AN EXTENSION OF THE HARDY-RAMANUJAN CIRCLE METHOD AND APPLICATIONS TO PARTITIONS WITHOUT SEQUENCES

KATHRIN BRINGMANN AND KARL MAHLBURG

Abstract. We develop a generalized version of the Hardy-Ramanujan "circle method" in order to derive asymptotic series expansions for the products of modular forms and mock theta functions. Classical asymptotic methods (including the circle method) do not work in this situation because such products are not modular, and in fact, the "error integrals" that occur in the transformations of the mock theta functions can (and often do) make a significant contribution to the asymptotic series. The resulting series include principal part integrals of Bessel functions, whereby the main asymptotic term can also be identified.

To illustrate the application of our method, we calculate the asymptotic series expansion for the number of partitions without sequences. Andrews showed that the generating function for such partitions is the product of the third order mock theta function χ and a (modular) infinite product series. The resulting asymptotic expansion for this example is particularly interesting because the error integrals in the modular transformation of the mock theta function component appear in the exponential main term.

1. Introduction

1.1. **Partition functions and asymptotics for harmonic Maass forms.** We begin with a general discussion of the history of the study of asymptotics for combinatorial generating functions and q-series before stating our main results in Section 1.2. Recall that an integer partition is a decomposition of a positive integer into the sum of weakly decreasing nonnegative integers, and that an *overpartition* is a partition in which the first occurrence of a part may also be overlined. Denote the number of integer partitions of n by $p(n)$, the number of partitions into distinct parts by $Q(n)$, and the number of overpartitions by $\overline{p}(n)$ (see [2] and [14] for more

---

2000 *Mathematics Subject Classification.* 11P82, 05A17, 33C10.

The first author was partially supported by NSF grant DMS-0757907 and by the Alfried Krupp prize. The second author was partially supported by NSA Grant 6917958.

combinatorial background). The corresponding generating functions are

$$\sum_{n\geq 0} p(n)q^n = \prod_{n\geq 0} \frac{1}{1-q^n} = \frac{1}{(q;q)_\infty}, \tag{1.1}$$
$$\sum_{n\geq 0} Q(n)q^n = \prod_{n\geq 0}(1+q^n) = (-q;q)_\infty,$$
$$\sum_{n\geq 0} \overline{p}(n)q^n = \prod_{n\geq 0} \frac{1+q^n}{1-q^n} = \frac{(-q;q)_\infty}{(q;q)_\infty},$$

where we use standard notation for the rising q-factorials $(a)_n = (a;q)_n := \prod_{i=0}^{n-1}(1-aq^i)$. In particular, the overpartitions are a convolution product of ordinary partitions and distinct parts partitions, so

$$\overline{p}(n) = \sum_{k=0}^{n} p(k)Q(n-k). \tag{1.2}$$

An important question in the theory of partitions is to determine exact formulas or asymptotics for functions such as $p(n)$ and its relatives. Indeed, since the generating functions in (1.1) are (essentially) meromorphic modular forms, these are special cases of the general question of determining the coefficients of modular forms. In fact, since many partition functions also have coefficients that grow monotonically, the Hardy-Ramanujan Tauberian Theorem [17] for eta-quotients shows that as $n \to \infty$, the following asymptotics hold:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}, \tag{1.3}$$
$$Q(n) \sim \frac{1}{4\sqrt[4]{3}n^{3/4}} e^{\pi\sqrt{\frac{n}{3}}},$$
$$\overline{p}(n) \sim \frac{1}{8n} e^{\pi\sqrt{n}}.$$

A key implication of Hardy and Ramanujan's result [17] is that the coefficients in a convolution product of modular forms such as the overpartition function will satisfy a logarithmic asymptotic of the form

$$(\log \overline{p}(n))^2 \sim (\log p(n))^2 + (\log Q(n))^2, \tag{1.4}$$

so that both summands from (1.2) make a predictable contribution to the overall asymptotic.

Building on Hardy and Ramanujan's earlier developments, Rademacher and Zuckerman later proved much more precise results about the coefficients of modular forms using the circle method, culminating in exact asymptotic series expansions for functions like $p(n)$ [21]. Such expansions look much like the one seen in Theorem 1.1, although the series for modular forms involve only Bessel functions rather than principal part integrals.

Another important example in the study of coefficients of hypergeometric series and automorphic forms is Ramanujan's third order *mock theta function*

$$f(q) = \sum_{n\geq 0} \alpha(n)q^n := 1 + \sum_{n\geq 1} \frac{q^{n^2}}{(-q;q)_n^2}, \tag{1.5}$$

which is famously not a modular form [22]. Instead, recent works of the first author, Ono, and Zwegers [10, 12, 24] show that $f(q)$ is best understood as the holomorphic part of a harmonic Maass form of half-integral weight. In terms of the practical application of the circle method, this means that the modular transformations of $f(q)$ yield another automorphic q=series object plus a Mordell-type integral (see Section 2); these integrals were absorbed into the error terms of the asymptotic series expansion for $\alpha(n)$ obtained by Dragonette [15] and Andrews [1].

Recent work of the first author and Ono essentially allows one to calculate *exact* series expansions for all of the above examples, and indeed for any harmonic Maass form of weight at most $1/2$ without using the circle method (although there are technical convergence issues in the case of weight equal to $1/2$) [11]. The series are derived from real analytic Maass-Poincaré series that are uniquely determined by the automorphic transformations and principal parts of the Maass forms. For example, this allowed the first author and Ono to completely prove the Andrews-Dragonnette conjecture, giving an exact formula for the coefficients $\alpha(n)$ in [10].

However, these very precise results do not apply to products of harmonic Maass forms (as the space of such automorphic forms is not closed under multiplication), and there has been recent interest in many functions of this type that arise in the study of probability, mathematical physics, and partition theory. Our main result uses calculations based on the circle method to find the asymptotic series expansion for such functions. Since our current state of knowledge does not include Poincaré series for functions in the space of harmonic Maass forms tensored with modular forms (and one should not necessarily even expect that such a basis exists), our method yields the best known asymptotics in this situation. In particular, we will use the important example of "partitions without sequences" to illustrate the application of our general results throughout the rest of the paper, but we emphasize that our approach of identifying principal part integrals in Sections 3 and 4 is widely applicable to other products of mock modular forms and modular forms.

1.2. **Partitions without sequences and the statement of the main results.**

In [3], Andrews considered partitions that do not contain any consecutive integers as parts, which had recently arisen in connection with certain probability models as well as in the study of threshold growth in cellular automata [18] (also see [6]). Adopting his notation, let $p_2(n)$ be the number of such partitions of size n. He derived the generating function

$$G_2(q) := \sum_{n\geq 0} p_2(n)q^n = \frac{(-q^3;q^3)_\infty}{(q^2;q^2)_\infty}\chi(q), \tag{1.6}$$

where

$$\chi(q) := \sum_{n\geq 0} \frac{q^{n^2}(-q;q)_n}{(-q^3;q^3)_n}$$

is another of Ramanujan's third-order mock theta functions [22].

Holroyd, Liggett and Romik [18] used clever combinatorial arguments to show that

$$r(n) \ll p_2(n) \ll nr(n), \tag{1.7}$$

where $r(n)$ are the coefficients of the infinite product from (1.6), namely

$$\xi(q) := \sum_{n\geq 0} r(n)q^n = \frac{(-q^3;q^3)_\infty}{(q^2;q^2)_\infty}.$$

This means that in the convolution product $G_2(q) = \xi(q)\chi(q)$, the exponential growth of the coefficients $p_2(n)$ is entirely due to the $\xi(q)$ factor, despite the fact that $\chi(q)$ (like any of the mock-theta functions) also has coefficients that grow exponentially. In other words, there must be a great deal of cancellation when these two series are multiplied.

Holroyd et al. also identified the main exponential growth factor, using Tauberian-type estimates to show that $\log r(n) \sim \frac{2\pi}{3}\sqrt{n}$. In fact, the circle method (or a Maass-Poincaré series decomposition as in [11]) applied to the weight $-1/2$ eta-quotient $\xi(q)$ would give the more precise estimate $r(n) \sim c \cdot \frac{1}{n} e^{\frac{2\pi}{3}\sqrt{n}}$ for some (explicit) constant c (as in Section 4).

Andrews [3] improved upon Holroyd et al's results by determining the cusp expansion of $G_2(q)$ as $q \to 1$. He rewrote the function by using a mock theta identity, replacing $\xi(q)$ by the sum of an eta-quotient and a different mock theta function. In particular, he obtained the decomposition

$$G_2(q) = \frac{(q^6;q^6)_\infty}{4(q^2;q^2)_\infty(q^3;q^3)_\infty} f(q) + \frac{3(q^3;q^3)_\infty^3}{4(q;q)_\infty(q^2;q^2)_\infty(q^6;q^6)_\infty}, \tag{1.8}$$

where $f(q)$ is as previously defined in (1.5). We denote the two terms on the right-side of equation (1.8) by

$$g_1(q) := \frac{(q^6;q^6)_\infty}{4(q^2;q^2)_\infty(q^3;q^3)_\infty} f(q), \qquad g_2(q) := \frac{3(q^3;q^3)_\infty^3}{4(q;q)_\infty(q^2;q^2)_\infty(q^6;q^6)_\infty}.$$

If $q = e^{-s}$, Andrews proved that as $s \downarrow 0$, $G_2(q)$ has the asymptotic behavior

$$G_2(q) \sim \frac{\sqrt{s}}{6\pi} \cdot e^{\frac{\pi^2}{9s}} + \frac{1}{2} \cdot e^{\frac{\pi^2}{9s}}, \tag{1.9}$$

where the two terms come from $g_1(q)$ and $g_2(q)$, respectively.

However, it requires more than the cuspidal estimate of (1.9) to determine $p_2(n)$ precisely (and it would not be enough to merely consider the other cusps). The chief technical issue is that although $G_2(q)$ essentially has weight zero modular transformation properties, it is not an automorphic form (or a holomorphic part thereof). Therefore it does not lie in the standard framework of of the circle method and/or Poincaré series, in which a modular or harmonic Maass form is determined by its "principal part", or cusp expansions. In fact, the theory developed by the first author, Ono, and Zwegers explains that $\chi(q)$ has an associated non-holomorphic part that is necessary to construct a "completed" harmonic Maass form. The principal part integrals that arise in our main theorem should be viewed as arising from the intermixing of the non-holomorphic part of $\chi(q)$ with the principal part of the eta-quotient.

