CLASS NUMBERS AND REPRESENTATIONS BY TERNARY QUADRATIC FORMS WITH CONGRUENCE CONDITIONS

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Abstract. In this paper, we are interested in the interplay between integral ternary quadratic forms and class numbers. This is partially motivated by a question of Petersson.

1. Introduction and statement of results

For $\boldsymbol{a} \in \mathbb{N}^3$ and $\boldsymbol{x} \in \mathbb{Z}^3$, we define

$$Q_{\boldsymbol{a}}(\boldsymbol{x}) := \sum_{j=1}^{3} a_j x_j^2.$$

Throughout we write vectors in bold letters and their components in non-bold and with subscripts. For $\boldsymbol{h} \in \mathbb{Z}^3$, $N \in \mathbb{N}$, and $n \in \mathbb{N}_0$, we let

$$r_{\boldsymbol{a},\boldsymbol{h},N}(n) := \#\left\{\boldsymbol{x} \in \mathbb{Z}^3 : Q_{\boldsymbol{a}}(\boldsymbol{x}) = n \text{ and } \boldsymbol{x} \equiv \boldsymbol{h} \pmod{N}\right\}.$$

If $N = 1$, then we omit both $\boldsymbol{h}$ and N, writing $r_{\boldsymbol{a}}(n)$ instead of $r_{\boldsymbol{a},\boldsymbol{h},N}(n)$.

It is well-known that $r_{\boldsymbol{1}}(n)$ (with $\boldsymbol{1} := (1\ 1\ 1)^T$) may be written in terms of *Hurwitz class numbers* $H(|D|)$ ($D < 0$ a discriminant) which count the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of integral binary quadratic forms of discriminant D, weighted by $\frac{1}{2}$ if the form is equivalent to $f(x_1^2 + x_2^2)$ and by $\frac{1}{3}$ if the form is equivalent to $f(x_1^2 - x_1x_2 + x_2^2)$, for some $f \in \mathbb{Z}$. By Gauss (see e.g. [11, Theorem 8.5]) $r_{\boldsymbol{1}}(n)$ is proportional to class numbers

$$r_{\boldsymbol{1}}(n) = \begin{cases} 12H(4n) & \text{if } n \equiv 1, 2 \pmod{4}, \\ 24H(n) & \text{if } n \equiv 3 \pmod{8}, \\ r_{\boldsymbol{1}}\left(\frac{n}{4}\right) & \text{if } 4 \mid n, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

In this paper, we extend (1.1) to include congruence conditions on $\boldsymbol{x}$ and to write such identities more uniformly. This is partially motivated by a question of Petersson [12], who observed that for many choices of $\boldsymbol{h}$, the numbers $r_{\boldsymbol{1},\boldsymbol{h},4}(n)$ satisfy formulas reminiscent of (1.1). His claims for $\boldsymbol{h} = (0\ 0\ 1)^T$ and $\boldsymbol{h} = (1\ 2\ 2)^T$ were later proved by Ebel [4]. To give a flavor of the types of identities that one obtains, for $\boldsymbol{h} = (0\ 1\ 1)^T$ we have

$$r_{\boldsymbol{1},\boldsymbol{h},4}(n) = \tfrac{1}{12}\delta_{n\equiv 2 \pmod{8}} r_{\boldsymbol{1}}(n) = \delta_{n \equiv 2 \pmod{8}} H(4n), \tag{1.2}$$

where here and throughout $\delta_S := 1$ if a statement S holds and $\delta_S := 0$ otherwise. Using (1.1) and the first identity in (1.2) as a guide, we undertake a two-part search; the first search yields choices of $\boldsymbol{a}$ for which identities like (1.1) hold, while the second search yields choices of $\boldsymbol{h}$ and N for which an identity resembling the first identity in (1.2) holds.

Prior work of Jones (see [9, Theorem 86]) immediately yields the choices of $\boldsymbol{a}$, namely those for which $Q_{\boldsymbol{a}}$ has *(genus) class number one*, which means that every integral quadratic form which is p-adically equivalent to $Q_{\boldsymbol{a}}$ is actually equivalent over $\mathbb{Z}$. Jones showed that, while the numbers

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$r_{\boldsymbol{a},\boldsymbol{h},N}(n)$ may not themselves be directly related to class numbers, a certain weighted average (known as the Siegel–Weil average) $r_{\mathrm{g}(Q_{\boldsymbol{a}})}(n)$ may indeed be written in terms of class numbers. If $Q_{\boldsymbol{a}}$ has class number one, then the average collapses to

$$r_{\mathrm{g}(Q_{\boldsymbol{a}})}(n) = r_{\boldsymbol{a}}(n).$$

The (finite set of) ternary integral quadratic forms with class number one is known (see the tables in [8]) and we let $\mathcal{C}$ denote the set of $\boldsymbol{a}$ for which $Q_{\boldsymbol{a}}$ has class number one. Note that $\mathcal{C}$ contains 82 elements. One can nicely package (see Lemma 4.1) the identities in (1.1) in the form

$$r_{\mathbf{1}}(n) = 12H(4n) - 24H(n). \tag{1.3}$$

For each $\boldsymbol{a} \in \mathcal{C}$, we obtain a similar formula relating $r_{\boldsymbol{a}}(n)$ to Hurwitz class numbers. For example, by Lemma 4.3, for $\boldsymbol{a} = (1\ 1\ 3)^T$ we have

$$r_{\boldsymbol{a}}(n) = 2\left(H(12n) + 2H(3n) - 3H\left(\tfrac{4n}{3}\right) - 6H\left(\tfrac{n}{3}\right)\right).$$

For each $\boldsymbol{a} \in \mathcal{C}$, we find a set $\mathcal{S}_{\boldsymbol{a}}$ consisting of $(\boldsymbol{h}, N)$ for which an identity resembling (1.2) holds. For each $(\boldsymbol{h}, N) \in \mathcal{S}_{\boldsymbol{a}}$, we show that $r_{\boldsymbol{a},\boldsymbol{h},N}(n)$ is proportional to $r_{\boldsymbol{a}}(n)$; the constant of proportionality $d_{\boldsymbol{a},\boldsymbol{h},N}(n)$ only depends on the residue class of $n \pmod M$ for some fixed M, mimicking (1.2). We list $d_{\boldsymbol{a},\boldsymbol{h},N}(n)$ in Appendix A; note that throughout Appendices A and B, the constant is taken to be zero if a given congruence class does not occur in the tables.

Theorem 1.1. *For each $\boldsymbol{a} \in \mathcal{C}$, $(\boldsymbol{h}, N) \in \mathcal{S}_{\boldsymbol{a}}$, and $n \in \mathbb{N}$, we have*

$$r_{\boldsymbol{a},\boldsymbol{h},N}(n) = d_{\boldsymbol{a},\boldsymbol{h},N}(n) r_{\boldsymbol{a}}(n).$$

Remarks 1.2.

(1) There is also a definition of genus for a quadratic form Q for which $\boldsymbol{x}$ is in a fixed congruence class $\boldsymbol{h} \pmod N$ and van der Blij [16] constructed a Siegel–Weil average $r_{\mathrm{g}(Q,\boldsymbol{h},N)}(n)$ for such quadratic forms with congruence conditions. For each prime p (including the prime ∞), there are so-called local densities $\beta_{(Q,\boldsymbol{h},N),p}(n)$ for which

$$r_{\mathrm{g}(Q,\boldsymbol{h},N)}(n) = \beta_{(Q,\boldsymbol{h},N),\infty}(n) \prod_p \beta_{(Q,\boldsymbol{h},N),p}(n), \tag{1.4}$$

the product running over all finite primes p. For all but finitely many p, one has

$$\beta_{(Q,\boldsymbol{h},N),p}(n) = \beta_{Q,p}(n). \tag{1.5}$$

From this, one can conclude that $r_{\mathrm{g}(Q,\boldsymbol{h},N)}(n)$ and $r_{\mathrm{g}(Q)}(n)$ are proportional; the constant of proportionality depends on p-adic properties of n for all of the primes p not satisfying (1.5). Hence, based on [9, Theorem 86], identities relating $r_{\boldsymbol{a},\boldsymbol{h},N}(n)$ to class numbers should only come from identities like the ones given in Theorem 1.1 in the special case that $(Q_{\boldsymbol{a}}, \boldsymbol{h}, N)$ has class number one; by [2, Corollary 4.4], this implies that $Q_{\boldsymbol{a}}$ has class number one.

(2) The identities in Theorem 1.1 were obtained via a large computer search. Such identities are not expected to exist for large N because the class number generally grows quickly as N increases (see for example [15]) and in our search we found no such examples for $N > 24$.

(3) In many cases, the identities given in Theorem 1.1 may be deduced by noting that if n is restricted to certain congruence classes, then $Q_{\boldsymbol{a}}(\boldsymbol{x}) = n$ implies that $\boldsymbol{x} \equiv \boldsymbol{h} \pmod N$ up to reordering of the variables (such calculations are carried out in [1]). However, this is not always the case for the identities proven in Theorem 1.1. For example, combining Lemma 4.1 with Theorem 1.1 (see the table in Appendix A), we obtain that for $\boldsymbol{a} = (1\ 2\ 2)^T$ and $\boldsymbol{h} \in \{(1\ 0\ 3)^T, (3\ 1\ 2)^T\}$

$$r_{\boldsymbol{a},\boldsymbol{h},6}(n) = \delta_{n \equiv 19 \pmod{24}} \left(H(4n) - 2H(n)\right).$$

Hence there are at least two distinct possibilities for the solutions $\boldsymbol{x} \pmod 6$. It would be interesting to try to directly prove that these are the same via a combinatorial argument.

(4) We prove Theorem 1.1 by obtaining a bound $n_{\boldsymbol{a},\boldsymbol{h},N}$ such that the identity holds for all $n \in \mathbb{N}$ if and only if it holds for $n \leq n_{\boldsymbol{a},\boldsymbol{h},N}$ and then checking $n \leq n_{\boldsymbol{a},\boldsymbol{h},N}$ directly with GP/Pari. We do not actually compute class numbers when proving Theorem 1.1; this is important because the algorithm used by GP/Pari to compute class numbers $H(D)$ for $D \geq 5 \cdot 10^5$ is only conditional on the Generalized Riemann Hypothesis. The identities in Theorem 1.1 are instead proven by computing $r_{\boldsymbol{a},\boldsymbol{h},N}(n)$ and $r_{\boldsymbol{a}}(n)$ directly from their definitions and then checking the constant of proportionality. Identities with class numbers are obtained by relating $r_{\boldsymbol{a}}(n)$ to these in Section 4. This does require the direct computation of the class numbers, but for all cases other than $\boldsymbol{a} \in \{(2\ 3\ 48)^T, (3\ 8\ 48)^T\}$ we only need to compute $H(D)$ for $D \leq 294\,912$, which is within the bound for which the computation of GP/Pari is unconditional. In the remaining cases, we use a relation from [3, p. 273] to calculate $H(D)$ unconditionally.

Another motivation for finding relations between class numbers and representations by ternary quadratic forms comes from computational number theory. Specifically, the algorithm of Jacobson and Mosunov [6] used to tabulate class groups of imaginary quadratic fields of all discriminants $|D| < 2^{40}$ (setting a record for unconditional computation) exploits (1.1) to evaluate $H(n)$ or $H(4n)$ for squarefree $n \not\equiv 7 \pmod 8$. However, since $r_{\boldsymbol{1}}(n) = 0$ for $n \equiv 7 \pmod 8$, the authors of [6] used another method to compute $H(n)$ for squarefree $n \equiv 7 \pmod 8$ (see [7]). One may use generating functions for identities resembling (1.3) for other choices of $\boldsymbol{a}$ to push the calculation further. Indeed, for $\boldsymbol{a} = (1\ 3\ 3)^T$, letting $(\frac{\cdot}{\cdot})$ denote the extended Legendre symbol, we may show, using the methods of this paper, that for $n \equiv 7 \pmod 8$ with $9 \nmid n$, we have

$$r_{\boldsymbol{a}}(n) = 8\left(1 + \left(\tfrac{n}{3}\right)\right) H(n).$$

Similarly, for $\boldsymbol{a} = (2\ 5\ 10)^T$, for $n \equiv 23 \pmod{24}$ with $25 \nmid n$, we have

$$r_{\boldsymbol{a}}(n) = 2\left(1 - \left(\tfrac{n}{5}\right)\right) H(n).$$

One may also use Theorem 1.1 and the lemmas in Section 4 to compute $H(n)$ quicker for special choices of n: since the shifted lattice defined by $\boldsymbol{x} \in \mathbb{Z}^3$ with $\boldsymbol{x} \equiv \boldsymbol{h} \pmod N$ is more "spread out" than the lattice $\mathbb{Z}^3$, the computation of $r_{\boldsymbol{a},\boldsymbol{h},N}(n)$ is faster than that of $r_{\boldsymbol{a}}(n)$.

Although we use [9, Theorem 86] as a guide for finding identities like (1.3) we do not employ it directly. Jones used a constructive proof to establish a bijection between (primitive) representations of integers and class numbers, but his theorem only applies to those n which are coprime to $2a_1a_2a_3$. In principle, one could take (1.4) and compute the local densities on the right-hand side in order to determine $r_{\boldsymbol{a},\boldsymbol{h},N}(n)$ if the class number is one. One could then realize a relationship between the right-hand side as a special value of a Dirichlet L-function for a real character and use Dirichlet's class number formula to obtain an identity involving class numbers. This is essentially done in [14, Theorem 1.5], although many of the constants would still need to be worked out to obtain an explicit identity and the calculation splits depending on the p-adic properties of n.

Instead, we investigate the representation numbers $r_{\boldsymbol{a},\boldsymbol{h},N}(n)$ by packaging them into generating functions generally referred to as *theta functions* ($q := e^{2\pi i \tau}$)

$$\Theta_{\boldsymbol{a},\boldsymbol{h},N}(\tau) := \prod_{j=1}^{3} \vartheta_{h_j,N}\left(2Na_j\tau\right) = \sum_{n\geq 0} r_{\boldsymbol{a},\boldsymbol{h},N}(n) q^n, \quad \text{where} \tag{1.6}$$

$$\vartheta_{h,N}(\tau) := \sum_{m \equiv h \pmod N} q^{\frac{m^2}{2N}}.$$

The functions $\vartheta_{h,N}$ are modular forms of weight $\frac{3}{2}$ (see Lemma 2.4), and we use the theory of non-holomorphic modular forms to recognize the generating functions of the right-hand sides of the equations similar to (1.3) as (holomorphic) modular forms of weight $\frac{3}{2}$. We then use the theory of modular forms to prove that these two modular forms are equal, and hence conclude that the coefficients used to build them are indeed equal.

The paper is organized as follows. In Section 2, we introduce and recall properties of non-holomorphic modular forms, theta functions, and class numbers. We then prove Theorem 1.1 in Section 3. In Section 4, we confirm identities resembling (1.3) for each $\boldsymbol{a} \in \mathcal{C}$, which, when combined with Theorem 1.1, yield the desired relations between class numbers representations by the form $Q_{\boldsymbol{a}}$ with $\boldsymbol{x} \equiv \boldsymbol{h} \pmod{N}$. We list all of the constants needed for stating Theorem 1.1 in Appendix A and the constants required for the proof of Theorem 1.1 are given in Appendix C. Finally, Appendix B (resp. D) gives the definitions of the constants needed to state (resp. prove) the lemmas in Section 4.

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2. Preliminaries

2.1. **Modular forms.** We briefly introduce modular forms below, but refer the reader to [11] for more details. As usual, for d odd, we set

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}, \\ i & \text{if } d \equiv -1 \pmod{4}. \end{cases}$$

A function $f : \mathbb{H} \to \mathbb{C}$ satisfies *modularity of weight* $\kappa \in \frac{1}{2} + \mathbb{Z}$ *on* $\Gamma \subseteq \Gamma_0(4)$ (Γ a congruence subgroup containing $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$) *with character* χ if for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have

$$f|_\kappa \gamma = \chi(d) f.$$

Here the weight κ *slash operator* is defined by

$$f\big|_\kappa \gamma(\tau) := \left(\tfrac{c}{d}\right) \varepsilon_d^{2k} (c\tau + d)^{-\kappa} f(\gamma\tau).$$

We call f a *(holomorphic) modular form* if f is holomorphic on $\mathbb{H}$ and $f(\tau)$ grows as most polynomially in v as $\tau = u + iv \to \mathbb{Q} \cup \{i\infty\}$. We call the equivalence classes of $\Gamma\backslash(\mathbb{Q} \cup \{i\infty\})$ the *cusps of* $\Gamma\backslash\mathbb{H}$. For a cusp ϱ we choose $M_\varrho \in \mathrm{SL}_2(\mathbb{Z})$ such that $M_\varrho(i\infty) = \varrho$. If f satisfies modularity of weight κ on Γ with some character χ, then $f_\varrho := f|_\kappa M_\varrho$ is invariant under T^{σ_ϱ} for some $\sigma_\varrho \in \mathbb{N}$ and hence has a Fourier expansion

$$f_\varrho(\tau) = \sum_{n \in \mathbb{Z}} c_{f,\varrho,v}(n) q^{\frac{n}{\sigma_\varrho}}.$$

The number σ_ϱ (chosen minimally) is the *cusp width* of ϱ; we drop the dependence on v in the notation if f is holomorphic and we drop ϱ from the notation if $\varrho = i\infty$. The growth conditions at the cusps for a holomorphic modular form are equivalent to the assumption that $c_{f,\varrho}(n) = 0$ for all $n < 0$. If f is a holomorphic modular form and $m \in \mathbb{Z}$ is minimal such that $c_{f,\varrho}(m) \neq 0$, then set

$$\mathrm{ord}_\varrho(f) := \tfrac{m}{\sigma_\varrho}.$$

For $\tau \in \mathbb{H}$ and a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, we let

$$\omega_\tau = \omega_{\Gamma,\tau} := \frac{\#\{\gamma \in \Gamma : \gamma\tau = \tau\}}{\#\{\gamma \in \Gamma : \gamma = \pm I\}}.$$

The following identity, known as the valence formula, is used throughout this paper to prove identities between coefficients of modular forms.

Lemma 2.1. *Suppose that $k \in \frac{1}{2}\mathbb{Z}$, $\Gamma \subseteq \Gamma_0(4)$ is a congruence subgroup containing $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and χ is a character. If $f \not\equiv 0$ is a (holomorphic) modular form of weight k on Γ with character χ, then*

$$\tfrac{k}{12}\left[\mathrm{SL}_2(\mathbb{Z}) : \Gamma\right] = \sum_{\tau \in \Gamma\backslash\mathbb{H}} \frac{\mathrm{ord}_\tau(f)}{\omega_\tau} + \sum_{\varrho \in \Gamma\backslash(\mathbb{Q}\cup\{i\infty\})} \mathrm{ord}_\varrho(f).$$

In particular, if $f(\tau) = \sum_{n \gg -\infty} c_f(n)q^n$ and $c_f(n) = 0$ for all $n \leq \frac{k}{12}\left[\mathrm{SL}_2(\mathbb{Z}) : \Gamma\right]$, then $f \equiv 0$.

We require indices of certain subgroups of $\mathrm{SL}_2(\mathbb{Z})$. As usual φ denotes Euler totient function and we set

$$\Gamma_{N,M} := \Gamma_0(N) \cap \Gamma_1(M).$$

Lemma 2.2. *If $N, M \in \mathbb{N}$ with $M \mid N$, then we have*

$$\left[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_{N,M}\right] = N \prod_{p \mid N}\left(1 + \tfrac{1}{p}\right)\varphi(M),$$

where the product runs over all primes divisors of N. In particular

$$\left[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)\right] = N \prod_{p \mid N}\left(1 + \tfrac{1}{p}\right).$$

Proof. We write

$$\left[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_{N,M}\right] = \left[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)\right]\left[\Gamma_0(N) : \Gamma_{N,M}\right].$$

We have, by [11, Proposition 1.7],

$$\left[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)\right] = N \prod_{p \mid N}\left(1 + \tfrac{1}{p}\right).$$

Since $M \mid N$, there is a natural surjective homomorphism from $\Gamma_0(N)$ to $(\mathbb{Z}/M\mathbb{Z})^\times$ with kernel $\Gamma_{N,M}$, from which we conclude that

$$\left[\Gamma_0(N) : \Gamma_{N,M}\right] = \varphi(M). \qquad \square$$

2.2. **Operators on non-holomorphic modular forms.** For some of the theta functions, we directly prove that they match generating functions for certain class numbers by showing that the generating functions are modular forms and then using Lemma 2.1. For this, we require some basic facts about operators on non-holomorphic modular forms. For $d, M \in \mathbb{N}$, and $m \in \mathbb{Z}$, the operators on $f(\tau) = \sum_{n\in\mathbb{Z}} c_{f,v}(n)q^n$ are defined as

$$f\big|U_d(\tau) := \sum_{n\in\mathbb{Z}} c_{f,\frac{v}{d}}(dn)\,q^n, \qquad f\big|V_d(\tau) := f(d\tau) = \sum_{n\in\mathbb{Z}} c_{f,dv}(n)q^{dn},$$

$$f\big|S_{M,m}(\tau) := \sum_{n\equiv m \pmod{M}} c_{f,v}(n)q^n.$$

Note that V_dU_d is the identity. The *radical* for $n \in \mathbb{N}$ is $\mathrm{rad}(n) := \prod_{p\mid n} p$. For a discriminant D, let $\chi_D(\cdot) := \left(\frac{D}{\cdot}\right)$. The proof of the following lemma is standard; however, for the reader's convenience, we give a proof.

Lemma 2.3. *Suppose that $f:\mathbb{H}\to\mathbb{C}$ satisfies modularity of weight $k\in\frac{1}{2}+\mathbb{Z}$ on $\Gamma_0(N)$ $(4\mid N)$ with character χ of conductor N_χ.*

(1) *The function $f|U_d$ satisfies modularity of weight k on $\Gamma_0(4\operatorname{lcm}(\frac{N}{4},\operatorname{rad}(d)))$ with character $\chi\chi_{4d}$. Moreover, if $f=g|V_d$, with g satisfying modularity of weight k on $\Gamma_0(\frac{N}{d})$ with character $\chi\chi_{4d}$, then $f|U_d$ satisfies modularity of weight k on $\Gamma_0(\frac{N}{d})$ with character $\chi\chi_{4d}$.*

(2) *Suppose that f satisfies modularity of weight k on $\Gamma_{N,L}$ $(L\in\mathbb{N})$ with character χ. For $M\not\equiv 2\pmod 4$ (resp. $M\equiv 2\pmod 4$), $f|S_{M,m}$ satisfies modularity of weight k on the group $\Gamma_{\operatorname{lcm}(N,M^2,N_\chi M,ML),\operatorname{lcm}(M,L)}$ (resp. $\Gamma_{\operatorname{lcm}(N,4M^2,N_\chi M,ML),\operatorname{lcm}(M,L)}$) with character χ. Moreover, if $M\mid 24$ and $M\not\equiv 2\pmod 4$ (resp. $M\equiv 2\pmod 4$), then $f|S_{M,m}$ satisfies modularity of weight k on $\Gamma_{\operatorname{lcm}(N,M^2,MN_\chi,ML),L}$ (resp. $\Gamma_{\operatorname{lcm}(N,4M^2,N_\chi M,ML),L}$) with character χ.*

(3) *The function $f|V_d$ satisfies modularity of weight k on $\Gamma_0(Nd)$ with character $\chi\chi_{4d}$.*

Proof. (1) For the first claim see for example [11, Proposition 3.7 (2)], where for $d=\prod_{p|n}p^{a_p}$ we write $U_d=\prod_{p|n}U_p^{a_p}$ and then apply the statement iteratively for U_p. The proof of the first claim closely follows the proof of [10, Lemma 1], which gives the argument for integral weight. The second statement follows directly from the fact that V_dU_d is the identity.

