular transformations as their holomorphic companions, but instead of being meromorphic they are annihilated by a hyperbolic Laplace operator. In most cases the arithmetically interesting part is the holomorphic part, the so-called *mock modular form*. Every mock modular form has a

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certain hidden companion, its *shadow*, which is necessary to fully understand the mock modular form. This shadow, which is a classical modular form of dual weight $2-k$, may be obtained from the associated harmonic Maass form of weight k by applying the differential operator $\xi_k := 2i\mathrm{Im}(\tau)^k \overline{\frac{\partial}{\partial\overline{\tau}}}$. To give one example of a mock modular form, Zagier [16] showed that the generating function of Hurwitz class numbers is a mock modular form of weight 3/2 with shadow $\Theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}$, where throughout $q := e^{2\pi i\tau}$.

In this paper, we show that (non-holomorphic) Jacobi forms also beautifully enter the picture. A (holomorphic) Jacobi form is a two variable generalization of modular forms satisfying a modular and elliptic transformation law. Furthermore, one requires certain growth conditions which can be stated in terms of its Fourier expansion. There is a well-developed theory of Jacobi forms due to Eichler and Zagier [13]. See also Section 2 for some basic facts on Jacobi forms required in this paper. In particular, there exists an important correspondence between them and modular forms of half-integral weight given by the so-called theta decomposition. The theory has since grown enormously, establishing deep connections to different types of automorphic forms and many other areas of mathematics and physics. In particular, interest for meromorphic Jacobi forms arose since coefficients of such Jacobi forms often encode interesting combinatorial statistics including Dyson's rank of partitions, and using properties of the Jacobi forms gives interesting combinatorial results. Furthermore, Jacobi forms also play a key role in understanding modularity of generating functions of quantum black holes [12].

Returning to Lie superalgebras, Kac and Wakimoto [18] have recently found a character formula for the affine Lie superalgebra $s\ell(m,n)^\wedge$ for $\mathrm{tr}_{L_{m,n}(\Lambda_{(\ell)})} q^{L_0}$ ($m > n \geq 1$ integers), where $L_{m,n}(\Lambda_{(\ell)})$ is the irreducible $s\ell(m,n)^\wedge$ module with highest weight $\Lambda_{(\ell)}$, and L_0 is the "energy operator". In important special cases the first author, in joint work with Ono [6] and then with Folsom [4], investigated a question raised by Kac concerning the "modularity" of the characters $\mathrm{tr}_{L_{m,n}(\Lambda_{(\ell)})} q^{L_0}$. To state these results, we recall the precise shape of the Kac-Wakimoto characters

$$\mathrm{ch}F = \sum_{\ell\in\mathbb{Z}} \mathrm{ch}F_\ell \zeta^\ell = e^{\Lambda_0} \prod_{k\geq 1} \frac{\prod_{r=1}^m \left(1+\zeta\beta_r q^{k-\frac{1}{2}}\right)\left(1+\zeta^{-1}\beta_r^{-1} q^{k-\frac{1}{2}}\right)}{\prod_{j=1}^n \left(1-\zeta\beta_{m+j} q^{k-\frac{1}{2}}\right)\left(1-\zeta^{-1}\beta_{m+j}^{-1} q^{k-\frac{1}{2}}\right)}, \tag{1.2}$$

where β_s are indeterminates. We note that the coefficient functions $\mathrm{ch}F_\ell$ depend upon the range in which ζ is taken. The change when moving to a different range is known as "wall crossing". We naturally decompose these coefficients as a "mock" piece and a piece exhibiting the wall-crossing behavior.

In previous works [4, 5, 6, 14] only the situation that all $\beta_s = 1$ has been treated. It has been shown that in this case the characters may be viewed in the framework of *almost harmonic weak Maass forms*, which are sums of harmonic weak Maass forms under iterates of the raising operator (themselves therefore non-harmonic weak Maass forms) multiplied by *almost holomorphic modular forms*, which are modular polynomials in $\mathrm{Im}(\tau)^{-1}$. In the simplest case when $n = 1$ one obtains the product of a usual modular form and a harmonic weak Maass form (the associated holomorphic part is called *mixed mock modular form*).

In this paper we show that considering the additional variables in (1.2) imposes extra structure which gives a cleaner picture for the specialized character as these are, by the present work, specializations of *mixed H-harmonic Maass-Jacobi forms*. These are, briefly speaking, non-holomorphic Jacobi forms annihilated by certain differential operators, multiplied by (weak) Jacobi forms. For the precise definitions see Section 2.

Theorem 1.1. *The multivariable Kac-Wakimoto characters chF$_\ell$ are the holomorphic parts of mixed H-harmonic Maass-Jacobi form.*

We next allow some of the Jacobi variables to be equal. This gives rise to a more complicated structure, due to the fact that the corresponding meromorphic Jacobi form has higher order poles. In order to describe this situation, we need to introduce *almost Maass-Jacobi forms*. Briefly speaking these are sums of H-harmonic Maass-Jacobi forms under iterates of the raising operator multiplied by almost holomorphic Jacobi forms under iterates of the lowering operator. To better describe the functions of interest, note that (1.2) can be written in term of Jacobi theta-functions. Therefore, after shifting variables and up to powers of the η-function, we can restrict to the study of the following meromorphic Jacobi form

$$\Phi(z,\mathbf{u};\tau) := \frac{\prod_{r=1}^{s} \vartheta\left(z+u_r+\frac{1}{2};\tau\right)^{m_r}}{\prod_{j=1}^{t} \vartheta\left(z-w_j;\tau\right)^{n_j}}, \tag{1.3}$$

where ϑ is the Jacobi theta function defined in (3.1). Moreover, s, t, m_r, $n_j \in \mathbb{N}$, $\tau \in \mathbb{H}$ is the modular variable, $z \in \mathbb{C}$, and $\mathbf{u} := (u_1, \cdots, u_s, w_1, \cdots, w_t) \in \mathbb{C}^{s+t}$ are the elliptic variables. We furthermore assume that $m := \sum_{r=1}^{s} m_r > n := \sum_{j=1}^{t} n_j$, where $m \equiv n \equiv 0 \pmod 2$ (the other residue classes can be treated similarly as demonstrated by the second author in [22], in the 1-variable case) and we fix $M := \frac{m-n}{2}$. Furthermore, one can also study the case $m = n$ as it has been done in [5] for the one-variable case by the first author, Folsom, and Mahlburg. We obtain the following modularity result.

Theorem 1.2. *The Fourier coefficients in z of the meromorphic Jacobi form Φ are the holomorphic parts of almost Maass-Jacobi forms.*

For a more precise version of the theorem, we refer the reader to Theorem 7.1.

The remainder of the paper is organized as follows. In Section 2, we recall the definition of certain non-holomorphic Jacobi forms and harmonic Maass-Jacobi forms, new automorphic objects that describe the structure of the Kac-Wakimoto characters. In Section 3, we extend the canonical decomposition of a meromorphic Jacobi form to the multivariable case. The one variable case was previously described by Dabholkar, Murty, and Zagier in [12] for poles of order at most 2. Any meromorphic Jacobi form then splits into a so-called finite part and a polar part. In Section 4, we describe the polar part in terms of well-known modular objects, namely almost holomorphic Jacobi forms. Each piece of the decomposition can be completed by adding a non-holomorphic term, in order to obtain a non-holomorphic Jacobi form. This is described in Section 5. In Section 6, we recall the definition of certain differential operators, namely the Maass raising and lowering operators, and we show how they naturally appear in the structure of the Kac-Wakimoto characters. We also describe the action of other differential operators on them. Theorem 1.1 is proved in Section 7.

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2. Holomorphic and non-holomorphic Jacobi forms

In this section we recall the definition of classical Jacobi forms [13, 26], as well as certain non-holomorphic generalization [7, 8].

2.1. **Classical Jacobi forms.** For $k \in \mathbb{Z}$, $L \in \mathrm{GL}_N(\mathbb{C})$ $(N \in \mathbb{N})$, and $[\gamma, (\boldsymbol{\lambda}, \boldsymbol{\mu})] \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^{2N}$, with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we define the slash operator acting on $\varphi : \mathbb{C}^N \times \mathbb{H} \to \mathbb{C}$ by

$$\varphi\big|_{k,L}[\gamma, (\boldsymbol{\lambda}, \boldsymbol{\mu})](\mathbf{u}; \tau) := (c\tau + d)^{-k} e^{\frac{1}{2}} \left(-\frac{c}{c\tau + d} L[\mathbf{u} + \boldsymbol{\lambda}\tau + \boldsymbol{\mu}] + L[\boldsymbol{\lambda}]\tau + 2\mathbf{u}^T L \boldsymbol{\lambda} \right) \times \varphi\left(\frac{\mathbf{u}}{c\tau + d}; \frac{a\tau + b}{c\tau + d} \right).$$

Here and throughout $L[x] := x^T L x$ and $e(x) := e^{2\pi i x}$.

Remark 2.1. Note that in the 1-variable case the index is given by $\frac{1}{2}L$.

Definition 2.2 (Weak Jacobi forms)**.** *A multivariable weak Jacobi form of index $L \in GL_N(\mathbb{Z})$ and weight $k \in \mathbb{Z}$ for a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, is a holomorphic function $\varphi : \mathbb{C}^N \times \mathbb{H} \to \mathbb{C}$ satisfying the following conditions:*

(1) *For all $[\gamma, (\boldsymbol{\lambda}, \boldsymbol{\mu})] \in \Gamma \ltimes \mathbb{Z}^{2N}$, we have*

$$\varphi\big|_{k,L}[\gamma, (\boldsymbol{\lambda}, \boldsymbol{\mu})](\mathbf{u}; \tau) = \varphi(\mathbf{u}; \tau).$$

(2) *For each fixed $\mathbf{u} = \boldsymbol{\alpha}\tau + \boldsymbol{\beta} \in \mathbb{C}^N$, the function $\varphi(\boldsymbol{\alpha}\tau + \boldsymbol{\beta}; \tau)$ is bounded, as $Im(\tau) \to \infty$.*

The space of weak (resp. holomorphic) Jacobi forms of weight k and index L is denoted by $\widetilde{J}_{k,L}$ (resp. $J_{k,L}$).