We assume the definition of $\omega_{h,k}$ from Section 2. For positive integers h, h', n, k, and ν, with $hh' \equiv -1 \pmod{k}$, define the roots of unity

$$\zeta(h,n,k,\nu) := (-1)^{\nu} e^{\frac{\pi i}{k}(-2hn + h'(-3\nu^2 + (-1)^k \nu))}$$

and

$$\alpha_r(h,k) := \frac{\omega_{h,k}\,\omega_{\frac{2h}{(2,r)},\frac{k}{(2,r)}}\,\omega_{\frac{3h}{(3,r)},\frac{k}{(3,r)}}}{\omega_{\frac{6h}{(6,r)},\frac{k}{(6,r)}}}.$$

Furthermore, for any $b > 0$, define the integral

$$\mathcal{I}_{b,k,\nu}(n) := \int_{-1}^{1} \frac{\sqrt{1-x^2}}{\cosh\left(\frac{\pi i(\nu - 1/6)}{k} - \frac{\pi x\sqrt{b}}{k\sqrt{3}}\right)} I_1\left(\frac{2\pi}{k}\sqrt{2bn(1-x^2)}\right) dx,$$

where $I_1(x)$ is a modified Bessel function of the first kind, which can be defined by the integral representation (3.12). We have the following asymptotic expansion.

Theorem 1.1. *Let $N := [n^{1/2}]$. The asymptotic expansion for $p_2(n)$ is given by*

$$\begin{aligned}
p_2(n) = {} & \frac{\pi}{6\sqrt{6n}} \sum_{\substack{0 \le h < k \le N \\ (h,k)=1 \\ (6,k)=3}} \frac{\alpha_3(h,k)}{k^2} \sum_{\nu \pmod{k}} \zeta(h,n,k,\nu)\mathcal{I}_{\frac{1}{6},k,\nu}(n) \\
& + \frac{5\pi}{36\sqrt{6n}} \sum_{\substack{0 \le h < k \le N \\ (h,k)=1 \\ (6,k)=2}} \frac{\alpha_2(h,k)}{k^2} \sum_{\nu \pmod{k}} \zeta(h,n,k,\nu)\mathcal{I}_{\frac{5}{36},k,\nu}(n) \\
& + \frac{\pi}{18\sqrt{6n}} \sum_{\substack{0 \le h < k \le N \\ (h,k)=1 \\ (6,k)=1}} \frac{\alpha_1(h,k)}{k^2} \sum_{\nu \pmod{k}} \zeta(h,n,k,\nu)\mathcal{I}_{\frac{1}{18},k,\nu}(n) \\
& + \frac{\pi}{6\sqrt{n}} \sum_{\substack{0 \le h < k \le N \\ (h,k)=1 \\ (6,k)=1}} \frac{\omega_{h,k}\omega_{2h,k}\omega_{6h,k}}{k\omega_{3h,k}^3} e^{-\frac{2\pi i h n}{k}} I_1\left(\frac{2\pi}{3k}\sqrt{n}\right) + O(\log n).
\end{aligned}$$

We can also isolate part of the leading exponential term to obtain the leading terms of the asymptotic expansion for $p_2(n)$ explicity.

Theorem 1.2. *For any $0 < c < 1/8$, as $n \to \infty$ we have*

$$p_2(n) = \left(\frac{1}{4\sqrt{3}n^{3/4}} + \frac{1}{18\sqrt{2}n}\right) e^{\frac{2\pi}{3}\sqrt{n}} + O\left(\frac{e^{\frac{2\pi}{3}\sqrt{n}}}{n^{1+c}}\right).$$

Remark. Recalling (1.7) and the subsequent discussion, this asymptotic is equivalent to $p_2(n) \sim C \cdot n^{\frac{1}{4}}\, r(n)$ (up to a constant scaling). This is markedly different from the behavior of modular partition functions seen in (1.3) and (1.4); although the coefficients of $\chi(q)$ grow asymptotically with some exponential factor $e^{C\sqrt{n}}$ (the same is true for any mock theta function due to their nontrivial principal parts [13, 24]), the exponential growth of $p_2(n)$ is the same as that of $r(n)$.

Remark. The second asymptotic term for $p_2(n)$ arises from one of the principal part integrals in Theorem 1.1, and the proof in Section 4 shows that there are also products of modular forms and mock theta functions in which the dominant exponential term in the coefficient asymptotics arises from such an integral (and thus from the non-holomorphic part of the mock theta function).

Although Theorem 1.2 is a consequence of Theorem 1.1 (the two terms correspond to the terms $k = 1$ in the two sums with $(6, k) = 1$), it is not an immediate corollary, as it requires some analysis to identify the principal part of the integrals $\mathcal{I}_{b,k,\nu}(n)$.

We now describe the structure of the paper. In Section 2, we record the modular transformation laws for $G_2(q)$. Section 3 contains the proofs of several technical integral estimates. We then apply the circle method in Section 4 and prove the asymptotic expansion of Theorem 1.1. In Section 5 we analyze the exponentially dominant terms in the expansion to prove Theorem 1.2.

Acknowledgments

The authors thank Todd Kemp for helpful discussions about principal part integrals. Moreover they thank the referee for many helpful comments.

2. Modular transformation properties

In this section we determine the modular transformations for $G_2(q)$. First, adopt the notation

$$P(q) := (q; q)_\infty^{-1}.$$

If h and k are coprime positive integers, then define h' so that $hh' \equiv -1 \pmod{k}$ (if k is even, we can assume this congruence holds modulo $4k$ and if k is odd we may assume that $8|h'$). Furthermore, we introduce a complex variable z with $\mathrm{Re}(z) > 0$ such that $q = e^{\frac{2\pi i}{k}(h+iz)}$, and define $q_1 := e^{\frac{2\pi i}{k}(h'+iz^{-1})}$. The classical modular transformation for $P(q)$ can then be written as

$$P(q) = \omega_{h,k} z^{1/2} e^{\frac{\pi(z^{-1}-z)}{12k}} P(q_1), \tag{2.1}$$

where $\omega_{h,k} := e^{\pi i s(h,k)}$, and $s(h, k)$ is the standard Dedekind sum [2].

If r is a positive integer, the transformation for $P(q^r)$ follows from (2.1). To compactly write the formula, we first let $g_r := (r,k)$ and define $\rho_r := \frac{r}{g_r}$, $k_r := \frac{k}{g_r}$. We also set $q_r := e^{\frac{2\pi i}{k_r}(h'_r + iz^{-1}/\rho_r)}$ with h'_r defined so that $h'_r\rho_r \equiv h' \pmod{k_r}$. The transformation law is then

$$P(q^r) = \omega_{h\rho_r,k_r}(\rho_r z)^{1/2} e^{\frac{\pi}{12k_r}\left(\frac{z^{-1}}{\rho_r} - \rho_r z\right)} P(q_r). \tag{2.2}$$

We write q_r only when it is clear that h and k are fixed. This nonstandard notation is appealing because we can (and will) always select h' such that $\rho_r \mid h'$ (since $(\rho_r, k_r) = 1$), and thus $q_r = q_1^{g_r/\rho_r}$.

The eta-quotient component of $g_1(q)$ is $\xi(q) := \frac{P(q^2)P(q^3)}{P(q^6)}$. Since $\xi(q)$ is an eta-quotient that is *essentially* modular with respect to a congruence subgroup of level 6, we need only consider the different transformations for all possible values of $(6,k)$. We begin with the case $6 \mid k$; we have

$$\xi(q) = \frac{\omega_{h,\frac{k}{2}}\omega_{h,\frac{k}{3}}}{\omega_{h,k/6}} \cdot z^{1/2} e^{\frac{-\pi}{12k}(z^{-1}-z)} \cdot \xi(q_1). \tag{2.3}$$

Next, if $(6,k) = 2$, then

$$\xi(q) = \frac{\omega_{h,\frac{k}{2}}\omega_{3h,k}}{\omega_{3h,\frac{k}{2}}} \cdot z^{1/2} e^{\frac{\pi z}{12k} + \frac{5\pi}{36kz}} \cdot \frac{P\left(q_1^2\right) P\left(q_1^{1/3}\right)}{P\left(q_1^{2/3}\right)}. \tag{2.4}$$

Similarly, if $(6,k) = 3$, then

$$\xi(q) = \frac{\omega_{2h,k}\omega_{h,\frac{k}{3}}}{\omega_{2h,\frac{k}{3}}} \cdot z^{1/2} e^{\frac{\pi z}{12k} + \frac{\pi}{6kz}} \cdot \frac{P\left(q_1^{1/2}\right) P\left(q_1^{3}\right)}{P\left(q_1^{3/2}\right)}. \tag{2.5}$$

Finally, if $(6,k) = 1$,

$$\xi(q) = \frac{\omega_{2h,k}\omega_{3h,k}}{\omega_{6h,k}} \cdot z^{1/2} e^{\frac{\pi z}{12k} + \frac{\pi}{18kz}} \cdot \frac{P\left(q_1^{1/2}\right) P\left(q_1^{1/3}\right)}{P\left(q_1^{1/6}\right)}. \tag{2.6}$$

Now we turn to $f(q)$, whose transformation law was studied by Andrews [1], and is *essentially* of level 2. If k is even we have

$$\begin{aligned} f(q) =& (-1)^{\frac{k}{2}+1} e^{\pi i\left(\frac{h'}{2} - \frac{3h'k}{4}\right)} \omega_{h,k} z^{-\frac{1}{2}} e^{\frac{\pi\left(z^{-1}-z\right)}{12k}} f(q_1) \\ &+ \frac{2\omega_{h,k}}{k} z^{\frac{1}{2}} e^{-\frac{\pi z}{12k}} \sum_{\nu \pmod{k}} (-1)^{\nu} e^{\frac{\pi i h'(-3\nu^2+\nu)}{k}} I_{k,\nu}(z), \end{aligned} \tag{2.7}$$

where

$$I_{k,\nu}(z) := \int_{\mathbb{R}} \frac{e^{-\frac{3\pi z x^2}{k}}}{\cosh\left(\frac{\pi i(\nu - \frac{1}{6})}{k} - \frac{\pi z x}{k}\right)}\, dx. \tag{2.8}$$

We note that there is a typo regarding the term $e^{\frac{\pi(z^{-1}-z)}{12k}}$ in this transformation as it is stated in Andrews' Theorem 2.2 (although the correct formula is stated in the proof); we have also replaced his $2k$ by an even k. Throughout we will use the residues $1 \leq \nu \leq k$ in all of our calculations.