(2) We start by rewriting

$$f\big|S_{M,m}(\tau)=\tfrac{1}{M}\sum_{r\pmod M}e^{\frac{2\pi imr}{M}}f\left(\tau-\tfrac{r}{M}\right). \tag{2.1}$$

For $r\in\mathbb{Z}$ and $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_{\operatorname{lcm}(N,M^2,ML),L}$, we choose $\varrho\in\mathbb{Z}$ such that $a\varrho\equiv dr\pmod M$ and compute, using the modularity of f,

$$f\left(\gamma\tau-\tfrac{r}{M}\right)=\left(\frac{c}{d+\varrho\frac{c}{M}}\right)\varepsilon_{d+\varrho\frac{c}{M}}^{-2k}\chi\left(d+\varrho\tfrac{c}{M}\right)(c\tau+d)^kf\left(\tau-\tfrac{\varrho}{M}\right).$$

Additionally assuming that $\gamma\in\Gamma_0(N_\chi M)$ (resp. $\gamma\in\Gamma_0(\operatorname{lcm}(4M^2,N_\chi M))$ if $M\not\equiv 2\pmod 4$ (resp. $M\equiv 2\pmod 4$)), a short calculation yields

$$f\left(\gamma\tau-\tfrac{r}{M}\right)=\left(\tfrac{c}{d}\right)\varepsilon_d^{-2k}\chi(d)(c\tau+d)^kf\left(\tau-\tfrac{\varrho}{M}\right).$$

Note that if $\gamma\in\Gamma_1(M)$, then we have $\varrho=r$. Plugging back into (2.1) yields the first claim.

Finally, assume that $M\mid 24$. Since $\gamma\in\Gamma_0(M)$, a is invertible $\pmod M$ and $a\equiv d\pmod M$. Hence $\varrho=r$, yielding the claimed modularity in this case as well.

(3) This is well-known (see e.g. [11, Proposition 3.7 (1)] and [10, Section 1] for a proof for integral weight). ☐

2.3. **Theta functions.** Let for $\boldsymbol{a}\in\mathbb{N}^3$

$$\ell=\ell_{\boldsymbol{a}}:=\operatorname{lcm}(\boldsymbol{a})\qquad\text{and}\qquad\mathcal{D}_{\boldsymbol{a}}:=a_1a_2a_3.$$

Using [13, Proposition 2.1], we obtain the following modularity of $\Theta_{\boldsymbol{a},\boldsymbol{h},N}$ from (1.6).

Lemma 2.4. *The function $\Theta_{\boldsymbol{a},\boldsymbol{h},N}$ is a modular form of weight $\frac{3}{2}$ on $\Gamma_{4\ell N^2,N}$ with character $\chi_{4\mathcal{D}_{\boldsymbol{a}}}$.*

2.4. **Modularity of class number generating functions.** Recall that Zagier [17] (see also [5, Theorem 2]) has proven that the class number generating function

$$\mathcal{H}(\tau):=\sum_{\substack{D\geq 0\\ D\equiv 0,3\pmod 4}}H(D)q^D$$

satisfies modular properties. To state his result, define for $x>0$ and $\alpha\in\mathbb{R}$ the *incomplete gamma function* $\Gamma(\alpha,x):=\int_x^\infty e^{-t}t^{\alpha-1}dt$.

Lemma 2.5. *The function*

$$\widehat{\mathcal{H}}(\tau) := \mathcal{H}(\tau) + \frac{1}{8\pi\sqrt{v}} + \frac{1}{4\sqrt{\pi}} \sum_{n\geq 1} n\Gamma\left(-\tfrac{1}{2}, 4\pi n^2 v\right) q^{-n^2}$$

transforms like a modular form of weight $\frac{3}{2}$ *on* $\Gamma_0(4)$.

The following lemma turns out to be useful for our purpose.

Lemma 2.6. *For* $r_1, r_2 \in \mathbb{N}$, *with* $\gcd(r_1, r_2) = 1$ *and* r_2 *squarefree, we have that*

$$\mathcal{H}_{r_1,r_2} := \mathcal{H}\big|(U_{r_1r_2} - r_2U_{r_1}V_{r_2})$$

is a modular form of weight $\frac{3}{2}$ *on* $\Gamma_0(4\,\mathrm{rad}(r_1)r_2)$ *with character* $\chi_{4r_1r_2}$.

Proof. We first prove modularity of $\widehat{\mathcal{H}}_{r_1,r_2} := \widehat{\mathcal{H}}|(U_{r_1r_2} - r_2U_{r_1}V_2)$. By Lemma 2.5, $\widehat{\mathcal{H}}$ satisfies modularity of weight $\frac{3}{2}$ on $\Gamma_0(4)$. Hence, by Lemma 2.3 (1) and (3), $\widehat{\mathcal{H}}_{r_1,r_2}$ is modular of weight $\frac{3}{2}$ on $\Gamma_0(4\,\mathrm{rad}(r_1)r_2)$ with character $\chi_{4r_1r_2}$.

We next show that this function does not have a non-holomorphic part. For this write

$$\widehat{\mathcal{H}}^-(\tau) := \widehat{\mathcal{H}}(\tau) - \mathcal{H}(\tau) =: \sum_{n\geq 0} c_v(n) q^{-n^2}.$$

We compute, for $r \in \mathbb{N}$,

$$\widehat{\mathcal{H}}^-\big|U_r(\tau) = \sum_{\substack{n\geq 0\\ r|n^2}} c_{\frac{v}{r}}(n) q^{-\frac{n^2}{r}}.$$

Thus writing $r_1 = \varrho\mu^2$ with ϱ squarefree, we obtain

$$\widehat{\mathcal{H}}^-\big|(U_{r_1r_2} - r_2U_{r_1}V_{r_2})(\tau) = \sum_{n\geq 0} c_{\frac{v}{r_1r_2}}(\varrho r_2\mu n) q^{-\varrho r_2 n^2} - r_2 \sum_{n\geq 0} c_{\frac{r_2 v}{r_1}}(\varrho\mu n) q^{-\varrho r_2 n^2}.$$

Plugging in the explicit formula for $c_v(n)$ from Lemma 2.6, the claim follows from the relation

$$c_{\frac{v}{r_1r_2}}(\varrho r_2\mu n) - r_2 c_{\frac{r_2 v}{r_1}}(\varrho\mu n) = 0. \qquad \square$$

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1, reducing the task of relating $r_{\boldsymbol{a},\boldsymbol{h},N}(n)$ to class numbers to finding such formulas for $r_{\boldsymbol{a}}(n)$.

Proof of Theorem 1.1. We let $\boldsymbol{a} \in \mathcal{C}$ and $(\boldsymbol{h}, N) \in \mathcal{S}_{\boldsymbol{a}}$. For each m and M appearing in the table in Appendix A, for $n \equiv m \pmod{M}$ we let $d = d_{\boldsymbol{a},\boldsymbol{h},N}(n)$ be the corresponding value of d appearing in that entry from the table.

Recall that for $D \mid M$ and $j \in \mathbb{Z}$ and a translation-invariant function f, we have

$$\sum_{\substack{m \pmod{M}\\ m\equiv j \pmod{D}}} f\big|S_{M,m} = f\big|S_{D,j}.$$

Thus if $d_{\boldsymbol{a},\boldsymbol{h},N}(n)$ only depends on $n \pmod{D}$, then

$$\sum_{\substack{m \pmod{M}\\ m\equiv j \pmod{D}}} d_{\boldsymbol{a},\boldsymbol{h},N}(m) f\big|S_{M,m} = d_{\boldsymbol{a},\boldsymbol{h},N}(j) \sum_{\substack{m \pmod{M}\\ m\equiv j \pmod{D}}} f\big|S_{M,m} = d_{\boldsymbol{a},\boldsymbol{h},N}(j) f\big|S_{D,j}.$$

This allows us to combine multiple cases by taking the lcm of the moduli of a number of cases that show up in the congruence conditions defining $d_{\boldsymbol{a},\boldsymbol{h},N}$.

Specifically, the claim is equivalent to showing that, for some M_N with $M \mid M_N$,

$$\Theta_{\boldsymbol{a},\boldsymbol{h},N} = \sum_{m \pmod{M_N}} d_{\boldsymbol{a},\boldsymbol{h},N}(m)\Theta_{\boldsymbol{a}}\Big|S_{M_N,m}. \tag{3.1}$$

We group many choices of N together in each entry in Appendix C by choosing M_N as the lcm of those M coming from each $(\boldsymbol{h},N) \in \mathcal{S}_{\boldsymbol{a}}$. For most cases, we prove (3.1) directly via Lemma 2.1. In particular if $(\boldsymbol{a},N) \notin \{((1\ 5\ 8)^T,20),\ ((1\ 6\ 16)^T,24),\ ((1\ 9\ 21)^T,21)\ ((1\ 9\ 24)^T,12),\ ((1\ 9\ 24)^T,24),$ $((1\ 16\ 24)^T,12),\ ((2\ 3\ 8)^T,24)\}$, then Lemma 2.4 implies that the left-hand side of (3.1) is a modular form of weight $\frac{3}{2}$ on $\Gamma_{4\ell N^2,N}$ with character $\chi_{4\mathcal{D}_{\boldsymbol{a}}}$. Combining Lemmas 2.4 and 2.3 (2), the right-hand side of (3.1) is a modular form of weight $\frac{3}{2}$ on $\Gamma_{\mathrm{lcm}(4\ell,M_N^2,M_NN_{\chi_{4\mathcal{D}_{\boldsymbol{a}}}}),M_N}$ with character $\chi_{4\mathcal{D}_{\boldsymbol{a}}}$. The intersection Γ of the two groups appearing from both sides of (3.1) is computed for each case and listed in the row of the table in Appendix C. Lemmas 2.1 and 2.2 yield a bound b given in the last entry of the row of the table in Appendix C, such that the identity holds if and only if it holds for the first b coefficients. We then check with a computer the identity for the first b coefficients, yielding the claim. For the remaining cases, b could theoretically be obtained by the same method, but the computer check to verify the claim would require too much time. In these cases, we show the claim by reducing it to one of the cases which is already done.

We start with $\boldsymbol{a} = (1\ 5\ 8)^T$ and $N = 20$ and reduce the claim to the claim for $N = 4$ that is already proven. Note that for $\boldsymbol{h} = (0\ 4\ 5)^T$ (resp. $\boldsymbol{h} = (0\ 8\ 5)^T$) we have $n \equiv 5 \pmod{25}$ (resp. $n \equiv -5 \pmod{25}$) and a direct calculation shows that

$$r_{\boldsymbol{a},\boldsymbol{h},20}(n) = \tfrac{1}{2}r_{\boldsymbol{a},\boldsymbol{h},4}(n).$$

Reducing $\boldsymbol{h} \pmod 4$ and plugging in the results for $N = 4$, we obtain the claims.

We next consider the case $\boldsymbol{a} = (1\ 6\ 16)^T$ and $N = 24$. For each of the choices in the table we may show that we have

$$r_{\boldsymbol{a},\boldsymbol{h},24}(n) = \tfrac{1}{2}r_{\boldsymbol{a},\boldsymbol{h},8}(n).$$

Reducing $\boldsymbol{h} \pmod 8$ then gives the claim.

We next assume that $\boldsymbol{a} = (1\ 9\ 21)^T$ and $N = 21$. With n restricted to the cases assumed in the table, elementary congruence considerations yield

$$r_{\boldsymbol{a},\boldsymbol{h},21}(n) = \tfrac{1}{2}r_{\boldsymbol{a},\boldsymbol{h},3}(n).$$

Reducing $\boldsymbol{h} \pmod 3$ then gives the claim using the table in Appendix A.

We next consider the case $\boldsymbol{a} = (1\ 9\ 24)^T$ and $12 \mid N$. We let $\boldsymbol{h} \in \{(12\ 4\ 3)^T, (12\ 4\ 9)^T\}$ be given and for $N = 12$ we claim that, noting that $\boldsymbol{h} \equiv (0\ 4\ \pm 3)^T \pmod{12}$,

$$r_{\boldsymbol{a},\boldsymbol{h},12}(n) = \begin{cases} \frac{1}{2}r_{\boldsymbol{a},\boldsymbol{h},3}(n) & \text{if } n \equiv 24 \pmod{32}, \\ \frac{1}{4}r_{\boldsymbol{a},\boldsymbol{h},3}(n) & \text{if } n \equiv 40 \pmod{64}, \\ 0 & \text{otherwise.} \end{cases}$$

In terms of theta functions, this claim may be written as

$$\Theta_{\boldsymbol{a},\boldsymbol{h},12} = \tfrac{1}{2}\Theta_{\boldsymbol{a},\boldsymbol{h},3}\big|S_{32,24} + \tfrac{1}{4}\Theta_{\boldsymbol{a},\boldsymbol{h},3}\big|S_{64,40}. \tag{3.2}$$

Note that by Lemma 2.4, the left-hand side of (3.2) is a modular form of weight $\frac{3}{2}$ on $\Gamma_{41472,12}$ with character χ_{24} For the right-hand side, we use that $\Theta_{\boldsymbol{a},\boldsymbol{h},3}$ is a modular form of weight $\frac{3}{2}$ on $\Gamma_{2592,3}$ with character χ_{24}. Applying Lemma 2.3 (2) yields a modular form of weight $\frac{3}{2}$ on $\Gamma_{331776,192}$ with character χ_{24}. By Lemmas 2.1 and 2.2 we need to check the first 5 308 416 coefficients. To complete the $N = 12$ case, we note that $\boldsymbol{h} \equiv (0\ 1\ 0)^T \pmod 3$, so we may plug in the identity from the $N = 3$ cases from the table.

We next claim that for $N=24$ we have

$$r_{\boldsymbol{a},\boldsymbol{h},24}(n)=\begin{cases} r_{\boldsymbol{a},\boldsymbol{h},12}(n) & \text{if } n\equiv 120 \pmod{128} \text{ and } \boldsymbol{h}=(12\ 4\ 3)^T\\ & \text{ or } n\equiv 56 \pmod{128} \text{ and } \boldsymbol{h}=(12\ 4\ 9)^T,\\ 0 & \text{otherwise.}\end{cases}$$

Note that the claim is equivalent to

$$\Theta_{\boldsymbol{a},\boldsymbol{h},24}=\begin{cases}\Theta_{\boldsymbol{a},\boldsymbol{h},12}\big|S_{128,120} & \text{if } \boldsymbol{h}=(12\ 4\ 3)^T,\\ \Theta_{\boldsymbol{a},\boldsymbol{h},12}\big|S_{128,56} & \text{if } \boldsymbol{h}=(12\ 4\ 9)^T.\end{cases} \tag{3.3}$$

The left-hand side is by Lemma 2.4 a modular form of weight $\frac{3}{2}$ on $\Gamma_{165888,24}$ with character χ_{24} which is trivial on that group. For the right-hand side $\Theta_{\boldsymbol{a},\boldsymbol{h},12}$ is a modular form of weight $\frac{3}{2}$ on $\Gamma_{41472,12}$ with character χ_{24}. By Lemma 2.3 (2), $\Theta_{\boldsymbol{a},\boldsymbol{h},12}|S_{128,m}$ is a modular form of weight $\frac{3}{2}$ on $\Gamma_{1327104,384}$. By Lemmas 2.1 and 2.2, we need to check the first 42 467 328 coefficients. Plugging in the identities for $N=12$ yields the claim.

For $\boldsymbol{a}=(1\ 16\ 24)^T$ and $N=12$, we can show that

$$r_{\boldsymbol{a},\boldsymbol{h},12}(n)=\tfrac{1}{2}r_{\boldsymbol{a},\boldsymbol{h},4}(n).$$

Reducing $\boldsymbol{h} \pmod 4$ and plugging in the formula for $N=4$ yields the claim.

For $\boldsymbol{a}=(2\ 3\ 8)^T$, $N=24$, and $n\equiv 3 \pmod 8$ we have

$$r_{\boldsymbol{a},\boldsymbol{h},24}(n)=\tfrac{1}{2}r_{\boldsymbol{a},\boldsymbol{h},8}(n).$$

The table in Appendix A for $\boldsymbol{a}=(2\ 3\ 8)^T$ gives the claim reducing $\boldsymbol{h} \pmod 8$. $\square$

4. Class number identities

In this section, we derive relations like (1.3) between $r_{\boldsymbol{a}}(n)$ and Hurwitz class numbers for each $\boldsymbol{a}\in\mathcal{C}$. The general idea is as follows: We use Lemmas 2.4, 2.6, and 2.3 to show that both sides of the identity are modular forms for some congruence subgroup Γ of $\Gamma_0(4)$ and then use Lemmas 2.1 and 2.2 to prove that the claim follows as long as it holds for the first b (given in Appendix C) Fourier coefficients. Finally, we check b coefficients with a computer. We give a detailed proof of one case in Lemma 4.1 in order to show how to carry out the details to obtain Γ and b and leave the details for the other cases to the reader. Setting $c_{(1\ 1\ 1)^T}(n):=12$ and taking $c_{\boldsymbol{a}}(n)$ from the tables in Appendix B otherwise, we show the following.

Lemma 4.1. *For $\boldsymbol{a}\in\{(1\ 1\ 1)^T,(1\ 1\ 4)^T,(1\ 2\ 2)^T,(1\ 2\ 8)^T,(1\ 4\ 4)^T,(1\ 8\ 8)^T\}$, we have*

$$r_{\boldsymbol{a}}(n)=c_{\boldsymbol{a}}(n)\left(H(4n)-2H(n)\right).$$

Proof of Lemma 4.1. For each choice of $\boldsymbol{a}$, we let M be the least common multiple of M's in the tables in Appendix B. It is then not hard to see that the claim is equivalent to

$$\Theta_{\boldsymbol{a}}=\sum_{m \pmod M} c_{\boldsymbol{a}}(m)\mathcal{H}_{1,2}\big|U_2\big|S_{M,m}.$$

To prove this, note that Lemma 2.4 implies that the left-hand side is a modular form of weight $\frac{3}{2}$ on $\Gamma_0(4\ell)$ ($\ell\mid 8$). By Lemma 2.4, $\mathcal{H}_{1,2}$ is a modular form of weight $\frac{3}{2}$ on $\Gamma_0(8)$ with character χ_8. Hitting with U_2 gives by Lemma 2.3 (1) a modular form of weight $\frac{3}{2}$ on $\Gamma_0(8)$. For simplicity we only consider $\boldsymbol{a}=(1\ 2\ 8)^T$. Hitting with $S_{16,m}$ gives a modular form of weight $\frac{3}{2}$ on $\Gamma_{256,16}$. By Lemmas 2.1 and 2.2, it suffices to check the identity for the first

$$\tfrac{1}{8}\left[\mathrm{SL}_2(\mathbb{Z}):\Gamma_{256,16}\right]=384$$

coefficients, which is easily verified. Note that the check requires the calculation of $H(4n)$ and $H(n)$ for every $n \leq 384$. Hence it suffices to compute $H(D)$ for $D \leq 4 \cdot 384 = 1\,536$ in order to verify the identity. Since $1\,536 < 5 \cdot 10^5$, the calculation built into GP/Pari is unconditional and we use this to verify the identity. For the other choices of $\boldsymbol{a}$, we need to similarly compute $H(D)$ for every non-negative D up to 4 times the valence formula bound given in Appendix D for the given choice of $\boldsymbol{a}$. A quick check reveals that the bound 384 from the valence formula for the cases $\boldsymbol{a} = (1\ 2\ 8)^T$ and $\boldsymbol{a} = (1\ 8\ 8)^T$ are the largest for the cases contained in this lemma and $4 \cdot 384 = 1\,536 < 5 \cdot 10^5$. Thus the calculation built into GP/Pari is unconditional for all of the cases covered by this lemma. $\square$

Lemma 4.2. *For* $\boldsymbol{a} \in \{(1\ 1\ 2)^T, (1\ 1\ 8)^T, (1\ 2\ 4)^T, (1\ 2\ 16)^T, (1\ 4\ 8)^T, (1\ 8\ 16)^T\}$*, we have*

$$r_{\boldsymbol{a}}(n) = c_{\boldsymbol{a}}(n)\left(H(8n) - 2H(2n)\right).$$

Proof. It is not hard to see that the formula is equivalent to

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod M} c_{\boldsymbol{a}}(m)\mathcal{H}_{1,2}\big|U_4\big|S_{M,m}.$$

The table in Appendix D, together with a short computer check, finishes, in this case, as well as in the following ones, the claim. In the cases where the calculation for the class numbers built into GP/Pari would be conditional, we add further comments explaining how the computations were done unconditionally, and otherwise (such as in this case) we simply list the identity that needs to be checked. $\square$

Letting $c_{(1\ 1\ 3)^T}(n) := 2$, we have the following.

Lemma 4.3. *For* $\boldsymbol{a} \in \{(1\ 1\ 3)^T, (1\ 3\ 9)^T\}$*, we have*

$$r_{\boldsymbol{a}}(n) = c_{\boldsymbol{a}}(n)\left(H(12n) + 2H(3n) - 3H\left(\tfrac{4n}{3}\right) - 6H\left(\tfrac{n}{3}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod M} c_{\boldsymbol{a}}(m)\mathcal{H}_{1,3}\big|\left(2 + U_4\right)\big|S_{M,m}.$$ $\square$

Lemma 4.4. *For* $\boldsymbol{a} = (1\ 1\ 5)^T$*, we have,*

$$r_{\boldsymbol{a}}(n) = 2H(20n) - 4H(5n) + 10H\left(\tfrac{4n}{5}\right) - 20H\left(\tfrac{n}{5}\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = 2\left(\mathcal{H}_{5,2} + 5\mathcal{H}_{1,2}\big|V_5\right)\big|U_2.$$ $\square$

Let $c_{(1\ 1\ 6)^T}(n) := 2(-1)^{n+1}$.