Classically [13, 26] Jacobi forms with $L > 0$ have been considered. This assumption allows a theta decomposition of the Jacobi form, meaning a decomposition into Jacobi theta functions multiplied by holomorphic functions (that are "essentially" the Fourier coefficients of the Jacobi form itself), which we call *theta coefficients*. To be more precise, define

$$\vartheta_{M,\ell}(z; \tau) := \sum_{\substack{\lambda \in \mathbb{Z} \\ \lambda \equiv \ell \pmod{2M}}} q^{\frac{\lambda^2}{4M}} e^{2\pi i \lambda z}.$$

Here

$$L := \begin{pmatrix} 2M & \mathbf{b}^T \\ \mathbf{b} & \widetilde{L} \end{pmatrix}, \tag{2.1}$$

where $2M \in \mathbb{N}$, $\mathbf{b} \in \mathbb{Z}^{N-1}$, and $\widetilde{L} \in \mathrm{GL}_{N-1}(\mathbb{Z})$. Thus, for a Jacobi form $\varphi(z, \mathbf{u}; \tau)$ that satisfies the hypotheses above, one may write

$$\varphi(z, \mathbf{u}; \tau) = \sum_{\ell \pmod{2M}} h_\ell(\mathbf{u}; \tau) \vartheta_{M,\ell}\left(z + \frac{1}{2M} \mathbf{u} \cdot \mathbf{b}; \tau \right),$$

where the Fourier coefficients $h_\ell(\mathbf{u}; \tau)$, defined in (3.4), transform as multivariable Jacobi forms of weight $k - \frac{1}{2}$ and index L^*, where

$$L^* := \widetilde{L} - \frac{1}{2M} \mathbf{b}\mathbf{b}^T. \tag{2.2}$$

2.2. **Almost holomorphic Jacobi forms.** One can generalize the notion of holomorphic Jacobi forms by allowing non-holomorphicity in both the elliptic and the modular variables. The simplest class of functions is given by so-called almost holomorphic Jacobi forms first considered by Libgober [20] in the 1-dimensional case for index 0.

Definition 2.3 (Almost holomorphic Jacobi forms)**.** *A function $\varphi : \mathbb{C}^N \times \mathbb{H} \to \mathbb{C}$ is called an almost holomorphic Jacobi form of index $L \in GL_N(\mathbb{Z})$ and weight $k \in \mathbb{Z}$ if it is a polynomial in $\frac{u_j - \overline{u_j}}{\tau - \overline{\tau}}$ and $\frac{1}{\tau - \overline{\tau}}$, with (weakly) holomorphic coefficients in $(\mathbf{u}, \tau)$ and satisfies the same transformation properties of a Jacobi form. The constant term of the polynomial is called a quasi Jacobi form.*

One such example is given by the weight 1 Jacobi Eisenstein series

$$E_1(z;\tau) := \sum_{(a,b)\in\mathbb{Z}^2}^{*} \frac{1}{(z + a\tau + b)},$$

where $\sum^*$ is the so-called Eisenstein summation, i.e.,

$$\sum_{(a,b)\in\mathbb{Z}^2}^{*} := \lim_{A\to\infty} \sum_{a=-A}^{A} \left(\lim_{B\to\infty} \sum_{b=-B}^{B} \right).$$

The associated almost holomorphic Jacobi form is given by

$$\widehat{E}_1(z;\tau) := E_1(z;\tau) + \frac{z - \overline{z}}{\tau - \overline{\tau}}.$$

Remark 2.4. Note that we correct a typo in the completion given in [20], dividing by $2\pi i$ the non-holomorphic part.

As in the case of almost holomorphic modular forms, almost holomorphic Jacobi forms are annihilated by powers of the lowering operator defined in (2.6).

2.3. ***H*-harmonic Maass-Jacobi forms.** Another non-holomorphic generalization of Jacobi forms has been recently introduced in [7] and [8], extending previous definitions given by Berndt and Schmidt [1] and Pitale [24]. In order to give the precise definition, we need to introduce some differential operators. Note that they can be given in terms of the raising and the lowering operators defined in Subsection 2.4. For $w \in \mathbb{C}$ and $\tau \in \mathbb{H}$, the *Casimir operator of weight $k \in \mathbb{Z}$ and index $N \in \mathbb{N}$* is defined (up to the constant $\frac{5}{8} + \frac{3k-k^2}{2}$) by

$$\begin{aligned}\mathcal{C}_{k,N} := & -2(\tau-\overline{\tau})^2\partial_\tau\partial_{\overline{\tau}} - (2k-1)(\tau-\overline{\tau})\partial_{\overline{\tau}} + \frac{(\tau-\overline{\tau})^2}{4\pi i N}\partial_{\overline{\tau}}\partial_w^2 + \frac{k(\tau-\overline{\tau})}{4\pi i N}\partial_w\partial_{\overline{w}} \\ & + \frac{(w-\overline{w})(\tau-\overline{\tau})}{4\pi i N}\partial_w^2\partial_{\overline{w}} - 2(w-\overline{w})(\tau-\overline{\tau})\partial_\tau\partial_{\overline{w}} + (1-k)(w-\overline{w})\partial_{\overline{w}} + \frac{(\tau-\overline{\tau})^2}{4\pi i N}\partial_\tau\partial_{\overline{w}}^2 \\ & + \left(\frac{(w-\overline{w})^2}{2} + \frac{k(\tau-\overline{\tau})}{4\pi i N}\right)\partial_{\overline{w}}^2 + \frac{(w-\overline{w})(\tau-\overline{\tau})}{4\pi i N}\partial_w\partial_{\overline{w}}\partial_{\overline{w}},\end{aligned}$$

where, here and throughout, $\partial_w := \frac{\partial}{\partial w}$. Note that in [7] there is a typo in the definition of $\mathcal{C}_{k,N}$.

Moreover, we define the *Heisenberg Laplace operator* [8] by

$$\Delta_N^H := \frac{\tau-\overline{\tau}}{2i}\partial_w\partial_{\overline{w}} + 2\pi N(w-\overline{w})\partial_{\overline{w}}.$$

Definition 2.5 (H-harmonic Maass-Jacobi forms [8])**.** *A real-analytic function $\varphi : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$ is called a Maass-Jacobi form of weight k and index N for $\Gamma \ltimes \mathbb{Z}^2$ (Γ a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$) if the following conditions are satisfied:*

1. *For all $[\gamma, (\lambda, \mu)] \in \Gamma \ltimes \mathbb{Z}^2$*

$$\varphi\big|_{k,2N}[\gamma,(\lambda,\mu)](u;\tau) = \varphi(u;\tau).$$

2. *There exists $\lambda \in \mathbb{C}$ such that $\mathcal{C}_{k,N}(\varphi) = \lambda\varphi$.*
3. *For each fixed $u = \alpha\tau + \beta \in \mathbb{C}$, the function $\varphi(\alpha\tau + \beta; \tau)$ is bounded, as $Im(\tau) \to \infty$.*

If λ in condition (2) equals 0, then we say that φ is harmonic. If in addition $\Delta_N^H(\varphi) = 0$, then φ is called Heisenberg harmonic (H-harmonic). Finally, we call φ a mixed (H-)harmonic Maass-Jacobi form if it can be written as a linear combination of (H-)harmonic Maass-Jacobi forms multiplied by weak Jacobi forms.

Note that our growth condition slightly differs from the one given in [8].

The functions of interest for this paper naturally occur as holomorphic parts of (mixed) H-harmonic Maass-Jacobi forms. We thus, in analogy to mock modular forms, call them *mock Jacobi forms*. A special example of a mock Jacobi form, which plays an important role in this paper, is the Appell-Lerch sum, defined for $N \in \mathbb{N}$,

$$f_N(z,u;\tau) := \sum_{\alpha\in\mathbb{Z}} \frac{q^{N\alpha^2} e(2N\alpha z)}{1-e(z-u)q^\alpha}. \tag{2.3}$$

In his Ph.D. thesis [28], Zwegers studied and used this function to relate meromorphic Jacobi forms and Ramanujan's mock theta functions. In particular, he found a non-holomorphic completion for f_N, in order to make it transform as a 2-variable Jacobi form. To describe this, for each $\ell \in \mathbb{Z}$, we need the real-analytic function $R_{N,\ell}$ defined by

$$R_{N,\ell}(u;\tau) := \sum_{\substack{\lambda\in\mathbb{Z}\\ \lambda\equiv\ell \pmod{2N}}} \left\{ \operatorname{sgn}\left(\lambda+\frac{1}{2}\right) - E\left(\left(\lambda + 2N\frac{\operatorname{Im}(u)}{\operatorname{Im}(\tau)}\right)\sqrt{\frac{\operatorname{Im}(\tau)}{N}}\right)\right\} \times e^{-\pi i \frac{\lambda^2}{2N}\tau - 2\pi i\lambda u}, \tag{2.4}$$

where $u \in \mathbb{C}$, and $E(z) := 2\int_0^z e^{-\pi t^2} dt$. The completion of f_N is the function $\widehat{f}_N$ defined by

$$\widehat{f}_N(z,u;\tau) := f_N(z,u;\tau) - \frac{1}{2}\sum_{\ell \pmod{2N}} R_{N,\ell}(u;\tau)\vartheta_{N,\ell}(z;\tau). \tag{2.5}$$

Zwegers proved the following.