If k is odd, then we have

$$f(q) = 2(-1)^{\frac{1}{2}(k-1)} e^{\frac{3\pi i h'}{4k}} \omega_{h,k} z^{-\frac{1}{2}} e^{-\frac{2\pi}{3kz} - \frac{\pi z}{12k}} \omega\left(q_1^{\frac{1}{2}}\right) + \frac{2\sqrt{z}}{k} e^{-\frac{\pi z}{12k}} \omega_{h,k} \sum_{\nu \pmod{k}} (-1)^{\nu} e^{-\frac{3\pi i h' \nu^2}{k} - \frac{\pi i h' \nu}{k}} I_{k,\nu}(z), \tag{2.9}$$

where

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}$$

is another one of Ramanujan's third-order mock theta functions.

If $6 \mid k$, then the transformation law of g_2 is given by

$$g_2(q) = \frac{3P(q)P(q^2)P(q^6)}{4P^3(q^3)} = \frac{\omega_{h,k}\, \omega_{h,\frac{k}{2}}\, \omega_{h,\frac{k}{6}}}{\omega_{h,\frac{k}{3}}^3} \cdot g_2(q_1). \tag{2.10}$$

If $(6,k) = 2$, then

$$g_2(q) = \frac{1}{4} \frac{\omega_{h,k}\, \omega_{h,\frac{k}{2}}\, \omega_{3h,\frac{k}{2}}}{\omega_{3h,k}^3} e^{\frac{2\pi}{9kz}} \frac{P(q_1)\, P(q_1^2)\, P\left(q_1^{2/3}\right)}{P^3\left(q_1^{1/3}\right)}. \tag{2.11}$$

If $(6,k) = 3$, then

$$g_2(q) = \frac{3}{2} \frac{\omega_{h,k}\, \omega_{2h,k}\, \omega_{2h,\frac{k}{3}}}{\omega_{h,\frac{k}{3}}^3} e^{-\frac{\pi}{2kz}} \frac{P(q_1)\, P\left(q_1^{\frac{1}{2}}\right) P\left(q_1^{3/2}\right)}{P^3(q_1^3)}. \tag{2.12}$$

Finally, if $(6,k) = 1$, then

$$g_2(q) = \frac{1}{2} \frac{\omega_{h,k}\, \omega_{2h,k}\, \omega_{6h,k}}{\omega_{3h,k}^3} e^{\frac{\pi}{18kz}} \frac{P(q_1)\, P\left(q_1^{\frac{1}{2}}\right) P\left(q_1^{\frac{1}{6}}\right)}{P^3\left(q_1^{\frac{1}{3}}\right)}. \tag{2.13}$$

3. Integral estimates

In this section we prove some of the technical bounds that we will need in order to apply the circle method and prove Theorem 1.1. Specifically, we show that integrating the transformation laws from Section 2 naturally leads to Bessel functions and our $\mathcal{I}_{b,k,\nu}$. Throughout we let $0 \leq h < k \leq N$ with $(h,k) = 1$, and $z = k\left(N^{-2} - i\Phi\right)$ with $-\vartheta'_{h,k} \leq \Phi \leq \vartheta''_{h,k}$. Here

$$\vartheta'_{h,k} := \frac{1}{k(k_1+k)}, \quad \vartheta''_{h,k} := \frac{1}{k(k_2+k)},$$

where $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ are adjacent Farey fractions in the Farey sequence of order N. ¿From the theory of Farey fractions it is known that

$$\frac{1}{k+k_j} \leq \frac{1}{N+1} \qquad (j=1,2). \tag{3.1}$$

Lemma 3.1. *If $b \in \mathbb{R}$, $\nu \in \mathbb{Z}$, with $0 < \nu \leq k$, let $\mathcal{J}_{b,k,\nu}(z) := ze^{\frac{\pi b}{kz}} I_{k,\nu}(z)$, and define the principal part truncation of $\mathcal{J}_{b,k,\nu}$ as*

$$\mathcal{J}'_{b,k,\nu}(z) := \sqrt{\frac{b}{3}} \int_{-1}^{1} \frac{e^{\frac{\pi b}{kz}(1-x^2)}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x\sqrt{b}}{k\sqrt{3}}\right)}\, dx.$$

As $z \to 0$, we have the following asymptotic behavior:

1. *If $b \leq 0$, then $|\mathcal{J}_{b,k,\nu}(z)| \ll \left|\frac{\pi}{2} - \frac{\pi(\nu-1/6)}{k}\right|^{-1}$.*
2. *If $b > 0$, then $\mathcal{J}_{b,k,\nu}(z) = \mathcal{J}'_{b,k,\nu}(z) + \mathcal{E}_{b,k,\nu}$, where the error term satisfies for $0 < \nu \leq k$*

$$|\mathcal{E}_{b,k,\nu}| \ll \left|\frac{\pi}{2} - \frac{\pi(\nu-1/6)}{k}\right|^{-1}.$$

Here all of the implied constants are allowed to depend on b.

Remark. We use the terminology "principal part truncation" for the integral defining $\mathcal{J}'_{b,k,\nu}$ as it is indeed a principal part integral, whose distribution is concentrated around $x = 0$. The phrase also serves as a reminder that we will view these integrals as a continuous analogue of the principal part of a q-series in the circle method.

Proof. We begin by making the substitution $x \mapsto x/(az)$ in $I_{k,\nu}(z)$, where a is some undetermined real constant which we will select later. This means that

$$\mathcal{J}_{b,k,\nu}(z) = \frac{1}{a} e^{\frac{\pi b}{kz}} \int\limits_{S} \frac{e^{-\frac{3\pi x^2}{a^2 kz}}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x}{ak}\right)}\, dx = \frac{1}{a} e^{\frac{\pi b}{kz}} \int\limits_{\mathbb{R}} \frac{e^{-\frac{3\pi x^2}{a^2 kz}}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x}{ak}\right)}\, dx, \tag{3.2}$$

where S is the line through the origin defined by $\arg(\pm x) = \arg(z)$. The last equality follows from the facts that e^s is entire and $\cosh(s)^{-1}$ has poles only at imaginary values of s. We also need the simple observation that for a fixed z the magnitude of the integrand can be bounded by e^{-Cx^2} as $|x| \to \infty$ for some constant $C > 0$. Thus the integral along a circular path of radius R that joins S and $\mathbb{R}$ vanishes as $R \to \infty$, and Cauchy's Theorem then allows us to shift S back to the real line.

We next require the bound

$$\mathrm{Re}(z^{-1}) = \frac{N^{-2}}{kN^{-4} + k\Phi^2} \geq \frac{N^2}{k + N^2k^{-1}} \geq \frac{k}{2}, \tag{3.3}$$

which grows as $k \to \infty$. This means that the asymptotic behavior of $|e^{bz^{-1}}|$ as $k \to \infty$ (and therefore $|\mathcal{J}_{b,k,\nu}(z)|$ as well) depends on the sign of b.

If $b \leq 0$, then we may take $a = 1$ for simplicity and proceed similarly as in [1, 8]. If $\alpha \geq 0$ and $\beta \in \mathbb{R}$, we use the simple bound

$$|\cosh(\alpha + i\beta)| = |\cosh(\alpha)\cos(\beta) + i\sinh(\alpha)\sin(\beta)| \geq |\cos(\beta)| \geq \left|\sin\left(\frac{\pi}{2} - \beta\right)\right|. \tag{3.4}$$

For $0 < \beta < \pi$, this yields a simple uniform bound throughout the range $1 \leq \nu < k$, namely that

$$\left|\cosh\left(\frac{\pi i(\nu - 1/6)}{k} - \frac{\pi x}{k}\right)\right| \geq \left|\sin\left(\frac{\pi}{2} - \frac{\pi(\nu - 1/6)}{k}\right)\right| \gg \left|\frac{\pi}{2} - \frac{\pi(\nu - 1/6)}{k}\right|. \tag{3.5}$$

Combing a simple bound for the Gaussian error function with (3.3) and (3.5) then completes the proof of (1), giving

$$\begin{aligned}|\mathcal{J}_{b,k,\nu}(z)| &\leq \left|\int_{\mathbb{R}} \frac{e^{-\frac{3\pi x^2}{kz}}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x}{k}\right)}\,dx\right| \ll \left|\frac{\pi}{2} - \frac{\pi(\nu - 1/6)}{k}\right|^{-1} \sqrt{\left(\frac{\mathrm{Re}(z^{-1})}{k}\right)^{-1}} \\ &\ll \left|\frac{\pi}{2} - \frac{\pi(\nu - 1/6)}{k}\right|^{-1}.\end{aligned} \tag{3.6}$$

In the case that $b > 0$, we follow (3.2) and write

$$\mathcal{J}_{b,k,\nu}(z) = \frac{1}{a}\int_{\mathbb{R}} \frac{e^{\frac{\pi}{kz}\left(b - \frac{3x^2}{a^2}\right)}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x}{ak}\right)}\,dx. \tag{3.7}$$

The asymptotic behavior of z^{-1} (recall (3.3)) implies that the integral in (3.7) naturally splits at $b = \frac{3x^2}{a^2}$. We therefore set $a = \sqrt{\frac{3}{b}}$ for convenience, which gives

$$\mathcal{J}_{b,k,\nu}(z) = \mathcal{J}'_{b,k,\nu}(z) + \mathcal{E}_{b,k,\nu}(z),$$

where

$$\mathcal{E}_{b,k,\nu}(z) := \sqrt{\frac{b}{3}} \int_{|x|>1} \frac{e^{\frac{\pi b}{kz}(1-x^2)}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x\sqrt{b}}{k\sqrt{3}}\right)}\,dx.$$