Lemma 4.5. *For* $\boldsymbol{a} \in \{(1\ 1\ 6)^T, (1\ 6\ 9)^T\}$*, we have*

$$r_{\boldsymbol{a}}(n) = c_{\boldsymbol{a}}(n)\left(H(24n) - 4H(6n) - 3H\left(\tfrac{8n}{3}\right) + 12H\left(\tfrac{2n}{3}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = -\sum_{m \pmod M} c_{\boldsymbol{a}}(m)\mathcal{H}_{2,3}\big|\left(4 - U_4\right)\big|S_{M,m}.$$ $\square$

Lemma 4.6. *We have, for* $\boldsymbol{a} = (1\ 1\ 9)^T$*,*

$$r_{\boldsymbol{a}}(n) = c(n)(H(4n) - 2H(n)) + 12H\left(\tfrac{4n}{9}\right) - 24H\left(\tfrac{n}{9}\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = 12\mathcal{H}_{1,2}\big|U_2\big|V_9 + \sum_{m \pmod 3} c(m)\mathcal{H}_{1,2}\big|U_2\big|S_{3,m}. \qquad \square$$

Lemma 4.7. *For $\boldsymbol{a} \in \{(1\ 1\ 12)^T, (1\ 4\ 12)^T, (1\ 9\ 12)^T\}$, we have*

$$r_{\boldsymbol{a}}(n) = c_{\boldsymbol{a}}(n)\left(H(12n) - 3H\left(\tfrac{4n}{3}\right) + 8H\left(\tfrac{3n}{4}\right) - 24H\left(\tfrac{n}{12}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod M} c_{\boldsymbol{a}}(m)\mathcal{H}_{1,3}\big|\left(U_4 + 8V_4\right)\big|S_{M,m}. \qquad \square$$

Lemma 4.8. *For $\boldsymbol{a} = (1\ 1\ 21)^T$, we have*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H(84n) - 7H\left(\tfrac{12n}{7}\right) - 3H\left(\tfrac{28n}{3}\right) + 21H\left(\tfrac{4n}{21}\right) - 4H\left(\tfrac{21n}{4}\right) + 28H\left(\tfrac{3n}{28}\right) + 12H\left(\tfrac{7n}{12}\right)\right.$$
$$\left. - 84H\left(\tfrac{n}{84}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 8} c(m)\left(\mathcal{H}_{12,7} - 3\mathcal{H}_{4,7}\big|V_3 - 4\mathcal{H}_{3,7}\big|V_4 + 12\mathcal{H}_{1,7}\big|V_{12}\right)\big|S_{8,m}. \qquad \square$$

Lemma 4.9. *For $\boldsymbol{a} \in \{(1\ 1\ 24)^T, (1\ 9\ 24)^T\}$, we have*

$$r_{\boldsymbol{a}}(n) = c_{\boldsymbol{a}}(n)\left(H(24n) - H(6n) - 3H\left(\tfrac{8n}{3}\right) + 3H\left(\tfrac{2n}{3}\right) + 12H\left(\tfrac{3n}{8}\right) - 36H\left(\tfrac{n}{24}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod M} c_{\boldsymbol{a}}(m)\left(\mathcal{H}_{8,3} - \mathcal{H}_{2,3} + 12\mathcal{H}_{1,3}\big|V_8\right)\big|S_{M,m}. \qquad \square$$

Lemma 4.10. *For $\boldsymbol{a} = (1\ 2\ 3)^T$, we have*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H(24n) - 2H\left(6n\right) + 3H\left(\tfrac{8n}{3}\right) - 6H\left(\tfrac{2n}{3}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 2} c(m)\mathcal{H}_{1,2}\big|\left(U_3 + 3V_3\right)\big|U_4\big|S_{2,m}. \qquad \square$$

Lemma 4.11. *For $\boldsymbol{a} = (1\ 2\ 5)^T$, we have*

$$r_{\boldsymbol{a}}(n) = (-1)^{n+1}\left(H(40n) - 4H(10n) - 5H\left(\tfrac{8n}{5}\right) + 20H\left(\tfrac{2n}{5}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 2} (-1)^{m+1}\left(\mathcal{H}_{8,5} - 4\mathcal{H}_{2,5}\right)\big|S_{2,m}. \qquad \square$$

Lemma 4.12. *We have, for $\boldsymbol{a} = (1\ 2\ 6)^T$,*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H(12n) - 2H(3n) + 3H\left(\tfrac{4n}{3}\right) - 6H\left(\tfrac{n}{3}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 4} c(m)\left(\mathcal{H}_{3,2} + 3\mathcal{H}_{1,2}\big|V_3\right)\big|U_2\big|S_{4,m}. \qquad \square$$

Lemma 4.13. *For $\boldsymbol{a} = (1\ 2\ 10)^T$, we have*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H(20n) - 4H(5n) - 5H\left(\tfrac{4n}{5}\right) + 20H\left(\tfrac{n}{5}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 4} c(m) \left(\mathcal{H}_{4,5} - 4\mathcal{H}_{1,5}\right) \big| S_{4,m}. \qquad \square$$

Lemma 4.14. *For* $\boldsymbol{a} = (1\ 3\ 3)^T$*, we have*

$$r_{\boldsymbol{a}}(n) = 2H(36n) + 4H(9n) - 6H(4n) - 12H(n).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = 2\left(\mathcal{H}_{4,3} + 2\mathcal{H}_{1,3}\right) \big| U_3. \qquad \square$$

Lemma 4.15. *For* $\boldsymbol{a} = (1\ 3\ 4)^T$*, we have*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H(12n) - 3H\left(\tfrac{4n}{3}\right) + 2H\left(\tfrac{3n}{4}\right) - 6H\left(\tfrac{n}{12}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 8} c(m) \left(\mathcal{H}_{4,3} + 2\mathcal{H}_{1,3} \big| V_4\right) \big| S_{8,m}. \qquad \square$$

Lemma 4.16. *For* $\boldsymbol{a} = (1\ 3\ 6)^T$*, we have*

$$r_{\boldsymbol{a}}(n) = c(n)\left(-H(72n) + 2H(18n) + 5H(8n) - 10H(2n)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 2} c(m) \left(5\mathcal{H}_{1,2} - \mathcal{H}_{9,2}\right) \big| U_4 \big| S_{2,m}. \qquad \square$$

Lemma 4.17. *For* $\boldsymbol{a} = (1\ 3\ 10)^T$*, we have*

$$r_{\boldsymbol{a}}(n) = \tfrac{1}{2}H(120n) - \tfrac{3}{2}H\left(\tfrac{40n}{3}\right) - 2H\left(\tfrac{15n}{2}\right) - \tfrac{5}{2}H\left(\tfrac{24n}{5}\right) + 6H\left(\tfrac{5n}{6}\right) + \tfrac{15}{2}H\left(\tfrac{8n}{15}\right) + 10H\left(\tfrac{3n}{10}\right) - 30H\left(\tfrac{n}{30}\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \tfrac{1}{2}\mathcal{H}_{1,3} \big| \left(U_5 - 5V_5\right) \big| \left(U_8 - 4V_2\right). \qquad \square$$

Lemma 4.18. *We have, for* $\boldsymbol{a} = (1\ 3\ 12)^T$*,*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H(36n) - 3H(4n) + 2H\left(\tfrac{9n}{4}\right) - 6H\left(\tfrac{n}{4}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 8} c(m)\mathcal{H}_{1,3} \big| \left(U_{12} + 2U_3V_4\right) \big| S_{8,m}. \qquad \square$$

Lemma 4.19. *For* $\boldsymbol{a} = (1\ 3\ 18)^T$*, we have*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H(24n) - 2H(6n) - 5H\left(\tfrac{8n}{3}\right) + 10H\left(\tfrac{2n}{3}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 6} c(m)\mathcal{H}_{1,2} \big| \left(U_3 - 5V_3\right) \big| U_4 \big| S_{6,m}. \qquad \square$$

Lemma 4.20. *We have, for* $\boldsymbol{a} = (1\ 3\ 30)^T$*,*

$$r_{\boldsymbol{a}}(n) = c(n)\Big(H(360n) - 2H(90n) - 3H(40n) + 6H(10n) - 5H\left(\tfrac{72n}{5}\right) + 10H\left(\tfrac{18n}{5}\right) + 15H\left(\tfrac{8n}{5}\right) - 30H\left(\tfrac{2n}{5}\right)\Big).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 2} c(m) \left(\mathcal{H}_{40,3} - 5\mathcal{H}_{8,3}\big|V_5 - 2\mathcal{H}_{20,3}|V_2 + 10\mathcal{H}_{4,3}|V_{10}\right)\big|U_3\big|S_{2,m}.$$

The claimed identity follows using that $\mathcal{H}(n) = 0$ if $2 \parallel n$. $\square$

Lemma 4.21. *For $\boldsymbol{a} \in \{(1\ 4\ 6)^T, (1\ 4\ 24)^T\}$, we have*

$$r_{\boldsymbol{a}}(n) = c_{\boldsymbol{a}}(n)\left(2H(24n) - 5H(6n) - 6H\left(\tfrac{8n}{3}\right) + 15H\left(\tfrac{2n}{3}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 8} c_{\boldsymbol{a}}(m)\mathcal{H}_{2,3}\big|\left(-5 + 2U_4\right)\big|S_{8,m}.$$ $\square$

Lemma 4.22. *For $\boldsymbol{a} = (1\ 5\ 5)^T$, we have*

$$r_{\boldsymbol{a}}(n) = -2H(100n) + 4H(25n) + 14H(4n) - 28H(n).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = -2\left(\mathcal{H}_{25,2} - 7\mathcal{H}_{1,2}\right)\big|U_2.$$ $\square$

Lemma 4.23. *We have, for $\boldsymbol{a} = (1\ 5\ 8)^T$,*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H(160n) - H(40n) - 5H\left(\tfrac{32n}{5}\right) + 5H\left(\tfrac{8n}{5}\right) - 12\delta_{8|n}\left(H\left(\tfrac{5n}{2}\right) - 5H\left(\tfrac{n}{10}\right)\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 8} c(m)\left(\mathcal{H}_{32,5} - \mathcal{H}_{8,5} - 12\mathcal{H}_{2,5}\big|V_4\right)\big|S_{8,m}.$$ $\square$

Lemma 4.24. *We have, for $\boldsymbol{a} = (1\ 5\ 10)^T$,*

$$r_{\boldsymbol{a}}(n) = (-1)^{n+1}\left(H(200n) - 4H(50n) - 5H(8n) + 20H(2n)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 2} (-1)^m\mathcal{H}_{2,5}\big|\left(4 - U_4\right)\big|U_5\big|S_{2,m}.$$ $\square$

Lemma 4.25. *We have for, $\boldsymbol{a} = (1\ 5\ 25)^T$,*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H(20n) - 2H(5n) - 7H\left(\tfrac{4n}{5}\right) + 14H\left(\tfrac{n}{5}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 5} c(m)\left(\mathcal{H}_{5,2} - 7\mathcal{H}_{1,2}\big|V_5\right)\big|U_2\big|S_{5,m}.$$ $\square$

Lemma 4.26. *We have, for $\boldsymbol{a} = (1\ 5\ 40)^T$,*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H(200n) - H(50n) - 5H(8n) + 5H(2n) - 6\delta_{8|n}\left(H\left(\tfrac{25n}{2}\right) - 5H\left(\tfrac{n}{2}\right)\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = -\sum_{m \pmod 8} c(m)\mathcal{H}_{2,5}\big|\left(1 - U_4 + 6V_4\right)\big|U_5\big|S_{8,m}.$$ $\square$

Lemma 4.27. *We have, for $\boldsymbol{a} \in \{(1\ 6\ 6)^T, (1\ 12\ 12)^T\}$,*

$$r_{\boldsymbol{a}}(n) = c_{\boldsymbol{a}}(n)\left(H(36n) - 4H(9n) - 3H(4n) + 12H(n)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 4} c_{\boldsymbol{a}}(n) \left(\mathcal{H}_{4,3} - 4\mathcal{H}_{1,3}\right) \big|U_3\big|S_{4,m}. \qquad \square$$

Lemma 4.28. *We have, for $\boldsymbol{a} \in \{(1\ 6\ 16)^T, (1\ 16\ 24)^T\}$,*

$$r_{\boldsymbol{a}}(n) = c_{\boldsymbol{a}}(n) \left(H(24n) - H(6n) - 3H\left(\tfrac{8n}{3}\right) - 12H\left(\tfrac{3n}{8}\right) + 3H\left(\tfrac{2n}{3}\right) + 36H\left(\tfrac{n}{24}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod{32}} c_{\boldsymbol{a}}(m)\mathcal{H}_{1,3}\big| \left(U_8 - U_2 - 12V_8\right) \big|S_{32,m}. \qquad \square$$

Lemma 4.29. *For $\boldsymbol{a} = (1\ 6\ 18)^T$, we have*

$$r_{\boldsymbol{a}}(n) = c(n) \left(H(12n) - 2H(3n) - 5H\left(\tfrac{4n}{3}\right) + 10H\left(\tfrac{n}{3}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod{12}} c(m) \left(\mathcal{H}_{3,2} - 5\mathcal{H}_{1,2}|V_3\right) \big|U_2\big|S_{12,m}. \qquad \square$$

Lemma 4.30. *For $\boldsymbol{a} = (1\ 6\ 24)^T$, we have*

$$r_{\boldsymbol{a}}(n) = c(n) \left(2H(36n) - 5H(9n) - 6H(4n) + 15H(n)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod{16}} c(m) \left(2\mathcal{H}_{4,3} - 5\mathcal{H}_{1,3}\right) \big|U_3\big|S_{16,m}. \qquad \square$$

Lemma 4.31. *We have, for $\boldsymbol{a} = (1\ 8\ 40)^T$,*

$$r_{\boldsymbol{a}}(n) = c(n) \left(2H(20n) - 5H(5n) - 10H\left(\tfrac{4n}{5}\right) + 25H\left(\tfrac{n}{5}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod{16}} c(m) \left(2\mathcal{H}_{4,5} - 5\mathcal{H}_{1,5}\right) \big|S_{16,m}. \qquad \square$$

Lemma 4.32. *We have, for $\boldsymbol{a} = (1\ 9\ 9)^T$,*

$$r_{\boldsymbol{a}}(n) = 12H\left(\tfrac{4n}{9}\right) - 24H\left(\tfrac{n}{9}\right) + 4\delta_{n\equiv 1 \pmod 3} \left(H(4n) - 2H(n)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = 4\mathcal{H}_{1,2}\big|U_2\big|(3V_9 + S_{3,1}). \qquad \square$$

Lemma 4.33. *We have, for $\boldsymbol{a} = (1\ 9\ 21)^T$,*

$$r_{\boldsymbol{a}}(n) = c(n)\Big(H(84n) - 2H(21n) - 3H\left(\tfrac{28n}{3}\right) + 6H\left(\tfrac{7n}{3}\right) - 7H\left(\tfrac{12n}{7}\right) + 14H\left(\tfrac{3n}{7}\right) + 21H\left(\tfrac{4n}{21}\right) - 42H\left(\tfrac{n}{21}\right)\Big).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = -\sum_{m \pmod 3} c(m)\mathcal{H}_{1,3}\big| \left(2 - U_4\right) \big| \left(U_7 - 7V_7\right) \big|S_{3,m}. \qquad \square$$

Lemma 4.34. *We have, for* $\boldsymbol{a} = (1\ 10\ 30)^T$,

$$r_{\boldsymbol{a}}(n) = c(n)\Big(H(300n) - 2H(75n) - 5H(12n) + 10H(3n) - 3H\left(\tfrac{100n}{3}\right) + 6H\left(\tfrac{25n}{3}\right) + 15H\left(\tfrac{4n}{3}\right) - 30H\left(\tfrac{n}{3}\right)\Big).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 4} c(m)\mathcal{H}_{1,3}\big|\,(U_{100} - 2U_{25} - 5U_4 + 10)\,\big|S_{4,m}. \qquad \square$$

Lemma 4.35. *We have, for* $\boldsymbol{a} = (1\ 21\ 21)^T$,

$$r_{\boldsymbol{a}}(n) = H(1764n) - 2H(441n) - 3H(196n) + 6H(49n) - 7H(36n) + 14H(9n) + 21H(4n) - 42H(n).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \mathcal{H}_{1,7}\big|\,(2 - U_4)\,\big|\,(3 - U_9)\,\big|U_7. \qquad \square$$

Lemma 4.36. *We have, for* $\boldsymbol{a} = (1\ 24\ 24)^T$,

$$r_{\boldsymbol{a}}(n) = c(n)\left(2H(36n) - 5H(9n) - 6H(4n) + 15H(n)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = -\sum_{m \pmod{16}} c(m)\mathcal{H}_{1,3}\big|\,(5 - 2U_4)\,\big|U_3\big|S_{16,m}. \qquad \square$$

Lemma 4.37. *For* $\boldsymbol{a} \in \{(2\ 2\ 3)^T, (2\ 3\ 18)^T, (3\ 4\ 4)^T, (3\ 4\ 36)^T\}$*, we have*

$$r_{\boldsymbol{a}}(n) = c_{\boldsymbol{a}}(n)\left(H(12n) - 4H(3n) - 3H\left(\tfrac{4n}{3}\right) + 12H\left(\tfrac{n}{3}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = -\sum_{m \pmod M} c_{\boldsymbol{a}}(m)\mathcal{H}_{1,3}\big|\,(4 - U_4)\,\big|S_{M,m}. \qquad \square$$

Lemma 4.38. *For* $\boldsymbol{a} = (2\ 3\ 3)^T$*, we have*

$$r_{\boldsymbol{a}}(n) = 2(-1)^n\left(H(72n) - 4H(18n) - 3H(8n) + 12H(2n)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = 2\sum_{m \pmod 2} (-1)^m\mathcal{H}_{4,3}\big|\,(U_2 - 4V_2)\,\big|U_3\big|S_{2,m}. \qquad \square$$

Lemma 4.39. *For* $\boldsymbol{a} = (2\ 3\ 6)^T$*, we have*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H(36n) - 2H(9n) - 5H(4n) + 10H(n)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = -\sum_{m \pmod 4} c(m)\mathcal{H}_{1,2}\big|\,(5 - U_9)\,\big|U_2\big|S_{4,m}. \qquad \square$$

Lemma 4.40. *We have, for* $\boldsymbol{a} \in \{(2\ 3\ 8)^T, (3\ 8\ 8)^T, (3\ 8\ 72)^T\}$,

$$r_{\boldsymbol{a}}(n) = c_{\boldsymbol{a}}(n)\left(2H(12n) - 5H(3n) - 6H\left(\tfrac{4n}{3}\right) + 15H\left(\tfrac{n}{3}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = -\sum_{m \pmod M} c_{\boldsymbol{a}}(m)\mathcal{H}_{1,3}\big|\,(5 - 2U_4)\,\big|S_{M,m}. \qquad \square$$

Lemma 4.41. *For $\boldsymbol{a} = (2\ 3\ 9)^T$, we have*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H(24n) - 2H(6n) - 5H\left(\tfrac{8n}{3}\right) + 10H\left(\tfrac{2n}{3}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 6} c(m)\mathcal{H}_{1,2}\big| \left(U_3 - 5V_3\right)\big|U_4\big|S_{6,m}. \qquad \square$$

Lemma 4.42. *For $\boldsymbol{a} \in \{(2\ 3\ 12)^T, (3\ 8\ 12)^T\}$, we have*

$$r_{\boldsymbol{a}}(n) = c_{\boldsymbol{a}}(n)\left(2H(72n) - 5H(18n) - 6H(8n) + 15H(2n)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 8} c_{\boldsymbol{a}}(m)\left(2\mathcal{H}_{8,3} - 5\mathcal{H}_{2,3}\right)\big|U_3\big|S_{8,m}. \qquad \square$$

Lemma 4.43. *For $\boldsymbol{a} = (2\ 3\ 48)^T$, we have*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H\left(72n\right) - H(18n) - 3H\left(8n\right) + 3H(2n) - 12H\left(\tfrac{9n}{8}\right) + 36H\left(\tfrac{n}{8}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod{32}} c(m)\left(\mathcal{H}_{8,3} - \mathcal{H}_{4,3}\big|V_2 - 12\mathcal{H}_{1,3}\big|V_8\right)\big|U_3\big|S_{32,m}.$$

To verify the identity, the table in Appendix D implies that we need to compute $H(72n)$, $H(18n)$, $H(8n)$, $H(2n)$, $H(\frac{9n}{8})$, and $H(\frac{n}{8})$ for every $n \leq 12\,288$. Using the code built into GP/Pari for computing class numbers, the proof would be conditional on the Generalized Riemann Hypothesis, so a slight modification of the calculation is necessary in order to obtain an unconditional result. Note that we can unconditionally compute $H(D)$ for $D \leq 18 \cdot 12\,288 = 221\,184 < 5 \cdot 10^5$ (in particular, we can compute $H(Cn)$ for any $C \leq 18$). It remains to compute $H(72n)$ for $6\,945 \leq n \leq 12\,288$; the method built into GP/Pari would give a result that is conditional on the Generalized Riemann Hypothesis. However, since $9 \mid 72n$ and $9 \mid 18n$, the corresponding fundamental discriminant $\Delta < 0$ in these cases is a divisor of $-8n$ and $8n \leq 8 \cdot 12\,288 = 98\,304$. Hence we can unconditionally compute $H(|\Delta|)$. Suppose that $72n = D = |\Delta|f^2$. In order to compute $H(D)$, we first compute $H(|\Delta|)$ using the code built into GP/Pari and then use the relation (see [3, p. 273])

$$H(D) = H\left(|\Delta|f^2\right) = H(|\Delta|)\sum_{1 \leq d | f} \mu(d)\left(\tfrac{\Delta}{d}\right)\sigma_1\left(\tfrac{f}{d}\right). \tag{4.1}$$

Both $H(|\Delta|)$ and the sum over divisors of f can be computed unconditionally, so $H(D)$ can be computed unconditionally. This was carried out for $D < 10^6$ in order to verify the claim. $\square$

Lemma 4.44. *We have, for $\boldsymbol{a} = (2\ 5\ 6)^T$,*

$$r_{\boldsymbol{a}}(n) = c(n)\Big(H(60n) - 2H(15n) - 3H\left(\tfrac{20n}{3}\right) - 5H\left(\tfrac{12n}{5}\right) + 6H\left(\tfrac{5n}{3}\right) + 10H\left(\tfrac{3n}{5}\right) + 15H\left(\tfrac{4n}{15}\right) - 30H\left(\tfrac{n}{15}\right)\Big).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 4} c(m)\left(\mathcal{H}_{12,5} - 3\mathcal{H}_{4,5}\big|V_3 - 2\mathcal{H}_{3,5} + 6\mathcal{H}_{1,5}\big|V_3\right)\big|S_{4,m}. \qquad \square$$

Lemma 4.45. *For $\boldsymbol{a} = (2\ 5\ 10)^T$, we have*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H(100n) - 4H(25n) - 5H(4n) + 20H(n)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = - \sum_{m \pmod 4} c(m)\mathcal{H}_{1,5}\big| \left(4 - U_4\right) \big|U_5\big|S_{4,m}. \qquad \square$$

Lemma 4.46. *For* $\boldsymbol{a} = (2\ 5\ 15)^T$ *we have*

$$r_{\boldsymbol{a}}(n) = c(n) \left(H(600n) - 2H(150n) - 5H(24n) + 10H(6n) - 3H\left(\tfrac{200n}{3}\right) + 6H\left(\tfrac{50n}{3}\right) + 15H\left(\tfrac{8n}{3}\right) \right.$$
$$\left. -30H\left(\tfrac{2n}{3}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 2} c(m)\mathcal{H}_{1,5}\big| \left(U_{24} - 3U_8V_3 - 2U_6 + 6U_2V_3\right) \big|U_5\big|S_{2,m}. \qquad \square$$

Lemma 4.47. *We have, for* $\boldsymbol{a} = (2\ 6\ 9)^T$,

$$r_{\boldsymbol{a}}(n) = c(n) \left(H(12n) - 2H(3n) - 5H\left(\tfrac{4n}{3}\right) + 10H\left(\tfrac{n}{3}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod{12}} c(m)\mathcal{H}_{1,2}\big| \left(U_3 - 5V_3\right) \big|U_2\big|S_{12,m}. \qquad \square$$

Lemma 4.48. *For* $\boldsymbol{a} = (2\ 6\ 15)^T$, *we have*

$$r_{\boldsymbol{a},\boldsymbol{h},N}(n) = c(n) \left(H(720n) - 2H(180n) - 3H(80n) + 6H(20n) - 5H\left(\tfrac{144n}{5}\right) + 10H\left(\tfrac{36n}{5}\right) \right.$$
$$\left. +15H\left(\tfrac{16n}{5}\right) - 30H\left(\tfrac{4n}{5}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 4} c(m) \left(\mathcal{H}_{80,3} - 2\mathcal{H}_{20,3} + 10\mathcal{H}_{4,3}\big|V_5 - 5\mathcal{H}_{16,3}\big|V_5\right) \big|U_3\big|S_{4,m}. \qquad \square$$

Lemma 4.49. *For* $\boldsymbol{a} = (3\ 3\ 4)^T$, *we have*

$$r_{\boldsymbol{a}}(n) = c(n) \left(H(36n) - 3H(4n) + 8H\left(\tfrac{9n}{4}\right) - 24H\left(\tfrac{n}{4}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 8} c(m) \left(\mathcal{H}_{4,3} + 8\mathcal{H}_{1,3}\big|V_4\right) |U_3\big|S_{8,m}. \qquad \square$$

Lemma 4.50. *For* $\boldsymbol{a} = (3\ 3\ 7)^T$, *we have*

$$r_{\boldsymbol{a}}(n) = H(252n) - 2H(63n) - 3H(28n) + 6H(7n) - 7H\left(\tfrac{36n}{7}\right) + 14H\left(\tfrac{9n}{7}\right) + 21H\left(\tfrac{4n}{7}\right) - 42H\left(\tfrac{n}{7}\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \mathcal{H}_{1,7}\big| \left(6 - 3U_4 - 2U_9 + U_{36}\right). \qquad \square$$

Lemma 4.51. *For* $\boldsymbol{a} = (3\ 3\ 8)^T$, *we have*

$$r_{\boldsymbol{a}}(n) = c(n) \left(H(72n) - H(18n) - 3H(8n) + 3H(2n) - 6\delta_{8|n}\left(H\left(\tfrac{9n}{2}\right) - 3H\left(\tfrac{n}{2}\right)\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 8} c(m) \left(\mathcal{H}_{8,3} - \mathcal{H}_{4,3}\big|V_2 - 6\mathcal{H}_{2,3}\big|V_4\right) \big|U_3\big|S_{8,m}. \qquad \square$$

Lemma 4.52. *For $\boldsymbol{a} = (3\ 4\ 12)^T$, we have*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H(9n) - 3H(n) + 2H\left(\tfrac{9n}{4}\right) - 6H\left(\tfrac{n}{4}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 8} c(m)\mathcal{H}_{1,3}\big|\left(1+2V_4\right)\big|U_3\big|S_{8,m}.$$ □

Setting $c_{(3\ 7\ 7)^T}(n) := 1$, we have the following.