Proposition 2.6 (Zwegers [28])**.** *For $N \in \mathbb{N}$, the function $\widehat{f}_N$ transforms like a Jacobi form on $\mathbb{C}^2 \times \mathbb{H}$ of weight 1 and index $\left(\begin{smallmatrix} 2N & 0 \\ 0 & -2N\end{smallmatrix}\right)$ for $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.*

2.4. **The lowering and the raising operator.** Here we define certain differential operators, first introduced by Maass [21] for 1-variable forms, and generalized by Berndt and Schmidt (see Section 3.5 in [1]) to a multivariable setting, acting on the spaces of Maass-Jacobi forms and almost holomorphic Jacobi forms in one elliptic variable, called the *raising operators* and the *lowering operators*. For $N \in \mathbb{N}$ and $k \in \frac{1}{2}\mathbb{Z}$, we define the operators

$$X_+^{k,N} := 2i\left(\partial_\tau + \tfrac{w-\overline{w}}{\tau-\overline{\tau}}\partial_w + 2\pi i N\tfrac{(w-\overline{w})^2}{(\tau-\overline{\tau})^2} + \tfrac{k}{\tau-\overline{\tau}}\right), \qquad Y_+^{k,N} := i\partial_w - 4\pi N\tfrac{w-\overline{w}}{\tau-\overline{\tau}},$$

$$X_-^{k,N} := -\tfrac{\tau-\overline{\tau}}{2i}\left((\tau-\overline{\tau})\partial_{\overline{\tau}} + (w-\overline{w})\partial_{\overline{w}}\right), \qquad Y_-^{k,N} := -\tfrac{\tau-\overline{\tau}}{2}\partial_{\overline{w}},$$

We recall the weight-moving properties of these operators in the following proposition.

Proposition 2.7 (Remark 3.5.2 in [1])**.** *For $G \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ and $\varphi \in \mathcal{C}^\infty(\mathbb{C}^\times\mathbb{H})$ we have*

$$X_\pm^{k,N}\left(\varphi|_{k,2N}G\right) = \left(X_\pm^{k,N}\varphi\right)|_{k\pm2,2N}G, \quad Y_\pm^{k,N}\left(\varphi|_{k,2N}G\right) = \left(Y_\pm^{k,N}\varphi\right)|_{k\pm1,2N}G.$$

We also need to define a multivariable version of $X_-^{k,L}$, where L is a positive definite matrix in $\mathrm{GL}_N(\mathbb{Z})$, that acts on functions on $\mathbb{C}^N \times \mathbb{H}$, namely

$$X_- = X_-^{k,L} := -\frac{\tau-\overline{\tau}}{2i}\left((\tau-\overline{\tau})\,\partial_{\overline{\tau}} + \langle \mathbf{u}-\overline{\mathbf{u}}, \partial_{\overline{\mathbf{u}}}\rangle\right), \tag{2.6}$$

where $<,>$ denotes the standard scalar product in $\mathbb{R}^N$.

Finally, we require the level $N \in \mathbb{N}$ *heat operator*

$$H_N := 8\pi i N \partial_\tau - \partial_w^2. \tag{2.7}$$

This operator acts on the space of Jacobi forms and preserves the elliptic transformation property, i.e., for $(\lambda,\mu) \in \mathbb{Z}^2$ [13]

$$H_N\left(\varphi\big|_{k,2N}(\lambda,\mu)\right) = H_N(\varphi)\big|_{k,2N}(\lambda,\mu).$$

In the special case of Jacobi forms of weight $k=\frac{1}{2}$, the operator H_N also preserves the modularity property, increasing the weight by 2 ($G \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$)

$$H_N\left(\varphi\big|_{\frac{1}{2},2N}G\right) = H_N(\varphi)\big|_{\frac{5}{2},2N}G.$$

An easy calculation rewrites the Casimir operator in terms of simpler operators. In order to state the result, we need to introduce the differential operator

$$\xi_{k,N} := \left(\frac{\tau-\overline{\tau}}{2i}\right)^{k-\frac{3}{2}}\left(-(\tau-\overline{\tau})\partial_{\overline{\tau}} - (z-\overline{z})\partial_{\overline{z}} + \frac{1}{4\pi N}\left(\frac{\tau-\overline{\tau}}{2i}\right)\partial_{\overline{z}\overline{z}}\right). \tag{2.8}$$

Lemma 2.8. *With the notation as above, the Casimir operator can be written as*

$$\mathcal{C}_{k,N} = \frac{X_-H_N}{2\pi M} + \frac{k}{2\pi N}\Delta_N^H - \frac{1}{2\pi N}\left(X_+^{\frac{1}{2},N}Y_-^2 - \frac{4i}{\tau-\overline{\tau}}Y_-^2\right) + (2k-1)\left(\frac{\tau-\overline{\tau}}{2i}\right)^{\frac{3}{2}-k}\xi_{k,N}.$$

We conclude this subsection by stating a commutator relation between $\mathcal{C}_{k,N}$ and $Y_+^{k,N}$, which can be verified by a straightforward calculation.

Lemma 2.9. *With the notation as above we have*

$$\mathcal{C}_{k,N}Y_+^{k-1,N} = Y_+^{k-1,N}\mathcal{C}_{k-1,N} - (k-2)Y_+^{k-1,N}.$$

2.5. **Almost harmonic Maass-Jacobi forms.** In order to describe the structure of the Fourier coefficients of multivariable meromorphic Jacobi forms, we introduce new automorphic objects which we call almost harmonic Maass-Jacobi forms, extending the definition of almost harmonic Maass forms introduced by the first author and Folsom in [4].

Definition 2.10. *An almost (H-harmonic) Maass-Jacobi form of weight $k \in \frac{1}{2}\mathbb{Z}$ and index $L \in GL_R(\mathbb{Z})$ for $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ with Nebentypus character χ, is a smooth function $\varphi : \mathbb{C}^R \times \mathbb{H} \to \mathbb{C}$ that can be decomposed as a linear combination of objects of the following shape:*

$$\sum_\lambda (X_-)^\lambda\,(g(\mathbf{u};\tau))\left(X_+^{\nu,-N}\right)^\lambda(f(w;\tau)), \tag{2.9}$$

where $\mathbf{u} = (u_j)_j$, w is a linear combination of the u_js, g is an almost holomorphic Jacobi form of weight $k-\nu$ and index $\widetilde{L} - \frac{1}{2N}\mathbf{b}\mathbf{b}^T$ (see (2.1)), and f is a (H-harmonic) Maass-Jacobi form of weight $\nu \in \frac{1}{2}\mathbb{Z}$ and index $-N$.

3. A canonical decomposition

In this section we describe the transformation properties and the canonical decomposition of

$$\Phi(z,\mathbf{u};\tau)=\frac{\prod_{r=1}^{s}\vartheta\left(z+u_r+\frac{1}{2};\tau\right)^{m_r}}{\prod_{j=1}^{t}\vartheta\left(z-w_j;\tau\right)^{n_j}},$$

where ϑ is the classical Jacobi theta function

$$\vartheta(z;\tau):=\sum_{\nu\in\frac{1}{2}+\mathbb{Z}}e^{\pi i\nu^2\tau+2\pi i\nu\left(z+\frac{1}{2}\right)}=-i\zeta^{-\frac{1}{2}}q^{\frac{1}{8}}\prod_{r\geq 1}\left(1-q^r\right)\left(1-\zeta q^{r-1}\right)\left(1-\zeta^{-1}q^r\right). \tag{3.1}$$

Here and throughout $\zeta:=e^{2\pi iz}$. Since Jacobi's theta function has simple zeros in $\mathbb{Z}+\tau\mathbb{Z}$, Φ has poles (modulo $\mathbb{Z}\tau+\mathbb{Z}$) at w_j of order n_j, for $j\in\{1,\cdots t\}$. In the following lemma we recall the transformation properties of ϑ.

Lemma 3.1 ([25], (80.31) and (80.8))**.** *For λ, $\mu\in\mathbb{Z}$, and $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, we have that*

$$\vartheta(z+\lambda\tau+\mu;\tau)=(-1)^{\lambda+\mu}q^{-\frac{\lambda^2}{2}}\zeta^{-\lambda}\vartheta(z;\tau),$$
$$\vartheta\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right)=\psi^3(\gamma)(c\tau+d)^{\frac{1}{2}}e^{\frac{\pi icz^2}{c\tau+d}}\vartheta(z;\tau),$$

where $\psi(\gamma)$ is the 24th root of unity occurring in the transformation law of η.

To describe the transformation law of the Jacobi form Φ, we define

$$L:=\begin{pmatrix}2M & m_1 & \cdots & m_s & n_1 & \cdots & n_t\\ m_1 & m_1 & 0 & & \cdots & & 0\\ \vdots & 0 & \ddots & & & & \\ m_s & & & m_s & \ddots & & \vdots\\ n_1 & \vdots & & \ddots & -n_1 & & \\ \vdots & & & & & \ddots & 0\\ n_t & 0 & & \cdots & & 0 & -n_t\end{pmatrix} \tag{3.2}$$

and denote by $\mathbf{b}$ the vector

$$\mathbf{b}=(\mathbf{b}_s,\mathbf{b}_t):=(m_1,\cdots,m_s,n_1,\cdots,n_t). \tag{3.3}$$

Lemma 3.1 immediately implies the following transformation properties for Φ.