One easily sees that

$$|\mathcal{E}_{b,k,\nu}(z)| \leq 2 \cdot \left|\sqrt{\frac{b}{3}} \int_1^\infty \frac{e^{\frac{\pi b}{kz}(1-x^2)}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x\sqrt{b}}{k\sqrt{3}}\right)}\,dx\right|. \tag{3.8}$$

Making the substitution $x \mapsto x+1$ and mimicking the arguments that led to (3.6) gives

$$|\mathcal{E}_{b,k,\nu}(z)| \ll \left| \sqrt{\frac{b}{3}} \int_0^\infty \frac{e^{\frac{\pi b}{kz}(-2x-x^2)}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi(x+1)\sqrt{b}}{k\sqrt{3}}\right)}\, dx \right| \ll \left| \frac{\pi}{2} - \frac{\pi(\nu-1/6)}{k} \right|^{-1}, \tag{3.9}$$

which completes the proof of (2). ☐

Proposition 3.2. *If $b > 0$ and $n \in \mathbb{N}$, then*

$$\begin{aligned}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi nz}{k}} \mathcal{J}'_{b,k,\nu}(z)\, d\Phi &= \frac{2\pi b}{k\sqrt{6n}} \int_{-1}^{1} \frac{\sqrt{1-x^2}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x\sqrt{b}}{k\sqrt{3}}\right)} I_1\left(\frac{2\pi}{k}\sqrt{2bn(1-x^2)}\right) dx + \mathcal{E}'_{b,k,\nu} \\ &= \frac{2\pi b}{k\sqrt{6n}} \cdot \mathcal{I}_{b,k,\nu}(n) + \mathcal{E}'_{b,k,\nu},\end{aligned}$$

with $|\mathcal{E}'_{b,k,\nu}| \ll \dfrac{1}{kN}\left|\dfrac{\pi}{2} - \dfrac{\pi(\nu-1/6)}{k}\right|^{-1}$. *Here all the implied constants may depend on b.*

Proof. We begin by symmetrizing the outer integral, writing

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{kN}}^{\frac{1}{kN}} - \int_{-\frac{1}{kN}}^{-\vartheta'_{h,k}} - \int_{\vartheta''_{h,k}}^{\frac{1}{kN}}. \tag{3.10}$$

In the second and third integrals of (3.10) (the "boundary errors"), the range is bounded away from zero as $\Phi \gg \frac{1}{kN}$. In this range, (3.3) implies that $\mathrm{Re}(z^{-1}) \ll k$ and thus $|e^{\frac{\pi b}{kz}}| = O_b(1)$. This means that even though b is positive, the bound from Lemma 3.1 part (1) still applies. The integrals are over an interval of length at most $\frac{1}{kN}$, so the overall boundary contribution is $O_b\left(\frac{1}{kN}\left|\frac{\pi}{2} - \frac{\pi(\nu-1/6)}{k}\right|^{-1}\right)$.

Now we expand the integral for $\mathcal{J}'$, recalling that $\Phi = i\left(\frac{z}{k} - \frac{1}{N^2}\right)$ and switching to the variable z:

$$\begin{aligned}\int_{-\frac{1}{kN}}^{\frac{1}{kN}} e^{\frac{2\pi nz}{k}} \mathcal{J}'_{b,k,\nu}(z)\, d\Phi &= \int_{-\frac{1}{kN}}^{\frac{1}{kN}} e^{\frac{2\pi nz}{k}} \sqrt{\frac{b}{3}} \int_{-1}^{1} \frac{e^{\frac{\pi b}{kz}(1-x^2)}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x\sqrt{b}}{k\sqrt{3}}\right)}\, dx\, d\Phi \\ &= \frac{\sqrt{b}}{ik\sqrt{3}} \int_{\frac{k}{N^2}-\frac{i}{N}}^{\frac{k}{N^2}+\frac{i}{N}} \int_{-1}^{1} \frac{e^{\frac{\pi b}{k}(1-x^2)z^{-1} + \frac{2\pi n}{k}z}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x\sqrt{b}}{k\sqrt{3}}\right)}\, dx\, dz.\end{aligned} \tag{3.11}$$

Next, we utilize a standard contour shift in the complex z-plane in order to better recognize the main term as a Bessel function. Let Γ be the counterclockwise circle that passes through the points $\frac{k}{N^2} \pm \frac{i}{N}$ and is tangent to the imaginary axis at the origin. The radius of this circle is $c = \frac{1}{2}(\frac{k}{N^2} + \frac{1}{k})$, and if $z = u + iv$, then the circle's equation is $u^2 + v^2 = 2cu$. This implies that $\mathrm{Re}(z^{-1}) = \frac{u}{u^2+v^2} = \frac{1}{2c} < k$ for all nonzero points of Γ.

In (3.11), the integrand's only pole in the z-variable is at $z = 0$, and thus Cauchy's theorem allows us to shift the original straight-line path to Γ_2, which is the portion of Γ that is to the right of $\mathrm{Re}(z) = \frac{k}{N^2}$. Let Γ_1 denote the arc of Γ to the left of this line. Along Γ_1, both z and z^{-1} have bounded real parts, and the numerator of the integrand is $O_b(1)$. Therefore (3.4) and (3.5) imply that

$$\frac{\sqrt{b}}{k\sqrt{3}} \left| \int_{\Gamma_1} \int_{-1}^{1} \frac{e^{\frac{\pi b}{k}(1-x^2)z^{-1} + \frac{2\pi n}{k} z}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x\sqrt{b}}{k\sqrt{3}}\right)} \, dx\, dz \right|$$
$$\ll \frac{1}{k} \int_{\Gamma_1} \int_{-1}^{1} \left| \frac{\pi}{2} - \frac{\pi(\nu-1/6)}{k} \right|^{-1} dx\, dz \ll \frac{1}{kN} \left| \frac{\pi}{2} - \frac{\pi(\nu-1/6)}{k} \right|^{-1}.$$

Excluding the error terms, we have now replaced the integral of $\mathcal{J}'$ by

$$\frac{\sqrt{b}}{ik\sqrt{3}} \int_{\Gamma} \int_{-1}^{1} \frac{e^{\frac{\pi b}{k}(1-x^2)z^{-1} + \frac{2\pi n}{k} z}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x\sqrt{b}}{k\sqrt{3}}\right)} \, dx\, dz$$
$$= \frac{\sqrt{b}}{ik\sqrt{3}} \int_{-1}^{1} \frac{1}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x\sqrt{b}}{k\sqrt{3}}\right)} \int_{\Gamma} e^{\frac{\pi b}{k}(1-x^2)z^{-1} + \frac{2\pi n}{k} z} \, dz\, dx.$$

The change of variables $Z = \frac{\pi b}{k}(1-x^2)z^{-1}$ takes Γ to the vertical line $\mathrm{Re}(z) = \gamma = \frac{\pi b}{2ck}(1-x^2) > 0$ and gives

$$\frac{\pi b\sqrt{b}}{ik^2\sqrt{3}} \int_{-1}^{1} \frac{(1-x^2)}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x\sqrt{b}}{k\sqrt{3}}\right)} \int_{\gamma-i\infty}^{\gamma+i\infty} Z^{-2} e^{Z + \frac{2\pi^2 nb}{k^2}(1-x^2)Z^{-1}} \, dZ\, dx.$$

Finally, we obtain the claimed formula by applying the integral representation for the modified Bessel functions I_σ (here $\sigma = \frac{2\pi^2 nb}{k^2}(1-x^2)$), namely

$$\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} t^{-r} e^{\sigma t^{-1} + t} \, dt = \sigma^{\frac{1-r}{2}} I_{r-1}(2\sqrt{\sigma}). \tag{3.12}$$

□

We next turn to the contribution of the non-holomorphic part.

Proposition 3.3. *Assuming the notation above, we have for $r > 0$ and as $n \to \infty$:*

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi}{k}\left(nz+\frac{r}{z}\right)}\, d\Phi = \frac{2\pi\sqrt{r}}{k\sqrt{n}} I_1\left(\frac{4\pi}{k}\sqrt{nr}\right) + O\left(\frac{1}{Nk}\right).$$

Proof sketch. The proof follows as in the work of Rademacher and Zuckerman [21, 20]; their setup was also used in the preceding proof of Proposition 3.2. Note that the integral in this result comes from the holomorphic part of the harmonic Maass form and has the same shape as in the case of classical modular forms. We skip the details proof here as it is essentially a known result, and is significantly easier than the above proof of Proposition 3.2 due to the absence of the ν-parameter and the $\mathcal{J}'$ function. □

4. The circle method and the proof of Theorem 1.1

4.1. **Set up.** To prove Theorem 1.1, we use the Hardy-Ramanujan method. By Cauchy's Theorem we have for $n > 0$

$$p_2(n) = \frac{1}{2\pi i} \int\limits_C \frac{G_2(q)}{q^{n+1}}\, dq,$$

where C is an arbitrary path inside the unit circle that loops around 0 in the counterclockwise direction. We chose the circle with radius $r = e^{-2\pi/N^2}$ with $N := \lfloor n^{1/2} \rfloor$, and use the parametrization $q = e^{-2\pi/N^2 + 2\pi i t}$ with $0 \le t \le 1$. This gives

$$p_2(n) = \int_0^1 G_2\left(e^{-\frac{2\pi}{N^2}+2\pi i t}\right) \cdot e^{\frac{2\pi n}{N^2} - 2\pi i n t}\, dt.$$

We let $h, k, \vartheta'_{h,k}, \vartheta''_{h,k}$ be defined as in Section 3.

We decompose the path of integration into paths along the Farey arcs $-\vartheta'_{h,k} \le \Phi \le \vartheta''_{h,k}$, where $\Phi = t - \frac{h}{k}$. Thus

$$\begin{aligned} p_2(n) &= \sum_{\substack{0 \le h < k \le N \\ (h,k)=1}} e^{-\frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} G_2\left(e^{\frac{2\pi i}{k}(h+iz)}\right) \cdot e^{\frac{2\pi n z}{k}}\, d\Phi \\ &= \sum_{\substack{0 \le h < k \le N \\ (h,k)=1}} e^{-\frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \left[g_1\left(e^{\frac{2\pi i}{k}(h+iz)}\right) + g_2\left(e^{\frac{2\pi i}{k}(h+iz)}\right)\right] \cdot e^{\frac{2\pi n z}{k}}\, d\Phi, \end{aligned} \tag{4.1}$$

where $z = k(N^{-2} - i\Phi)$ as before.