Lemma 4.53. *For $\boldsymbol{a} \in \{(3\ 7\ 7)^T, (3\ 7\ 63)^T\}$, we have*

$$r_{\boldsymbol{a}}(n) = c(n)\Big(H(588n) - 2H(147n) - 7H(12n) + 14H(3n) - 3H\left(\tfrac{196n}{3}\right) + 6H\left(\tfrac{49n}{3}\right) + 21H\left(\tfrac{4n}{3}\right) \\ -42H\left(\tfrac{n}{3}\right)\Big).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = -\sum_{m \pmod M} c_{\boldsymbol{a}}(m)\mathcal{H}_{1,7}\big|\left(2-U_4\right)\big|\left(U_3-3V_3\right)\big|U_7\big|S_{M,m}.$$ □

Lemma 4.54. *We have, for $\boldsymbol{a} = (3\ 8\ 48)^T$,*

$$r_{\boldsymbol{a}}(n) = c(n)\left(2H(72n) - 6H(8n) - 11H\left(\tfrac{9n}{2}\right) + 33H\left(\tfrac{n}{2}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod{32}} c(m)\mathcal{H}_{1,3}\big|\left(2U_8 - 11V_2\right)\big|U_3\big|S_{32,m}.$$

As in the case of $\boldsymbol{a} = (2\ 3\ 48)^T$, the computation built into GP/Pari would be conditional upon GRH. However, the same calculation as in Lemma 4.43, using (4.1), gives an unconditional proof in this case. □

Lemma 4.55. *For $\boldsymbol{a} = (3\ 10\ 30)^T$, we have*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H(900n) - 2H\left(225n\right) - 3H(100n) - 5H(36n) + 6H(25n) + 10H(9n) + 15H(4n) \\ -30H(n)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 4} c(m)\mathcal{H}_{1,5}\big|\left(6 - 3U_4 - 2U_9 + U_{36}\right)\big|U_5\big|S_{4,m}.$$ □

Lemma 4.56. *For $\boldsymbol{a} = (5\ 6\ 15)^T$, we have*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H(1800n) - 2H(450n) - 3H(200n) - 5H(72n) + 6H(50n) + 10H(18n) + 15H(8n) \\ -30H(2n)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod 2} c(m)\mathcal{H}_{2,3}\big|\left(5 - U_{25}\right)\big|\left(2-U_4\right)\big|U_3\big|S_{2,m}.$$ □

Lemma 4.57. *For $\boldsymbol{a} = (5\ 8\ 40)^T$, we have*

$$r_{\boldsymbol{a}}(n) = c(n)\left(H\left(100n\right) - 5H\left(4n\right) - 10H\left(\tfrac{25n}{4}\right) + 50H\left(\tfrac{n}{4}\right)\right).$$

Proof. The equivalent formula in this case is

$$\Theta_{\boldsymbol{a}} = \sum_{m \pmod{16}} c(m)\mathcal{H}_{1,5}\big|\left(U_4 - 10V_4\right)\big|U_5\big|S_{16,m}.$$ □

Appendix A. Tables for the constants $d_{\boldsymbol{a},\boldsymbol{h},N}(m)$.

Here we list $d_{\boldsymbol{a},\boldsymbol{h},N}(m)$ from the statement of Theorem 1.1. For each $\boldsymbol{a} \in \mathcal{C}$, we give a table listing for $(\boldsymbol{h}, N) \in \mathcal{S}_{\boldsymbol{a}}$ the corresponding m, M, and $d = d_{\boldsymbol{a},\boldsymbol{h},N}(n)$ valid for every $n \equiv m \pmod M$ such that $\Theta_{\boldsymbol{a},\boldsymbol{h},N} = \sum_{m \pmod M} d_{\boldsymbol{a},\boldsymbol{h},N}(m)\Theta_{\boldsymbol{a}}\big|S_{M,m}$.