Lemma 3.2. *The function Φ defined in* (1.3) *satisfies the following transformation properties:*

1. *For all $(\lambda,\boldsymbol{\lambda})\in\mathbb{Z}^{s+t+1}$ and $(\mu,\boldsymbol{\mu})\in\mathbb{Z}^{s+t+1}$, we have*
$$\Phi\left(z+\lambda\tau+\mu,\mathbf{u}+\boldsymbol{\lambda}\tau+\boldsymbol{\mu};\tau\right)=(-1)^{\mathbf{b}\cdot\boldsymbol{\mu}+(\mathbf{0},\mathbf{b}_t)\cdot\boldsymbol{\lambda}}q^{-\frac{1}{2}L[(\lambda,\boldsymbol{\lambda})]}e^{-2\pi i(z,\mathbf{u})^TL(\lambda,\boldsymbol{\lambda})}\Phi\left(z,\mathbf{u};\tau\right).$$
2. *For all $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(2)$, we have*
$$\Phi\left(\frac{z}{c\tau+d},\frac{\mathbf{u}}{c\tau+d};\gamma\tau\right)=\chi^*(\gamma)(c\tau+d)^Me^{\frac{1}{2}}\left(\frac{c}{c\tau+d}L[(z,\mathbf{u})]\right)\Phi(z,\mathbf{u};\tau).$$
Here $\chi^*(\gamma):=\psi(\gamma)^{6M}(-1)^{\frac{mc}{4}}$, $m=\sum_{r=1}^{s}m_r$, $M=\frac{m-n}{2}$, and $\psi(\gamma)$ as in Lemma 3.1.

In the remainder of this section, we provide a decomposition of the multivariable Jacobi form Φ into a *finite* and a *polar part*. We follow the procedure used in [4] for the one-variable case. This technique was originally introduced by Dabholkar, Murthy, and Zagier [12] in the context of 1-variable Jacobi forms with poles of order at most 2, which arose when studying generating functions for quantum black holes.

For a fixed $\omega \in \mathbb{C}$, we define

$$h_\ell^{(\omega)}(\mathbf{u};\tau) := q^{-\frac{\ell^2}{4M}} e^{-2\pi i \frac{\ell}{2M}\mathbf{b}\cdot\mathbf{u}} \int_\omega^{\omega+1} \Phi(z,\mathbf{u};\tau) e^{-2\pi i \ell z} dz. \tag{3.4}$$

If Φ would be holomorphic, then the integral in the definition of $h_\ell^{(\omega)}$ would neither depend on the path of integration, nor on the choice of ω, and the function $h_\ell^{(\omega)}$ would also just depend on $\ell \pmod{2M}$. In our situation this is no longer true. Following the argument in [12], we define the *canonical Fourier coefficients of* Φ as

$$h_\ell(\mathbf{u};\tau) := h_\ell^{\left(-\frac{\ell\tau}{2M}\right)}(\mathbf{u};\tau), \tag{3.5}$$

where now the path of integration is the straight line, with an appropriate modification if a pole lies on it. With this choice, using Lemma 3.2, it is easy to show that

$$h_\ell(\mathbf{u};\tau) = h_{\ell+2M}(\mathbf{u};\tau).$$

As in the 1-variable case, we construct the finite part of Φ out of h_ℓ,

$$\Phi^F(z,\mathbf{u};\tau) := \sum_{\ell \pmod{2M}} h_\ell(\mathbf{u};\tau)\vartheta_{M,\ell}\left(z + \frac{1}{2M}\mathbf{b}\cdot\mathbf{u};\tau\right).$$

Moreover, we define the polar part of Φ as

$$\begin{aligned}\Phi^P(z,\mathbf{u};\tau) := -\sum_{j=1}^{t}\sum_{\lambda=1}^{n_j} \frac{\widetilde{D}_{\lambda,j}(\mathbf{u};\tau)}{(\lambda-1)!}\delta_\varepsilon^{\lambda-1}\Bigg[f_M\left(z+\frac{1}{2M}\mathbf{b}\cdot\mathbf{u}, \frac{1}{2M}\mathbf{b}\cdot\mathbf{u}+w_j+\varepsilon;\tau\right) \\ - \sum_{\ell \pmod{2M}} \mathcal{E}_{M,\ell}\left(\frac{1}{2M}\mathbf{b}\cdot\mathbf{u}+w_j+\varepsilon;\tau\right)\vartheta_{M,\ell}\left(z+\frac{1}{2M}\mathbf{b}\cdot\mathbf{u};\tau\right)\Bigg]_{\varepsilon=0},\end{aligned} \tag{3.6}$$

where $\delta_\varepsilon := \frac{1}{2\pi i}\frac{\partial}{\partial\varepsilon}$, f_M is the Appell-Lerch sum defined in (2.3), and the function $\mathcal{E}_{M,\ell}$ is given by

$$\mathcal{E}_{M,\ell}(\varepsilon;\tau) := \sum_{\substack{\lambda\in\mathbb{Z} \\ \lambda\equiv\ell \pmod{2M}}} \frac{1}{2}\left(\operatorname{sgn}\left(\lambda+\frac{1}{2}\right) - \operatorname{sgn}\left(\lambda + 2M\operatorname{Im}(\varepsilon)\right)\right) q^{-\frac{\lambda^2}{4M}} e^{-2\pi i\lambda\varepsilon}. \tag{3.7}$$

Note that $\mathcal{E}_{M,\ell}$ is a polynomial as a function of q. Furthermore, the $\widetilde{D}_{\lambda,j}$s are the Laurent coefficients of Φ at the pole w_j, namely, as $\varepsilon \to 0$, we have

$$\Phi(\varepsilon + w_j, \mathbf{u};\tau) = \sum_{\lambda=1}^{n_j} \frac{\widetilde{D}_{\lambda,j}(\mathbf{u};\tau)}{(2\pi i\varepsilon)^\lambda} + O(1). \tag{3.8}$$

Similarly to the proof in the 1-variable case (see for instance [4], [12], and [23]), we obtain the decomposition for Φ.

Proposition 3.3. *With the notation as above, we have*

$$\Phi(z,\mathbf{u};\tau) = \Phi^F(z,\mathbf{u};\tau) + \Phi^P(z,\mathbf{u};\tau).$$

4. Almost holomorphic Jacobi forms

A special role in this paper is played by a certain elementary real-analytic function G (see (4.3)), that allows us to rewrite (3.6) in terms of functions described in Section 2. It turns out that (3.6) can be written in terms of certain functions which only depend on τ and a single elliptic variable

$$w = w^{(j)} := \frac{1}{2M}\mathbf{b}\cdot\mathbf{u} + w_j, \tag{4.1}$$

where $\mathbf{b}$ is defined in (3.3). We omit the index j from the notation when it is clear from the context.

Let F be the function defined by

$$F(\varepsilon, w; \tau) := e^M\left(\frac{(\varepsilon + w - \overline{w})^2}{\tau - \overline{\tau}}\right), \tag{4.2}$$

and G its normalization

$$G(\varepsilon, w; \tau) := \frac{F(\varepsilon, w; \tau)}{F(0, w; \tau)}. \tag{4.3}$$

Remark 4.1. Note that the new elliptic variable w is just a linear combination of the elliptic variables $\mathbf{u}$. In particular, if we look at F as a function of $\mathbf{u}$, applying the transformation $\mathbf{u} \mapsto \mathbf{u} + \boldsymbol{\lambda}\tau + \boldsymbol{\mu}$ is equivalent of applying $w \mapsto w + \lambda\tau + \mu$, with $\lambda := \frac{1}{2M}\mathbf{b}\cdot\boldsymbol{\lambda} + \lambda_{s+j}$ and $\mu := \frac{1}{2M}\mathbf{b}\cdot\boldsymbol{\mu} + \mu_{s+j}$.

As we see in Theorem 4.2, the Laurent coefficients $\widetilde{D}_{r,j}$ (see (3.8)) can be completed to an almost holomorphic modular form $D_{r,j}$ defined as

$$D_{r,j}(\mathbf{u}; \tau) := \sum_{\kappa=0}^{n_j - r} \widetilde{D}_{r+\kappa,j}(\mathbf{u}; \tau)\frac{1}{\kappa!}\delta_\varepsilon^\kappa \left[G(\varepsilon, w; \tau)\right]_{\varepsilon=0}. \tag{4.4}$$

Since the $\widetilde{D}_{r,j}$s are the Laurent coefficients of Φ in w_j, it follows immediately that the $D_{r,j}$s are the Laurent coefficients of $G\Phi$, namely, as $\varepsilon \to 0$, we have

$$G(\varepsilon, w; \tau)\Phi(\varepsilon + w_j, \mathbf{u}; \tau) = \sum_{r=1}^{n_j} \frac{D_{r,j}(\mathbf{u}; \tau)}{(2\pi i\varepsilon)^r} + O(1). \tag{4.5}$$

Theorem 4.2. *The functions $D_{r,j}$ are almost holomorphic Jacobi forms.*

In order to prove Theorem 4.2, we need to understand the transformation properties of G.

Lemma 4.3. *The function G satisfies the following transformation properties.*

(1) *For all λ, $\mu \in \mathbb{Z}$, we have*

$$G(\varepsilon, w + \lambda\tau + \mu; \tau) = e^M(2\lambda\varepsilon)\, G(\varepsilon, w; \tau).$$

(2) *For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have*

$$G\left(\frac{\varepsilon}{c\tau + d}, \frac{w}{c\tau + d}; \gamma\tau\right) = e^{-M}\left(\frac{c}{c\tau + d}\left(\varepsilon^2 + 2\varepsilon w\right)\right) G(\varepsilon, w; \tau).$$

Proof: The proof is just a direct computation and follows immediately from the following transformation properties of F, which are easily verified:

(1) For all λ, $\mu \in \mathbb{Z}$, we have

$$F(\varepsilon, w+\lambda\tau+\mu;\tau) = e^M\left(\lambda^2(\tau-\overline{\tau})+2\lambda(\varepsilon+w-\overline{w})\right)F(\varepsilon,w;\tau).$$

(2) For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$F\left(\frac{\varepsilon}{c\tau+d}, \frac{w}{c\tau+d}; \gamma\tau\right) = e^M\left(-\frac{c}{c\tau+d}\left(\varepsilon+w\right)^2+\frac{c}{c\overline{\tau}+d}\overline{w}^2\right)F\left(\varepsilon,w;\tau\right).$$

□

We now have all the ingredients to prove Theorem 4.2.

