A Note on Andrews' Partitions with Parts Separated by Parity

Kathrin Bringmann and Chris Jennings-Shaffer

Dedicated to George Andrews in honor of his 80th birthday

Abstract. In this note we give three identities for partitions with parts separated by parity, which were recently introduced by Andrews.

Mathematics Subject Classification (2010). Primary 11P81, 11P84.

Keywords. Number theory, partitions, parity, modular forms, mock theta functions.

1. Introduction

Recently Andrews [1] studied integer partitions in which all parts of a given parity are smaller than those of the opposite parity. Furthermore, he considered eight subcases based on the parity of the smaller parts and parts of a given parity appearing at most once or an unlimited number of times. Following Andrews, we use "ed" for evens distinct, "eu" for evens unlimited, "od" for odds distinct, and "ou" for odds unlimited. With "zw" and "xy" from the four choices above, we let $F_{\mathrm{xy}}^{\mathrm{zw}}(q)$ denote the generating function of partitions where zw specifies the parity and condition of the larger parts and xy specifies the parity and condition of the smaller parts.

The eight relevant generating functions are

$$F_{\mathrm{eu}}^{\mathrm{ou}}(q) := \sum_{n=0}^{\infty} \frac{q^{2n}}{\left(q^2;q^2\right)_n \left(q^{2n+1};q^2\right)_\infty},$$

$$F_{\mathrm{eu}}^{\mathrm{od}}(q) := \sum_{n=0}^{\infty} \frac{q^{2n}\left(-q^{2n+1};q^2\right)_\infty}{\left(q^2;q^2\right)_n},$$

The research of the first author is supported by the Alfried Krupp Prize for Young University Teachers of the Krupp foundation and the research leading to these results receives funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant agreement n. 335220 - AQSER.

$$F_{\mathrm{ed}}^{\mathrm{ou}}(q) := \sum_{n=0}^{\infty} \frac{\left(-q^2;q^2\right)_n q^{2n+2}}{\left(q^{2n+3};q^2\right)_\infty},$$

$$F_{\mathrm{ed}}^{\mathrm{od}}(q) := \sum_{n=0}^{\infty} q^{2n+2} \left(-q^2,q^2\right)_n \left(-q^{2n+3};q^2\right)_\infty,$$

$$F_{\mathrm{ou}}^{\mathrm{eu}}(q) := \sum_{n=0}^{\infty} \frac{q^{2n+1}}{\left(q;q^2\right)_{n+1} \left(q^{2n+2};q^2\right)_\infty},$$

$$F_{\mathrm{ou}}^{\mathrm{ed}}(q) := \sum_{n=0}^{\infty} \frac{q^{2n+1}\left(-q^{2n+2};q^2\right)_\infty}{\left(q;q^2\right)_{n+1}},$$

$$F_{\mathrm{od}}^{\mathrm{eu}}(q) := \sum_{n=0}^{\infty} \frac{q^{2n+1}\left(-q;q^2\right)_n}{\left(q^{2n+2};q^2\right)_\infty},$$

$$F_{\mathrm{od}}^{\mathrm{ed}}(q) := \sum_{n=0}^{\infty} q^{2n+1} \left(-q;q^2\right)_n \left(-q^{2n+2};q^2\right)_\infty.$$

Here we are using the standard product notation $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$ for $n \in \mathbb{N}_0 \cup \{\infty\}$. We note that with the exception of $F_{\mathrm{eu}}^{\mathrm{ou}}(q)$ and $F_{\mathrm{eu}}^{\mathrm{od}}(q)$, we do not allow the subpartition consisting of the smaller parts to be empty.

Andrews' identities (after minor corrections) can be stated as

$$F_{\mathrm{eu}}^{\mathrm{ou}}(q) = \frac{1}{(1-q)\left(q^2;q^2\right)_\infty},$$

$$F_{\mathrm{eu}}^{\mathrm{od}}(q) = \frac{1}{2}\left(\frac{1}{\left(q^2;q^2\right)_\infty} + \left(-q;q^2\right)_\infty^2\right),$$

$$F_{\mathrm{ed}}^{\mathrm{ou}}(-q) = \frac{1}{2\left(-q;q^2\right)_\infty}\left((-q;q)_\infty - 1 - \sum_{n=0}^{\infty} q^{\frac{n(3n-1)}{2}}\left(1-q^n\right)\right),$$

$$F_{\mathrm{ou}}^{\mathrm{eu}}(q) = \frac{1}{1-q}\left(\frac{1}{\left(q;q^2\right)_\infty} - \frac{1}{\left(q^2;q^2\right)_\infty}\right),$$