For notational convenience we group the terms based on the divisibility properties of k, writing

$$p_2(n) = \sum\nolimits_6 + \sum\nolimits_3 + \sum\nolimits_2 + \sum\nolimits_1,$$

where $\sum_d$ denotes the sum over all terms $0 \le h < k \le N$ with $(h,k) = 1$ and $(6,k) = d$.

4.2. **Estimation of $\sum_6$.** Our strategy for estimating $\sum_6$, as well as the other sums, is inspired by the classical circle method, where the asymptotic contributions are largely determined by the leading powers of q_1. Specifically, since $|q_1| = e^{-\frac{2\pi}{k}\mathrm{Re}(z^{-1})}$ and we are considering $z \to 0$, positive exponents will be absorbed into the global error. In contrast, negative exponents will lead to principal part integrals or Bessel functions depending on whether or not the term involves a Mordell integral (using Proposition 3.3 and Lemma 3.1, respectively).

Combining the transformations (2.3), (2.7), and (2.10), we have

$$\sum\nolimits_6 = S_{61} + S_{62} + S_{63} \tag{4.2}$$

$$:= \sum_{\substack{h,k \\ 6|k}} \frac{\omega_{h,k}\,\omega_{h,\frac{k}{2}}\omega_{h,\frac{k}{3}}\,(-1)^{\frac{k}{2}+1}}{\omega_{h,\frac{k}{6}}} e^{\frac{\pi i h'}{2}\left(1-\frac{3k}{2}\right)-\frac{2\pi i h n}{k}} \int\limits_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k}} g_1(q_1)\, d\Phi$$

$$+ \sum_{\substack{h,k \\ 6|k}} \frac{\omega_{h,k}\,\omega_{h,\frac{k}{2}}\,\omega_{h,\frac{k}{6}}}{\omega^3_{h,\frac{k}{3}}} e^{-\frac{2\pi i h n}{k}} \int\limits_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k}} \cdot g_2(q_1) d\Phi$$

$$+ \frac{1}{2}\sum_{\substack{h,k \\ 6|k}} \frac{\omega_{h,k}\,\omega_{h,\frac{k}{2}}\,\omega_{h,\frac{k}{3}}\, e^{\frac{-2\pi i h n}{k}}}{k\,\omega_{h,\frac{k}{6}}} \sum_{\nu \pmod k} (-1)^{\nu} e^{\frac{\pi i h'(-3\nu^2+\nu)}{k}}$$

$$\times \int\limits_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k}}\, z\, e^{-\frac{\pi}{12kz}} \xi(q_1) I_{k,\nu}(z) d\Phi.$$

Throughout the remainder of the paper, we write $\sum\limits_{h,k}$ as a shorthand for the summation conditions in (4.1).

We now estimate each of the S_{6i} and show that they are part of the error term. Using the trivial bound for all of the roots of unity we obtain the estimate

$$|S_{61}| \leq \sum_{\substack{h,k \\ 6|k}} \int\limits_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \left| e^{\frac{2\pi n z}{k}} \right| \cdot |g_1(q_1)|\; d\Phi = \sum_{\substack{h,k \\ 6|k}} \int\limits_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n}{N^2}} \cdot |g_1(q_1)|\; d\Phi. \tag{4.3}$$

Furthermore, (3.3) implies that $g_1(q_1)$ is uniformly bounded over the outer sum. Therefore, by (4.3) we have

$$|S_{61}| \ll \sum_{\substack{h,k \\ 6|k}} \frac{e^{\frac{2\pi n}{N^2}}}{kN} \ll e^{2\pi n/N^2} = O(1). \tag{4.4}$$

The same arguments also imply that $|S_{62}| = O(1)$.

This leaves S_{63}, which is complicated by the presence of the error integral $I_{k,\nu}(z)$ within the integrand. Again we use that in the domain of integration ξ is uniformly bounded. We have

$$\begin{aligned}
|S_{63}| &\leq \sum_{\substack{h,k\\6|k}} \frac{1}{2k} \sum_{\nu \pmod k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \left|e^{\frac{2\pi n z}{k}}\right| \cdot |\xi(q_1)| \cdot \left|z e^{-\frac{\pi}{12kz}}\, I_{k,\nu}(z)\right| d\Phi \\
&\ll e^{2\pi n/N^2} \sum_{\substack{h,k\\6|k}} \frac{1}{k} \sum_{\nu=1}^{k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \left|\mathcal{J}_{-\frac{1}{12},k,\nu}(z)\right| d\Phi.
\end{aligned}$$

Lemma 3.1 implies that

$$\begin{aligned}
|S_{63}| &\ll e^{2\pi n/N^2} \sum_{\substack{h,k\\6|k}} \frac{1}{k} \sum_{\nu=1}^{k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \left|\frac{\pi}{2} - \frac{\pi(\nu - 1/6)}{k}\right|^{-1} d\Phi \\
&\ll \sum_{\substack{h,k\\6|k}} \frac{1}{k^2 N} \sum_{\nu=1}^{k} \left|\frac{\pi}{2} - \frac{\pi(\nu - 1/6)}{k}\right|^{-1} d\Phi \\
&\ll \sum_{\substack{h,k\\6|k}} \frac{1}{kN} \cdot \log k = O\left(\log N\right).
\end{aligned} \tag{4.5}$$

Overall, we have proven that $\left|\sum_6\right| = O(\log N)$.

4.3. **Estimation of $\sum_2$.** By (2.4), (2.7), and (2.11), we have

$$\begin{aligned}
\sum\nolimits_2 &= S_{21}+S_{22}+S_{23}\\
&:= \frac{1}{4}\sum_{\substack{h,k\\(6,k)=2}} \frac{\omega_{h,k}\,\omega_{h,\frac{k}{2}}\,\omega_{3h,k}\,(-1)^{\frac{k}{2}+1}}{\omega_{3h,\frac{k}{2}}} e^{\frac{\pi i h'}{2}\left(1-\frac{3k}{2}\right)-\frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k}+\frac{2\pi}{9kz}} \frac{P\left(q_1^2\right)\,P\left(q_1^{1/3}\right)f(q_1)}{P\left(q_1^{2/3}\right)}d\Phi\\
&+\frac{1}{4}\sum_{\substack{h,k\\(6,k)=2}} \frac{\omega_{h,k}\,\omega_{h,\frac{k}{2}}\,\omega_{3h,\frac{k}{2}}}{\omega_{3h,k}^3} e^{-\frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k}+\frac{2\pi}{9kz}} \frac{P(q_1)\,P\left(q_1^2\right)P\left(q_1^{2/3}\right)}{P^3\left(q_1^{1/3}\right)}d\Phi\\
&+\sum_{\substack{h,k\\(6,k)=2}} \frac{\omega_{h,k}\,\omega_{h,\frac{k}{2}}\,\omega_{3h,k}}{2k\,\omega_{3h,\frac{k}{2}}} e^{\frac{-2\pi i h n}{k}} \sum_{\nu \pmod{k}} (-1)^\nu e^{\frac{\pi i h'(-3\nu^2+\nu)}{k}}\\
&\qquad\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k}}\, z\, e^{\frac{5\pi}{36kz}} \frac{P\left(q_1^2\right)\,P\left(q_1^{1/3}\right)}{P\left(q_1^{2/3}\right)} I_{k,\nu}(z)d\Phi.
\end{aligned}$$

Although S_{21} and S_{22} both have $e^{\pi/kz}$ to the positive exponent $2/9$ (and thus would individually make a nontrivial asymptotic contribution according to Lemma 3.1), we now show that the leading term cancels for all h and k in $S_{21}+S_{22}$, leaving only negative exponents (which lie in the asymptotic error term).

For odd h, we use the explicit Dedekind sum evaluation [2]

$$\omega_{h,k} = \left(\frac{-k}{h}\right)\exp\left(-\pi i\left(\frac{1}{4}(2-hk-h)+\frac{1}{12}\left(k-k^{-1}\right)\left(2h-h'+h^2h'\right)\right)\right). \tag{4.6}$$

Recalling that at the beginning of Section 2 we chose h' so that $3|h'$, it is not hard to see that

$$\frac{\omega_{h,k}\,\omega_{h,\frac{k}{2}}\,\omega_{3h,k}\,(-1)^{\frac{k}{2}+1}}{\omega_{3h,\frac{k}{2}}}\cdot e^{\frac{\pi i h'}{2}\left(1-\frac{3k}{2}\right)} = -\frac{\omega_{h,k}\,\omega_{h,\frac{k}{2}}\,\omega_{3h,\frac{k}{2}}}{\omega_{3h,k}^3}. \tag{4.7}$$

Indeed (4.7) is equivalent to the identity

$$\frac{\omega_{3h,k}^4}{\omega_{3h,\frac{k}{2}}^2} = (-1)^{k/2} e^{\frac{\pi i h'}{2}\left(\frac{3k}{2}-1\right)}.$$

By inserting (4.6) we find that the left-hand side is

$$\frac{\omega_{3h,k}^4}{\omega_{3h,\frac{k}{2}}^2} = \frac{\exp\left(-4\pi i\left(\frac{1}{4}(2-3hk-3h)+\frac{1}{12}\left(k-k^{-1}\right)\left(6h-h'/3+3h^2h'\right)\right)\right)}{\exp\left(-2\pi i\left(\frac{1}{4}(2-3hk/2-3h)+\frac{1}{12}\left(k/2-2k^{-1}\right)\left(6h-h'/3+3h^2h'\right)\right)\right)} = (-1)^{\frac{k}{2}+1} e^{\frac{\pi i h}{2}\left(\frac{k}{2}+3\right)}.$$

Now (4.7) follows by using the fact that $h \equiv -h' \pmod 4$ and k is even.