Proof of Theorem 4.2: In what follows, the matrix L^* is defined in (2.2), while $\mathbf{b}$, $\widetilde{L}$, and M are the blocks of the matrix L as described in (2.1). Similarly to the 1-dimensional case (see [4, 13]) one obtains the following transformation properties

(1) For all $\boldsymbol{\lambda}$, $\boldsymbol{\mu} \in \mathbb{Z}^{s+t}$ such that $\frac{1}{2M}\mathbf{b}\cdot\boldsymbol{\lambda}$ and $\frac{1}{2M}\mathbf{b}\cdot\boldsymbol{\lambda} \in \mathbb{Z}$, one has

$$D_{r,j}\left(\mathbf{u}+\boldsymbol{\lambda}\tau+\boldsymbol{\mu};\tau\right) = (-1)^{\mathbf{b}\cdot\boldsymbol{\mu}+(\mathbf{0},\mathbf{b}_t)\cdot\boldsymbol{\lambda}}e^{-M}\left(\lambda^2\tau+2\lambda w\right)q^{-\frac{1}{2}L^*[\boldsymbol{\lambda}]}e\left(-\mathbf{u}^TL^*\boldsymbol{\lambda}\right)D_{r,j}\left(\mathbf{u};\tau\right),$$

where λ and μ are defined in Remark 4.1.

(2) For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$D_{r,j}\left(\frac{\mathbf{u}}{c\tau+d};\gamma\tau\right) = \chi^*(\gamma)(c\tau+d)^{M-r}e^{\frac{1}{2}}\left(\frac{c}{c\tau+d}L^*[\mathbf{u}]\right)e^M\left(\frac{c}{c\tau+d}w^2\right)D_{r,j}\left(\mathbf{u};\tau\right),$$

where χ^* is as in Lemma 3.2.

The polynomial structure with holomorphic coefficients follows from (4.4), since the terms $\delta_\varepsilon^r\left[G\left(\varepsilon,w;\tau\right)\right]_{\varepsilon=0}$ are polynomials in $\frac{w-\overline{w}}{\tau-\overline{\tau}}$ and $\frac{1}{\overline{\tau}-\tau}$ by construction. □

We conclude the section by rewriting Φ^P in terms of $D_{r,j}$. For this, we use a similar trick as in the proof of the 1-variable case [4].

Proposition 4.4. *With the notation as above, the polar part of Φ can be written as*

$$\Phi^P(z,\mathbf{u};\tau) = -\sum_{j=1}^{t}\sum_{r=1}^{n_j}\frac{D_{r,j}(\mathbf{u};\tau)}{(r-1)!}\delta_\varepsilon^{r-1}\left[\frac{f_M\left(z+\frac{1}{2M}\mathbf{b}\cdot\mathbf{u},w^{(j)}+\varepsilon;\tau\right)}{G\left(\varepsilon,w^{(j)};\tau\right)}\right.$$
$$\left.-\sum_{\ell \pmod{2M}}\frac{\mathcal{E}_{M,\ell}\left(w^{(j)}+\varepsilon;\tau\right)\vartheta_{M,\ell}\left(z+\frac{1}{2M}\mathbf{b}\cdot\mathbf{u};\tau\right)}{G\left(\varepsilon,w^{(j)};\tau\right)}\right]_{\varepsilon=0}. \tag{4.6}$$

5. The completions of Φ^F and Φ^P

In this section we complete the two functions that occur in the canonical decomposition of Φ, namely Φ^F and Φ^P. To be more precise, we define the completion $\widehat{\Phi}^P$ of Φ^P by

$$\widehat{\Phi}^P(z,\mathbf{u};\tau) := \Phi^P(z,\mathbf{u};\tau) + \sum_{\ell \pmod{2M}}\sum_{j=1}^{t}\sum_{r=1}^{n_j}\frac{D_{r,j}(\mathbf{u};\tau)}{(r-1)!}$$
$$\times\,\delta_\varepsilon^{r-1}\left[\frac{\mathcal{R}_{M,\ell}\left(\varepsilon+w^{(j)};\tau\right)}{G\left(\varepsilon,w^{(j)};\tau\right)}\right]_{\varepsilon=0}\vartheta_{M,\ell}\left(z+\frac{1}{2M}\mathbf{b}\cdot\mathbf{u};\tau\right), \tag{5.1}$$

where

$$\mathcal{R}_{M,\ell}(u;\tau) := \frac{1}{2} R_{M,\ell}(u;\tau) - \mathcal{E}_{M,\ell}(u;\tau). \tag{5.2}$$

Here $D_{\lambda,j}$ is the completion of the Laurent coefficients of Φ at w_j given in (4.4), $R_{M,\ell}$ is the real analytic function defined in (2.4), and $\mathcal{E}_{M,\ell}$ is the function given in (3.7). Furthermore, we define the completion $\widehat{\Phi}^F$ of Φ^F by

$$\widehat{\Phi}^F(z,\mathbf{u};\tau) := \sum_{\ell \pmod{2M}} \widehat{h}_\ell(\mathbf{u};\tau)\vartheta_{M,\ell}\left(z + \frac{1}{2M}\mathbf{b}\cdot\mathbf{u};\tau\right),$$

where

$$\widehat{h}_\ell(\mathbf{u};\tau) := h_\ell(\mathbf{u};\tau) - \sum_{j=1}^{t}\sum_{r=1}^{n_j} \frac{D_{r,j}(\mathbf{u};\tau)}{(r-1)!}\delta_\varepsilon^{r-1}\left[\frac{\mathcal{R}_{M,\ell}\left(\varepsilon + w^{(j)};\tau\right)}{G\left(\varepsilon, w^{(j)};\tau\right)}\right]_{\varepsilon=0}. \tag{5.3}$$

In the following proposition we show that the completed function $\widehat{\Phi}^P$ just depends on the poles of Φ, as well as Φ^P. Furthermore, we prove that both $\widehat{\Phi}^P$ and $\widehat{\Phi}^F$ transform as multivariable Jacobi forms.

Proposition 5.1. *With the notation as above, one has*

$$\widehat{\Phi}^P(z,\mathbf{u};\tau) = -\sum_{j=1}^{t}\sum_{r=1}^{n_j} \frac{D_{r,j}(\mathbf{u};\tau)}{(r-1)!}\delta_\varepsilon^{r-1}\left[\frac{\widehat{f}_M\left(z + \frac{1}{2M}\mathbf{b}\cdot\mathbf{u}, w^{(j)} + \varepsilon;\tau\right)}{G\left(\varepsilon, w^{(j)};\tau\right)}\right]_{\varepsilon=0}.$$

Furthermore, the functions $\widehat{\Phi}^F$ and $\widehat{\Phi}^P$ satisfy the same modular and elliptic transformation properties as Φ (see Lemma 3.2).

Proof: The new expression for $\widehat{\Phi}^P$ follows immediately by plugging (2.5) into (4.6) and (5.1). In order to show that $\widehat{\Phi}^P$ transforms in the same way as Φ, it is enough to note that Proposition 2.6, Theorem 4.2, and Lemma 4.3 imply that each summand in the definition of $\widehat{\Phi}^P$ gives the same automorphy factor as Φ.

To prove the transformation properties of $\widehat{\Phi}^F$, note that splitting $\widehat{h}_\ell$ according to (5.3) we may rewrite $\widehat{\Phi}^F$ as

$$\widehat{\Phi}^F(z,\mathbf{u};\tau) = \Phi^F(z,\mathbf{u};\tau) - \sum_{\ell \pmod{2M}}\sum_{j=1}^{t}\sum_{r=1}^{n_j}\frac{D_{r,j}(\mathbf{u};\tau)}{(r-1)!}$$
$$\times\, \delta_\varepsilon^{r-1}\left[\frac{\mathcal{R}_{M,\ell}\left(\varepsilon + w^{(j)};\tau\right)}{G\left(\varepsilon, w^{(j)};\tau\right)}\right]_{\varepsilon=0}\vartheta_{M,\ell}\left(z + \frac{1}{2M}\mathbf{b}\cdot\mathbf{u};\tau\right).$$

From (5.1) it is clear that $\Phi = \Phi^F + \Phi^P = \widehat{\Phi}^F + \widehat{\Phi}^P$, and as a consequence we obtain the transformation properties of $\widehat{\Phi}^F$. □

As a consequence of Proposition 5.1, we can describe the modular transformation law of the canonical Fourier coefficients $\widehat{h}_\ell$. This will be used in Section 7 in order to prove Theorem 1.2.

Corollary 5.2. *The canonical Fourier coefficients $\widehat{h}_\ell$ transform as (vector-valued) Jacobi forms of weight $M - \frac{1}{2}$ and index L^* for $\Gamma_0(2)$.*

Proof: Writing $\widehat{\Phi}^F$ as a theta decomposition, one uses the modularity (resp. elliptic) property of both $\widehat{\Phi}^F$ and $\vartheta_{M,\ell}$. The conclusion follows from the linear independence of $(\vartheta_{M,\ell})_{\ell \pmod{2M}}$. Furthermore, the multiplier system of the vector-valued transformation is the standard one for the theta coefficients of a Jacobi form. □

6. Action of some operators

In this section, we describe the functions appearing in (5.3) in terms of the lowering and the raising operators introduced in Section 2.4. We first state the relevant results and prove them later in this section.

Proposition 6.1. *For all $\lambda \in \{1, \cdots, n_j\}$, we have*

$$X_-\left(D_{\lambda,j}(\mathbf{u};\tau)\right) = \frac{M}{4\pi} D_{\lambda+2,j}(\mathbf{u};\tau).$$

In particular, for $\lambda \in \{1,2\}$ and $n \in \mathbb{N}_0$, we have

$$X_-^n\left(D_{\lambda,j}(\mathbf{u};\tau)\right) = \left(\frac{M}{4\pi}\right)^n D_{\lambda+2n,j}(\mathbf{u};\tau).$$

Moreover, we show how to use the function F defined in (4.2) to pass from the differential operator δ_ε to the raising operator $X_+^{k,M}$. This extends previous work of the authors [4, 23].