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (1\ 1\ 1)^T$				
$(0\ 0\ 1)^T$	3	1	3	$\frac{1}{6}$
$(0\ 1\ 1)^T$	3	2	3	$\frac{1}{12}$
$(0\ 1\ 1)^T$	4	2	8	$\frac{1}{12}$
$(0\ 1\ 2)^T$	4	5	8	$\frac{1}{12}$
$(1\ 1\ 1)^T$	4	3	8	$\frac{1}{8}$
$(1\ 1\ 2)^T$	4	6	8	$\frac{1}{12}$
$(1\ 1\ 3)^T$	6	11	24	$\frac{1}{12}$
$(1\ 3\ 3)^T$	6	19	24	$\frac{1}{6}$
$(1\ 2\ 3)^T$	8	14	16	$\frac{1}{48}$
$\boldsymbol{a} = (1\ 1\ 2)^T$				
$(0\ 0\ 1)^T$	4	2	16	$\frac{1}{6}$
$(1\ 0\ 1)^T$	4	3	8	$\frac{1}{8}$
$(1\ 1\ 1)^T$	4	4	8	$\frac{1}{12}$
$(1\ 2\ 1)^T$	4	7	8	$\frac{1}{8}$
$(2\ 2\ 1)^T$	4	10	16	$\frac{1}{6}$
$(4\ 2\ 3)^T$	4	6	16	$\frac{1}{4}$
$(2\ 2\ 1)^T$	8	10	32	$\frac{1}{24}$
$(2\ 2\ 3)^T$	8	26	32	$\frac{1}{24}$
$(4\ 2\ 3)^T$	12	14	24	$\frac{1}{24}$
$\boldsymbol{a} = (1\ 1\ 3)^T$				
$(0\ 0\ 1)^T$	3	3	9	$\frac{1}{2}$
$(0\ 2\ 1)^T$	4	7	8	$\frac{1}{4}$
$(1\ 1\ 0)^T$	4	2	8	$\frac{1}{4}$
$(1\ 1\ 1)^T$	4	5	8	$\frac{1}{16}$
$(1\ 1\ 2)^T$	4	6	8	$\frac{1}{4}$
$(0\ 0\ 1)^T$	6	3	36	$\frac{1}{2}$
$(0\ 3\ 2)^T$	6	21	72	$\frac{1}{8}$
$(0\ 3\ 2)^T$	6	57	72	$\frac{1}{4}$
$(1\ 2\ 2)^T$	6	17	24	$\frac{1}{24}$
$(3\ 3\ 1)^T$	6	21	72	$\frac{1}{4}$
$(3\ 3\ 2)^T$	6	30	72	$\frac{1}{2}$
$(1\ 3\ 2)^T$	8	6	8	$\frac{1}{16}$
$(3\ 3\ 2)^T$	12	30	72	$\frac{1}{8}$
$(3\ 3\ 4)^T$	12	66	72	$\frac{1}{8}$
$\boldsymbol{a} = (1\ 1\ 4)^T$				
$(0\ 0\ 1)^T$	4	4	16	$\frac{1}{6}$
$(0\ 1\ 1)^T$	4	5	8	$\frac{1}{16}$
$(0\ 2\ 1)^T$	4	8	16	$\frac{1}{6}$
$(1\ 1\ 1)^T$	4	6	8	$\frac{1}{8}$
$(2\ 2\ 1)^T$	4	12	16	$\frac{1}{2}$
$(0\ 4\ 1)^T$	8	20	64	$\frac{1}{12}$
$(0\ 4\ 3)^T$	8	52	64	$\frac{1}{16}$
$(2\ 2\ 3)^T$	12	44	48	$\frac{1}{24}$
$(2\ 6\ 3)^T$	12	28	48	$\frac{1}{16}$
$\boldsymbol{a} = (1\ 1\ 5)^T$				
$(1\ 1\ 1)^T$	4	7	8	$\frac{1}{8}$
$(0\ 0\ 1)^T$	5	5	25	$\frac{1}{10}$
$(0\ 0\ 3)^T$	5	20	25	$\frac{1}{10}$
$(5\ 5\ 1)^T$	10	55	200	$\frac{1}{10}$
$(5\ 5\ 3)^T$	10	95	200	$\frac{1}{10}$
$\boldsymbol{a} = (1\ 1\ 6)^T$				
$(0\ 0\ 1)^T$	3	6	9	$\frac{1}{2}$
$(0\ 0\ 1)^T$	4	6	16	$\frac{1}{2}$
$(0\ 1\ 1)^T$	4	7	8	$\frac{1}{8}$
$(0\ 2\ 1)^T$	4	10	16	$\frac{1}{8}$
$(2\ 1\ 1)^T$	4	3	8	$\frac{1}{8}$
$(2\ 2\ 1)^T$	4	14	16	$\frac{1}{2}$
$(0\ 0\ 1)^T$	6	6	72	$\frac{1}{2}$
$(0\ 0\ 1)^T$	6	42	144	$\frac{1}{4}$
$(0\ 3\ 1)^T$	6	15	36	$\frac{1}{4}$
$(0\ 3\ 2)^T$	6	33	36	$\frac{1}{4}$
$(3\ 3\ 2)^T$	6	42	144	$\frac{1}{4}$
$(3\ 3\ 2)^T$	6	114	144	$\frac{1}{2}$
$(2\ 2\ 1)^T$	8	14	32	$\frac{1}{8}$
$(2\ 2\ 3)^T$	8	30	32	$\frac{1}{8}$
$\boldsymbol{a} = (1\ 1\ 8)^T$				
$(0\ 0\ 1)^T$	4	8	32	$\frac{1}{6}$
$(0\ 0\ 1)^T$	4	24	32	$\frac{1}{2}$
$(0\ 2\ 1)^T$	4	12	16	$\frac{1}{4}$
$(2\ 2\ 1)^T$	4	16	32	$\frac{1}{3}$
$\boldsymbol{a} = (1\ 1\ 8)^T$ (cont.)				
$(4\ 4\ 1)^T$	8	40	128	$\frac{1}{6}$
$(4\ 4\ 3)^T$	8	104	128	$\frac{1}{6}$
$\boldsymbol{a} = (1\ 1\ 9)^T$				
$(1\ 1\ 1)^T$	4	3	8	$\frac{1}{8}$
$(0\ 0\ 1)^T$	9	9	81	$\frac{1}{6}$
$(0\ 0\ 2)^T$	9	36	81	$\frac{1}{6}$
$(0\ 0\ 4)^T$	9	63	81	$\frac{1}{6}$
$(9\ 9\ 1)^T$	18	171	324	$\frac{1}{6}$
$(9\ 9\ 5)^T$	18	63	324	$\frac{1}{6}$
$(9\ 9\ 7)^T$	18	279	324	$\frac{1}{6}$
$\boldsymbol{a} = (1\ 1\ 12)^T$				
$(0\ 0\ 1)^T$	3	3	9	$\frac{1}{2}$
$(0\ 0\ 1)^T$	4	12	16	$\frac{1}{2}$
$(1\ 1\ 1)^T$	4	6	8	$\frac{1}{8}$
$(2\ 2\ 1)^T$	4	20	32	$\frac{1}{4}$
$(3\ 3\ 1)^T$	6	30	72	$\frac{1}{2}$
$(3\ 3\ 2)^T$	6	66	72	$\frac{1}{2}$
$(0\ 4\ 1)^T$	8	28	64	$\frac{1}{4}$
$(0\ 4\ 3)^T$	8	60	64	$\frac{1}{4}$
$\boldsymbol{a} = (1\ 1\ 21)^T$				
$(0\ 0\ 1)^T$	3	3	9	$\frac{1}{2}$
$(1\ 1\ 1)^T$	4	7	8	$\frac{1}{8}$
$(3\ 3\ 1)^T$	6	39	72	$\frac{1}{2}$
$(0\ 0\ 1)^T$	7	21	49	$\frac{1}{2}$
$(0\ 0\ 2)^T$	7	35	49	$\frac{1}{2}$
$(0\ 0\ 3)^T$	7	42	49	$\frac{1}{2}$
$(7\ 7\ 1)^T$	14	119	392	$\frac{1}{2}$
$(7\ 7\ 3)^T$	14	287	392	$\frac{1}{2}$
$(7\ 7\ 5)^T$	14	231	392	$\frac{1}{2}$
$\boldsymbol{a} = (1\ 1\ 24)^T$				
$(0\ 0\ 1)^T$	3	6	9	$\frac{1}{2}$
$(0\ 0\ 1)^T$	4	24	32	$\frac{1}{2}$
$(0\ 0\ 1)^T$	4	40	64	$\frac{1}{4}$
$(0\ 2\ 1)^T$	4	12	16	$\frac{1}{4}$
$(4\ 4\ 1)^T$	8	56	128	$\frac{1}{2}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (1\ 1\ 24)^T$ (cont.)				
$(4\ 4\ 3)^T$	8	120	128	$\frac{1}{2}$
$\boldsymbol{a} = (1\ 2\ 2)^T$				
$(0\ 0\ 1)^T$	4	2	16	$\frac{1}{4}$
$(0\ 1\ 1)^T$	4	4	16	$\frac{1}{6}$
$(0\ 1\ 2)^T$	4	10	16	$\frac{1}{4}$
$(1\ 1\ 1)^T$	4	5	8	$\frac{1}{8}$
$(2\ 0\ 1)^T$	4	6	16	$\frac{1}{4}$
$(2\ 1\ 1)^T$	4	8	16	$\frac{1}{6}$
$(2\ 2\ 1)^T$	4	14	16	$\frac{1}{4}$
$(1\ 0\ 3)^T$	6	19	24	$\frac{1}{12}$
$(1\ 1\ 2)^T$	6	11	24	$\frac{1}{24}$
$(3\ 1\ 2)^T$	6	19	24	$\frac{1}{12}$
$(0\ 1\ 3)^T$	8	20	32	$\frac{1}{24}$
$(2\ 1\ 2)^T$	8	14	32	$\frac{1}{16}$
$(2\ 1\ 3)^T$	8	24	32	$\frac{1}{24}$
$(2\ 3\ 2)^T$	8	30	32	$\frac{1}{16}$
$\boldsymbol{a} = (1\ 2\ 3)^T$				
$(0\ 0\ 1)^T$	3	3	9	$\frac{1}{6}$
$(0\ 1\ 2)^T$	4	14	16	$\frac{1}{6}$
$(1\ 1\ 1)^T$	4	6	8	$\frac{1}{12}$
$(2\ 1\ 0)^T$	4	6	16	$\frac{1}{6}$
$(0\ 3\ 2)^T$	6	30	72	$\frac{1}{18}$
$(0\ 3\ 2)^T$	6	66	72	$\frac{1}{6}$
$(3\ 0\ 1)^T$	6	12	72	$\frac{1}{9}$
$(3\ 3\ 1)^T$	6	30	72	$\frac{1}{9}$
$(0\ 3\ 2)^T$	12	30	144	$\frac{1}{36}$
$(6\ 3\ 4)^T$	12	102	144	$\frac{1}{36}$
$\boldsymbol{a} = (1\ 2\ 4)^T$				
$(0\ 0\ 1)^T$	4	4	16	$\frac{1}{4}$
$(0\ 1\ 1)^T$	4	6	16	$\frac{1}{8}$
$(0\ 2\ 1)^T$	4	12	16	$\frac{1}{4}$
$(1\ 1\ 1)^T$	4	7	8	$\frac{1}{8}$
$(2\ 0\ 1)^T$	4	8	32	$\frac{1}{3}$
$(2\ 1\ 1)^T$	4	10	16	$\frac{1}{4}$
$(2\ 2\ 1)^T$	4	16	32	$\frac{1}{3}$
$(0\ 2\ 1)^T$	8	12	64	$\frac{1}{8}$
$(0\ 2\ 3)^T$	8	44	64	$\frac{1}{8}$
$(4\ 2\ 1)^T$	8	28	64	$\frac{1}{8}$
$(4\ 2\ 3)^T$	8	60	64	$\frac{1}{8}$
$\boldsymbol{a} = (1\ 2\ 5)^T$				
$(0\ 1\ 0)^T$	4	2	16	$\frac{1}{2}$
$(2\ 1\ 2)^T$	4	10	16	$\frac{1}{2}$
$(0\ 0\ 1)^T$	5	5	25	$\frac{1}{2}$
$(0\ 0\ 2)^T$	5	20	25	$\frac{1}{2}$
$(2\ 1\ 2)^T$	8	26	32	$\frac{1}{8}$
$(2\ 3\ 2)^T$	8	10	32	$\frac{1}{8}$
$(0\ 5\ 2)^T$	10	70	400	$\frac{1}{4}$
$(0\ 5\ 2)^T$	10	170	400	$\frac{1}{2}$
$(0\ 5\ 2)^T$	10	370	400	$\frac{1}{2}$
$(0\ 5\ 4)^T$	10	130	400	$\frac{1}{2}$
$(0\ 5\ 4)^T$	10	230	400	$\frac{1}{4}$
$(0\ 5\ 4)^T$	10	330	400	$\frac{1}{2}$
$(5\ 0\ 1)^T$	10	30	400	$\frac{1}{2}$
$(5\ 0\ 1)^T$	10	230	400	$\frac{1}{4}$
$(5\ 0\ 3)^T$	10	70	400	$\frac{1}{4}$
$(5\ 0\ 3)^T$	10	270	400	$\frac{1}{2}$
$(0\ 5\ 4)^T$	20	130	400	$\frac{1}{4}$
$(0\ 5\ 8)^T$	20	370	400	$\frac{1}{4}$
$(10\ 5\ 2)^T$	20	170	400	$\frac{1}{4}$
$(10\ 5\ 6)^T$	20	330	400	$\frac{1}{4}$
$\boldsymbol{a} = (1\ 2\ 6)^T$				
$(0\ 0\ 1)^T$	3	6	9	$\frac{1}{6}$
$(0\ 1\ 1)^T$	4	8	16	$\frac{1}{6}$
$(2\ 1\ 1)^T$	4	12	16	$\frac{1}{6}$
$(0\ 3\ 1)^T$	6	24	144	$\frac{1}{9}$
$(0\ 3\ 1)^T$	6	60	144	$\frac{1}{9}$
$(3\ 0\ 1)^T$	6	15	144	$\frac{1}{6}$
$(3\ 0\ 1)^T$	6	87	144	$\frac{1}{6}$
$(3\ 3\ 2)^T$	6	51	144	$\frac{1}{6}$
$\boldsymbol{a} = (1\ 2\ 8)^T$				
$(0\ 0\ 1)^T$	4	8	16	$\frac{1}{4}$
$(0\ 1\ 1)^T$	4	10	16	$\frac{1}{4}$
$(0\ 2\ 1)^T$	4	16	64	$\frac{1}{3}$
$(0\ 2\ 1)^T$	4	32	64	$\frac{1}{3}$
$(2\ 0\ 1)^T$	4	12	32	$\frac{1}{4}$
$(2\ 1\ 1)^T$	4	14	16	$\frac{1}{4}$
$(2\ 2\ 1)^T$	4	20	32	$\frac{1}{2}$
$(4\ 0\ 1)^T$	8	24	128	$\frac{1}{4}$
$(4\ 0\ 3)^T$	8	88	128	$\frac{1}{4}$
$(4\ 4\ 1)^T$	8	56	128	$\frac{1}{4}$
$\boldsymbol{a} = (1\ 2\ 8)^T$ (cont.)				
$(4\ 4\ 3)^T$	8	120	128	$\frac{1}{4}$
$(2\ 0\ 3)^T$	12	76	96	$\frac{1}{24}$
$\boldsymbol{a} = (1\ 2\ 10)^T$				
$(0\ 1\ 1)^T$	4	12	32	$\frac{1}{8}$
$(0\ 1\ 1)^T$	4	28	32	$\frac{1}{4}$
$(1\ 1\ 1)^T$	4	5	8	$\frac{1}{8}$
$(0\ 0\ 1)^T$	5	10	25	$\frac{1}{2}$
$(0\ 0\ 2)^T$	5	15	25	$\frac{1}{2}$
$(5\ 0\ 2)^T$	10	65	200	$\frac{1}{2}$
$(5\ 0\ 4)^T$	10	185	200	$\frac{1}{2}$
$(5\ 5\ 1)^T$	10	85	200	$\frac{1}{2}$
$(5\ 5\ 3)^T$	10	165	200	$\frac{1}{2}$
$\boldsymbol{a} = (1\ 2\ 16)^T$				
$(0\ 0\ 1)^T$	4	32	128	$\frac{1}{3}$
$(0\ 0\ 1)^T$	4	64	128	$\frac{1}{3}$
$(0\ 0\ 1)^T$	4	16	32	$\frac{1}{4}$
$(0\ 2\ 1)^T$	4	40	64	$\frac{1}{2}$
$(0\ 2\ 1)^T$	4	24	64	$\frac{1}{4}$
$(2\ 0\ 1)^T$	4	20	32	$\frac{1}{2}$
$(2\ 2\ 1)^T$	4	28	32	$\frac{1}{2}$
$(0\ 4\ 1)^T$	8	48	256	$\frac{1}{4}$
$(0\ 4\ 1)^T$	8	112	256	$\frac{1}{4}$
$(0\ 4\ 3)^T$	8	176	256	$\frac{1}{4}$
$(0\ 4\ 3)^T$	8	240	256	$\frac{1}{4}$
$\boldsymbol{a} = (1\ 3\ 3)^T$				
$(0\ 0\ 1)^T$	3	3	9	$\frac{1}{4}$
$(0\ 1\ 1)^T$	3	6	9	$\frac{1}{4}$
$(0\ 1\ 1)^T$	4	6	8	$\frac{1}{4}$
$(1\ 0\ 2)^T$	4	5	8	$\frac{1}{4}$
$(1\ 1\ 1)^T$	4	7	8	$\frac{1}{16}$
$(2\ 1\ 1)^T$	4	2	8	$\frac{1}{4}$
$(0\ 1\ 1)^T$	6	6	36	$\frac{1}{4}$
$(0\ 1\ 2)^T$	6	15	72	$\frac{1}{16}$
$(0\ 1\ 2)^T$	6	51	72	$\frac{1}{8}$
$(0\ 1\ 3)^T$	6	30	36	$\frac{1}{4}$
$(3\ 0\ 2)^T$	6	21	36	$\frac{1}{4}$
$(3\ 1\ 1)^T$	6	15	72	$\frac{1}{8}$
$(3\ 1\ 3)^T$	6	39	72	$\frac{1}{8}$
$(3\ 2\ 2)^T$	6	33	36	$\frac{1}{4}$
$(2\ 1\ 3)^T$	8	2	16	$\frac{1}{16}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (1\ 3\ 3)^T$ (cont.)				
$(3\ 2\ 4)^T$	12	69	72	$\frac{1}{16}$
$\boldsymbol{a} = (1\ 3\ 4)^T$				
$(0\ 1\ 0)^T$	3	3	9	$\frac{1}{2}$
$(0\ 0\ 1)^T$	4	20	32	$\frac{1}{12}$
$(0\ 0\ 1)^T$	4	4	32	$\frac{1}{8}$
$(0\ 1\ 1)^T$	4	7	8	$\frac{1}{8}$
$(2\ 0\ 1)^T$	4	8	16	$\frac{1}{6}$
$(2\ 2\ 1)^T$	4	20	32	$\frac{1}{6}$
$(3\ 1\ 0)^T$	6	12	144	$\frac{1}{3}$
$(3\ 1\ 0)^T$	6	84	288	$\frac{1}{6}$
$(3\ 1\ 0)^T$	6	228	288	$\frac{1}{4}$
$(0\ 4\ 3)^T$	12	84	288	$\frac{1}{24}$
$(0\ 4\ 3)^T$	12	228	288	$\frac{1}{16}$
$(6\ 2\ 3)^T$	12	84	288	$\frac{1}{12}$
$(6\ 4\ 3)^T$	12	120	144	$\frac{1}{12}$
$\boldsymbol{a} = (1\ 3\ 6)^T$				
$(0\ 0\ 1)^T$	3	6	9	$\frac{1}{2}$
$(0\ 1\ 0)^T$	3	3	9	$\frac{1}{2}$
$(0\ 2\ 1)^T$	4	2	16	$\frac{1}{6}$
$(1\ 1\ 1)^T$	4	2	8	$\frac{1}{12}$
$(2\ 0\ 1)^T$	4	10	16	$\frac{1}{6}$
$(0\ 0\ 1)^T$	6	42	72	$\frac{1}{6}$
$(0\ 0\ 1)^T$	6	6	144	$\frac{1}{2}$
$(0\ 2\ 3)^T$	6	66	72	$\frac{1}{6}$
$(0\ 2\ 3)^T$	6	30	72	$\frac{1}{2}$
$(3\ 1\ 0)^T$	6	12	72	$\frac{1}{3}$
$(3\ 1\ 3)^T$	6	66	72	$\frac{1}{3}$
$(3\ 3\ 1)^T$	6	42	72	$\frac{1}{3}$
$(3\ 3\ 2)^T$	6	60	72	$\frac{1}{3}$
$(0\ 2\ 3)^T$	12	66	144	$\frac{1}{12}$
$(6\ 4\ 3)^T$	12	138	144	$\frac{1}{12}$
$\boldsymbol{a} = (1\ 3\ 9)^T$				
$(0\ 0\ 1)^T$	3	9	27	$\frac{1}{4}$
$(0\ 0\ 1)^T$	3	18	27	$\frac{1}{2}$
$(1\ 0\ 1)^T$	4	2	8	$\frac{1}{4}$
$(1\ 1\ 1)^T$	4	5	8	$\frac{1}{16}$
$(1\ 2\ 1)^T$	4	6	8	$\frac{1}{4}$
$(0\ 3\ 2)^T$	6	63	108	$\frac{1}{4}$
$(0\ 3\ 2)^T$	6	99	108	$\frac{1}{2}$
$(3\ 0\ 1)^T$	6	18	108	$\frac{1}{2}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (1\ 3\ 9)^T$ (cont.)				
$(3\ 0\ 1)^T$	6	90	108	$\frac{1}{4}$
$(3\ 3\ 1)^T$	6	45	216	$\frac{1}{4}$
$(3\ 3\ 1)^T$	6	117	216	$\frac{1}{8}$
$\boldsymbol{a} = (1\ 3\ 10)^T$				
$(0\ 1\ 0)^T$	3	3	9	$\frac{1}{2}$
$(0\ 2\ 1)^T$	4	6	16	$\frac{1}{6}$
$(1\ 1\ 1)^T$	4	6	8	$\frac{1}{12}$
$(2\ 0\ 1)^T$	4	14	16	$\frac{1}{6}$
$(0\ 0\ 1)^T$	5	10	25	$\frac{1}{2}$
$(0\ 0\ 2)^T$	5	15	25	$\frac{1}{2}$
$(0\ 2\ 3)^T$	6	30	72	$\frac{1}{6}$
$(0\ 2\ 3)^T$	6	66	72	$\frac{1}{2}$
$(3\ 1\ 0)^T$	6	12	72	$\frac{1}{3}$
$(3\ 1\ 3)^T$	6	30	72	$\frac{1}{3}$
$(0\ 0\ 1)^T$	10	10	100	$\frac{1}{2}$
$(0\ 0\ 1)^T$	10	110	200	$\frac{1}{6}$
$(0\ 0\ 3)^T$	10	90	200	$\frac{1}{2}$
$(0\ 0\ 3)^T$	10	190	200	$\frac{1}{6}$
$(5\ 5\ 1)^T$	10	110	200	$\frac{1}{3}$
$(5\ 5\ 2)^T$	10	140	200	$\frac{1}{3}$
$(5\ 5\ 3)^T$	10	190	200	$\frac{1}{3}$
$(5\ 5\ 4)^T$	10	60	200	$\frac{1}{3}$
$(0\ 2\ 3)^T$	12	102	144	$\frac{1}{12}$
$(6\ 4\ 3)^T$	12	30	144	$\frac{1}{12}$
$(0\ 5\ 3)^T$	15	165	225	$\frac{1}{4}$
$(0\ 5\ 6)^T$	15	210	225	$\frac{1}{4}$
$\boldsymbol{a} = (1\ 3\ 12)^T$				
$(0\ 1\ 1)^T$	3	6	9	$\frac{1}{4}$
$(0\ 0\ 1)^T$	4	12	32	$\frac{1}{8}$
$(0\ 0\ 1)^T$	4	28	32	$\frac{1}{12}$
$(0\ 2\ 1)^T$	4	8	16	$\frac{1}{6}$
$(1\ 0\ 1)^T$	4	5	8	$\frac{1}{8}$
$(2\ 2\ 1)^T$	4	28	32	$\frac{1}{6}$
$(3\ 1\ 2)^T$	6	60	288	$\frac{1}{12}$
$(3\ 1\ 2)^T$	6	132	144	$\frac{1}{6}$
$(3\ 1\ 2)^T$	6	204	288	$\frac{1}{8}$
$(0\ 2\ 3)^T$	12	120	144	$\frac{1}{24}$
$(6\ 2\ 3)^T$	12	156	288	$\frac{1}{24}$
$\boldsymbol{a} = (1\ 3\ 18)^T$				
$(0\ 2\ 1)^T$	4	14	16	$\frac{1}{6}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (1\ 3\ 18)^T$ (cont.)				
$(1\ 1\ 1)^T$	4	6	8	$\frac{1}{12}$
$(2\ 0\ 1)^T$	4	6	16	$\frac{1}{6}$
$\boldsymbol{a} = (1\ 3\ 30)^T$				
$(0\ 1\ 1)^T$	3	6	9	$\frac{1}{4}$
$(0\ 2\ 1)^T$	4	10	32	$\frac{1}{6}$
$(1\ 1\ 1)^T$	4	2	8	$\frac{1}{12}$
$(2\ 0\ 1)^T$	4	2	16	$\frac{1}{6}$
$(0\ 0\ 1)^T$	5	5	25	$\frac{1}{2}$
$(0\ 0\ 2)^T$	5	20	25	$\frac{1}{2}$
$(0\ 2\ 1)^T$	6	2	72	$\frac{1}{4}$
$(0\ 2\ 1)^T$	6	42	72	$\frac{1}{12}$
$(3\ 1\ 1)^T$	6	42	72	$\frac{1}{6}$
$(3\ 1\ 2)^T$	6	60	72	$\frac{1}{6}$
$(0\ 0\ 1)^T$	10	30	200	$\frac{1}{2}$
$(0\ 0\ 1)^T$	10	130	200	$\frac{1}{6}$
$(0\ 0\ 3)^T$	10	70	200	$\frac{1}{2}$
$(0\ 0\ 3)^T$	10	170	200	$\frac{1}{6}$
$(5\ 5\ 1)^T$	10	130	200	$\frac{1}{3}$
$(5\ 5\ 2)^T$	10	20	200	$\frac{1}{3}$
$(5\ 5\ 3)^T$	10	170	200	$\frac{1}{3}$
$(5\ 5\ 4)^T$	10	180	200	$\frac{1}{3}$
$\boldsymbol{a} = (1\ 4\ 4)^T$				
$(0\ 1\ 1)^T$	4	8	16	$\frac{1}{12}$
$(2\ 0\ 1)^T$	4	8	32	$\frac{1}{6}$
$(2\ 1\ 1)^T$	4	12	32	$\frac{1}{4}$
$(2\ 1\ 2)^T$	4	24	32	$\frac{1}{6}$
$(0\ 2\ 1)^T$	8	20	64	$\frac{1}{24}$
$(0\ 3\ 1)^T$	8	40	64	$\frac{1}{24}$
$(0\ 3\ 2)^T$	8	52	64	$\frac{1}{24}$
$(4\ 3\ 1)^T$	8	56	64	$\frac{1}{24}$
$(2\ 3\ 3)^T$	12	76	96	$\frac{1}{24}$
$\boldsymbol{a} = (1\ 4\ 6)^T$				
$(0\ 0\ 1)^T$	3	6	9	$\frac{1}{2}$
$(0\ 1\ 0)^T$	4	4	16	$\frac{1}{4}$
$(0\ 1\ 1)^T$	4	10	16	$\frac{1}{8}$
$(0\ 1\ 2)^T$	4	12	16	$\frac{1}{4}$
$(1\ 1\ 1)^T$	4	3	8	$\frac{1}{8}$
$(2\ 1\ 0)^T$	4	8	64	$\frac{1}{2}$
$(2\ 1\ 0)^T$	4	40	64	$\frac{1}{4}$
$(2\ 1\ 1)^T$	4	14	16	$\frac{1}{4}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (1\ 4\ 6)^T$ (cont.)				
$(0\ 0\ 1)^T$	6	6	144	$\frac{1}{2}$
$(0\ 0\ 3)^T$	6	42	144	$\frac{1}{4}$
$(0\ 3\ 1)^T$	6	42	144	$\frac{1}{4}$
$(0\ 3\ 1)^T$	6	78	144	$\frac{1}{2}$
$(3\ 0\ 1)^T$	6	15	72	$\frac{1}{2}$
$(3\ 0\ 2)^T$	6	33	72	$\frac{1}{2}$
$(3\ 3\ 1)^T$	6	51	72	$\frac{1}{2}$
$(3\ 3\ 2)^T$	6	69	72	$\frac{1}{2}$
$(0\ 1\ 2)^T$	8	28	64	$\frac{1}{8}$
$(0\ 3\ 2)^T$	8	60	64	$\frac{1}{8}$
$(4\ 1\ 2)^T$	8	44	64	$\frac{1}{8}$
$(4\ 3\ 2)^T$	8	12	64	$\frac{1}{8}$
$(0\ 3\ 2)^T$	12	60	144	$\frac{1}{8}$
$(0\ 3\ 4)^T$	12	132	144	$\frac{1}{8}$
$(6\ 3\ 4)^T$	12	168	576	$\frac{1}{8}$
$(6\ 3\ 4)^T$	12	456	576	$\frac{1}{4}$
$\boldsymbol{a} = (1\ 4\ 8)^T$				
$(0\ 0\ 1)^T$	4	24	64	$\frac{1}{4}$
$(0\ 0\ 1)^T$	4	8	64	$\frac{1}{6}$
$(0\ 1\ 1)^T$	4	12	16	$\frac{1}{8}$
$(0\ 2\ 1)^T$	4	24	64	$\frac{1}{4}$
$(0\ 2\ 1)^T$	4	40	64	$\frac{1}{6}$
$(2\ 0\ 1)^T$	4	12	32	$\frac{1}{4}$
$(2\ 1\ 1)^T$	4	16	32	$\frac{1}{6}$
$(2\ 2\ 1)^T$	4	28	32	$\frac{1}{4}$
$(4\ 2\ 1)^T$	8	40	128	$\frac{1}{12}$
$(4\ 2\ 3)^T$	8	104	128	$\frac{1}{12}$
$\boldsymbol{a} = (1\ 4\ 12)^T$				
$(0\ 0\ 1)^T$	3	3	9	$\frac{1}{2}$
$(2\ 0\ 1)^T$	4	16	128	$\frac{1}{6}$
$(2\ 0\ 1)^T$	4	48	64	$\frac{1}{5}$
$(2\ 0\ 1)^T$	4	80	128	$\frac{1}{8}$
$(2\ 1\ 0)^T$	4	8	32	$\frac{1}{2}$
$(2\ 1\ 1)^T$	4	20	32	$\frac{1}{8}$
$(2\ 1\ 2)^T$	4	24	32	$\frac{1}{2}$
$(0\ 3\ 2)^T$	6	120	144	$\frac{1}{2}$
$(0\ 3\ 2)^T$	6	84	288	$\frac{1}{8}$
$(0\ 3\ 2)^T$	6	228	288	$\frac{1}{4}$
$(0\ 2\ 1)^T$	8	28	64	$\frac{1}{8}$
$(0\ 2\ 3)^T$	8	60	64	$\frac{1}{8}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (1\ 4\ 12)^T$ (cont.)				
$(6\ 3\ 2)^T$	12	120	288	$\frac{1}{4}$
$(6\ 3\ 4)^T$	12	264	288	$\frac{1}{4}$
$\boldsymbol{a} = (1\ 4\ 24)^T$				
$(0\ 0\ 1)^T$	3	6	9	$\frac{1}{2}$
$(0\ 0\ 1)^T$	4	24	64	$\frac{1}{2}$
$(0\ 0\ 1)^T$	4	40	64	$\frac{1}{8}$
$(0\ 1\ 1)^T$	4	12	16	$\frac{1}{8}$
$(0\ 2\ 1)^T$	4	40	64	$\frac{1}{8}$
$(0\ 2\ 1)^T$	4	56	64	$\frac{1}{2}$
$(2\ 0\ 1)^T$	4	28	32	$\frac{1}{4}$
$(2\ 2\ 1)^T$	4	12	32	$\frac{1}{4}$
$(0\ 0\ 1)^T$	6	60	144	$\frac{1}{4}$
$(0\ 0\ 1)^T$	6	24	288	$\frac{1}{2}$
$(0\ 0\ 1)^T$	6	168	576	$\frac{1}{4}$
$(0\ 3\ 2)^T$	6	132	144	$\frac{1}{4}$
$(0\ 3\ 2)^T$	6	168	576	$\frac{1}{4}$
$(0\ 3\ 2)^T$	6	456	576	$\frac{1}{2}$
$(4\ 2\ 1)^T$	8	56	128	$\frac{1}{4}$
$(4\ 2\ 3)^T$	8	120	128	$\frac{1}{4}$
$\boldsymbol{a} = (1\ 5\ 5)^T$				
$(1\ 1\ 1)^T$	4	3	8	$\frac{1}{8}$
$(0\ 0\ 1)^T$	5	5	25	$\frac{1}{4}$
$(0\ 0\ 2)^T$	5	20	25	$\frac{1}{4}$
$(0\ 1\ 1)^T$	5	10	25	$\frac{1}{4}$
$(0\ 2\ 2)^T$	5	15	25	$\frac{1}{4}$
$(5\ 1\ 1)^T$	10	35	200	$\frac{1}{4}$
$(5\ 1\ 5)^T$	10	155	200	$\frac{1}{4}$
$(5\ 3\ 3)^T$	10	115	200	$\frac{1}{4}$
$(5\ 3\ 5)^T$	10	195	200	$\frac{1}{4}$
$\boldsymbol{a} = (1\ 5\ 8)^T$				
$(0\ 0\ 1)^T$	4	8	32	$\frac{1}{2}$
$(0\ 0\ 1)^T$	4	24	64	$\frac{1}{4}$
$(0\ 1\ 0)^T$	5	5	25	$\frac{1}{2}$
$(0\ 2\ 0)^T$	5	20	25	$\frac{1}{2}$
$(4\ 4\ 1)^T$	8	104	128	$\frac{1}{2}$
$(4\ 4\ 3)^T$	8	40	128	$\frac{1}{2}$
$(0\ 4\ 5)^T$	20	680	800	$\frac{1}{4}$
$(0\ 4\ 5)^T$	20	280	1600	$\frac{1}{8}$
$(0\ 8\ 5)^T$	20	520	800	$\frac{1}{4}$
$(0\ 8\ 5)^T$	20	920	1600	$\frac{1}{8}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (1\ 5\ 10)^T$				
$(0\ 0\ 1)^T$	4	10	16	$\frac{1}{2}$
$(2\ 2\ 1)^T$	4	2	16	$\frac{1}{2}$
$(2\ 2\ 1)^T$	8	2	32	$\frac{1}{8}$
$(2\ 2\ 3)^T$	8	18	32	$\frac{1}{8}$
$\boldsymbol{a} = (1\ 5\ 25)^T$				
$(1\ 1\ 1)^T$	4	7	8	$\frac{1}{8}$
$\boldsymbol{a} = (1\ 5\ 40)^T$				
$(0\ 0\ 1)^T$	4	56	64	$\frac{1}{4}$
$(0\ 0\ 1)^T$	4	8	32	$\frac{1}{2}$
$(4\ 4\ 1)^T$	8	8	128	$\frac{1}{2}$
$(4\ 4\ 3)^T$	8	72	128	$\frac{1}{2}$
$\boldsymbol{a} = (1\ 6\ 6)^T$				
$(0\ 0\ 1)^T$	3	6	9	$\frac{1}{4}$
$(0\ 1\ 1)^T$	3	3	9	$\frac{1}{4}$
$(0\ 0\ 1)^T$	4	6	16	$\frac{1}{4}$
$(0\ 1\ 1)^T$	4	12	32	$\frac{1}{4}$
$(0\ 1\ 1)^T$	4	28	32	$\frac{1}{8}$
$(0\ 1\ 2)^T$	4	14	16	$\frac{1}{4}$
$(1\ 1\ 1)^T$	4	5	8	$\frac{1}{8}$
$(2\ 0\ 1)^T$	4	10	16	$\frac{1}{4}$
$(2\ 1\ 2)^T$	4	2	16	$\frac{1}{4}$
$(0\ 1\ 2)^T$	6	30	36	$\frac{1}{8}$
$(3\ 0\ 2)^T$	6	33	72	$\frac{1}{4}$
$(3\ 1\ 1)^T$	6	21	72	$\frac{1}{4}$
$(3\ 1\ 2)^T$	6	39	72	$\frac{1}{8}$
$(3\ 1\ 3)^T$	6	69	72	$\frac{1}{4}$
$(3\ 2\ 2)^T$	6	57	72	$\frac{1}{4}$
$(2\ 1\ 2)^T$	8	2	32	$\frac{1}{16}$
$(2\ 3\ 2)^T$	8	18	32	$\frac{1}{16}$
$\boldsymbol{a} = (1\ 6\ 9)^T$				
$(0\ 0\ 1)^T$	3	9	27	$\frac{1}{4}$
$(0\ 0\ 1)^T$	3	18	27	$\frac{1}{2}$
$(0\ 1\ 0)^T$	4	6	16	$\frac{1}{2}$
$(2\ 1\ 2)^T$	4	14	16	$\frac{1}{2}$
$(0\ 3\ 2)^T$	6	126	216	$\frac{1}{2}$
$(0\ 3\ 2)^T$	6	198	216	$\frac{1}{4}$
$(0\ 3\ 2)^T$	6	90	432	$\frac{1}{8}$
$(0\ 3\ 2)^T$	6	234	432	$\frac{1}{4}$
$(3\ 0\ 1)^T$	6	18	432	$\frac{1}{2}$
$(3\ 0\ 1)^T$	6	90	432	$\frac{1}{8}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (1\ 6\ 9)^T$				
$(0\ 1\ 2)^T$	6	138	144	$\frac{1}{4}$
$(3\ 0\ 1)^T$	6	306	432	$\frac{1}{4}$
$(2\ 1\ 2)^T$	8	14	32	$\frac{1}{8}$
$(2\ 3\ 2)^T$	8	30	32	$\frac{1}{8}$
$(0\ 3\ 4)^T$	12	198	432	$\frac{1}{8}$
$(0\ 3\ 4)^T$	12	342	432	$\frac{1}{4}$
$(6\ 3\ 2)^T$	12	126	432	$\frac{1}{4}$
$(6\ 3\ 2)^T$	12	414	432	$\frac{1}{8}$
$\boldsymbol{a} = (1\ 6\ 16)^T$				
$(0\ 1\ 0)^T$	3	6	9	$\frac{1}{2}$
$(0\ 2\ 1)^T$	4	40	64	$\frac{1}{4}$
$(0\ 2\ 1)^T$	4	56	64	$\frac{1}{2}$
$(2\ 0\ 1)^T$	4	20	32	$\frac{1}{2}$
$(2\ 2\ 1)^T$	4	12	32	$\frac{1}{2}$
$(0\ 4\ 1)^T$	8	112	256	$\frac{1}{4}$
$(0\ 4\ 1)^T$	8	176	256	$\frac{1}{4}$
$(0\ 4\ 3)^T$	8	48	256	$\frac{1}{4}$
$(0\ 4\ 3)^T$	8	240	256	$\frac{1}{4}$
$(0\ 2\ 3)^T$	12	168	576	$\frac{1}{8}$
$(0\ 2\ 3)^T$	12	312	576	$\frac{1}{4}$
$(6\ 2\ 3)^T$	12	204	288	$\frac{1}{4}$
$(6\ 4\ 3)^T$	12	276	288	$\frac{1}{4}$
$(0\ 4\ 3)^T$	24	240	2304	$\frac{1}{8}$
$(0\ 4\ 3)^T$	24	816	2304	$\frac{1}{8}$
$(0\ 4\ 9)^T$	24	1392	2304	$\frac{1}{8}$
$(0\ 4\ 9)^T$	24	1968	2304	$\frac{1}{8}$
$\boldsymbol{a} = (1\ 6\ 18)^T$				
$(0\ 1\ 1)^T$	4	8	16	$\frac{1}{6}$
$(2\ 1\ 1)^T$	4	12	16	$\frac{1}{6}$
$\boldsymbol{a} = (1\ 6\ 24)^T$				
$(0\ 1\ 1)^T$	3	3	9	$\frac{1}{4}$
$(0\ 0\ 1)^T$	4	8	16	$\frac{1}{4}$
$(0\ 1\ 1)^T$	4	14	16	$\frac{1}{4}$
$(2\ 0\ 1)^T$	4	12	16	$\frac{1}{4}$
$(2\ 1\ 1)^T$	4	2	16	$\frac{1}{2}$
$(2\ 2\ 1)^T$	4	20	32	$\frac{1}{2}$
$(0\ 1\ 1)^T$	6	30	144	$\frac{1}{4}$
$(0\ 1\ 1)^T$	6	66	144	$\frac{1}{4}$
$(0\ 1\ 2)^T$	6	102	144	$\frac{1}{4}$
$(3\ 0\ 1)^T$	6	234	432	$\frac{1}{4}$
$\boldsymbol{a} = (1\ 6\ 24)^T$ (cont.)				
$(4\ 0\ 1)^T$	8	40	128	$\frac{1}{4}$
$(4\ 0\ 3)^T$	8	104	128	$\frac{1}{4}$
$(4\ 4\ 1)^T$	8	8	128	$\frac{1}{4}$
$(4\ 4\ 3)^T$	8	72	128	$\frac{1}{4}$
$(6\ 2\ 3)^T$	12	276	288	$\frac{1}{8}$
$\boldsymbol{a} = (1\ 8\ 8)^T$				
$(0\ 0\ 1)^T$	4	8	64	$\frac{1}{4}$
$(0\ 0\ 1)^T$	4	24	64	$\frac{1}{4}$
$(0\ 1\ 1)^T$	4	16	64	$\frac{1}{6}$
$(0\ 1\ 1)^T$	4	32	64	$\frac{1}{6}$
$(0\ 1\ 2)^T$	4	40	64	$\frac{1}{4}$
$(0\ 1\ 2)^T$	4	56	64	$\frac{1}{4}$
$(2\ 1\ 1)^T$	4	20	32	$\frac{1}{4}$
$(4\ 1\ 2)^T$	8	56	128	$\frac{1}{8}$
$(4\ 1\ 3)^T$	8	96	128	$\frac{1}{12}$
$(4\ 2\ 3)^T$	8	120	128	$\frac{1}{8}$
$\boldsymbol{a} = (1\ 8\ 16)^T$				
$(0\ 0\ 1)^T$	4	16	64	$\frac{1}{4}$
$(0\ 0\ 1)^T$	4	32	128	$\frac{1}{3}$
$(0\ 1\ 1)^T$	4	24	64	$\frac{1}{8}$
$(0\ 1\ 1)^T$	4	40	64	$\frac{1}{4}$
$(0\ 2\ 1)^T$	4	48	64	$\frac{1}{4}$
$(0\ 2\ 1)^T$	4	64	128	$\frac{1}{3}$
$(2\ 1\ 1)^T$	4	28	32	$\frac{1}{4}$
$(0\ 2\ 1)^T$	8	112	256	$\frac{1}{8}$
$(0\ 2\ 1)^T$	8	48	256	$\frac{1}{8}$
$(0\ 2\ 3)^T$	8	176	256	$\frac{1}{8}$
$(0\ 2\ 3)^T$	8	240	256	$\frac{1}{8}$
$\boldsymbol{a} = (1\ 8\ 40)^T$				
$(2\ 1\ 1)^T$	4	20	32	$\frac{1}{4}$
$(0\ 0\ 1)^T$	5	15	25	$\frac{1}{2}$
$(0\ 0\ 2)^T$	5	10	25	$\frac{1}{2}$
$\boldsymbol{a} = (1\ 9\ 9)^T$				
$(0\ 1\ 0)^T$	3	9	27	$\frac{1}{6}$
$(0\ 1\ 0)^T$	3	18	27	$\frac{1}{6}$
$(1\ 1\ 1)^T$	4	3	8	$\frac{1}{8}$
$(3\ 1\ 3)^T$	6	99	216	$\frac{1}{6}$
$(3\ 1\ 3)^T$	6	171	216	$\frac{1}{6}$
$\boldsymbol{a} = (1\ 9\ 12)^T$				
$(0\ 1\ 0)^T$	3	9	27	$\frac{1}{4}$
$\boldsymbol{a} = (1\ 9\ 12)^T$ (cont.)				
$(0\ 1\ 0)^T$	3	18	27	$\frac{1}{2}$
$(0\ 0\ 1)^T$	4	12	16	$\frac{1}{2}$
$(1\ 1\ 1)^T$	4	6	8	$\frac{1}{8}$
$(2\ 2\ 1)^T$	4	20	32	$\frac{1}{4}$
$(3\ 1\ 0)^T$	6	18	216	$\frac{1}{2}$
$(3\ 1\ 0)^T$	6	90	216	$\frac{1}{4}$
$(3\ 1\ 3)^T$	6	126	216	$\frac{1}{2}$
$(3\ 1\ 3)^T$	6	198	216	$\frac{1}{4}$
$(0\ 4\ 3)^T$	12	252	432	$\frac{1}{8}$
$(0\ 4\ 3)^T$	12	396	432	$\frac{1}{4}$
$(6\ 2\ 3)^T$	12	180	864	$\frac{1}{8}$
$(6\ 2\ 3)^T$	12	468	864	$\frac{1}{16}$
$\boldsymbol{a} = (1\ 9\ 21)^T$				
$(0\ 1\ 0)^T$	3	9	27	$\frac{1}{4}$
$(0\ 1\ 0)^T$	3	18	27	$\frac{1}{2}$
$(1\ 1\ 1)^T$	4	7	8	$\frac{1}{8}$
$(3\ 1\ 3)^T$	6	63	216	$\frac{1}{4}$
$(3\ 1\ 3)^T$	6	207	216	$\frac{1}{2}$
$(0\ 0\ 1)^T$	7	21	49	$\frac{1}{2}$
$(0\ 0\ 2)^T$	7	35	49	$\frac{1}{2}$
$(0\ 0\ 3)^T$	7	42	49	$\frac{1}{2}$
$(7\ 7\ 1)^T$	14	119	196	$\frac{1}{2}$
$(7\ 7\ 3)^T$	14	91	196	$\frac{1}{2}$
$(7\ 7\ 5)^T$	14	35	196	$\frac{1}{2}$
$(0\ 7\ 3)^T$	21	630	1323	$\frac{1}{8}$
$(0\ 7\ 3)^T$	21	1071	1323	$\frac{1}{4}$
$(0\ 7\ 6)^T$	21	315	1323	$\frac{1}{4}$
$(0\ 7\ 6)^T$	21	1197	1323	$\frac{1}{8}$
$(0\ 7\ 9)^T$	21	819	1323	$\frac{1}{8}$
$(0\ 7\ 9)^T$	21	1260	1323	$\frac{1}{4}$
$\boldsymbol{a} = (1\ 9\ 24)^T$				
$(0\ 1\ 0)^T$	3	9	27	$\frac{1}{4}$
$(0\ 1\ 0)^T$	3	18	27	$\frac{1}{2}$
$(0\ 0\ 1)^T$	4	24	64	$\frac{1}{2}$
$(0\ 0\ 1)^T$	4	40	64	$\frac{1}{4}$
$(4\ 4\ 1)^T$	8	56	128	$\frac{1}{2}$
$(4\ 4\ 3)^T$	8	120	128	$\frac{1}{2}$
$(0\ 4\ 3)^T$	12	504	864	$\frac{1}{4}$
$(0\ 4\ 3)^T$	12	792	864	$\frac{1}{8}$
$(0\ 4\ 3)^T$	12	360	1728	$\frac{1}{16}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (1\ 9\ 24)^T$ (cont.)				
$(0\ 4\ 3)^T$	12	936	1728	$\frac{1}{8}$
$(12\ 4\ 3)^T$	24	504	3456	$\frac{1}{4}$
$(12\ 4\ 3)^T$	24	1656	3456	$\frac{1}{8}$
$(12\ 4\ 9)^T$	24	2232	3456	$\frac{1}{4}$
$(12\ 4\ 9)^T$	24	3384	3456	$\frac{1}{8}$
$\boldsymbol{a} = (1\ 10\ 30)^T$				
$(0\ 0\ 1)^T$	3	3	9	$\frac{1}{2}$
$(0\ 1\ 1)^T$	4	8	16	$\frac{1}{6}$
$(2\ 1\ 1)^T$	4	12	16	$\frac{1}{6}$
$(0\ 3\ 1)^T$	6	12	144	$\frac{1}{3}$
$(0\ 3\ 1)^T$	6	120	144	$\frac{1}{3}$
$(3\ 0\ 1)^T$	6	39	72	$\frac{1}{2}$
$(3\ 3\ 2)^T$	6	3	72	$\frac{1}{2}$
$\boldsymbol{a} = (1\ 12\ 12)^T$				
$(0\ 0\ 1)^T$	3	3	9	$\frac{1}{4}$
$(0\ 1\ 1)^T$	3	6	9	$\frac{1}{4}$
$(2\ 0\ 1)^T$	4	16	64	$\frac{1}{5}$
$(2\ 0\ 1)^T$	4	48	128	$\frac{1}{6}$
$(2\ 0\ 1)^T$	4	112	128	$\frac{1}{8}$
$(2\ 1\ 1)^T$	4	28	32	$\frac{1}{8}$
$(0\ 1\ 1)^T$	6	24	144	$\frac{1}{4}$
$(0\ 1\ 1)^T$	6	60	288	$\frac{1}{8}$
$(0\ 1\ 3)^T$	6	120	144	$\frac{1}{4}$
$(0\ 1\ 3)^T$	6	156	288	$\frac{1}{8}$
$(3\ 1\ 1)^T$	6	69	72	$\frac{1}{4}$
$(0\ 1\ 3)^T$	8	56	64	$\frac{1}{8}$
$(4\ 1\ 3)^T$	8	8	64	$\frac{1}{8}$
$\boldsymbol{a} = (1\ 16\ 24)^T$				
$(0\ 0\ 1)^T$	3	6	9	$\frac{1}{2}$
$(0\ 1\ 0)^T$	4	16	64	$\frac{1}{4}$
$(0\ 1\ 0)^T$	4	32	256	$\frac{1}{2}$
$(0\ 1\ 0)^T$	4	160	256	$\frac{1}{4}$
$(0\ 1\ 1)^T$	4	40	64	$\frac{1}{8}$
$(0\ 1\ 1)^T$	4	56	64	$\frac{1}{4}$
$(2\ 1\ 1)^T$	4	12	32	$\frac{1}{4}$
$(0\ 0\ 1)^T$	6	60	288	$\frac{1}{2}$
$(0\ 0\ 1)^T$	6	168	288	$\frac{1}{4}$
$(0\ 0\ 1)^T$	6	24	576	$\frac{1}{2}$
$(0\ 3\ 1)^T$	6	168	288	$\frac{1}{4}$
$(0\ 3\ 1)^T$	6	204	288	$\frac{1}{2}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (1\ 16\ 24)^T$ (cont.)				
$(0\ 3\ 1)^T$	6	312	576	$\frac{1}{2}$
$(0\ 1\ 2)^T$	8	112	256	$\frac{1}{8}$
$(0\ 1\ 2)^T$	8	176	256	$\frac{1}{8}$
$(0\ 3\ 2)^T$	8	48	256	$\frac{1}{8}$
$(0\ 3\ 2)^T$	8	240	256	$\frac{1}{8}$
$(0\ 3\ 4)^T$	12	528	576	$\frac{1}{8}$
$(0\ 3\ 4)^T$	12	672	2304	$\frac{1}{8}$
$(0\ 3\ 4)^T$	12	1824	2304	$\frac{1}{4}$
$\boldsymbol{a} = (1\ 21\ 21)^T$				
$(0\ 0\ 1)^T$	3	3	9	$\frac{1}{4}$
$(0\ 1\ 1)^T$	3	6	9	$\frac{1}{4}$
$(1\ 1\ 1)^T$	4	3	8	$\frac{1}{8}$
$(3\ 1\ 1)^T$	6	51	72	$\frac{1}{4}$
$(3\ 1\ 3)^T$	6	3	72	$\frac{1}{4}$
$(0\ 1\ 2)^T$	7	7	49	$\frac{1}{8}$
$(0\ 1\ 3)^T$	7	14	49	$\frac{1}{8}$
$(0\ 2\ 3)^T$	7	28	49	$\frac{1}{8}$
$(7\ 1\ 3)^T$	14	259	392	$\frac{1}{8}$
$(7\ 1\ 5)^T$	14	203	392	$\frac{1}{8}$
$(7\ 3\ 5)^T$	14	371	392	$\frac{1}{8}$
$\boldsymbol{a} = (1\ 24\ 24)^T$				
$(0\ 0\ 1)^T$	3	6	9	$\frac{1}{4}$
$(0\ 1\ 1)^T$	3	3	9	$\frac{1}{4}$
$(0\ 0\ 1)^T$	4	24	64	$\frac{1}{4}$
$(0\ 0\ 1)^T$	4	40	64	$\frac{1}{4}$
$(0\ 1\ 2)^T$	4	8	64	$\frac{1}{4}$
$(0\ 1\ 2)^T$	4	56	64	$\frac{1}{4}$
$(2\ 1\ 1)^T$	4	20	32	$\frac{1}{4}$
$(0\ 1\ 2)^T$	6	12	144	$\frac{1}{8}$
$(0\ 1\ 2)^T$	6	120	144	$\frac{1}{8}$
$(4\ 1\ 2)^T$	8	8	128	$\frac{1}{8}$
$(4\ 2\ 3)^T$	8	72	128	$\frac{1}{8}$
$\boldsymbol{a} = (2\ 2\ 3)^T$				
$(0\ 0\ 1)^T$	3	3	9	$\frac{1}{2}$
$(0\ 1\ 0)^T$	4	2	16	$\frac{1}{4}$
$(0\ 1\ 2)^T$	4	14	16	$\frac{1}{4}$
$(1\ 1\ 0)^T$	4	4	32	$\frac{1}{4}$
$(1\ 1\ 0)^T$	4	20	32	$\frac{1}{8}$
$(1\ 1\ 1)^T$	4	7	8	$\frac{1}{8}$
$(1\ 2\ 0)^T$	4	10	16	$\frac{1}{4}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (2\ 2\ 3)^T$ (cont.)				
$(1\ 2\ 2)^T$	4	6	16	$\frac{1}{4}$
$(0\ 0\ 1)^T$	6	3	72	$\frac{1}{2}$
$(0\ 3\ 1)^T$	6	21	72	$\frac{1}{4}$
$(0\ 3\ 2)^T$	6	30	36	$\frac{1}{4}$
$(3\ 3\ 1)^T$	6	39	72	$\frac{1}{2}$
$(1\ 2\ 2)^T$	8	22	32	$\frac{1}{16}$
$(2\ 3\ 2)^T$	8	6	32	$\frac{1}{16}$
$(0\ 3\ 2)^T$	12	30	144	$\frac{1}{8}$
$(0\ 3\ 4)^T$	12	66	144	$\frac{1}{8}$
$(3\ 3\ 4)^T$	12	84	288	$\frac{1}{16}$
$(3\ 3\ 4)^T$	12	228	288	$\frac{1}{8}$
$(3\ 6\ 2)^T$	12	102	144	$\frac{1}{8}$
$(3\ 6\ 4)^T$	12	138	144	$\frac{1}{8}$
$\boldsymbol{a} = (2\ 3\ 3)^T$				
$(0\ 0\ 1)^T$	3	3	9	$\frac{1}{4}$
$(0\ 1\ 1)^T$	3	6	9	$\frac{1}{4}$
$(1\ 0\ 0)^T$	4	2	16	$\frac{1}{2}$
$(1\ 0\ 1)^T$	4	5	8	$\frac{1}{8}$
$(1\ 0\ 2)^T$	4	14	16	$\frac{1}{8}$
$(1\ 1\ 2)^T$	4	1	8	$\frac{1}{8}$
$(1\ 2\ 2)^T$	4	10	16	$\frac{1}{2}$
$(0\ 1\ 1)^T$	6	6	144	$\frac{1}{4}$
$(0\ 1\ 1)^T$	6	78	144	$\frac{1}{8}$
$(0\ 1\ 2)^T$	6	15	36	$\frac{1}{8}$
$(0\ 1\ 3)^T$	6	102	144	$\frac{1}{4}$
$(0\ 1\ 3)^T$	6	30	144	$\frac{1}{8}$
$(3\ 0\ 2)^T$	6	30	144	$\frac{1}{8}$
$(3\ 0\ 2)^T$	6	66	72	$\frac{1}{4}$
$(3\ 1\ 2)^T$	6	33	36	$\frac{1}{8}$
$(3\ 2\ 2)^T$	6	42	72	$\frac{1}{4}$
$(3\ 2\ 2)^T$	6	78	144	$\frac{1}{8}$
$(1\ 2\ 2)^T$	8	26	32	$\frac{1}{8}$
$(3\ 2\ 2)^T$	8	10	32	$\frac{1}{8}$
$(3\ 0\ 4)^T$	12	66	144	$\frac{1}{8}$
$(3\ 2\ 2)^T$	12	42	144	$\frac{1}{8}$
$(3\ 2\ 4)^T$	12	75	144	$\frac{1}{32}$
$(3\ 2\ 6)^T$	12	138	144	$\frac{1}{8}$
$(3\ 4\ 4)^T$	12	114	144	$\frac{1}{8}$
$\boldsymbol{a} = (2\ 3\ 6)^T$				
$(0\ 0\ 1)^T$	3	6	9	$\frac{1}{2}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (2\ 3\ 6)^T$ (cont.)				
$(0\ 1\ 0)^T$	3	3	9	$\frac{1}{2}$
$(1\ 0\ 1)^T$	4	8	16	$\frac{1}{6}$
$(1\ 2\ 1)^T$	4	4	16	$\frac{1}{6}$
$(0\ 1\ 3)^T$	6	57	72	$\frac{1}{2}$
$(0\ 3\ 1)^T$	6	33	72	$\frac{1}{2}$
$(3\ 0\ 1)^T$	6	24	144	$\frac{1}{3}$
$(3\ 0\ 1)^T$	6	132	144	$\frac{1}{3}$
$(3\ 1\ 0)^T$	6	21	72	$\frac{1}{2}$
$(3\ 2\ 3)^T$	6	84	144	$\frac{1}{3}$
$(3\ 2\ 3)^T$	6	120	144	$\frac{1}{3}$
$(3\ 3\ 2)^T$	6	69	72	$\frac{1}{2}$
$(3\ 2\ 3)^T$	12	84	144	$\frac{1}{12}$
$(3\ 4\ 3)^T$	12	120	144	$\frac{1}{12}$
$\boldsymbol{a} = (2\ 3\ 8)^T$				
$(0\ 1\ 0)^T$	3	3	9	$\frac{1}{2}$
$(0\ 0\ 1)^T$	4	8	16	$\frac{1}{4}$
$(0\ 2\ 1)^T$	4	4	16	$\frac{1}{4}$
$(1\ 0\ 1)^T$	4	10	16	$\frac{1}{4}$
$(1\ 2\ 1)^T$	4	6	16	$\frac{1}{4}$
$(2\ 2\ 1)^T$	4	28	32	$\frac{1}{2}$
$(3\ 2\ 0)^T$	6	30	144	$\frac{1}{2}$
$(3\ 2\ 0)^T$	6	66	144	$\frac{1}{2}$
$(3\ 2\ 3)^T$	6	102	144	$\frac{1}{2}$
$(3\ 2\ 3)^T$	6	138	144	$\frac{1}{2}$
$(0\ 4\ 1)^T$	8	56	128	$\frac{1}{4}$
$(0\ 4\ 3)^T$	8	120	128	$\frac{1}{4}$
$(4\ 4\ 1)^T$	8	88	128	$\frac{1}{4}$
$(4\ 4\ 3)^T$	8	24	128	$\frac{1}{4}$
$(0\ 2\ 3)^T$	12	84	288	$\frac{1}{8}$
$(0\ 4\ 3)^T$	12	120	288	$\frac{1}{8}$
$(3\ 2\ 3)^T$	12	102	144	$\frac{1}{8}$
$(3\ 4\ 3)^T$	12	138	144	$\frac{1}{8}$
$(6\ 2\ 3)^T$	12	156	288	$\frac{1}{4}$
$(0\ 4\ 3)^T$	24	120	1152	$\frac{1}{8}$
$(0\ 4\ 9)^T$	24	696	1152	$\frac{1}{8}$
$(12\ 4\ 3)^T$	24	408	1152	$\frac{1}{8}$
$(12\ 4\ 9)^T$	24	984	1152	$\frac{1}{8}$
$\boldsymbol{a} = (2\ 3\ 9)^T$				
$(1\ 0\ 2)^T$	4	6	16	$\frac{1}{6}$
$(1\ 1\ 1)^T$	4	6	8	$\frac{1}{12}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (2\ 3\ 9)^T$ (cont.)				
$(1\ 2\ 0)^T$	4	14	16	$\frac{1}{6}$
$\boldsymbol{a} = (2\ 3\ 12)^T$				
$(0\ 1\ 1)^T$	3	6	9	$\frac{1}{4}$
$(0\ 0\ 1)^T$	4	12	16	$\frac{1}{4}$
$(0\ 2\ 1)^T$	4	24	64	$\frac{1}{2}$
$(0\ 2\ 1)^T$	4	56	64	$\frac{1}{4}$
$(1\ 0\ 1)^T$	4	14	16	$\frac{1}{8}$
$(1\ 1\ 1)^T$	4	1	8	$\frac{1}{8}$
$(1\ 2\ 1)^T$	4	10	16	$\frac{1}{4}$
$(2\ 0\ 1)^T$	4	4	16	$\frac{1}{4}$
$(0\ 1\ 1)^T$	6	15	72	$\frac{1}{4}$
$(0\ 1\ 2)^T$	6	51	72	$\frac{1}{4}$
$(3\ 1\ 1)^T$	6	33	72	$\frac{1}{4}$
$(3\ 1\ 2)^T$	6	69	72	$\frac{1}{4}$
$(3\ 2\ 1)^T$	6	6	72	$\frac{1}{8}$
$(3\ 2\ 1)^T$	6	42	144	$\frac{1}{4}$
$(3\ 2\ 2)^T$	6	6	72	$\frac{1}{8}$
$(3\ 2\ 2)^T$	6	114	144	$\frac{1}{4}$
$(2\ 0\ 1)^T$	8	20	64	$\frac{1}{8}$
$(2\ 0\ 3)^T$	8	52	64	$\frac{1}{8}$
$(2\ 4\ 1)^T$	8	4	64	$\frac{1}{8}$
$(2\ 4\ 3)^T$	8	36	64	$\frac{1}{8}$
$(0\ 2\ 3)^T$	12	120	576	$\frac{1}{16}$
$(0\ 2\ 3)^T$	12	408	576	$\frac{1}{8}$
$(3\ 2\ 3)^T$	12	138	144	$\frac{1}{16}$
$\boldsymbol{a} = (2\ 3\ 18)^T$				
$(0\ 0\ 1)^T$	3	9	27	$\frac{1}{2}$
$(0\ 0\ 1)^T$	3	18	27	$\frac{1}{4}$
$(1\ 0\ 1)^T$	4	4	32	$\frac{1}{4}$
$(1\ 0\ 1)^T$	4	20	32	$\frac{1}{8}$
$(1\ 1\ 1)^T$	4	7	8	$\frac{1}{8}$
$(0\ 3\ 2)^T$	6	99	216	$\frac{1}{4}$
$(0\ 3\ 2)^T$	6	171	216	$\frac{1}{2}$
$(3\ 3\ 1)^T$	6	63	216	$\frac{1}{2}$
$(3\ 3\ 1)^T$	6	207	216	$\frac{1}{4}$
$\boldsymbol{a} = (2\ 3\ 48)^T$				
$(0\ 1\ 1)^T$	3	6	9	$\frac{1}{4}$
$(0\ 2\ 1)^T$	4	28	32	$\frac{1}{2}$
$(2\ 0\ 1)^T$	4	24	32	$\frac{1}{4}$
$(2\ 0\ 1)^T$	4	40	64	$\frac{1}{2}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (2\ 3\ 48)^T$ (cont.)				
$(2\ 2\ 1)^T$	4	4	32	$\frac{1}{2}$
$(4\ 0\ 1)^T$	8	16	256	$\frac{1}{4}$
$(4\ 0\ 1)^T$	8	80	256	$\frac{1}{4}$
$(4\ 0\ 3)^T$	8	44	256	$\frac{1}{4}$
$(4\ 0\ 3)^T$	8	208	256	$\frac{1}{4}$
$\boldsymbol{a} = (2\ 5\ 6)^T$				
$(0\ 0\ 1)^T$	3	6	9	$\frac{1}{2}$
$(1\ 0\ 1)^T$	4	8	16	$\frac{1}{6}$
$(1\ 2\ 1)^T$	4	12	16	$\frac{1}{6}$
$(0\ 1\ 0)^T$	5	5	25	$\frac{1}{2}$
$(0\ 2\ 0)^T$	5	20	25	$\frac{1}{2}$
$(0\ 3\ 1)^T$	6	51	72	$\frac{1}{2}$
$(3\ 0\ 1)^T$	6	24	144	$\frac{1}{3}$
$(3\ 0\ 1)^T$	6	60	144	$\frac{1}{3}$
$(3\ 3\ 2)^T$	6	15	72	$\frac{1}{2}$
$(0\ 1\ 5)^T$	10	155	200	$\frac{1}{2}$
$(0\ 3\ 5)^T$	10	195	200	$\frac{1}{2}$
$(5\ 1\ 0)^T$	10	55	200	$\frac{1}{2}$
$(5\ 2\ 5)^T$	10	120	400	$\frac{1}{3}$
$(5\ 2\ 5)^T$	10	220	400	$\frac{1}{3}$
$(5\ 3\ 0)^T$	10	95	200	$\frac{1}{2}$
$(5\ 4\ 5)^T$	10	280	400	$\frac{1}{3}$
$(5\ 4\ 5)^T$	10	380	400	$\frac{1}{3}$
$(0\ 3\ 5)^T$	15	195	225	$\frac{1}{4}$
$(0\ 6\ 5)^T$	15	105	225	$\frac{1}{4}$
$(5\ 2\ 5)^T$	20	220	400	$\frac{1}{12}$
$(5\ 4\ 5)^T$	20	280	400	$\frac{1}{12}$
$(5\ 6\ 5)^T$	20	380	400	$\frac{1}{12}$
$(5\ 8\ 5)^T$	20	120	400	$\frac{1}{12}$
$\boldsymbol{a} = (2\ 5\ 10)^T$				
$(1\ 0\ 1)^T$	4	12	32	$\frac{1}{4}$
$(1\ 0\ 1)^T$	4	28	32	$\frac{1}{8}$
$(1\ 1\ 1)^T$	4	1	8	$\frac{1}{8}$
$\boldsymbol{a} = (2\ 5\ 15)^T$				
$(0\ 0\ 1)^T$	3	6	9	$\frac{1}{2}$
$(1\ 0\ 2)^T$	4	14	16	$\frac{1}{6}$
$(1\ 1\ 1)^T$	4	6	8	$\frac{1}{12}$
$(1\ 2\ 0)^T$	4	6	16	$\frac{1}{6}$
$(0\ 3\ 1)^T$	6	60	72	$\frac{1}{3}$
$(3\ 0\ 2)^T$	6	6	72	$\frac{1}{6}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (2~5~15)^T$ (cont.)				
$(3~0~2)^T$	6	42	72	$\frac{1}{2}$
$(3~3~1)^T$	6	6	72	$\frac{1}{3}$
$(3~0~2)^T$	12	78	144	$\frac{1}{12}$
$(3~6~4)^T$	12	6	144	$\frac{1}{12}$
$\boldsymbol{a} = (2~6~9)^T$				
$(1~1~0)^T$	4	8	16	$\frac{1}{6}$
$(1~1~2)^T$	4	12	16	$\frac{1}{6}$
$\boldsymbol{a} = (2~6~15)^T$				
$(0~1~1)^T$	3	3	9	$\frac{1}{4}$
$(1~1~0)^T$	4	8	16	$\frac{1}{6}$
$(1~1~2)^T$	4	4	16	$\frac{1}{6}$
$(0~0~1)^T$	5	15	25	$\frac{1}{2}$
$(0~0~2)^T$	5	10	25	$\frac{1}{2}$
$(0~1~1)^T$	6	21	72	$\frac{1}{4}$
$(3~1~2)^T$	6	84	144	$\frac{1}{6}$
$(3~1~2)^T$	6	120	144	$\frac{1}{6}$
$(3~2~1)^T$	6	57	72	$\frac{1}{4}$
$(0~5~1)^T$	10	165	200	$\frac{1}{2}$
$(0~5~3)^T$	10	85	200	$\frac{1}{2}$
$(5~0~1)^T$	10	65	200	$\frac{1}{2}$
$(5~0~3)^T$	10	185	200	$\frac{1}{2}$
$(5~5~2)^T$	10	260	400	$\frac{1}{3}$
$(5~5~2)^T$	10	360	400	$\frac{1}{3}$
$(5~5~4)^T$	10	40	400	$\frac{1}{3}$
$(5~5~4)^T$	10	340	400	$\frac{1}{3}$
$(5~5~2)^T$	20	260	400	$\frac{1}{12}$
$(5~5~4)^T$	20	40	400	$\frac{1}{12}$
$(5~5~6)^T$	20	340	400	$\frac{1}{12}$
$(5~5~8)^T$	20	360	400	$\frac{1}{12}$
$\boldsymbol{a} = (3~3~4)^T$				
$(0~1~0)^T$	3	3	9	$\frac{1}{4}$
$(1~1~0)^T$	3	6	9	$\frac{1}{4}$
$(0~0~1)^T$	4	4	16	$\frac{1}{2}$
$(1~1~1)^T$	4	2	8	$\frac{1}{8}$
$(2~2~1)^T$	4	28	32	$\frac{1}{4}$
$(1~1~0)^T$	6	6	72	$\frac{1}{4}$
$(1~1~3)^T$	6	42	72	$\frac{1}{4}$
$(1~3~0)^T$	6	30	72	$\frac{1}{4}$
$(1~3~3)^T$	6	66	72	$\frac{1}{4}$
$(0~4~1)^T$	8	52	64	$\frac{1}{4}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (3~3~4)^T$ (cont.)				
$(0~4~3)^T$	8	20	64	$\frac{1}{4}$
$(0~4~3)^T$	12	84	144	$\frac{1}{8}$
$(2~2~3)^T$	12	60	288	$\frac{1}{16}$
$(2~6~3)^T$	12	156	288	$\frac{1}{16}$
$(4~4~3)^T$	12	132	144	$\frac{1}{8}$
$(4~8~3)^T$	24	276	576	$\frac{1}{16}$
$(4~8~9)^T$	24	564	576	$\frac{1}{16}$
$\boldsymbol{a} = (3~3~7)^T$				
$(0~1~0)^T$	3	3	9	$\frac{1}{4}$
$(1~1~0)^T$	3	6	9	$\frac{1}{4}$
$(1~1~1)^T$	4	5	8	$\frac{1}{8}$
$(1~1~3)^T$	6	69	72	$\frac{1}{4}$
$(1~3~3)^T$	6	21	72	$\frac{1}{4}$
$(0~0~1)^T$	7	7	49	$\frac{1}{2}$
$(0~0~2)^T$	7	28	49	$\frac{1}{2}$
$(0~0~3)^T$	7	14	49	$\frac{1}{2}$
$(7~7~1)^T$	14	105	196	$\frac{1}{2}$
$(7~7~3)^T$	14	161	196	$\frac{1}{2}$
$(7~7~5)^T$	14	77	196	$\frac{1}{2}$
$(0~7~3)^T$	21	210	441	$\frac{1}{8}$
$(0~7~6)^T$	21	399	441	$\frac{1}{8}$
$(0~7~9)^T$	21	273	441	$\frac{1}{8}$
$(7~7~3)^T$	21	357	441	$\frac{1}{8}$
$(7~7~6)^T$	21	105	441	$\frac{1}{8}$
$(7~7~9)^T$	21	420	441	$\frac{1}{8}$
$\boldsymbol{a} = (3~3~8)^T$				
$(0~1~0)^T$	3	3	9	$\frac{1}{4}$
$(1~1~0)^T$	3	6	9	$\frac{1}{4}$
$(0~0~1)^T$	4	8	32	$\frac{1}{2}$
$(0~0~1)^T$	4	56	64	$\frac{1}{4}$
$(0~2~1)^T$	4	4	16	$\frac{1}{4}$
$(4~4~1)^T$	8	104	128	$\frac{1}{2}$
$(4~4~3)^T$	8	40	128	$\frac{1}{2}$
$(0~4~3)^T$	12	264	288	$\frac{1}{8}$
$(0~4~3)^T$	12	120	576	$\frac{1}{16}$
$(2~4~3)^T$	12	132	144	$\frac{1}{16}$
$(4~4~3)^T$	12	168	288	$\frac{1}{8}$
$(4~4~3)^T$	12	312	576	$\frac{1}{16}$
$(4~4~3)^T$	24	168	1152	$\frac{1}{8}$
$(4~4~9)^T$	24	744	1152	$\frac{1}{8}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (3~3~8)^T$ (cont.)				
$(4~12~3)^T$	24	552	1152	$\frac{1}{8}$
$(4~12~9)^T$	24	1128	1152	$\frac{1}{8}$
$\boldsymbol{a} = (3~4~4)^T$				
$(1~0~0)^T$	3	3	9	$\frac{1}{2}$
$(0~1~1)^T$	4	8	16	$\frac{1}{4}$
$(2~0~1)^T$	4	48	64	$\frac{1}{5}$
$(2~0~1)^T$	4	16	128	$\frac{1}{6}$
$(2~0~1)^T$	4	80	128	$\frac{1}{8}$
$(2~1~1)^T$	4	20	32	$\frac{1}{8}$
$(1~0~3)^T$	6	39	72	$\frac{1}{4}$
$(2~3~3)^T$	6	120	144	$\frac{1}{2}$
$(2~3~3)^T$	6	84	288	$\frac{1}{4}$
$(0~1~3)^T$	8	40	64	$\frac{1}{8}$
$(4~1~3)^T$	8	24	64	$\frac{1}{8}$
$(2~0~3)^T$	12	48	576	$\frac{1}{10}$
$(2~0~3)^T$	12	336	1152	$\frac{1}{16}$
$(2~0~3)^T$	12	912	1152	$\frac{1}{12}$
$(2~3~3)^T$	12	84	288	$\frac{1}{16}$
$(4~3~3)^T$	12	120	144	$\frac{1}{8}$
$(4~3~9)^T$	24	408	576	$\frac{1}{16}$
$(8~3~9)^T$	24	552	576	$\frac{1}{16}$
$\boldsymbol{a} = (3~4~12)^T$				
$(1~0~1)^T$	3	6	9	$\frac{1}{4}$
$(2~0~1)^T$	4	24	32	$\frac{1}{2}$
$(2~1~0)^T$	4	16	64	$\frac{1}{5}$
$(2~1~0)^T$	4	48	128	$\frac{1}{6}$
$(2~1~0)^T$	4	112	128	$\frac{1}{8}$
$(2~1~1)^T$	4	28	32	$\frac{1}{8}$
$(2~2~1)^T$	4	8	32	$\frac{1}{2}$
$(2~0~1)^T$	6	24	144	$\frac{1}{4}$
$(2~0~1)^T$	6	60	288	$\frac{1}{16}$
$(2~0~1)^T$	6	204	288	$\frac{1}{8}$
$(0~1~2)^T$	8	52	64	$\frac{1}{8}$
$(0~3~2)^T$	8	20	64	$\frac{1}{8}$
$(2~0~3)^T$	12	120	288	$\frac{1}{8}$
$(2~3~3)^T$	12	156	288	$\frac{1}{32}$
$(2~3~4)^T$	12	528	576	$\frac{1}{20}$
$(2~3~4)^T$	12	240	1152	$\frac{1}{32}$
$(2~3~4)^T$	12	816	1152	$\frac{1}{24}$
$(2~6~3)^T$	12	264	288	$\frac{1}{8}$