Using (4.7), $S_{21}+S_{22}$ simplifies to

$$S_{21}+S_{22}=\frac{1}{4}\sum_{\substack{h,k\\(6,k)=2}}\frac{\omega_{h,k}\,\omega_{h,\frac{k}{2}}\,\omega_{3h,\frac{k}{2}}}{\omega_{3h,k}^3}e^{-\frac{2\pi ihn}{k}}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}}e^{\frac{2\pi nz}{k}}e^{\frac{2\pi}{9kz}}P_2(q_1)d\Phi, \tag{4.8}$$

where

$$P_2(q_1):=\frac{P(q_1)\,P(q_1^2)\,P\left(q_1^{2/3}\right)}{P^3\left(q_1^{1/3}\right)}-\frac{P(q_1^2)\ P\left(q_1^{1/3}\right)f(q_1)}{P\left(q_1^{2/3}\right)}\in O\left(\left|q_1^{1/3}\right|\right). \tag{4.9}$$

Since $e^{\frac{2\pi}{9kz}}=O\left(\left|q_1^{-1/9}\right|\right)$, the integrand in (4.8) is $O\left(\left|q_1^{2/9}\right|\right)$ as a whole, and is thus uniformly bounded over the sum. In analogy with the bounds for S_{61} in Section 4.2, we can show that

$$|S_{21}+S_{22}|=O(1).$$

For the term S_{23}, we identify the portion that contributes exponential growth to the asymptotic expansion. Write

$$\begin{aligned} S_{23}&=S_{23}^e+S'_{23} \\ &:=\sum_{\substack{h,k\\(6,k)=2}}\frac{\omega_{h,k}\,\omega_{h,\frac{k}{2}}\,\omega_{3h,k}}{2k\,\omega_{3h,\frac{k}{2}}}e^{\frac{-2\pi ihn}{k}}\sum_{\nu\pmod k}(-1)^\nu e^{\frac{\pi ih'(-3\nu^2+\nu)}{k}}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}}e^{\frac{2\pi nz}{k}}\ z\,e^{\frac{5\pi}{36kz}} \\ &\qquad\times\left(\frac{P(q_1^2)\ P\left(q_1^{1/3}\right)}{P\left(q_1^{2/3}\right)}-1\right)I_{k,\nu}(z)d\Phi \\ &+\sum_{\substack{h,k\\(6,k)=2}}\frac{\omega_{h,k}\,\omega_{h,\frac{k}{2}}\,\omega_{3h,k}}{2k\,\omega_{3h,\frac{k}{2}}}e^{\frac{-2\pi ihn}{k}}\sum_{\nu\pmod k}(-1)^\nu e^{\frac{\pi ih'(-3\nu^2+\nu)}{k}}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}}e^{\frac{2\pi nz}{k}}\ z\,e^{\frac{5\pi}{36kz}}I_{k,\nu}(z)d\Phi. \end{aligned} \tag{4.10}$$

Excluding the terms z and $I_{k,\nu}$, the remaining portion of the integrand in S_{23}^e is $O\left(\left|q_1^{19/72}\right|\right)$, and thus the integrand as a whole has magnitude bounded by $|\mathcal{J}_{-\frac{19}{36},k,\nu}(z)|$. As in (4.5), Lemma 3.1 part (1) implies that $|S_{23}^e|=O(\log N)$.

For S'_{23}, we apply Lemma 3.1 part (2) and the bounds of Section 4.2 to obtain

(4.11)
$$S'_{23} = \sum_{\substack{h,k \\ (6,k)=2}} \frac{\omega_{h,k}\,\omega_{h,\frac{k}{2}}\,\omega_{3h,k}}{2k\,\omega_{3h,\frac{k}{2}}} e^{\frac{-2\pi i h n}{k}} \sum_{\nu \pmod k} (-1)^\nu e^{\frac{\pi i h'(-3\nu^2+\nu)}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k}} \mathcal{J}_{\frac{5}{36},k,\nu}(z) d\Phi$$
$$= \sum_{\substack{h,k \\ (6,k)=2}} \frac{\omega_{h,k}\,\omega_{h,\frac{k}{2}}\,\omega_{3h,k}}{2k\,\omega_{3h,\frac{k}{2}}} e^{\frac{-2\pi i h n}{k}} \sum_{\nu \pmod k} (-1)^\nu e^{\frac{\pi i h'(-3\nu^2+\nu)}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k}} \mathcal{J}'_{\frac{5}{36},k,\nu}(z) d\Phi + O(\log N).$$

Proposition 3.2 and the bounds from Section 4.2 give an asymptotic expansion for S'_{23} in Bessel functions,

$$S_{23} = \frac{5\pi}{36\sqrt{6n}} \sum_{\substack{h,k \\ (6,k)=2}} \frac{\omega_{h,k}\,\omega_{h,\frac{k}{2}}\,\omega_{3h,k}}{k^2\,\omega_{3h,\frac{k}{2}}} e^{\frac{-2\pi i h n}{k}} \sum_{\nu \pmod k} (-1)^\nu e^{\frac{\pi i h'(-3\nu^2+\nu)}{k}}$$
$$\times \int_{-1}^{1} \frac{\sqrt{1-x^2}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x\sqrt{5}}{6k\sqrt{3}}\right)} I_1\left(\frac{\pi\sqrt{10}}{3k}\sqrt{n(1-x^2)}\right) dx + O(\log N). \tag{4.12}$$

4.4. **Estimation of $\sum_3$.** Once again, we combine the transformations (2.5), (2.9), and (2.12) to write

$$\sum\nolimits_3 = S_{31} + S_{32} + S_{33}$$
$$:= \frac{1}{2} \sum_{\substack{h,k \\ (6,k)=3}} \frac{\omega_{h,k}\,\omega_{2h,k}\,\omega_{h,\frac{k}{3}}\,(-1)^{\frac{1}{2}(k-1)}}{\omega_{2h,\frac{k}{3}}} e^{\frac{3\pi i h'}{4k} - \frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k}} e^{-\frac{\pi}{2kz}} \frac{P\left(q_1^{1/2}\right) P\left(q_1^3\right) \omega\left(q_1^{1/2}\right)}{P\left(q_1^{3/2}\right)} d\Phi$$
$$+ \frac{3}{2} \sum_{\substack{h,k \\ (6,k)=3}} \frac{\omega_{h,k}\,\omega_{2h,k}\,\omega_{2h,\frac{k}{3}}}{\omega^3_{h,\frac{k}{3}}} e^{-\frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k}} e^{-\frac{\pi}{2kz}} \frac{P\left(q_1\right) P\left(q_1^{1/2}\right) P\left(q_1^{3/2}\right)}{P^3\left(q_1^3\right)} d\Phi$$
$$+ \sum_{\substack{h,k \\ (6,k)=3}} \frac{\omega_{h,k}\,\omega_{2h,k}\,\omega_{h,\frac{k}{3}}}{2k\,\omega_{2h,\frac{k}{3}}} e^{\frac{-2\pi i h n}{k}} \sum_{\nu \pmod k} (-1)^\nu e^{\frac{\pi i h'(-3\nu^2-\nu)}{k}}$$
$$\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z e^{\frac{2\pi n z}{k} + \frac{\pi}{6kz}} \frac{P\left(q_1^{1/2}\right) P\left(q_1^3\right)}{P\left(q_1^{3/2}\right)} I_{k,\nu}(z) d\Phi.$$

Following Section 4.2, both S_{31} and S_{32} are both $O(1)$.

For S_{33}, we imitate (4.10), (4.11), and (4.12), using Lemma 3.1 and Proposition 3.2 to find the asymptotic expansion

(4.13)

$$\begin{aligned} S_{33} = &\sum_{\substack{h,k \\ (6,k)=3}} \frac{\omega_{h,k}\,\omega_{2h,k}\,\omega_{h,\frac{k}{3}}}{2k\,\omega_{2h,\frac{k}{3}}} e^{\frac{-2\pi i h n}{k}} \sum_{\nu \pmod k} (-1)^{\nu} e^{\frac{\pi i h'(-3\nu^2-\nu)}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k}} \mathcal{J}'_{\frac{1}{6},k,\nu}(z) d\Phi + O(\log N) \\ = &\frac{\pi}{6\sqrt{6n}} \sum_{\substack{h,k \\ (6,k)=3}} \frac{\omega_{h,k}\,\omega_{2h,k}\,\omega_{h,\frac{k}{3}}}{k^2\,\omega_{2h,\frac{k}{3}}} e^{\frac{-2\pi i h n}{k}} \sum_{\nu \pmod k} (-1)^{\nu} e^{\frac{\pi i h'(-3\nu^2-\nu)}{k}} \\ &\times \int_{-1}^{1} \frac{\sqrt{1-x^2}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x}{3\sqrt{2}k}\right)} I_1\left(\frac{2\pi}{k\sqrt{3}}\sqrt{n(1-x^2)}\right) dx + O(\log N). \end{aligned}$$

4.5. **Estimation of $\sum_1$.** Finally, (2.6), (2.9), and (2.13) give

$$\begin{aligned} \textstyle\sum_1 &= S_{11} + S_{12} + S_{13} \\ &:= \frac{1}{2} \sum_{\substack{h,k \\ (6,k)=1}} \frac{\omega_{h,k}\,\omega_{2h,k}\,\omega_{3h,k}\,(-1)^{\frac{1}{2}(k-1)}}{\omega_{6h,k}} \cdot e^{\frac{3\pi i h'}{4k} - \frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k}} e^{-\frac{11\pi}{18kz}} \frac{P\left(q_1^{\frac{1}{2}}\right) P\left(q_1^{1/3}\right) \omega\left(q_1^{1/2}\right)}{P\left(q_1^{1/6}\right)} d\Phi \\ &+ \frac{1}{2} \sum_{\substack{h,k \\ (6,k)=1}} \frac{\omega_{h,k}\,\omega_{2h,k}\,\omega_{6h,k}}{\omega_{3h,k}^3} e^{-\frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k}} e^{\frac{\pi}{18kz}} \frac{P(q_1)\,P\left(q_1^{1/2}\right) P\left(q_1^{1/6}\right)}{P^3\left(q_1^{1/3}\right)} d\Phi \\ &+ \sum_{\substack{h,k \\ (6,k)=1}} \frac{\omega_{h,k}\,\omega_{2h,k}\,\omega_{3h,k}}{2k\,\omega_{6h,k}} e^{\frac{-2\pi i h n}{k}} \sum_{\nu \pmod k} (-1)^{\nu} e^{\frac{\pi i h'(-3\nu^2-\nu)}{k}} \\ &\qquad \times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k}}\, z\, e^{\frac{\pi}{18kz}} \frac{P\left(q_1^{1/2}\right) P\left(q_1^{1/3}\right)}{P\left(q_1^{1/6}\right)} I_{k,\nu}(z) d\Phi. \end{aligned}$$

The previous arguments and Lemma 3.1 part (1) again imply that $|S_{11}| = O(1)$. For S_{12}, we isolate the main term by writing

$$\frac{P(q_1)\,P\left(q_1^{1/2}\right) P\left(q_1^{1/6}\right)}{P^3\left(q_1^{1/3}\right)} = 1 + \left(\frac{P(q_1)\,P\left(q_1^{1/2}\right) P\left(q_1^{1/6}\right)}{P^3\left(q_1^{1/3}\right)} - 1\right).$$

As in (4.9), the total contribution of the second summand to S_{12} is $O(1)$.