Proposition 6.2. *For all $\lambda \in \mathbb{N}_0$, we have*

$$F(w;\tau)\delta_w^{2\lambda}\left[\frac{\mathcal{R}_{M,\ell}(w;\tau)}{F(w;\tau)}\right] = \left(\frac{M}{\pi}\right)^\lambda \left(X_+^{\frac{1}{2},-M}\right)^\lambda \left(\mathcal{R}_{M,\ell}(w;\tau)\right),$$

$$F(w;\tau)\delta_w^{2\lambda+1}\left[\frac{\mathcal{R}_{M,\ell}(w;\tau)}{F(w;\tau)}\right] = -\frac{1}{2\pi}\left(\frac{M}{\pi}\right)^\lambda \left(X_+^{\frac{3}{2},-M}\right)^\lambda \left(Y_+^{\frac{1}{2},-M}\left(\mathcal{R}_{M,\ell}(w;\tau)\right)\right).$$

We conclude this section by stating a fundamental property satisfied by $\mathcal{R}_{M,\ell}$. It turns out that it is annihilated by various operators introduced in Section 2.

Proposition 6.3. *The function $\mathcal{R}_{M,\ell}$ is annihilated by the heat operator H_{-M}, the Heisenberg operator Δ_{-M}^H, and the Casimir operator $\mathcal{C}_{\frac{1}{2},-M}$. Moreover, the function $Y_+^{\frac{1}{2},-M}\mathcal{R}_{M,\ell}$ is an eigenfunction with respect to $\mathcal{C}_{\frac{3}{2},-M}$ of eigenvalue $\frac{1}{2}$.*

6.1. **Laurent coefficients and the lowering operator.** In this subsection, we prove Proposition 6.1. The main step is the proof of the following lemma.

Lemma 6.4. *For each positive integer $r > 1$, one has*

$$X_-\left(\delta_\varepsilon^r\left[G(\varepsilon,w;\tau)\right]_{\varepsilon=0}\right) = \frac{r(r-1)M}{4\pi}\delta_\varepsilon^{r-2}\left[G(\varepsilon,w;\tau)\right]_{\varepsilon=0}.$$

Moreover, for $r \in \{0,1\}$,

$$X_-\left(\delta_\varepsilon^r\left[G(\varepsilon,w;\tau)\right]_{\varepsilon=0}\right) = 0. \tag{6.1}$$

Proof: We proceed by induction on r. For $r \in \{0,1\}$, the proof is straightforward. It is enough to check that 1 and $\frac{w-\overline{w}}{\tau-\overline{\tau}}$ are annihilated by X_-.

We denote by $G^{(r)} := \delta_\varepsilon^r\left[G(\varepsilon,w;\tau)\right]_{\varepsilon=0}$. First, we note that for all $r > 1$ one has

$$G^{(r)} = G^{(r-1)}G^{(1)} + (r-1)\frac{M}{\pi i(\tau-\overline{\tau})}G^{(r-2)}. \tag{6.2}$$

Letting $r = 2$, then applying the lowering operator and using (6.1) yields

$$X_-\left(G^{(2)}\right) = X_-\left(G^{(1)}\right)G^{(1)} + \frac{M}{\pi i(\tau-\overline{\tau})}X_-\left(G^{(0)}\right) + \frac{M}{2\pi}G^{(0)}.$$

Using (6.1) we prove the claim.

Assume that the statement is true for all $0 \leq s < r$. Applying the lowering operator and using (6.1) yields

$$X_-\left(G^{(r)}\right) = X_-\left(G^{(r-1)}\right) G^{(1)} + (r-1)\frac{M}{\pi i(\tau-\overline{\tau})} X_-\left(G^{(r-2)}\right) + (r-1)\frac{M}{2\pi} G^{(r-2)}. \tag{6.3}$$

By induction, we may rewrite the right-hand side of (6.3) as

$$\begin{aligned}&\frac{M(r-1)(r-2)}{4\pi} G^{(r-3)} G^{(1)} + (r-1)\frac{M}{\pi i(\tau-\overline{\tau})}\frac{M(r-2)(r-3)}{4\pi} G^{(r-4)} + (r-1)\frac{M}{2\pi} G^{(r-2)}\\ &= \frac{M(r-1)(r-2)}{4\pi}\left(G^{(r-3)} G^{(1)} + \frac{M(r-3)}{i\pi(\tau-\overline{\tau})} G^{(r-4)}\right) + \frac{M(r-1)}{2\pi} G^{(r-2)}.\end{aligned}$$

Again using (6.2), this equals

$$\frac{M(r-1)(r-2)}{4\pi} G^{(r-2)} + \frac{M(r-1)}{2\pi} G^{(r-2)} = \frac{r(r-1)M}{4\pi} G^{(r-2)}.$$

This conclude the inductive step. □

Remark 6.5. In the previous lemma, we have applied the lowering operator with respect to the elliptic variable w. In fact, w is a function of $\mathbf{u}$. More precisely, it is a linear combination of the components of $\mathbf{u}$. Therefore, one can easily see that Lemma 6.4 remains true using the lowering operator with respect to $\mathbf{u}$.

The previous lemma immediately implies Proposition 6.1.

Proof of Proposition 6.1: We just prove the first claim. The second statement follows trivially. Using (4.4) and the fact that $\widetilde{D}_{\lambda,j}$ are holomorphic functions, we can write

$$X_-\left(D_{\lambda,j}(\mathbf{u};\tau)\right) = \sum_{r=0}^{n_j-\lambda} \widetilde{D}_{\lambda+r,j}(\mathbf{u};\tau)\frac{1}{r!} X_-\left(\delta_\varepsilon^r \left[G\left(\varepsilon,w;\tau\right)\right]_{\varepsilon=0}\right).$$

By Lemma 6.4 and Remark 6.5, this equals

$$\frac{M}{4\pi}\sum_{r=2}^{n_j-\lambda} \widetilde{D}_{\lambda+r,j}(\mathbf{u};\tau)\frac{1}{(r-2)!}\delta_\varepsilon^{r-2}\left[G\left(\varepsilon,w;\tau\right)\right]_{\varepsilon=0}.$$

Changing r into $r+2$ and applying again (4.4), we conclude the proof. □

6.2. **The non-holomorphic part $\mathcal{R}_{M,\ell}$ and the raising operator.** Our goal is to write for all $\lambda \geq 1$

$$\delta_\varepsilon^{\lambda-1}\left[\frac{\mathcal{R}_{M,\ell}\left(\varepsilon+w;\tau\right)}{G(\varepsilon,w;\tau)}\right]_{\varepsilon=0} \tag{6.4}$$

in terms of the raising operator.

Remark 6.6. Note that we can write (6.4) as

$$F(0,w;\tau)\delta_z^{\lambda-1}\left[\frac{\mathcal{R}_{M,\ell}\left(z;\tau\right)}{F(z-w,w;\tau)}\right]_{z=w},$$

where F is the function defined in (4.2). Moreover, it is straightforward to see that this equals

$$F(w;\tau)\delta_w^{\lambda-1}\left[\frac{\mathcal{R}_{M,\ell}\left(w;\tau\right)}{F(w;\tau)}\right], \tag{6.5}$$

with $F(w;\tau) := F(0,w;\tau)$. Depending on the situation, we use the forms (6.4) or (6.5).

In the next proposition we show how the operator $X_+^{k+2,-M}$ commutes with ∂_w.

Proposition 6.7. *For any $f(w;\tau) \in \mathcal{C}^\infty(\mathbb{C}\times\mathbb{H})$, one has*

$$F(w;\tau)\partial_w^2\left[\frac{X_+^{k,-M}\left(f\left(w;\tau\right)\right)}{F(w;\tau)}\right] = X_+^{k+2,-M}\left(F(w;\tau)\partial_w^2\left[\frac{f\left(w;\tau\right)}{F(w;\tau)}\right]\right). \tag{6.6}$$

Proof: We proceed by comparing the left and the right hand side of (6.6). In order to simplify the notation, for $\lambda \in \mathbb{N}$, we define

$$\mathcal{F}_\lambda = \mathcal{F}_\lambda(w;\tau) := F(w;\tau)\partial_w^\lambda\left[\frac{1}{F(w;\tau)}\right],$$
$$\partial_w^\lambda\left[f\right] := \partial_w^\lambda\left[f(w;\tau)\right].$$

The left hand side of (6.6) explicitly becomes

$$\mathcal{F}_2 X_+^{k,-M}(f) + 2\mathcal{F}_1\partial_w\left[X_+^{k,-M}(f)\right] + \partial_w^2\left[X_+^{k,-M}(f)\right]. \tag{6.7}$$

Similarly, the right hand side may be written as

$$X_+^{k+2,-M}\left(\mathcal{F}_2 f + 2\mathcal{F}_1\partial_w\left[f\right] + \partial_w^2\left[f\right]\right). \tag{6.8}$$

Using the general fact that for two functions g and h,

$$X_+^{k,-M}(gh) = gX_+^{k,-M}(h) + X_+^{0,0}(g)h,$$

and noting that

$$X_+^{0,0}\left(\mathcal{F}_1\right) = 0,$$

gives that (6.8) can be written as

$$\mathcal{F}_2 X_+^{k+2,-M}\left(f\right) + X_+^{0,0}\left(\mathcal{F}_2\right)f + 2\mathcal{F}_1 X_+^{k+2,-M}\left(\partial_w\left[f\right]\right) + X_+^{k+2,-M}\left(\partial_w^2\left[f\right]\right). \tag{6.9}$$

Subtracting (6.7) from (6.8) gives that the difference between the right and the left hand side of (6.6) equals

$$\begin{aligned}\mathcal{F}_2&\left(X_+^{k+2,-M}\left(f\right) - X_+^{k,-M}(f)\right) + X_+^{0,0}\left(\mathcal{F}_2\right)f \\ &+ 2\mathcal{F}_1\left(X_+^{k+2,-M}\left(\partial_w\left[f\right]\right) - \partial_w\left[X_+^{k,-M}(f)\right]\right) + X_+^{k+2,-M}\left(\partial_w^2\left[f\right]\right) - \partial_w^2\left[X_+^{k,-M}(f)\right].\end{aligned} \tag{6.10}$$