$$F_{\mathrm{ou}}^{\mathrm{ed}}(-q) = -\frac{\left(-q^2;q^2\right)_\infty}{2}\left(2 - \frac{1}{(-q;q)_\infty} - \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q)_n^2\left(1+q^{n+1}\right)}\right),$$

$$F_{\mathrm{od}}^{\mathrm{eu}}(-q) = -\frac{1}{\left(q^2;q^2\right)_\infty}\sum_{j=1}^{\infty}\sum_{n=j}^{\infty}(-1)^{n+j} q^{\frac{n(3n+1)}{2}-j^2}\left(1-q^{2n+1}\right).$$

Surprisingly, these identities are derived with little more than the q-binomial theorem, Heine's transformation, and the Rogers-Fine identity. In the following theorem, we give new identities for $F_{\mathrm{ed}}^{\mathrm{od}}(q)$, $F_{\mathrm{od}}^{\mathrm{ed}}(q)$, and $F_{\mathrm{ou}}^{\mathrm{ed}}(-q)$.

Theorem 1.1. *The following identities hold,*

$$F_{\mathrm{ed}}^{\mathrm{od}}(q) = \frac{q\left(-q;q^2\right)_\infty}{1-q}\left(1 - \frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty}\right), \tag{1.1}$$

$$F_{\mathrm{od}}^{\mathrm{ed}}(q) = \frac{q(-q^2;q^2)_\infty}{1-q}\left(2 - \frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}\right), \tag{1.2}$$

$$F_{\mathrm{ou}}^{\mathrm{ed}}(-q) = -\frac{(-q^2;q^2)_\infty}{2}\left(2 - \frac{1}{(-q;q)_\infty} - \frac{2}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{1+q^n}\right). \tag{1.3}$$

Remark. The functions $F_{\mathrm{ed}}^{\mathrm{od}}(q)$ and $F_{\mathrm{od}}^{\mathrm{ed}}(q)$ are basically modular functions. Also we find that $F_{\mathrm{ou}}^{\mathrm{ed}}(-q)$ is related to Ramanujan's third order mock theta function $f(q)$, as

$$\begin{aligned} f(q) := \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2} &= \frac{2}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n} \\ &= 2 - \frac{2}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{1+q^n}, \end{aligned}$$

where the final equality uses Euler's pentagonal numbers theorem.

2. Proof of Theorem 1.1

To prove equations (1.1) and (1.2), we require the following q-series identity,

$$\sum_{n=0}^{\infty}\frac{(x;q)_n q^n}{(y;q)_n} = \frac{q(x;q)_\infty}{y(y;q)_\infty\left(1-\frac{xq}{y}\right)} + \frac{\left(1-\frac{q}{y}\right)}{\left(1-\frac{xq}{y}\right)}. \tag{2.1}$$

We note that (2.1) is (4.1) from [3] and was proved with Heine's transformation [4, page 241, (III.2)]. To prove equation (1.3) we require the concept of a Bailey pair and Bailey's Lemma, which are described in [2, Chapter 3]. A pair of sequences (α,β) is called a *Bailey pair relative* to $a=q$ if

$$\beta_n = \sum_{j=0}^{n}\frac{\alpha_j}{(q;q)_{n-j}(q^2;q)_{n+j}}.$$

A limiting form of Bailey's Lemma states that if (α_n,β_n) is a Bailey pair relative to q, then

$$\sum_{n=0}^{\infty} q^{n^2+n}\beta_n = \frac{1}{(q^2;q)_\infty}\sum_{n=0}^{\infty} q^{n^2+n}\alpha_n. \tag{2.2}$$

The Bailey pair we use is given by

$$\beta_n' := \frac{1}{(-q;q)_n^2(1+q^{n+1})}, \qquad \alpha_n' := \frac{2(-1)^n q^{\frac{n(n+1)}{2}}(1-q^{2n+1})}{(1-q)(1+q^n)(1+q^{n+1})}, \tag{2.3}$$

which follows from taking the Bailey pair from Theorem 8 of [5] with $a\to q$, $b=-1$, $c=-q$, and $d=-1$ and dividing both α_n and β_n by $(1+q)$.