Thus we are left with

$$S_{12} = \frac{1}{2} \sum_{\substack{h,k \\ (6,k)=1}} \frac{\omega_{h,k}\, \omega_{2h,k}\, \omega_{6h,k}}{\omega_{3h,k}^3} e^{-\frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k} + \frac{\pi}{18k} z^{-1}} d\Phi + O(1).$$

Using Proposition 3.3 yields

$$S_{12} = \frac{\pi}{6\sqrt{n}} \sum_{\substack{h,k \\ (6,k)=1}} \frac{\omega_{h,k}\, \omega_{2h,k}\, \omega_{6h,k}}{k\omega_{3h,k}^3} e^{-\frac{2\pi i h n}{k}} I_1\left(\frac{2\pi}{3k}\sqrt{n}\right) + O(1). \tag{4.14}$$

Finally, the asymptotic expansion for S_{13} also follows as before, and has the form
(4.15)

$$\begin{aligned} S_{13} = &\sum_{\substack{h,k \\ (6,k)=1}} \frac{\omega_{h,k}\, \omega_{2h,k}\, \omega_{3h,k}}{2k\, \omega_{6h,k}} e^{\frac{-2\pi i h n}{k}} \sum_{\nu \pmod k} (-1)^\nu e^{\frac{\pi i h'(-3\nu^2-\nu)}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi n z}{k}} \mathcal{J}'_{\frac{1}{18},k,\nu}(z) d\Phi + O(\log N) \\ = &\frac{\pi}{18\sqrt{6n}} \sum_{\substack{h,k \\ (6,k)=1}} \frac{\omega_{h,k}\, \omega_{2h,k}\, \omega_{3h,k}}{k^2\, \omega_{6h,k}} e^{\frac{-2\pi i h n}{k}} \sum_{\nu \pmod k} (-1)^\nu e^{\frac{\pi i h'(-3\nu^2-\nu)}{k}} \\ &\times \int_{-1}^{1} \frac{\sqrt{1-x^2}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x}{3\sqrt{6}k}\right)} I_1\left(\frac{2\pi}{3k}\sqrt{n(1-x^2)}\right) dx + O(\log N). \end{aligned}$$

Altogether, we have shown that $p_2(n) = S_{23} + S_{33} + S_{12} + S_{13} + O(\log N)$, and the formulas in (4.12) – (4.15) finish the proof of Theorem 1.1.

5. Principal part integrals and the proof of Theorem 1.2

Using the approximations from Section 3, we still need to find the asymptotic expansion of the integrals $\mathcal{I}_{b,k,\nu}(n)$ in order to find the asymptotic expansion for $p_2(n)$. As a first simplification, we will use the Bessel function asymptotic for $x \to \infty$ (see (4.12.7) in [5])

$$I_\ell(x) = \frac{e^x}{\sqrt{2\pi x}} + O\left(\frac{e^x}{x^{\frac{3}{2}}}\right). \tag{5.1}$$

This implies that
(5.2)

$$\mathcal{I}_{b,k,\nu}(n) = \frac{\sqrt{k}}{2\pi(2bn)^{1/4}} \int_{-1}^{1} \frac{(1-x^2)^{1/4}}{\cosh\left(\frac{\pi i(\nu-1/6)}{k} - \frac{\pi x\sqrt{b}}{k\sqrt{3}}\right)} e^{\frac{2\pi}{k}\sqrt{2bn(1-x^2)}} \left(1 + O\left(\frac{k}{\sqrt{n(1-x^2)}}\right)\right) dx.$$

We will address asymptotics in a more general setting, so we first identify the key properties of the $\mathcal{I}_{b,k,\nu}$.

Proposition 5.1. *Let* $g(x) := \dfrac{(1-x^2)^{1/4}}{\cosh(ai+bx)}$ *for* $a>0$, $b\in\mathbb{R}$. *Then* $h(x)=g(x)+g(-x)$ *is a monotonically decreasing real function for* $x\in[0,1]$.

Proof. First, $1-x^2$ is monotonically decreasing as $|x|$ increases, so we need only address the denominators. Basic algebra gives the simplification

$$\begin{aligned}\frac{1}{\cosh(ai+bx)}+\frac{1}{\cosh(ai-bx)} &= \frac{\cosh(ai-bx)+\cosh(ai+bx)}{\cosh(ai+bx)\cosh(ai-bx)}\\ &= \frac{2\cos(a)\cosh(bx)}{\cos(2a)+\cosh(2bx)}.\end{aligned}\tag{5.3}$$

Thus we need only show that $\dfrac{\cosh(bx)}{\cos(2a)+\cosh(2bx)}$ is monotonic. The derivative of this function is

$$\frac{b\cdot\sinh(bx)(\cos(2a)+\cosh(2bx))-\cosh(bx)\cdot 2b\cdot\sinh(2bx)}{(\cos(2a)+\cosh(2bx))^2}.$$

When $x=0$, this is zero, and for $x>0$, it is not difficult to show that the numerator is always negative. Thus the function is monotonically decreasing, as claimed. ☐

Note that $h(x)$ as in the preceding proposition has a Taylor series in a neighborhood of $x=0$. Furthermore, the function

$$\frac{h(x)}{\sqrt{1-x^2}}$$

is also monotonically decreasing in a neighborhood of 0, which will be helpful later in bounding error terms.

To set further notation, let

$$I^*_{f,a}(n) := 2\int_0^1 f(x)\,e^{a\pi\sqrt{n(1-x^2)}}\,dx.\tag{5.4}$$

Proposition 5.2. *Suppose that* $f(x)$ *is a positive function defined on* $[0,1]$ *that is bounded, differentiable, and monotonically decreasing, and that also has a Taylor series on some neighborhood of* 0. *If* $a>0$, *then we have for all* $0<c<\frac{1}{8}$ *as* $n\to\infty$

$$I^*_{f,a}(n)=\sqrt{\frac{2}{a}}f(0)n^{-\frac{1}{4}}e^{\pi a\sqrt{n}}+O\left(n^{-\frac{1}{4}-c}e^{\pi a\sqrt{n}}\right)$$

Proof. To finish the proof it is enough to show that

$$I^*_a(n)+J^*_a(n)\le I^*_{f,a}(n)\le I^*_a(n)\tag{5.5}$$

with

$$I^*_a(n):=\sqrt{\frac{2}{a}}f(0)n^{-\frac{1}{4}}e^{\pi a\sqrt{n}}$$

and

$$J^*_a(n)\ll n^{-\frac{1}{4}-c}e^{\pi a\sqrt{n}}.$$

We first show the upper bound. We have

$$I^*_{f,a}(n) \le 2f(0)e^{\pi a\sqrt{n}} \int_0^1 e^{\pi a\sqrt{n}\left(\sqrt{1-x^2}-1\right)}dx.$$

Using the upper bound $\sqrt{1-x^2}-1 \le -\frac{x^2}{2}$ gives

$$\begin{aligned} I^*_{f,a}(n) &\le 2f(0)e^{\pi a\sqrt{n}} \int_0^1 e^{-\frac{\pi a\sqrt{n}x^2}{2}}dx = 2f(0)e^{\pi a\sqrt{n}}n^{-\frac14}\int_0^{n^{\frac14}} e^{-\frac{\pi a x^2}{2}}dx \\ &\le 2f(0)e^{\pi a\sqrt{n}}n^{-\frac14}\int_{\mathbb{R}} e^{-\frac{\pi a x^2}{2}}dx = \sqrt{\frac{2}{a}}f(0)e^{\pi a\sqrt{n}}n^{-\frac14} = I^*_a(n). \end{aligned}$$

We next consider the lower bound. We have

$$\begin{aligned} I^*_{f,a}(n) &= 2e^{\pi a\sqrt{n}}n^{-\frac14}\int_0^{n^{\frac14}} f\left(\frac{y}{n^{\frac14}}\right)e^{\pi a\sqrt{n}\left(\sqrt{1-\frac{y^2}{\sqrt{n}}}-1\right)}dy \\ &\ge 2e^{\pi a\sqrt{n}}n^{-\frac14}\int_0^{n^c} f\left(\frac{y}{n^{\frac14}}\right)e^{\pi a\sqrt{n}\left(\sqrt{1-\frac{y^2}{\sqrt{n}}}-1\right)}dy \end{aligned} \tag{5.6}$$

for any $0 < c < \frac14$. For $|x| < \frac{1}{\sqrt{2}}$, the Taylor series expansion

$$\sqrt{1-x^2} = 1 - \sum_{n\ge1}\frac{(2n-2)!}{2^{2n-1}(n-1)!n!}x^{2n}$$

easily implies the bound

$$\sqrt{1-x^2}-1 \ge -\frac{x^2}{2} - \sum_{n\ge2}\frac{x^{2n}}{2} \ge -\frac{x^2}{2} - x^4. \tag{5.7}$$

The estimate (5.7) and the monotonicity of $f(x)$ applied to (5.6) then give

$$I^*_{f,a}(n) \ \ge\ 2e^{\pi a\sqrt{n}}n^{-\frac14}f\left(n^{c-\frac14}\right)e^{-\pi a n^{4c-\frac12}}\int_0^{n^c} e^{-\frac{\pi a y^2}{2}}dy. \tag{5.8}$$

We will now use the Taylor series expansion of $f(x)$ to write $f(n^{c-1/4}) = f(0) - E_1(n)$, where $E_1(n) = O(n^{c-1/4})$, and we also write the exponential function $e^{-\pi a n^{4c-1/2}} = 1 - E_2(n)$, where $E_2(n) = O(n^{4c-1/2})$. Note that both E_1 and E_2 are positive functions. Furthermore, we use the following bound for the error function:

$$\int_{n^c}^\infty e^{-\frac{\pi a y^2}{2}}dy \le \int_{n^c}^\infty e^{-\frac{\pi a y}{2}}dy = \frac{2}{\pi a}e^{-\frac{\pi a n^c}{2}}.$$

Plugging in to (5.8), we obtain

$$I^*_{f,a}(n) \ge 2e^{\pi a\sqrt{n}}n^{-\frac14}\left(f(0)-E_1(n)\right)\left(1-E_2(n)\right)\left(\frac{1}{\sqrt{2a}} - \frac{2}{\pi a}e^{-\frac{\pi a n^c}{2}}\right) \tag{5.9}$$

In other words, recalling 5.5 we have shown that

$$J_a^*(n) \ll n^{-\frac{1}{4}-c} e^{\pi a\sqrt{n}}$$

for any $0 < c < \frac{1}{8}$.