A direct computation yields

$$X_+^{k+2,-M}\left(f\right) - X_+^{k,-M}(f) = \frac{4i}{\tau-\overline{\tau}}f,$$

$$X_+^{k+2,-M}\left(\partial_w\left[f\right]\right) - \partial_w\left[X_+^{k,-M}(f)\right] = \frac{2i}{\tau-\overline{\tau}}\partial_w\left[f\right] - 8\pi M\frac{w-\overline{w}}{(\tau-\overline{\tau})^2}f,$$

$$X_+^{k+2,-M}\left(\partial_w^2\left[f\right]\right) - \partial_w^2\left[X_+^{k,-M}(f)\right] = -16\pi M\frac{w-\overline{w}}{(\tau-\overline{\tau})^2}\partial_w\left[f\right] - \frac{8\pi M}{(\tau-\overline{\tau})^2}f.$$

Thus, we can write (6.10) as

$$\begin{aligned}\mathcal{F}_2&\frac{4i}{\tau-\overline{\tau}}f + X_+^{0,0}\left(\mathcal{F}_2\right)f \\ &+ 2\mathcal{F}_1\left(\frac{2i}{\tau-\overline{\tau}}\partial_w\left[f\right] - 8\pi M\frac{w-\overline{w}}{(\tau-\overline{\tau})^2}f\right) - 16\pi M\frac{w-\overline{w}}{(\tau-\overline{\tau})^2}\partial_w\left[f\right] - \frac{8\pi M}{(\tau-\overline{\tau})^2}f.\end{aligned} \tag{6.11}$$

To conclude the proof it is enough to show that (6.11) equals 0. This can be done with a direct computation using

$$\mathcal{F}_1 = -4\pi i M \frac{w-\overline{w}}{\tau-\overline{\tau}}, \tag{6.12}$$

$$\mathcal{F}_2 = (4\pi i M)^2 \frac{(w-\overline{w})^2}{(\tau-\overline{\tau})^2} - \frac{4\pi i M}{(\tau-\overline{\tau})}, \tag{6.13}$$

$$X_+^{0,0}\left(\mathcal{F}_2\right) = -\frac{8\pi M}{(\tau-\overline{\tau})^2}.$$

□

We now have all the ingredients to prove Proposition 6.2.

Proof of Proposition 6.2: The $\lambda = 0$ case is trivial for both of the statements. For $\lambda \geq 1$ we proceed by induction. To simplify the notation, throughout the proof we omit the variables when writing the functions.

We start by proving the first claim. For $\lambda = 1$, the left-hand side equals

$$\delta_w^2\left[\mathcal{R}_{M,\ell}\right] + 2\delta_w\left[\mathcal{R}_{M,\ell}\right] F \delta_w \left[\frac{1}{F}\right] + \mathcal{R}_{M,\ell} F \delta_w^2 \left[\frac{1}{F}\right]. \tag{6.14}$$

Using (6.12), (6.13), and the identity

$$\delta_\varepsilon^2\left[\mathcal{R}_{M,\ell}\left(\varepsilon + w;\tau\right)\right]_{\varepsilon=0} = -4M\delta_\tau\left[\mathcal{R}_{M,\ell}\left(w;\tau\right)\right],$$

which follows by Proposition 6.3, equation (6.14) equals

$$-\frac{2M}{\pi i}\left(\partial_\tau + \frac{w-\overline{w}}{\tau-\overline{\tau}}\partial_w - 2\pi i M \frac{(w-\overline{w})^2}{(\tau-\overline{\tau})^2} + \frac{1}{2(\tau-\overline{\tau})}\right)\mathcal{R}_{M,\ell}.$$

This, by definition of X_+, gives the claim for $\lambda = 1$.

Assume now that the statement is true for $\lambda - 1$, then

$$F\delta_w^{2\lambda}\left[\frac{1}{F}\mathcal{R}_{M,\ell}\right] = \left(\frac{M}{\pi}\right)^{\lambda-1} F\delta_w^2\left[\frac{1}{F}\left(X_+^{\frac{1}{2},-M}\right)^{\lambda-1}\left(\mathcal{R}_{M,\ell}\right)\right].$$

Applying Proposition 6.7 λ times, this expression equals

$$\left(\frac{M}{\pi}\right)^{\lambda-1}\left(X_+^{\frac{5}{2},-M}\right)^{\lambda-1}\left(F\delta_w^2\left[\frac{1}{F}\mathcal{R}_{M,\ell}\right]\right) = \left(\frac{M}{\pi}\right)^{\lambda}\left(X_+^{\frac{1}{2},-M}\right)^{\lambda}\left(\mathcal{R}_{M,\ell}\right),$$

as claimed.

Next we prove the second statement. For $\lambda = 1$, we have

$$F\delta_w^3\left(\frac{\mathcal{R}_{M,\ell}}{F}\right) = -\frac{1}{2\pi}F\left(\delta_w^2\left[\frac{1}{F}\right]Y_+(R) + 2\delta_w\left[\frac{1}{F}\right]\delta_w\left[Y_+(\mathcal{R}_{M,\ell})\right] + \frac{1}{F}\delta_w^2\left[Y_+(\mathcal{R}_{M,\ell})\right]\right),$$

where we have used the claim for $\lambda = 0$. Using (6.12) and (6.13), we write this as

$$-\frac{1}{2\pi}\left(\frac{iM}{\pi(\tau-\overline{\tau})} + (2M)^2\frac{(w-\overline{w})^2}{(\tau-\overline{\tau})^2} - 4M\frac{w-\overline{w}}{\tau-\overline{\tau}}\delta_w + \delta_w^2\right)\left(Y_+(\mathcal{R}_{M,\ell})\right). \tag{6.15}$$

A direct computation gives

$$\delta_w^2\left[Y_+(\mathcal{R}_{M,\ell})\right] = -4M\delta_\tau\left[Y_+(\mathcal{R}_{M,\ell})\right] + \frac{2Mi}{\pi(\tau-\overline{\tau})}Y_+(\mathcal{R}_{M,\ell}).$$

Thus, we rewrite (6.15) as

$$\frac{M}{\pi^2 i}\left(\frac{3}{2(\tau-\overline{\tau})} - 2\pi i M\frac{(w-\overline{w})^2}{(\tau-\overline{\tau})^2} + \frac{w-\overline{w}}{\tau-\overline{\tau}}\partial_w + \partial_\tau\right)\left(Y_+(\mathcal{R}_{M,\ell})\right).$$

This, by definition of $X_+^{\frac{3}{2},-M}$, concludes the proof for $\lambda = 1$. Assume now that the statement is true for $\lambda - 1$. Then

$$F\delta_w^{2\lambda+1}\left[\frac{1}{F}\mathcal{R}_{M,\ell}\right] = -\frac{1}{2\pi}\left(\frac{M}{\pi}\right)^{\lambda-1} F\delta_w^2\left[\frac{1}{F}\left(X_+^{\frac{3}{2},-M}\right)^{\lambda-1}\left(Y_+\left(\mathcal{R}_{M,\ell}\right)\right)\right].$$

Applying Proposition 6.7 λ times, we rewrite this as

$$-\frac{1}{2\pi}\left(\frac{M}{\pi}\right)^{\lambda-1}\left(X_+^{\frac{3}{2}+2,-M}\right)^{\lambda-1}\left(F\delta_w^2\left[\frac{1}{F}Y_+\left(\mathcal{R}_{M,\ell}\right)\right]\right). \tag{6.16}$$

By induction, firstly using the claim for $\lambda = 0$ and then the claim for $\lambda = 1$, we have

$$F\delta_w^2\left(\frac{1}{F}Y_+\left(\mathcal{R}_{M,\ell}\right)\right) = -2\pi F\delta_w^3\left[\frac{1}{F}\mathcal{R}_{M,\ell}\right] = \frac{M}{\pi}X_+^{\frac{3}{2},-M}\left(Y_+(\mathcal{R}_{M,\ell})\right).$$

Thus, as claimed, (6.16) equals

$$-\frac{1}{2\pi}\left(\frac{M}{\pi}\right)^{\lambda}\left(\left(X_+^{\frac{3}{2},-M}\right)^{\lambda}\left(Y_+(\mathcal{R}_{M,\ell})\right)\right).$$

□

Proof of Proposition 6.3: A special case of the fact that $\mathcal{R}_{M,\ell}$ is annihilated by H_{-M} is proved in [9] by the first author and Zwegers. In [23], the second author showed the general case.

Next, we prove that $\mathcal{R}_{M,\ell}$ is annihilated by the Heisenberg operator. In order to do so, we recall that by Lemma 1.8 of [28], we have

$$\partial_{\overline{w}}\left[\mathcal{R}_{M,\ell+M}(w;\tau)\right] = -2i\sqrt{iM}e^{-2\pi i\ell\overline{w}-\frac{\ell^2}{4M}\overline{\tau}}\vartheta\left(2M\overline{w}-\frac{1}{2}+\ell\overline{\tau};-2M\overline{\tau}\right)\frac{F(w;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}}. \tag{6.17}$$

Note that the result in Lemma 1.8 of [28] is stated in terms of a slightly different function R. We point out that $R_{M,\ell}$ can be written in terms of R with a slight modification of the elliptic variable. The operator ∂_w acts trivially on anti-holomorphic functions, thus, using (6.17), we have that $\Delta_M^H\left(\mathcal{R}_{M,\ell+M}(w;\tau)\right)$ equals

$$-\sqrt{iM}e^{-2\pi i\ell\overline{w}-\frac{\ell^2}{4M}\overline{\tau}}\vartheta\left(2M\overline{w}-\frac{1}{2}+\ell\overline{\tau};-2M\overline{\tau}\right)\left(-\frac{\tau-\overline{\tau}}{8\pi iM}\partial_w+\frac{1}{2}(w-\overline{w})\right)\left(\frac{F(w;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}}\right).$$

A direct computation gives that

$$\left(-\frac{\tau-\overline{\tau}}{8\pi iM}\partial_w+\frac{1}{2}(w-\overline{w})\right)\left(\frac{F(w;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}}\right) = 0,$$

therefore $\mathcal{R}_{M,\ell}$ is annihilated by Δ_{-M}^H.