$\boldsymbol{h}$	N	m	M	d
$\boldsymbol{a} = (3\ 4\ 36)^T$				
$(0\ 0\ 1)^T$	3	9	27	$\frac{1}{4}$
$(0\ 0\ 1)^T$	3	18	27	$\frac{1}{2}$
$(0\ 1\ 1)^T$	4	8	16	$\frac{1}{4}$
$(2\ 1\ 1)^T$	4	20	32	$\frac{1}{8}$
$(0\ 3\ 1)^T$	6	72	432	$\frac{1}{2}$
$(0\ 3\ 1)^T$	6	180	864	$\frac{1}{4}$
$(0\ 3\ 1)^T$	6	360	432	$\frac{1}{4}$
$(0\ 3\ 1)^T$	6	468	864	$\frac{1}{8}$
$\boldsymbol{a} = (3\ 7\ 7)^T$				
$(1\ 0\ 0)^T$	3	3	9	$\frac{1}{2}$
$(1\ 1\ 1)^T$	4	1	8	$\frac{1}{8}$
$(1\ 3\ 3)^T$	6	21	36	$\frac{1}{2}$
$(0\ 1\ 2)^T$	7	35	49	$\frac{1}{8}$
$(0\ 1\ 3)^T$	7	21	49	$\frac{1}{8}$
$(0\ 2\ 3)^T$	7	42	49	$\frac{1}{8}$
$(7\ 1\ 3)^T$	14	21	196	$\frac{1}{8}$
$(7\ 1\ 5)^T$	14	133	196	$\frac{1}{8}$
$(7\ 3\ 5)^T$	14	189	196	$\frac{1}{8}$
$(7\ 3\ 6)^T$	21	21	441	$\frac{1}{16}$
$(7\ 3\ 9)^T$	21	336	441	$\frac{1}{16}$
$(7\ 6\ 9)^T$	21	84	441	$\frac{1}{16}$
$\boldsymbol{a} = (3\ 7\ 63)^T$				
$(0\ 0\ 1)^T$	3	9	27	$\frac{1}{4}$
$(0\ 0\ 1)^T$	3	18	27	$\frac{1}{2}$
$(1\ 1\ 1)^T$	4	1	8	$\frac{1}{8}$
$(3\ 3\ 1)^T$	6	9	108	$\frac{1}{4}$
$(3\ 3\ 1)^T$	6	45	108	$\frac{1}{2}$
$\boldsymbol{a} = (3\ 8\ 8)^T$				
$(1\ 0\ 0)^T$	3	3	9	$\frac{1}{2}$
$(0\ 0\ 1)^T$	4	8	64	$\frac{1}{4}$
$(0\ 0\ 1)^T$	4	56	64	$\frac{1}{4}$
$(0\ 1\ 2)^T$	4	24	64	$\frac{1}{4}$
$(0\ 1\ 2)^T$	4	40	64	$\frac{1}{4}$
$(2\ 1\ 1)^T$	4	28	32	$\frac{1}{4}$
$(2\ 0\ 3)^T$	6	84	144	$\frac{1}{4}$
$(2\ 0\ 3)^T$	6	120	144	$\frac{1}{4}$
$(4\ 1\ 2)^T$	8	88	128	$\frac{1}{8}$
$\boldsymbol{a} = (3\ 8\ 8)^T$ (cont.)				
$(4\ 3\ 2)^T$	8	24	128	$\frac{1}{8}$
$(2\ 3\ 3)^T$	12	156	288	$\frac{1}{8}$
$(4\ 0\ 3)^T$	12	120	576	$\frac{1}{8}$
$(4\ 0\ 3)^T$	12	264	576	$\frac{1}{8}$
$(4\ 3\ 6)^T$	12	408	576	$\frac{1}{8}$
$(4\ 3\ 6)^T$	12	552	576	$\frac{1}{8}$
$(4\ 3\ 6)^T$	24	408	1152	$\frac{1}{16}$
$(4\ 6\ 9)^T$	24	984	1152	$\frac{1}{16}$
$\boldsymbol{a} = (3\ 8\ 12)^T$				
$(1\ 0\ 1)^T$	3	6	9	$\frac{1}{4}$
$(0\ 1\ 0)^T$	4	8	64	$\frac{1}{2}$
$(0\ 1\ 0)^T$	4	56	64	$\frac{1}{8}$
$(0\ 1\ 1)^T$	4	4	16	$\frac{1}{8}$
$(0\ 1\ 2)^T$	4	40	64	$\frac{1}{2}$
$(0\ 1\ 2)^T$	4	56	64	$\frac{1}{8}$
$(2\ 1\ 0)^T$	4	20	32	$\frac{1}{4}$
$(2\ 1\ 2)^T$	4	4	32	$\frac{1}{4}$
$(2\ 0\ 1)^T$	6	24	576	$\frac{1}{4}$
$(2\ 0\ 1)^T$	6	60	144	$\frac{1}{8}$
$(2\ 0\ 1)^T$	6	312	576	$\frac{1}{8}$
$(2\ 3\ 2)^T$	6	132	144	$\frac{1}{8}$
$(2\ 3\ 2)^T$	6	168	288	$\frac{1}{4}$
$(4\ 3\ 2)^T$	8	40	128	$\frac{1}{4}$
$(2\ 3\ 2)^T$	6	312	576	$\frac{1}{8}$
$(4\ 1\ 2)^T$	8	104	128	$\frac{1}{4}$
$(2\ 3\ 2)^T$	12	132	288	$\frac{1}{16}$
$(2\ 3\ 4)^T$	12	276	288	$\frac{1}{16}$
$(4\ 3\ 2)^T$	12	168	576	$\frac{1}{8}$
$(4\ 3\ 2)^T$	12	312	576	$\frac{1}{32}$
$(4\ 3\ 4)^T$	12	312	576	$\frac{1}{32}$
$(4\ 3\ 4)^T$	12	456	576	$\frac{1}{8}$
$(4\ 3\ 6)^T$	24	552	1152	$\frac{1}{16}$
$(4\ 9\ 6)^T$	24	1128	1152	$\frac{1}{16}$
$\boldsymbol{a} = (3\ 8\ 48)^T$				
$(1\ 0\ 1)^T$	3	6	9	$\frac{1}{4}$
$(0\ 0\ 1)^T$	4	48	64	$\frac{1}{4}$
$(0\ 0\ 1)^T$	4	224	256	$\frac{1}{4}$
$\boldsymbol{a} = (3\ 8\ 48)^T$ (cont.)				
$(0\ 0\ 1)^T$	4	96	256	$\frac{1}{2}$
$(0\ 1\ 1)^T$	4	40	64	$\frac{1}{4}$
$(0\ 1\ 1)^T$	4	56	64	$\frac{1}{8}$
$(2\ 1\ 1)^T$	4	4	32	$\frac{1}{4}$
$(2\ 3\ 1)^T$	6	24	288	$\frac{1}{8}$
$(2\ 3\ 1)^T$	6	132	288	$\frac{1}{4}$
$(2\ 3\ 1)^T$	6	168	576	$\frac{1}{4}$
$(2\ 3\ 2)^T$	6	24	288	$\frac{1}{8}$
$(2\ 3\ 2)^T$	6	276	288	$\frac{1}{4}$
$(2\ 3\ 2)^T$	6	456	576	$\frac{1}{4}$
$(0\ 2\ 1)^T$	8	16	256	$\frac{1}{8}$
$(0\ 2\ 1)^T$	8	80	256	$\frac{1}{8}$
$(0\ 2\ 3)^T$	8	144	256	$\frac{1}{8}$
$(0\ 2\ 3)^T$	8	208	256	$\frac{1}{8}$
$\boldsymbol{a} = (3\ 8\ 72)^T$				
$(0\ 0\ 1)^T$	3	9	27	$\frac{1}{2}$
$(0\ 0\ 1)^T$	3	18	27	$\frac{1}{4}$
$(2\ 1\ 1)^T$	4	28	32	$\frac{1}{4}$
$\boldsymbol{a} = (3\ 10\ 30)^T$				
$(1\ 0\ 1)^T$	3	6	9	$\frac{1}{4}$
$(0\ 1\ 1)^T$	4	8	16	$\frac{1}{6}$
$(2\ 1\ 1)^T$	4	4	16	$\frac{1}{6}$
$(1\ 0\ 1)^T$	6	33	72	$\frac{1}{4}$
$(1\ 3\ 2)^T$	6	69	72	$\frac{1}{4}$
$(2\ 3\ 1)^T$	6	24	144	$\frac{1}{6}$
$(2\ 3\ 1)^T$	6	132	144	$\frac{1}{6}$
$\boldsymbol{a} = (5\ 6\ 15)^T$				
$(0\ 1\ 1)^T$	3	3	9	$\frac{1}{4}$
$(0\ 1\ 2)^T$	4	2	16	$\frac{1}{6}$
$(1\ 1\ 1)^T$	4	2	8	$\frac{1}{12}$
$(2\ 1\ 0)^T$	4	10	16	$\frac{1}{6}$
$(0\ 1\ 2)^T$	6	66	72	$\frac{1}{12}$
$(0\ 1\ 2)^T$	6	102	144	$\frac{1}{4}$
$(3\ 1\ 1)^T$	6	66	72	$\frac{1}{6}$
$(3\ 2\ 1)^T$	6	12	72	$\frac{1}{6}$
$\boldsymbol{a} = (5\ 8\ 40)^T$				
$(2\ 1\ 1)^T$	4	4	32	$\frac{1}{4}$