Proof of Theorem 1.1. We find that

$$F_{\mathrm{ed}}^{\mathrm{od}}(q) = (-q;q^2)_\infty \sum_{n=1}^{\infty}\frac{(-q^2;q^2)_{n-1}q^{2n}}{(-q;q^2)_n}$$

$$= \frac{\left(-q;q^2\right)_\infty}{2}\left(-1+\sum_{n=0}^{\infty}\frac{\left(-1;q^2\right)_n q^{2n}}{\left(-q;q^2\right)_n}\right).$$

With $q \mapsto q^2$, $x=-1$, and $y=-q$, equation (2.1) implies that

$$\sum_{n=0}^{\infty}\frac{(-1;q^2)q^{2n}}{(-q;q^2)_n} = -\frac{q\left(-1;q^2\right)_\infty}{\left(-q;q^2\right)_\infty(1-q)}+\frac{1+q}{1-q}.$$

Equation (1.1) then follows after elementary simplifications.

Similarly, we have that

$$F_{\mathrm{od}}^{\mathrm{ed}}(q) = \left(-q^2;q^2\right)_\infty \sum_{n=0}^{\infty}\frac{\left(-q;q^2\right)_n q^{2n+1}}{\left(-q^2;q^2\right)_n}.$$

By applying (2.1) with $q \mapsto q^2$, $x=-q$, and $y=-q^2$, we find that

$$\sum_{n=0}^{\infty}\frac{\left(-q;q^2\right)q^{2n}}{\left(-q^2;q^2\right)_n} = -\frac{\left(-q;q^2\right)_\infty}{\left(-q^2;q^2\right)_\infty(1-q)}+\frac{2}{1-q},$$

and (1.2) follows.

For $F_{\mathrm{ou}}^{\mathrm{ed}}(q)$, we begin with Andrews' identity [1]

$$F_{\mathrm{ou}}^{\mathrm{ed}}(-q) = -\frac{\left(-q^2;q^2\right)_\infty}{2}\left(2-\frac{1}{(-q;q)_\infty}-\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q)_n^2\left(1+q^{n+1}\right)}\right).$$

By applying (2.2) to the Bailey pair (α', β') in (2.3), we have that

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q)_n^2\left(1+q^{n+1}\right)} = \frac{2}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{3n(n+1)}{2}}\left(1-q^{2n+1}\right)}{(1+q^n)\left(1+q^{n+1}\right)}.$$

We use the partial fraction decomposition

$$\frac{1-q^{2n+1}}{(1+q^n)\left(1+q^{n+1}\right)} = \frac{1}{1+q^n}-\frac{q^{n+1}}{1+q^{n+1}},$$

to deduce that

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{3n(n+1)}{2}}\left(1-q^{2n+1}\right)}{(1+q^n)\left(1+q^{n+1}\right)} = \sum_{n=0}^{\infty}(-1)^n q^{\frac{3n(n+1)}{2}}\left(\frac{1}{1+q^n}-\frac{q^{n+1}}{1+q^{n+1}}\right)$$
$$= \sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{1+q^n}.$$

Altogether this implies equation (1.3). □

By applying Theorem 1.1 part 3 of [6] to the Bailey pair $E(3)$ of [7], we find that

$$F_{\mathrm{od}}^{\mathrm{ed}}(-q) = -\frac{q\,(q;q)_\infty\left(-q^2;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty^2} \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m q^{\frac{n(n+3)}{2}+2nm+2m^2+2m}\left(1+q^{2m+1}\right).$$

As such, we have that

$$\left(\sum_{n,m\geq 0}-\sum_{n,m<0}\right)(-1)^m q^{\frac{n(n+3)}{2}+2nm+2m(m+1)} = \frac{2\left(q^2;q^2\right)_\infty}{(1+q)\left(q;q^2\right)_\infty}-\frac{\left(q^2;q^2\right)_\infty}{(1+q)\left(-q^2;q^2\right)_\infty}.$$

We note that the corresponding quadratic form is degenerate, and so a priori the modularity properties of this theta function are unclear. More generally, one can prove directly that, for $c \in \mathbb{N}$,

$$\sum_{n,m\geq 0} z^n w^m q^{n^2+2cnm+c^2m^2} = \frac{1}{1-\frac{w}{z^c}}\sum_{k=0}^{c-1}\sum_{n=0}^{\infty} z^{cn+k} q^{(cn+k)^2}\left(1-\frac{w^{n+1}}{z^{cn+c}}\right).$$

The above is a sum of partial theta functions, which sometimes combine to give a modular form.

Acknowledgments

The authors thank George Andrews, Karl Mahlburg, and the anonymous referee for their careful reading and comments on an earlier version of this manuscript.

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Kathrin Bringmann
Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
e-mail: kbringma@math.uni-koeln.de

Chris Jennings-Shaffer
Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
e-mail: cjenning@math.uni-koeln.de