We proceed by showing that $\mathcal{C}_{\frac{1}{2},-M}(\mathcal{R}_{M,\ell}) = 0$. For this, we use Lemma 2.8. In particular, since $\xi_{\frac{1}{2},-M}\left(\mathcal{R}_{M,\ell}\right) = 0$ (which can be seen easily by a direct computation), it remains to prove that

$$X_+^{\frac{1}{2},-M}\left(\partial_{\overline{w}}\partial_{\overline{w}}\left(\mathcal{R}_{M,\ell+M}(w;\tau)\right)\right) = 0.$$

To see this, using (6.17), we compute

$$\partial_{\overline{w}}\partial_{\overline{w}}\left(\mathcal{R}_{M,\ell+M}(w;\tau)\right) = \partial_{\overline{w}}\left[H(\overline{w};-\overline{\tau})\right]\frac{F(w;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}} + H(\overline{w};-\overline{\tau})\partial_{\overline{w}}\left[\frac{F(w;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}}\right], \tag{6.18}$$

where

$$H(w;\tau) := -2i\sqrt{iM}e^{-2\pi i\ell w+\frac{\ell^2}{4M}\tau}\vartheta\left(2Mw-\frac{1}{2}+\ell\tau;2M\tau\right).$$

Since $H(\overline{w};-\overline{\tau})$ and $\partial_{\overline{w}}\left[H(\overline{w};-\overline{\tau})\right]$ are anti-holomorphic, $X_+^{\frac{1}{2},-M}$ acts trivially on them. Therefore, applying $X_+^{\frac{1}{2},-M}$ to equation (6.18), we obtain

$$\partial_{\overline{w}}\left[H(\overline{w};-\overline{\tau})\right]X_+^{\frac{1}{2},-M}\left(\frac{F(w;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}}\right)+H(\overline{w};-\overline{\tau})X_+^{\frac{1}{2},-M}\left(\partial_{\overline{w}}\left[\frac{F(w;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}}\right]\right).$$

To conclude, a direct computation shows that

$$X_+^{\frac{1}{2},-M}\left(\frac{F(w;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}}\right)=X_+^{\frac{1}{2},-M}\left(\partial_{\overline{w}}\left[\frac{F(w;\tau)}{(\tau-\overline{\tau})^{\frac{1}{2}}}\right]\right)=0.$$

The last statement follows from the previous ones, using Lemma 2.9. □

6.3. **The action of $\xi_{k,M}$ and $\xi_{k,M}^H$.** Consider the ξ-operator (see (2.8)) introduced in [7] and the Heisenberg generalization (see equation (3.1) in [8])

$$\xi_{k,M}^H := -\sqrt{\frac{\tau-\overline{\tau}}{2iM}}e^{-2\pi iM\frac{(z-\overline{z})^2}{\tau-\overline{\tau}}}\overline{\partial_{\overline{z}}}.$$

In the following proposition we show the image of $\widehat{h}_\ell$ under the action of these two operators.

Proposition 6.8. *Let $\widehat{h}_\ell$ be the completion of the canonical Fourier coefficients of Φ as in* (5.3). *Then*

$$\xi_{\frac{1}{2},-M}\left(\widehat{h}_\ell\right)=0,$$

$$\xi_{\frac{1}{2},-M}^H\left(\widehat{h}_\ell(\mathbf{u};\tau)\right)=-\frac{1}{2}\sum_{j=1}^{t}\sum_{\lambda=1}^{n_j}\frac{\overline{\widetilde{D}_{\lambda,j}(\mathbf{u};\tau)}}{(\lambda-1)!}\delta_\varepsilon^{\lambda-1}\left[\vartheta_{M,\ell}\left(w^{(j)}+\varepsilon;\tau\right)\right]_{\varepsilon=0},$$

where $w^{(j)}$ is given in (6.16).

Proof: Using the same trick used to rewrite (3.6) as (4.6), one can see that (5.3) equals

$$\widehat{h}_\ell(\mathbf{u};\tau)=h_\ell(\mathbf{u};\tau)-\sum_{j=1}^{t}\sum_{\lambda=1}^{n_j}\frac{\widetilde{D}_{\lambda,j}(\mathbf{u};\tau)}{(\lambda-1)!}\delta_\varepsilon^{\lambda-1}\left[\frac{1}{2}R_{M,\ell}\left(w^{(j)}+\varepsilon;\tau\right)-\mathcal{E}_{M,\ell}\left(w^{(j)}+\varepsilon;\tau\right)\right]_{\varepsilon=0}.$$

Since $\xi_{\frac{1}{2},-M}$ and $\xi_{\frac{1}{2},-M}^H$ annihilate holomorphic functions and commute with δ_ε, it is enough to compute the action of these operators on $R_{M,\ell}$. A direct calculation shows that

$$\xi_{\frac{1}{2},-M}\left(R_{M,\ell}(z;\tau)\right)=0,$$

$$\xi_{\frac{1}{2},-M}^H\left(R_{M,\ell}(z;\tau)\right)=\vartheta_{M,\ell}(z;\tau),$$

proving the proposition. □

7. Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.2: The transformation properties of the functions $\widehat{h}_\ell$ are given in Corollary 5.2. It remains to show that they can be written as in (2.9). Applying Propositions 6.1 and 6.2, we rewrite (5.3) as

$$\widehat{h}_\ell(\mathbf{u};\tau) = h_\ell(\mathbf{u};\tau) - \sum_{j=1}^{t} \sum_{\lambda=0}^{\left[\frac{n_j-1}{2}\right]} \frac{4^\lambda}{(2\lambda)!} X_-^\lambda \left(D_{1,j}(\mathbf{u};\tau)\right) \left(X_+^{\frac{1}{2}}\right)^\lambda \left(\mathcal{R}_{M,\ell}(w;\tau)\right)$$
$$+ \frac{1}{2\pi} \sum_{j=1}^{t} \sum_{\lambda=0}^{\left[\frac{n_j-2}{2}\right]} \frac{4^\lambda}{(2\lambda+1)!} X_-^\lambda \left(D_{2,j}(\mathbf{u};\tau)\right) \left(X_+^{\frac{3}{2}}\right)^\lambda \left(Y_+^{\frac{1}{2},M}\left[\mathcal{R}_{M,\ell}(w;\tau)\right]\right).$$

In order to show that $\widehat{h}_\ell$ is an almost Maass-Jacobi form, it remains to prove that the function $R_{M,\ell}$ is the non-holomorphic part of a certain vector-valued H-harmonic Maass-Jacobi form. Since both $\mathcal{C}_{\frac{1}{2},-M}$ and Δ^H_{-M} annihilate holomorphic functions, Proposition 6.3 implies that it suffices to construct a real-analytic Jacobi form whose non-holomorphic part is $R_{M,\ell}$. This has been proved already in Proposition 5.1 of [8] using multivariable Appell sums introduced by Zwegers in [27].

□

We conclude this section recalling and proving Theorem 1.1.

Theorem 7.1. *The multivariable Kac-Wakimoto characters chF_ℓ are the holomorphic parts of mixed harmonic Maass-Jacobi forms $\widehat{chF}_\ell$ of weight $M-\frac{1}{2}$ and index L^* for $\Gamma_0(2)$. Furthermore, the functions $\widehat{chF}_\ell$ are annihilated by $\xi_{\frac{1}{2},-M}$, and their image under $\xi^H_{\frac{1}{2},-M}$ lies in $\widetilde{J}_{M-1,L^*}\otimes J_{\frac{1}{2},M}$.*

Proof: Using Jacobi's triple product identity (3.1), we can rewrite the generating function $\mathrm{ch}F$ for the Kac-Wakimoto characters in term of Φ, given in (1.3) (choosing $m_r = n_j = 1$ for all r and j), namely

$$\mathrm{ch}F = e^{\Lambda_0}(-1)^m i^{-n}\zeta^M q^{\frac{M}{3}}\eta(\tau)^{-2M}\left(\prod_{r=1}^{m}\beta_r^{\frac{1}{2}}\right)\left(\prod_{j=1}^{n}\beta_{m+j}^{-\frac{1}{2}}\right)\Phi\left(z+\frac{\tau}{2},\mathbf{u};\tau\right),$$

where η is the Dedekind eta function $\eta(\tau) := q^{\frac{1}{24}}\prod_{k\geq 1}\left(1-q^k\right)$. Therefore, the Kac-Wakimoto characters correspond to the canonical Fourier coefficients h_ℓ of Φ choosing $n_j = m_r = 1$ for all j and r (up to multiplication by η-powers and shifting of variables). In this case, we have

$$\widehat{h}_\ell(\mathbf{u};\tau) = h_\ell(\mathbf{u};\tau) - \sum_{j=1}^{n} D_{1,j}(\mathbf{u};\tau)\mathcal{R}_{M,\ell}(w;\tau),$$

where the Laurent coefficients $D_{1,j}(\mathbf{u};\tau)$ are weak Jacobi forms. To conclude the proof, we recall that $\mathcal{R}_{M,\ell}$ is the non-holomorphic part of a vector-valued H-harmonic Maass-Jacobi form. This follows from Theorem 4.5 in [27] and Proposition 6.3.

The images under $\xi_{\frac{1}{2},-M}$ and $\xi^H_{\frac{1}{2},-M}$ follow from Proposition 6.8. □

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