Appendix B. Tables for the constants $c_{\boldsymbol{a}}(n)$

In this appendix, we give the constants $c_{\boldsymbol{a}}(n)$ occurring in the lemmas in Section 4 for $n \equiv m \pmod{M}$. For this, define $\varepsilon_{m,d} := (-1)^{\delta_{d|m}}$.

$\boldsymbol{a}$	m	M	c
$(1\ 1\ 2)^T$	0	2	12
	1	2	4
$(1\ 1\ 4)^T$	0	4	12
	1	4	8
	2	4	4
$(1\ 1\ 8)^T$	$1, 4, 5$	8	4
	2	8	8
	0	8	12
$(1\ 1\ 9)^T$	1	3	8
	2	3	4
$(1\ 1\ 12)^T$	$\pm 2, 5$	8	2
	1	8	3
	0	4	$\frac{2}{3}$
$(1\ 1\ 21)^T$	$1, 2$	4	1
	0	4	$\frac{1}{3}$
	7	8	$\frac{1}{2}$
$(1\ 1\ 24)^T$	$1, 2, 5$	8	2
	4	8	1
	0	8	$\frac{1}{2}$
$(1\ 2\ 2)^T$	$1, 2$	4	4
	$0, 3$	4	12
$(1\ 2\ 3)^T$	0	2	3
	1	2	1
$(1\ 2\ 4)^T$	1	2	2
	$2, 4$	8	4
	$0, 6$	8	12
$(1\ 2\ 6)^T$	$2, 3$	4	1
	$0, 1$	4	3
$(1\ 2\ 8)^T$	$1, 4, 8, 9$	16	4
	2	4	2
	3	8	6
	$0, 12$	16	12
$(1\ 2\ 10)^T$	$1, 2$	4	1
	$0, 3$	4	-1
$(1\ 2\ 16)^T$	$1, 3, 4$	8	2
	6	8	6
	$0, 24$	32	12
	$2, 8, 16, 18$	32	4

$\boldsymbol{a}$	m	M	c
$(1\ 3\ 4)^T$	1	8	$\frac{3}{2}$
	5	8	1
	$0, 3$	4	2
$(1\ 3\ 6)^T$	1	2	1
	0	2	3
$(1\ 3\ 9)^T$	1	3	1
	0	3	2
$(1\ 3\ 12)^T$	$0, 1$	4	2
	7	8	1
	3	8	$\frac{3}{2}$
$(1\ 3\ 18)^T$	$1, 3$	6	$\varepsilon_{m,3}$
	$0, 4$	6	$3\varepsilon_{m,3}$
$(1\ 3\ 30)^T$	1	2	$\frac{1}{2}$
	0	2	$\frac{3}{2}$
$(1\ 4\ 4)^T$	1	4	4
	0	4	12
$(1\ 4\ 6)^T$	$0, 2$	8	-2
	$4, 6$	8	2
	1	2	$\frac{1}{2}$
$(1\ 4\ 8)^T$	1	4	2
	4	8	4
	0	8	12
$(1\ 4\ 12)^T$	1	8	$\frac{3}{2}$
	0	4	$\frac{2}{3}$
	5	8	1
$(1\ 4\ 24)^T$	1	4	$\frac{1}{2}$
	$0, 4$	8	$2\varepsilon_{m,8}$
$(1\ 5\ 8)^T$	$1, 5, 6$	8	$\frac{1}{2}$
	0	4	$\frac{1}{4}\varepsilon_{m,8}$
$(1\ 5\ 25)^T$	± 1	5	1
	0	5	-2
$(1\ 5\ 40)^T$	$1, 5, 6$	8	1
	0	4	$\frac{1}{2}\varepsilon_{m,8}$
$(1\ 6\ 6)^T$	$1, 2$	4	2
	$0, 3$	4	-2
$(1\ 6\ 9)^T$	$1, 4$	6	$\varepsilon_{m,2}$
	$0, 3$	6	$2\varepsilon_{m,2}$
$(1\ 6\ 16)^T$	$\pm 1, \pm 6, \pm 7$	16	1
	$\pm 4, \pm 12, 16, 24$	32	$\frac{1}{2}$

$\boldsymbol{a}$	m	M	c
$(1\ 6\ 16)^T$	$0, 8$	32	$-\frac{1}{2}$
$(1\ 6\ 18)^T$	$3, 6, 7, 10$	12	$\varepsilon_{m,3}$
	$0, 1, 4, 9$	12	$3\varepsilon_{m,3}$
$(1\ 6\ 24)^T$	1	8	1
	2	4	$\frac{1}{2}$
	$0, 7, 2, 15$	16	-2
	$4, 8$	16	2
$(1\ 8\ 8)^T$	$1, 4, 8, 9$	16	4
	$0, 12$	16	12
$(1\ 8\ 16)^T$	$1, \pm 4, 9, \pm 12, 17, 25$	32	2
	$8, 16$	32	4
	$0, 24$	32	12
$(1\ 8\ 40)^T$	1	8	$\frac{1}{2}$
	$4, 8$	16	1
	$0, 12$	16	-1
$(1\ 9\ 12)^T$	1	24	$\frac{3}{2}$
	4	12	$\frac{1}{3}$
	0	12	$\frac{2}{3}$
	$6, 18, 21$	24	2
	9	24	3
	$10, 13, 22$	24	1
$(1\ 9\ 21)^T$	1	3	$\frac{1}{2}$
	0	3	1
$(1\ 9\ 24)^T$	$1, 4, 13$	24	1
	$9, 12, 21$	24	2
	10	24	3
	18	24	6
	16	24	-1
	0	24	-2
$(1\ 10\ 30)^T$	$0, 1$	4	$\frac{3}{2}$
	$2, 3$	4	$\frac{1}{2}$
$(1\ 12\ 12)^T$	$0, 1$	4	$\varepsilon_{m,2}$
$(1\ 16\ 24)^T$	1	8	1
	$\pm 4, \pm 12, 16, 24$	32	1