$\square$

Remark. Since f is uniformly bounded, Proposition 5.2 could also be proven through a reformulation in terms of distributions. In particular, similar arguments as those used in the proof also imply that as $N \to \infty$,

$$\mu_N := N^{\frac{1}{4}} e^{\frac{2\pi}{3}\sqrt{N}\left(\sqrt{1-x^2}-1\right)}\, dx \to \sqrt{\frac{1}{2a}}\, \delta_0, \tag{5.10}$$

where δ_0 is the Dirac delta measure at $x = 0$.

Proof of Theorem 1.2. We prove the expansion for $p_2(n)$ by determining the main terms in Theorem 1.1, and estimating the remainder. First, consider the last sum. The contribution coming from the term $k = 1$ is given by

$$\frac{\pi}{6\sqrt{n}} I_1\left(\frac{2\pi}{3}\sqrt{n}\right) = \frac{1}{4\sqrt{3}n^{\frac{3}{4}}} e^{\frac{2\pi\sqrt{n}}{3}} + O\left(\frac{e^{\frac{2\pi\sqrt{n}}{3}}}{n^{\frac{5}{4}}}\right)$$

by (5.1). Next we estimate, again using (5.1), the terms coming from $k > 1$ (up to a constant) against

$$n^{-\frac{3}{4}} \sum_{\substack{5\le k\le N\\ 0\le h<k}} \frac{1}{\sqrt{k}} e^{\frac{2\pi\sqrt{n}}{3k}} \ll e^{\frac{2\pi}{15}\sqrt{n}},$$

which is exponentially smaller than the main term. The first and second sums in Theorem 1.1 are estimated in exactly the same way, and also contribute only to the error term.

This leaves the third sum. Again we start with the term $k = 1$, which by (5.2) can be estimated against

$$\frac{\pi}{18\sqrt{6}n} \mathcal{I}_{\frac{1}{18},1,1}(n) = \frac{1}{36\sqrt{2}n^{\frac{3}{4}}} \int_{-1}^{1} \frac{(1-x^2)^{1/4}}{\cosh\left(-\frac{\pi i}{6} - \frac{\pi x}{3\sqrt{6}}\right)} e^{\frac{2\pi}{3}\sqrt{n(1-x^2)}} \left(1 + O\left(\frac{1}{\sqrt{n(1-x^2)}}\right)\right) dx. \tag{5.11}$$

We first consider the main term in this expression. Using Propositions 5.1 and 5.2 this yields the bound

$$\frac{1}{12\sqrt{6}n} \frac{1}{\cosh\left(\frac{\pi i}{6}\right)} e^{\frac{2\pi\sqrt{n}}{3}} + O\left(\frac{e^{\frac{2\pi\sqrt{n}}{3}}}{n^{1+c}}\right) = \frac{1}{18\sqrt{2}n} e^{\frac{2\pi\sqrt{n}}{3}} + O\left(\frac{e^{\frac{2\pi\sqrt{n}}{3}}}{n^{1+c}}\right).$$

The big-O term in (5.11) can similarly be estimated against

$$n^{-\frac{3}{2}} e^{\frac{2\pi\sqrt{n}}{3}}.$$

This follows by recalling that $\frac{h(x)}{\sqrt{1-x^2}}$ is initially decreasing, as well as the fact that (5.11) is a principal part integral.

Finally we treat the $k > 1$ terms from the third sum of Theorem 1.1. These may be estimated against (up to a constant)

$$\begin{aligned}
(5.12)\quad & \frac{1}{\sqrt{n}} \sum_{\substack{5 \leq k < N \\ 0 \leq h < k}} \frac{1}{k} \max_{1 \leq \nu \leq k} \left| \mathcal{I}_{\frac{1}{18},k,\nu}(n) \right| \\
& \ll \frac{1}{n^{\frac{3}{4}}} \sum_{\substack{5 \leq k < N \\ 0 \leq h < k}} \frac{1}{\sqrt{k}} \max_{1 \leq \nu \leq k} \left| \int_{-1}^{1} \frac{(1-x^2)^{1/4}}{\cosh\left(\frac{\pi i \left(\nu - \frac{1}{6}\right)}{k} - \frac{\pi x}{3k\sqrt{6}}\right)} e^{\frac{2\pi}{3k}\sqrt{2n(1-x^2)}} \, dx \right| .
\end{aligned}$$

For a given k, Proposition 5.2 again bounds the integrals (up to a constant that is uniform in ν) by

$$\left| \frac{1}{\cosh\left(\frac{\pi i \left(\nu - \frac{1}{6}\right)}{k}\right)} \right| n^{-\frac{1}{4}} e^{\frac{2\pi}{3k}\sqrt{n}} \ll k n^{-\frac{1}{4}} e^{\frac{2\pi}{15}\sqrt{n}}.$$

This yields that (5.12) may be estimated against

$$n^{\frac{1}{4}} e^{\frac{2\pi}{15}\sqrt{n}},$$

which is again an exponential error, and thus the proof is complete. $\square$

References

[1] G. Andrews, *On the theorems of Watson and Dragonette for Ramanujan's mock theta functions*, Amer. J. Math. **88** No. 2 (1966), 454–90.

[2] G. Andrews, *The theory of partitions*, Cambridge University Press, Cambridge, 1998.

[3] G. Andrews, *Partitions with short sequences and mock theta functions*, Proc. Nat. Acad. Sci. **102** No. 13 (2005), 4666–71.

[4] G. Andrews, *A survey of multipartitions: Congruences and identities*, Surveys in Number Theory, K. Alladi, ed., Developments in Mathematics **17**, pp. 1–19, Springer, 2008.

[5] G. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, **71**. Cambridge University Press, Cambridge, 1999.

[6] G. Andrews, H. Eriksson, F. Petrov, and D. Romik, *Integrals, partitions and MacMahon's theorem*, J. Comb. Theory (A) **114** (2007), 545–554.

[7] K. Bringmann, *On the explicit construction of higher deformations of partition statistics*, Duke Math. J. **144** (2008), 195–233.

[8] K. Bringmann, *Asymptotics for rank partition functions*, Transactions of the AMS, accepted for publication.

[9] K. Bringmann, F. Garvan and K. Mahlburg, *Partition statistics and quasiweak Maass forms*, Int. Math. Res. Not. (2008), rmn 124.

[10] K. Bringmann and K. Ono, *The $f(q)$ mock theta function conjecture and partition ranks*, Invent. Math. **165** (2006), 243–266.

[11] K. Bringmann and K. Ono, *Coefficients of harmonic weak Maass forms*, to appear in Proc. of the 2008 Univ. of Florida Conference on partitions, q-series, and modular forms.
[12] K. Bringmann and K. Ono, *Dyson's ranks and Maass forms*, Annals of Math. **171** (2010), 419–449.
[13] J. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. Journal **125** (2004), 45–90.
[14] S. Corteel and J. Lovejoy, *Overpartitions*, Trans. Amer. Math. Soc. **356** (2004), 1623–35.
[15] L. Dragonette, *Some asymptotic formulae for the mock theta series of Ramanujan*, Trans. Amer. Math. Soc. **72** (1952), 474–500.
[16] G.H. Hardy, *Ramanujan*, Cambridge University Press, 1940.
[17] G.H. Hardy and S. Ramanujan, *Asmptotic formulae for the distribution of integers of various types,* Proc. London Math. Soc. Series 2, **16** (1918), 112–32.
[18] A. Holroyd, T. Liggett and D. Romik, *Integrals, partitions, and cellular automata*, Trans. Amer. Math. Soc. **356** (2004) 356, 3349–3368.
[19] K. Ono, *The Web of Modularity*, CBMS Regional Conference Series in Mathematics 102 (American Mathematical Society), 2004.
[20] H. Rademacher, *On the expansion of the partition function in a series*, Ann. of Math. (2) **44**, (1943), 416–422.
[21] H. Rademacher and H. S. Zuckerman, *On the Fourier coefficients of certain modular forms of positive dimension*, Ann. of Math. (2) **39** No. 2 (1938), 433–462.
[22] G. Watson, *The final problem: An account of the mock theta functions*, J. London Math. Soc. **11** (1936), 55–80.
[23] D. Zagier, *Ramanujan' mock theta functions and their applications [d'aprés Zwegers and Bringmann-Ono]* (2006), Séminaire Bourbaki, No. 986.
[24] S. Zwegers, *Mock theta-functions and real analytic modular forms,* q-series with applications to combinatorics, number theory, and physics (Urbana, IL, 2000), 269–277, Contemp. Math. **291**, Amer. Math. Soc., Providence, RI, 2001.
[25] S. Zwegers, *Mock theta functions*, Ph.D. Thesis, Universiteit Utrecht, 2002.

Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
E-mail address: kbringma@math.uni-koeln.de

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.
E-mail address: mahlburg@math.mit.edu

Department of Mathematics, Princeton University, Princeton, NJ 08544, U.S.A.
E-mail address: mahlburg@math.princeton.edu