$\boldsymbol{a}$	m	M	c
$(1\ 16\ 24)^T$	$0,8$	32	$-\frac{1}{2}$
$(1\ 24\ 24)^T$	1	8	1
	$4,8$	16	2
	$0,12$	16	-2
$(2\ 2\ 3)^T$	$2,3$	4	2
	$0,1$	4	-2
$(2\ 3\ 6)^T$	$1,2$	4	-1
	$0,3$	4	-3
$(2\ 3\ 8)^T$	2	4	$\frac{1}{2}$
	$0,4$ $5,13$	16	-2
	3	8	1
	$8,12$	16	2
$(2\ 3\ 9)^T$	$3,5$	6	$\varepsilon_{m,3}$
	$0,2$	6	$3\varepsilon_{m,3}$
$(2\ 3\ 12)^T$	1	2	$\frac{1}{2}$
	$2,4$	8	2
	$0,6$	8	-2
$(2\ 3\ 18)^T$	$2,11$	12	1
	$5,8$	12	-1
	$3,6$	12	2
	$0,9$	12	-2
$(2\ 3\ 48)^T$	$2,\pm 3,$ ± 5	16	1
	4	8	$\frac{1}{2}$
	$8,16$	32	$\frac{1}{2}$

$\boldsymbol{a}$	m	M	c
$(2\ 3\ 48)^T$	$0,24$	32	$-\frac{1}{2}$
$(2\ 5\ 6)^T$	$0,1$	4	$\frac{3}{2}$
	$2,3$	4	$\frac{1}{2}$
$(2\ 5\ 10)^T$	$0,3$	4	-1
	$1,2$	4	1
$(2\ 5\ 15)^T$	1	2	$\frac{1}{2}$
	0	2	$\frac{3}{2}$
$(2\ 6\ 9)^T$	$0,5$ $8,9$	12	$3\varepsilon_{m,3}$
	$2,3$	12	$\varepsilon_{m,3}$
	$6,11$	12	$\varepsilon_{m,3}$
$(2\ 6\ 15)^T$	$1,2$	4	$\frac{1}{2}$
	$0,3$	4	$\frac{3}{2}$
$(3\ 3\ 4)^T$	$2,7$	8	2
	3	8	3
	0	4	$\frac{2}{3}$
$(3\ 3\ 8)^T$	$3,6$ 7	8	2
	0	4	$\varepsilon_{m,8}$
$(3\ 4\ 4)^T$	$0,3$ $4,7$	8	$2\varepsilon_{m,4}$
$(3\ 4\ 12)^T$	3	8	6
	$0,4$ 7	8	2
$(3\ 4\ 36)^T$	$0,3$	12	$2\varepsilon_{m,2}$
	$4,7$	12	$\varepsilon_{m,2}$

$\boldsymbol{a}$	m	M	c
$(3\ 7\ 63)^T$	1	3	$\frac{1}{2}$
	0	3	1
$(3\ 8\ 8)^T$	3	8	1
	$0,4$	16	-2
	$8,12$	16	2
$(3\ 8\ 12)^T$	3	4	$\frac{1}{2}$
	$0,4$	8	$2\varepsilon_{m,8}$
$(3\ 8\ 48)^T$	3	8	$\frac{1}{2}$
	4	8	$\frac{1}{6}$
	$8,16$	32	$\frac{2}{3}$
	$0,24$	32	$-\frac{2}{3}$
$(3\ 8\ 72)^T$	3	24	$\frac{1}{3}$
	$8,44$	48	$\frac{1}{3}$
	$6,35$	48	$\frac{1}{6}$
	$12,24$	48	$\frac{2}{3}$
	$20,32$	48	$-\frac{1}{3}$
	$0,36$	48	$-\frac{2}{3}$
$(3\ 10\ 30)^T$	$1,2$	4	$\frac{1}{2}$
	$0,3$	4	$\frac{3}{2}$
$(5\ 6\ 15)^T$	1	2	$\frac{1}{2}$
	0	2	$\frac{3}{2}$
$(5\ 8\ 40)^T$	5	8	1
	$4,8$	16	$\frac{1}{3}$
	$0,12$	16	$-\frac{1}{3}$

Appendix C. Bounds for the valence formula

Here we give tables for the bounds obtained from the valence formula. We then list how many coefficients have been computed with a computer in order to obtain the claim. The calculations were done using GP/Pari and run in parallel on 6 cores split between one desktop (Dell Inspiron, i5 processor) and one laptop (Samsung NP900X3L, i5-6200U processor); the GP code may be found on the second author's website (see `https://hkumath.hku.hk/~bkane/papers/PeterssonQF/polygonal-Petersson.gp`). Most of the calculations took approximately 6 real-time hours (i.e., 36 core-hours) per $\boldsymbol{a}$, while those where the bound from the valence formula was larger than 10^8 required a longer calculation of approximately 2-3 real-time days (i.e., approximately 15 core-days) each.

$\boldsymbol{a}$	N	subgroup	coeff.	$\boldsymbol{a}$	N	subgroup	coeff.
$(1\ 1\ 1)^T$	$N \nmid 6$	$\Gamma_{256,16}$	384	$(1\ 1\ 4)^T$	$N \neq 12$	$\Gamma_{4096,64}$	24 576
	$N \mid 6$	$\Gamma_{576,24}$	1 152		$N = 12$	$\Gamma_{2304,48}$	9 216
$(1\ 1\ 2)^T$	$N \neq 12$	$\Gamma_{1024,32}$	3 072	$(1\ 1\ 5)^T$	all	$\Gamma_{40000,200}$	720 000
	$N = 12$	$\Gamma_{1152,24}$	2 304	$(1\ 1\ 6)^T$	$N \neq 8$	$\Gamma_{41472,144}$	497 664
$(1\ 1\ 3)^T$	all	$\Gamma_{20736,72}$	124 416		$N = 8$	$\Gamma_{3072,32}$	12 288

$\boldsymbol{a}$	N	subgroup	coeff.
$(1\ 1\ 8)^T$	all	$\Gamma_{16384,128}$	196 608
$(1\ 1\ 9)^T$	$N \neq 4$	$\Gamma_{104976,324}$	2 834 352
	$N = 4$	$\Gamma_{576,8}$	576
$(1\ 1\ 12)^T$	$N \nmid 6$	$\Gamma_{12288,64}$	98 304
	$N \mid 6$	$\Gamma_{5184,72}$	31 104
$(1\ 1\ 21)^T$	$N \nmid 14$	$\Gamma_{36288,72}$	248 832
	$N \mid 14$	$\Gamma_{460992,392}$	22 127 616
$(1\ 1\ 24)^T$	$N \neq 3$	$\Gamma_{49152,128}$	786 432
	$N = 3$	$\Gamma_{2592,9}$	3 888
$(1\ 2\ 2)^T$	$N \neq 6$	$\Gamma_{1024,32}$	3 072
	$N = 6$	$\Gamma_{576,24}$	1 152
$(1\ 2\ 3)^T$	all	$\Gamma_{20736,144}$	248 832
$(1\ 2\ 4)^T$	all	$\Gamma_{4096,64}$	24 756
$(1\ 2\ 5)^T$	$N \neq 8$	$\Gamma_{160000,400}$	5 760 000
	$N = 8$	$\Gamma_{5120,32}$	18 432
$(1\ 2\ 6)^T$	all	$\Gamma_{20736,144}$	248 832
$(1\ 2\ 8)^T$	$N \neq 12$	$\Gamma_{16384,128}$	196 608
	$N = 12$	$\Gamma_{9216,96}$	73 728
$(1\ 2\ 10)^T$	$N \neq 4$	$\Gamma_{40000,200}$	720 000
	$N = 4$	$\Gamma_{5120,32}$	18 432
$(1\ 2\ 16)^T$	all	$\Gamma_{65536,256}$	1 572 864
$(1\ 3\ 3)^T$	$N \neq 8$	$\Gamma_{5184,72}$	31 104
	$N = 8$	$\Gamma_{768,16}$	1 536
$(1\ 3\ 4)^T$	all	$\Gamma_{82944,288}$	1 990 656
$(1\ 3\ 6)^T$	all	$\Gamma_{20736,144}$	248 832
$(1\ 3\ 9)^T$	all	$\Gamma_{46656,216}$	839 808
$(1\ 3\ 10)^T$	$N \mid 12$	$\Gamma_{103680,144}$	1 492 992
	$N \mid 10$	$\Gamma_{120000,200}$	2 880 000
	$N = 15$	$\Gamma_{405000,225}$	14 580 000
$(1\ 3\ 12)^T$	all	$\Gamma_{82944,288}$	1 990 656
$(1\ 3\ 18)^T$	all	$\Gamma_{2304,16}$	4 608
$(1\ 3\ 30)^T$	$N \mid 10$	$\Gamma_{120000,200}$	2 880 000
	$N \mid 6$	$\Gamma_{25920,72}$	186 624
	$N = 4$	$\Gamma_{15360,32}$	73 728
$(1\ 4\ 4)^T$	$N \neq 8$	$\Gamma_{9216,96}$	73 728
	$N = 8$	$\Gamma_{4096,64}$	24 576
$(1\ 4\ 6)^T$	$N \nmid 8$	$\Gamma_{331776,576}$	15 925 248
	$N \mid 8$	$\Gamma_{12288,64}$	98 304
$(1\ 4\ 8)^T$	all	$\Gamma_{16384,128}$	196 608
$(1\ 4\ 12)^T$	$N \neq 8$	$\Gamma_{82944,288}$	1 990 656
	$N = 8$	$\Gamma_{12288,64}$	98 304
$(1\ 4\ 24)^T$	$N \nmid 6$	$\Gamma_{49152,128}$	786 432
	$N \mid 6$	$\Gamma_{331776,576}$	15 925 248
$(1\ 5\ 5)^T$	all	$\Gamma_{40000,200}$	720 000
$(1\ 5\ 8)^T$	$N \mid 8$	$\Gamma_{81920,128}$	1 179 648

$\boldsymbol{a}$	N	subgroup	coeff.
$(1\ 5\ 8)^T$	$N = 5$	$\Gamma_{20000,25}$	90 000
$(1\ 5\ 10)^T$	all	$\Gamma_{5120,32}$	18 432
$(1\ 5\ 25)^T$	all	$\Gamma_{1600,8}$	1 440
$(1\ 5\ 40)^T$	all	$\Gamma_{81920,128}$	1 179 648
$(1\ 6\ 6)^T$	$N \nmid 8$	$\Gamma_{5184,72}$	31 104
	$N \mid 8$	$\Gamma_{3072,32}$	12 288
$(1\ 6\ 9)^T$	$N \neq 8$	$\Gamma_{186624,432}$	6 718 464
	$N = 8$	$\Gamma_{9216,32}$	36 864
$(1\ 6\ 16)^T$	$N \mid 12$	$\Gamma_{331776,576}$	15 925 248
	$N = 8$	$\Gamma_{196608,256}$	6 291 456
$(1\ 6\ 18)^T$	all	$\Gamma_{2304,16}$	4 608
$(1\ 6\ 24)^T$	$N \neq 8$	$\Gamma_{82944,288}$	1 990 656
	$N = 8$	$\Gamma_{49152,128}$	786 432
$(1\ 8\ 8)^T$	all	$\Gamma_{16384,128}$	196 608
$(1\ 8\ 16)^T$	all	$\Gamma_{65536,256}$	1 572 864
$(1\ 8\ 40)^T$	$N = 4$	$\Gamma_{5120,32}$	18 432
	$N = 5$	$\Gamma_{20000,25}$	90 000
$(1\ 9\ 9)^T$	all	$\Gamma_{46656,216}$	839 808
$(1\ 9\ 12)^T$	all	$\Gamma_{746496,864}$	53 747 712
$(1\ 9\ 21)^T$	$N \mid 14$	$\Gamma_{345744,196}$	8 297 856
	$N \mid 6$	$\Gamma_{326592,216}$	6 728 464
	$N = 4$	$\Gamma_{4032,8}$	4 608
$(1\ 9\ 24)^T$	$N \mid 8$	$\Gamma_{147456,128}$	2 359 296
	$N = 3$	$\Gamma_{23328,27}$	104 976
$(1\ 10\ 30)^T$	all	$\Gamma_{103680,144}$	1 492 992
$(1\ 12\ 12)^T$	$N \nmid 8$	$\Gamma_{82944,288}$	1 990 656
	$N \mid 8$	$\Gamma_{49152,128}$	786 432
$(1\ 16\ 24)^T$	$N \mid 6$	$\Gamma_{331776,576}$	15 925 248
	$N \mid 8$	$\Gamma_{196608,256}$	6 291 456
$(1\ 21\ 21)^T$	$N \nmid 14$	$\Gamma_{36288,72}$	248 832
	$N \mid 14$	$\Gamma_{460992,392}$	22 127 616
$(1\ 24\ 24)^T$	$N \nmid 8$	$\Gamma_{20736,144}$	248 832
	$N \mid 8$	$\Gamma_{49152,128}$	786 432
$(2\ 2\ 3)^T$	all	$\Gamma_{82944,288}$	1 990 656
$(2\ 3\ 3)^T$	$N \neq 8$	$\Gamma_{20736,144}$	248 836
	$N = 8$	$\Gamma_{3072,32}$	12 288
$(2\ 3\ 6)^T$	all	$\Gamma_{20736,144}$	248 832
$(2\ 3\ 8)^T$	$N \mid 12$	$\Gamma_{82944,288}$	1 990 656
	$N = 8$	$\Gamma_{49152,128}$	786 432
$(2\ 3\ 9)^T$	all	$\Gamma_{2304,16}$	4 608
$(2\ 3\ 12)^T$	all	$\Gamma_{331776,576}$	15 925 248
$(2\ 3\ 18)^T$	$N \neq 4$	$\Gamma_{46656,216}$	839 808
	$N = 4$	$\Gamma_{9216,32}$	36 864
$(2\ 3\ 48)^T$	$N \neq 3$	$\Gamma_{196608,256}$	6 291 456
	$N = 3$	$\Gamma_{5184,9}$	7 776

$\boldsymbol{a}$	N	subgroup	coeff.
$(2\ 5\ 6)^T$	$N \mid 20$	$\Gamma_{480000,400}$	23 040 000
	$N \mid 6$	$\Gamma_{103680,144}$	1 492 992
	$N = 15$	$\Gamma_{40500,225}$	14 580 000
$(2\ 5\ 10)^T$	all	$\Gamma_{5120,32}$	18 432
$(2\ 5\ 15)^T$	all	$\Gamma_{103680,144}$	1 492 992
$(2\ 6\ 9)^T$	all	$\Gamma_{2304,16}$	4 608
$(2\ 6\ 15)^T$	$N \nmid 20$	$\Gamma_{103680,144}$	1 492 992
	$N \mid 20$	$\Gamma_{480000,400}$	23 040 000
$(3\ 3\ 4)^T$	all	$\Gamma_{331776,576}$	15 925 248
$(3\ 3\ 7)^T$	$N = 14$	$\Gamma_{115248,196}$	2 765 952
	$N \mid 21$	$\Gamma_{777924,441}$	56 010 528
	$N \in \{4, 6\}$	$\Gamma_{36288,72}$	248 832
$(3\ 3\ 8)^T$	$N \mid 8$	$\Gamma_{49152,128}$	786 432
	$N \in \{12, 24\}$	$\Gamma_{1327104,1152}$	127 401 984
$(3\ 4\ 4)^T$	all	$\Gamma_{1327104,1152}$	127 401 984
$(3\ 4\ 12)^T$	all	$\Gamma_{1327104,1152}$	127 401 984

$\boldsymbol{a}$	N	subgroup	coeff.
$(3\ 4\ 36)^T$	all	$\Gamma_{746496,864}$	53 747 712
$(3\ 7\ 7)^T$	$N \mid 21$	$\Gamma_{777924,441}$	56 010 528
	$N = 14$	$\Gamma_{115248,169}$	2 765 952
	$N = 4$	$\Gamma_{1344,8}$	1 536
	$N = 6$	$\Gamma_{9072,36}$	31 104
$(3\ 7\ 63)^T$	$N \neq 4$	$\Gamma_{81648,108}$	839 808
	$N = 4$	$\Gamma_{4032,8}$	4 608
$(3\ 8\ 8)^T$	all	$\Gamma_{1327104,1152}$	127 401 984
$(3\ 8\ 12)^T$	all	$\Gamma_{1327104,1152}$	127 401 984
$(3\ 8\ 48)^T$	$N \nmid 8$	$\Gamma_{331776,576}$	15 925 248
	$N \mid 8$	$\Gamma_{196608,256}$	6 291 456
$(3\ 8\ 72)^T$	$N = 3$	$\Gamma_{23328,27}$	104 976
	$N = 4$	$\Gamma_{9216,32}$	36 864
$(3\ 10\ 30)^T$	all	$\Gamma_{103680,144}$	1 492 992
$(5\ 6\ 15)^T$	all	$\Gamma_{103680,144}$	1 492 992
$(5\ 8\ 40)^T$	all	$\Gamma_{5120,32}$	18 432

Appendix D. Bounds for the proofs of the lemmas

Here we give the bounds from the valence formula calculations in Section 4.

$\boldsymbol{a}$	$(1\ 1\ 1)^T$	$(1\ 1\ 2)^T$	$(1\ 1\ 3)^T$	$(1\ 1\ 4)^T$	$(1\ 1\ 5)^T$	$(1\ 1\ 6)^T$	$(1\ 1\ 8)^T$
group	$\Gamma_0(8)$	$\Gamma_0(16)$	$\Gamma_0(24)$	$\Gamma_0(16)$	$\Gamma_0(40)$	$\Gamma_0(24)$	$\Gamma_0(64)$
coeff.	2	3	6	3	9	6	12
$\boldsymbol{a}$	$(1\ 1\ 9)^T$	$(1\ 1\ 12)^T$	$(1\ 1\ 21)^T$	$(1\ 1\ 24)^T$	$(1\ 2\ 2)^T$	$(1\ 2\ 3)^T$	$(1\ 2\ 4)^T$
group	$\Gamma_0(72)$	$\Gamma_0(192)$	$\Gamma_0(1344)$	$\Gamma_0(192)$	$\Gamma_0(16)$	$\Gamma_0(48)$	$\Gamma_0(64)$
coeff.	18	48	384	48	3	12	12
$\boldsymbol{a}$	$(1\ 2\ 5)^T$	$(1\ 2\ 6)^T$	$(1\ 2\ 8)^T$	$(1\ 2\ 10)^T$	$(1\ 2\ 16)^T$	$(1\ 3\ 3)^T$	$(1\ 3\ 4)^T$
group	$\Gamma_0(80)$	$\Gamma_0(48)$	$\Gamma_0(256)$	$\Gamma_0(80)$	$\Gamma_{1024,32}$	$\Gamma_0(24)$	$\Gamma_0(192)$
coeff.	18	12	384	18	3 072	6	48
$\boldsymbol{a}$	$(1\ 3\ 6)^T$	$(1\ 3\ 9)^T$	$(1\ 3\ 10)^T$	$(1\ 3\ 12)^T$	$(1\ 3\ 18)^T$	$(1\ 3\ 30)^T$	$(1\ 4\ 4)^T$
group	$\Gamma_0(48)$	$\Gamma_0(72)$	$\Gamma_0(120)$	$\Gamma_0(192)$	$\Gamma_0(144)$	$\Gamma_0(240)$	$\Gamma_0(16)$
coeff.	12	18	36	48	36	72	3
$\boldsymbol{a}$	$(1\ 4\ 6)^T$	$(1\ 4\ 8)^T$	$(1\ 4\ 12)^T$	$(1\ 4\ 24)^T$	$(1\ 5\ 5)^T$	$(1\ 5\ 8)^T$	$(1\ 5\ 10)^T$
group	$\Gamma_0(192)$	$\Gamma_0(64)$	$\Gamma_0(192)$	$\Gamma_0(192)$	$\Gamma_0(40)$	$\Gamma_0(320)$	$\Gamma_0(80)$
coeff.	48	12	48	48	9	72	18
$\boldsymbol{a}$	$(1\ 5\ 25)^T$	$(1\ 5\ 40)^T$	$(1\ 6\ 6)^T$	$(1\ 6\ 9)^T$	$(1\ 6\ 16)^T$	$(1\ 6\ 18)^T$	$(1\ 6\ 24)^T$
group	$\Gamma_{200,5}$	$\Gamma_0(320)$	$\Gamma_0(48)$	$\Gamma_0(144)$	$\Gamma_{3072,32}$	$\Gamma_0(144)$	$\Gamma_{768,16}$
coeff.	180	72	12	36	12 288	36	1 536
$\boldsymbol{a}$	$(1\ 8\ 8)^T$	$(1\ 8\ 16)^T$	$(1\ 8\ 40)^T$	$(1\ 9\ 9)^T$	$(1\ 9\ 12)^T$	$(1\ 9\ 21)^T$	$(1\ 9\ 24)^T$
group	$\Gamma_{256,16}$	$\Gamma_{1024,32}$	$\Gamma_{1280,16}$	$\Gamma_0(72)$	$\Gamma_0(576)$	$\Gamma_0(504)$	$\Gamma_0(576)$
coeff.	512	3 072	2 304	18	144	144	144
$\boldsymbol{a}$	$(1\ 10\ 30)^T$	$(1\ 12\ 12)^T$	$(1\ 16\ 24)^T$	$(1\ 21\ 21)^T$	$(1\ 24\ 24)^T$	$(2\ 2\ 3)^T$	$(2\ 3\ 3)^T$
group	$\Gamma_0(240)$	$\Gamma_0(48)$	$\Gamma_{3072,32}$	$\Gamma_0(168)$	$\Gamma_{768,16}$	$\Gamma_0(48)$	$\Gamma_0(48)$
coeff.	72	12	12 288	48	1 536	12	12

$\boldsymbol{a}$	$(2\ 3\ 6)^T$	$(2\ 3\ 8)^T$	$(2\ 3\ 9)^T$	$(2\ 3\ 12)^T$	$(2\ 3\ 18)^T$	$(2\ 3\ 48)^T$	$(2\ 5\ 6)^T$
group	$\Gamma_0(48)$	$\Gamma_{768,16}$	$\Gamma_0(144)$	$\Gamma_0(192)$	$\Gamma_0(144)$	$\Gamma_{3072,32}$	$\Gamma_0(240)$
coeff.	12	1 536	36	48	36	12 288	72
$\boldsymbol{a}$	$(2\ 5\ 10)^T$	$(2\ 5\ 15)^T$	$(2\ 6\ 9)^T$	$(2\ 6\ 15)^T$	$(3\ 3\ 4)^T$	$(3\ 3\ 7)^T$	$(3\ 3\ 8)^T$
group	$\Gamma_0(80)$	$\Gamma_0(240)$	$\Gamma_0(576)$	$\Gamma_0(240)$	$\Gamma_0(192)$	$\Gamma_0(168)$	$\Gamma_0(192)$
coeff.	18	72	144	72	48	48	48
$\boldsymbol{a}$	$(3\ 4\ 4)^T$	$(3\ 4\ 12)^T$	$(3\ 4\ 36)^T$	$(3\ 7\ 7)^T$	$(3\ 7\ 63)^T$	$(3\ 8\ 8)^T$	$(3\ 8\ 12)^T$
group	$\Gamma_0(192)$	$\Gamma_0(192)$	$\Gamma_0(144)$	$\Gamma_0(168)$	$\Gamma_0(504)$	$\Gamma_{768,16}$	$\Gamma_0(192)$
coeff.	48	48	36	48	144	1 536	48
$\boldsymbol{a}$	$(3\ 8\ 48)^T$	$(3\ 8\ 72)^T$	$(3\ 10\ 30)^T$	$(5\ 6\ 15)^T$	$(5\ 8\ 40)^T$		
group	$\Gamma_{3072,32}$	$\Gamma_{2304,48}$	$\Gamma_0(240)$	$\Gamma_0(240)$	$\Gamma_{1280,16}$		
coeff.	12 288	9 216	72	72	2 304		

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